The friable mean value of Erdős-Hooley's function

Journées arithmétiques Nancy

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Erdős-Hooley's function

Erdős-Hooley's function (Hooley, 1979)

Let $n \geqslant 1$,

$$\Delta(n,v) := \sum_{\substack{d|n \\ \mathrm{e}^v < d \leqslant \mathrm{e}^{v+1}}} 1 \quad (v \in \mathbb{R}), \quad \Delta(n) := \sup_{v \in \mathbb{R}} \Delta(n,v).$$

Arithmetic function which measures the logarithmic concentration of the set of divisors of an integer.

Recall that we denote by τ the function "number of divisors". We have the following trivial lower and upper bound

$$\max\left(1; \left\lfloor \frac{\tau(n)}{\log n} \right\rfloor\right) \leqslant \Delta(n) \leqslant \tau(n) \quad (n > 1).$$

We are interested in the mean value of such a function, i.e. the quantity

$$\frac{1}{x}\sum_{n\leqslant x}\Delta(n).$$

Its order of magnitude is not known exactly at this time.

Hall & Tenenbaum (1982)

$$\frac{1}{x}\sum_{n\leqslant x}\Delta(n)\gg\log_2 x\qquad(x\geqslant 3).$$

Hooley (1979)

$$\frac{1}{x}\sum_{n\leqslant x}\Delta(n)\ll(\log x)^{4/\pi-1}\qquad(x\geqslant 3).$$

- Tenenbaum (1985): There exists $c_0 > 0$ such that

$$\frac{1}{x}\sum_{n\leqslant x}\Delta(n) \ll \exp(c_0\sqrt{\log_2 x \log_3 x}) \qquad (x \geqslant 16).$$

- La Bretèche & Tenenbaum (2022): Given any $a > \sqrt{2}\log 2 \approx 0.98026$, we have

$$\frac{1}{x}\sum_{n\leqslant x}\Delta(n) \ll \exp(a\sqrt{\log_2 x}) \qquad (x \geqslant 3).$$

- Koukoulopoulos & Tao (2023): We have

$$\frac{1}{x}\sum_{n\leqslant x}\Delta(n) \ll (\log_2 x)^{11/4} \qquad (x \geqslant 3).$$

Friable integers

Definition

An integer $n > 1$ is said to be y-friable if its largest prime factor, denoted $P^+(n)$, does not exceed y.

$$n = \underbrace{p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}.}_{\text{prime factorization}}$$

The integer n is y-friable $\iff p_j \leqslant y \quad \forall j \in \{1, \cdots, k\}$.

We traditionally set

$$S(x,y) := \{n : n \leqslant x, P(n) \leqslant y\},$$

and also

$$\Psi(x,y) := |S(x,y)|.$$

Remark

When $y \geqslant x$, $\Psi(x,y) = \lfloor x \rfloor$.

Typical problem: study the average behaviour of an arithmetic function over $S(x,y)$.
Question: What is the proportion of y-friable integers among the integers less than x ?

Definition

The Dickman function ϱ is the unique function, continuous on $]0,\infty[$, differentiable on $]1,\infty[$ satisfying

$$\begin{cases} \varrho(u) = 1 & (0 \leqslant u \leqslant 1), \\ u\varrho'(u) + \varrho(u-1) = 0 & (u > 1). \end{cases}$$

The function ϱ has over-exponential decay.

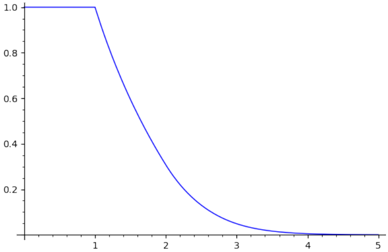

Figure: Graph of the function $\varrho(u)$

Dickman (1930)

For all fixed $u > 0$,

$$\lim_{x \to +\infty} \frac{\Psi(x, x^{1/u})}{x} = \varrho(u).$$

$\varrho(u)$ is the probability that an integer $\leqslant x$ is $x^{1/u}$-friable.

This still holds when $u = \dfrac{\log x}{\log y}$ tends to infinity and it is possible to get an asymptotic formula.

Let $\varepsilon > 0$, we define the following domain

$$H_\varepsilon := \Big\{(x,y) : \; x \geqslant 3, \; \exp\Big((\log_2 x)^{5/3+\varepsilon}\Big) \leqslant y \leqslant x\Big\}.$$

Notation $u := \dfrac{\log x}{\log y} \qquad (x \geqslant y \geqslant 2).$

Hildebrand (1986)

For all $\varepsilon > 0$ and uniformly for $(x,y) \in H_\varepsilon$, we have

$$\Psi(x,y) = x\varrho(u)\Big\{1 + O\Big(\frac{\log(u+1)}{\log y}\Big)\Big\}.$$

H_ε is the largest domain in which the previous result is known to be true.

Let

$$\zeta(s, y) := \prod_{p \leqslant y} \left(1 - \frac{1}{p^s}\right)^{-1},$$

the Dirichlet series of the indicator function of y-integers. By Perron's formula we have for $x \notin \mathbb{N}$, $\alpha > 0$,

$$\Psi(x, y) = \frac{1}{2i\pi} \int_{\alpha - i\infty}^{\alpha + i\infty} \zeta(s, y) \frac{x^s}{s} \, \mathrm{d}s.$$

Let

$$\varphi_y(s) := -\frac{\zeta'(s,y)}{\zeta(s,y)} = \sum_{p \leqslant y} \frac{\log p}{p^s - 1} \quad (y \geqslant 2,\ \Re s > 0).$$

$\alpha = \alpha(x,y)$ the unique real positive solution of $\varphi_y(\alpha) = \log x$ is the saddle-point which appears using the saddle-point method.

$$u := \frac{\log x}{\log y} \quad (x \geqslant y \geqslant 2)\,; \quad \bar{u} := \min\left(\frac{y}{\log y}, u\right).$$

Hildebrand & Tenenbaum (1986)

Uniformly for $x \geqslant y \geqslant 2$, we have

$$\Psi(x,y) = \frac{x^\alpha \zeta(\alpha,y)}{\alpha\sqrt{2\pi|\varphi_y'(\alpha)|}}\left\{1 + O\left(\frac{1}{\bar{u}}\right)\right\}.$$

The friable mean value of Erdős-Hooley's function

Define $\Psi(x, y; f) := \sum_{n \in S(x,y)} f(n)$.
Goal : estimate

$$\mathfrak{S}(x, y) := \frac{\Psi(x, y; \Delta)}{\Psi(x, y)} \qquad (x \geqslant y \geqslant 2).$$

Let $\varrho_2(u) = \int_0^u \varrho(v)\varrho(u - v)\,\mathrm{d}v = 2^{u+O(u/\log 2u)}\varrho(u)$. The trivial bounds $\max(1, \lfloor\tau(n)/\log n\rfloor) \leqslant \Delta(n) \leqslant \tau(n)$ and the estimate

$$\Psi(x, y; \tau) \sim x\varrho_2(u)\log y \quad (x \to \infty, (x, y) \in H_\varepsilon)$$

due to Tenenbaum and Wu (2003) give

$$\frac{2^{u+O(u/\log 2u)}}{\sqrt{u}} \ll \mathfrak{S}(x, y) \ll 2^{u+O(u/\log 2u)} \log y \qquad ((x, y) \in H_\varepsilon).$$

Result on H_ε

$$\frac{2^{u+O(u/\log 2u)}}{\sqrt{u}} \ll \mathfrak{S}(x,y) \ll 2^{u+O(u/\log 2u)} \log y \qquad ((x,y)\in H_\varepsilon).$$

Theorem (Martin, Tenenbaum, W. (2023))

Let $\varepsilon > 0$. *For* $(x,y)\in H_\varepsilon$, *we have*

$$\log_2 y + 2^{u+O(u/\log 2u)} \ll \mathfrak{S}(x,y) \ll 2^{u+O(u/\log 2u)} \mathrm{e}^{c\sqrt{\log_2 y \log_3 y}}.$$

Remark

For $\exp\left((\log_2 x)^{5/3+\varepsilon}\right) < y < x^{1/(2\log_3 x)}$, *we have*

$$\mathfrak{S}(x,y) = 2^{u+O(u/\log 2u)}.$$

A first lower bound in H_ε

Recall that

$$\mathfrak{S}(x,y) \gg \frac{2^{u+O(u/\log 2u)}}{\sqrt{u}} \qquad \big((x,y) \in H_\varepsilon\big).$$

Proposition 1

Let $\varepsilon > 0$. Uniformly for $(x,y) \in H_\varepsilon$, we have

$$\mathfrak{S}(x,y) \gg 2^{u+O(u/\log 2u)}.$$

- Probabilistic argument: we introduce a random variable which takes the values $\log d$, when d goes through the set of divisors of n, with uniform probability $1/\tau(n)$. We denote by σ_n^2 its variance.
- By Bienaymé-Tchebychev's inequality, we get

$$\frac{1}{\tau(n)} \sum_{\substack{d|n \\ |\log d - \frac{1}{2}\log n| \geqslant 2\sigma_n}} 1 \leqslant \tfrac{1}{4}.$$

- Then, we have for $n \geqslant 1$

$$\Delta(n) \geqslant \frac{3\tau(n)}{16\sigma_n + 4}$$

- Restriction to square-free integers. Uniformly for $x \geqslant y \geqslant 2$, we have

$$\sum_{n \in S(x,y)} \Delta(n) \gg \frac{1}{\sqrt{\log y \log(2x)}} \sum_{n \in S(x,y)} \tau(n)\mu(n)^2.$$

The above result is not good when u is too small.

Proposition 2

Uniformly for $(x, y) \in \{x \geqslant y > x^{1/(2\log_3 x)}\}$, *we have*

$$\mathfrak{S}(x, y) \gg \log_2 y + 2^{u+O(u/\log 2u)}.$$

Hall & Tenenbaum (1988)

$$\Delta(n) \geqslant \frac{1}{2\tau(n)} \sum_{\substack{dd'|n \\ |\log(d/d')|\leqslant 1}} 1 \quad (n > 0).$$

This yields

$$\sum_{n\in S(x,y)} \Delta(n) \gg \sum_{\substack{d\leqslant\sqrt{x} \\ P(d)\leqslant y}} 2^{-\Omega(d)}\Psi\Big(\frac{x}{d^2}, y\Big)\big(S_{d,y}(2d) - S_{d,y}(d)\big),$$

where

$$S_{d,y}(D) := \sum_{\substack{(d,d')=1, P(d')\leqslant y \\ d'\leqslant D}} 2^{-\Omega(d')}.$$

Set

$$C_\kappa(f) := \prod_p (1 - 1/p)^\kappa \sum_{\nu \geqslant 0} f(p^\nu)/p^\nu.$$

We use a theorem of Tenenbaum and Wu. If f is a multiplicative function satisfying some hypotheses, we have the following asymptotic formula for $(x, y) \in H_\varepsilon$:

$$\Psi(x, y; f) := \sum_{n \in S(x,y)} f(n) = C_\kappa(f) x \varrho_\kappa(u) (\log y)^{\kappa - 1} \{1 + R\},$$

where R is an error term.
To handle the condition $(d, d') = 1$, we need a parametrized version of this theorem with some uniformity in f.

We have

$$S_{d,y}(2d) - S_{d,y}(d) = \sum_{\substack{(d,d')=1, P(d')\leqslant y \\ d<d'\leqslant D}} 2^{-\Omega(d')}.$$

An effective lower bound for this furnishes

$$\mathfrak{S}(x,y) \gg \frac{1}{\varrho(u)\sqrt{\log y}} \sum_{\substack{d\leqslant\sqrt{x} \\ P(d)\leqslant y}} \frac{\varrho(u-2u_d)\varrho_{1/2}(u_d)}{d2^{\Omega(d)}}\left(\frac{\varphi(d)}{d}\right)^{1/2}.$$

Partial summation leads to the stated lower bound.

What about an upper bound in H_ε ?

Recall the trivial milestone

$$\mathfrak{S}(x,y) \ll 2^{u+O(u/\log 2u)} \log y \qquad \big((x,y)\in H_\varepsilon\big).$$

Theorem (Martin, Tenenbaum, W. (2023))

Let $\varepsilon > 0$. For $(x,y)\in H_\varepsilon$, we have

$$\mathfrak{S}(x,y) \ll 2^{u+O(u/\log 2u)} \mathrm{e}^{c\sqrt{\log_2 y \log_3 y}}.$$

We adapt the method developed by Tenenbaum in 1985 to the friable case. Let

$$\Delta(n,v) := |\{d : d \mid n,\ \mathrm{e}^v < d \leqslant \mathrm{e}^{v+1}\}| \qquad (n\geqslant 1,\ v\in\mathbb{R}).$$

Note that

$$\Delta(n) := \max_{v\in\mathbb{R}} \Delta(n,v).$$

Let

$$M_q(n) = \int_{\mathbb{R}} \Delta(n; v)^q du \qquad (q \geqslant 1).$$

- We have $\lim_{q \to \infty} M_q(n)^{1/q} = \Delta(n)$.
- We denote by n_k the product of the first k prime factors of n. Then, we evaluate

$$L(\sigma; k, q) := \sum_{n \geqslant 1} \mu(n)^2 \frac{M_q(n_k)^{1/q}}{n^\sigma} \qquad (\sigma > 1).$$

In the friable case we evaluate

$$L(\sigma; k, q, y) := \sum_{P(n) \leqslant y} \mu(n)^2 \frac{M_q(n_k)^{1/q}}{n^\beta},$$

where β is the saddle-point related to the friable mean-value of $\tau(n)$.

Result on $\{2 \leqslant y \leqslant \exp\left((\log_2 x)^2\right)\}$

Let

$$g(t) := \log\left\{\frac{(1+2t)^{1+2t}}{(1+t)^{1+t}(4t)^t}\right\} \qquad (t > 0).$$

Note that g is positive and strictly increasing on $(0, +\infty)$. The asymptotic behaviour of this function is given by

$$g(t) = \begin{cases} \log 2 - \dfrac{1}{4t} + O\Big(\dfrac{1}{t^2}\Big) & \text{as } t \to \infty, \\ t \log\Big(\dfrac{1}{t}\Big) - t(\log 4 - 1) + O(t^2) & \text{as } t \to 0. \end{cases}$$

Theorem (Martin, Tenenbaum, W. (2023))

For $2 \leqslant y \leqslant \exp\left((\log_2 x)^2\right)$, with $\varepsilon_y := 1/\sqrt{\log y}$ and $\lambda = y/\log x$, we have

$$\mathfrak{S}(x,y) = \left(\frac{\Psi(x,y;\tau)}{\Psi(x,y)}\right)^{1+O(\varepsilon_y)} \asymp \mathrm{e}^{ug(\lambda)\left(1+O(\varepsilon_y)\right)}.$$

Idea of the proof on $\{2 \leqslant y \leqslant \mathrm{e}^{(\log_2 x)^2}\}$

Theorem (Martin, Tenenbaum, W. (2023))

For $2 \leqslant y \leqslant \exp\big((\log_2 x)^2\big)$, *with* $\varepsilon_y := 1/\sqrt{\log y}$ *and* $\lambda = y/\log x$, *we have*

$$\mathfrak{S}(x,y) = \Big(\frac{\Psi(x,y;\tau)}{\Psi(x,y)}\Big)^{1+O(\varepsilon_y)} \asymp \mathrm{e}^{ug(\lambda)\left(1+O(\varepsilon_y)\right)}.$$

- Recall that $\max\Big(1; \Big\lfloor \dfrac{\tau(n)}{\log n} \Big\rfloor\Big) \leqslant \Delta(n) \leqslant \tau(n) \quad (n > 1)$.
- In this domain, the factor $1/\log n$ is negligible. We have

$$\mathfrak{S}(x,y) = \Big(\frac{\Psi(x,y;\tau)}{\Psi(x,y)}\Big)^{1+O(\varepsilon_y)} \qquad (2 \leqslant y \leqslant \mathrm{e}^{(\log_2 x)^2}).$$

- From Drappeau (2016) and Tenenbaum (2022) we have

$$\frac{\Psi(x,y;\tau)}{\Psi(x,y)} \asymp \zeta(\alpha,y)\mathrm{e}^{-\overline{u}(\log 4-1)\{1+o(1)\}}.$$

A refinement gives the stated conclusion.