Global series for height 1 multiple zeta functions

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July 4, 2023

Euler's constant

The *Euler constant* $\gamma = 0.5772156649\cdots$, given by

$$\gamma = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{1}{j} - \log n\right) = \lim_{s\to 0}\left(\Gamma(s) - \frac{1}{s}\right) = -\lim_{s\to 1}\left(\zeta(s) - \frac{1}{s-1}\right),$$

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- It has classical series representations

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- Natural generalizations of these classical series describe the singular behavior of *height 1 multiple zeta functions* and the Ramanujan summation of *multiple harmonic star sums*.

For positive integers $s_2, \ldots, s_j$, the *multiple zeta function* $\zeta(s, s_2, \ldots, s_j)$ may be considered as a single-variable function defined for $\Re(s) > 1$ by

$$\zeta(s, s_2, \ldots, s_j) = \sum_{n_1 > n_2 > \cdots > n_j > 0} \frac{1}{n_1^s n_2^{s_2} \cdots n_j^{s_j}}.$$

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- When $s = s_1 > 1$ is an integer, the value $\zeta(s_1, s_2, \ldots, s_j)$ is known as a *multiple zeta value* of weight $s_1 + \cdots + s_j$, of depth j, and of height $\#\{i : s_i > 1\}$.

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- The *height 1 zeta function* $\zeta(s, \{1\}^{j-1}) := \zeta(s, \underbrace{1, \ldots, 1}_{j-1})$ of depth j has a meromorphic continuation to $\mathbb{C}$ with poles among $s = 1, 0, -1, -2, \ldots$. We will determine the coefficients $\gamma_i^{[j]}(k)$ in the Laurent series

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$$\zeta(s, \{1\}^{j-1}) = \sum_{i=-N}^{-1} \gamma_i^{[j]}(k)(s-k)^i + \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \gamma_i^{[j]}(k)(s-k)^i$$

for all degrees $i \leq 1$ and all poles $k = 1, 0, -1, -2, \ldots$; we will refer to the coefficients $\gamma_i^{[j]}(k)$ for $i \geq 0$ as "height 1 Stieltjes constants". The poles at $s = 0$ and at $s = 1$ seem to be the most interesting.

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$$\zeta(s) = \frac{1}{s-1} + \sum_{i=0}^{\infty} \frac{(-1)^i \gamma_i}{i!} (s-1)^i,$$

  for all degrees $i \leq 1$ and all poles $k = 1, 0, -1, -2, \ldots$; we will refer to the coefficients $\gamma_i^{[j]}(k)$ for $i \geq 0$ as "height 1 Stieltjes constants". The poles at $s = 0$ and at $s = 1$ seem to be the most interesting.
- Thus the classical Stieltjes constants are denoted $\gamma_i := \gamma_i^{[1]}(1)$.

Theorem (2022)

For any positive integer j, the series representation

$$\zeta(s+1,\{1\}^{j-1}) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}}{n!(n+s)^{j+1}}$$

is absolutely convergent for all $s \in \mathbb{C}$, except where s is zero or a negative integer, and provides a meromorphic continuation of $\zeta(s+1,\{1\}^{j-1})$ to the entire complex plane. Alternately, for each nonnegative integer j we have

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- The coefficients of these global series are complex-order Bernoulli polynomials, defined by

$$\left(\frac{t}{e^t-1}\right)^z e^{xt} = \sum_{n=0}^{\infty} B_n^{(z)}(x)\frac{t^n}{n!}.$$

Sketch of proof

For $\Re(a) > 0$ the *multiple Hurwitz zeta function* $\zeta_r(s, a)$ of order r may be defined by the expressions

$$\zeta_r(s, a) = \sum_{m=0}^{\infty} \binom{m + r - 1}{m} (m + a)^{-s} = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}(a)}{n!(n + s - r)},$$

where the first converges for $\Re(s) > r$, while the second is convergent for all $s \in \mathbb{C}$.

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- Using these expressions, we evaluate

$$D_r^j \zeta_r(s,a)\Big|_{r=0,a\to 0} = j!\zeta(s+1,\{1\}^{j-1}) = \frac{j!}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{(-1)^n B_n^{(n+s)}}{n!(n+s)^{j+1}},$$

and a similar evaluation at $(r,a) = (1,1)$.

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- This also exhibits the multiple Hurwitz zeta function $\zeta_r(s,1)$ as the ordinary generating function of the sequence $\{\zeta(s,\{1\}^j)\}_{j=0}^{\infty}$ of height 1 zeta functions.

Height 1 zeta functions at $s = 1$

Corollary (2022)

For any positive integer j, the height 1 zeta function $\zeta(s, \{1\}^{j-1})$ of depth j has a pole of order j at $s = 1$. The singular part (degree $i \leq 0$) of its Laurent series is described by

$$s^j \zeta(s+1, \{1\}^{j-1}) \equiv \Gamma(s+1)^{-1} \pmod{s^{j+1}\mathbb{C}[[s]]},$$

and the linear coefficient is given by the series

$$-\gamma_1^{[j]}(1) = \sum_{n=1}^{\infty} |b_n| H_{n-1}^{(j+1)} + [s^{j+1}](\Gamma(s+1)^{-1}),$$

where $H_n^{(m)} := \sum_{k=1}^{n} 1/k^m$ is the generalized harmonic number.

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Sketch of proof. Start with the global series for $\zeta(s+1, \{1\}^{j-1})$, multiply both sides by $\Gamma(s)$, subtract the (singular) $n = 0$ term. Then the limit as $s \to 0$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^n B_n^{(n)}}{n! n^{j+1}} = \sum_{n=1}^{\infty} \sum_{k=0}^{n} (-1)^k b_k \frac{1}{n^{j+1}} = \sum_{k=0}^{\infty} (-1)^k b_k \sum_{n=\max(k,1)}^{\infty} \frac{1}{n^{j+1}}$$

$$= \sum_{k=0}^{\infty} (-1)^k b_k (\zeta(j+1) - H_{k-1}^{(j+1)}) = \sum_{k=1}^{\infty} |b_k| H_{k-1}^{(j+1)}.$$

Stieltjes constants at $s = 1$

For $j = 1, 2$ we have the slowly convergent series

$$\sum_{n=1}^{\infty} |b_n| H_{n-1}^{(2)} = -\gamma_1 - \frac{\gamma^2}{2} + \frac{\pi^2}{12} \qquad (\gamma_1 = \gamma_1^{[1]}(1)),$$

$$\sum_{n=1}^{\infty} |b_n| H_{n-1}^{(3)} = -\gamma_1^{[2]}(1) - \frac{\gamma^3}{6} + \frac{\gamma\pi^2}{12} - \frac{\zeta(3)}{3}.$$

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- Coppo and Candelpergher recently gave an evaluation equivalent to

$$\gamma_1^{[2]}(1) = \frac{1}{2}K_2 - \frac{\gamma_2}{2} + \frac{\gamma\pi^2}{12} - \frac{\gamma^3}{2} - \gamma\gamma_1,$$

where K_n is defined by

$$K_n = \frac{i\pi}{2} \int_{-1}^{1} x \mathrm{Log}^n(\mathrm{Log}(1 + e^{i\pi x}))dx.$$

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- We also note that $\gamma_1^{[j]}(1) < 0$ for all j, with

$$\lim_{j\to\infty} \gamma_1^{[j]}(1) = -|b_1| = -\frac{1}{2},$$

"Stieltjes constants" at $s = 0$

Corollary (2022)

For any positive integer j, the height 1 zeta function $\zeta(s,\{1\}^{j-1})$ of depth j has a pole of order $j-1$ at $s=0$. The singular part (degree $i \leq 0$) of its Laurent series is described by

$$s^{j-1}\zeta(s,\{1\}^{j-1}) \equiv \frac{s-1}{2\Gamma(s+1)} \pmod{s^j\mathbb{C}[[s]]},$$

and the linear coefficient is given by the series

$$-\gamma_1^{[j]}(0) = (-1)^j - \sum_{n=2}^{\infty} \frac{|b_n|}{(n-1)^j} + [s^j]\left(\frac{s-1}{2\Gamma(s+1)}\right).$$

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Sketch of proof. Use the second of the global series from the theorem; this time it is the $n = 1$ term that is singular, use similar identities, such as

$$b_n = \frac{B_n^{(n)}(1)}{n!}, \qquad B_1^{(s+1)}(1) = \frac{1-s}{2}.$$

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- For $j = 1$ this is a shifted version of the classical Mascheroni series; such shifted series are intimately connected to the values $\zeta'(-k)$ for integers k, and with the Ramanujan summation of hyperharmonic numbers.

“Stieltjes constants” at $s = 0$

For depth $j = 1$, $\zeta(s)$ has no pole at $s = 0$, but this corollary gives $\zeta(0) = -1/2$ and

$$\gamma_1^{[1]}(0) = -\zeta'(0) = \log\sqrt{2\pi} = \sum_{n=2}^{\infty} \frac{|b_n|}{n-1} + \frac{\gamma}{2} + \frac{1}{2}.$$

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$$\gamma_{1-j}^{[j]}(0) = -\frac{1}{2}, \qquad \gamma_{2-j}^{[j]}(0) = \frac{1}{2} - \frac{\gamma}{2},$$

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- We calculate

$$\gamma_1^{[2]}(0) = \sum_{n=2}^{\infty} \frac{|b_n|}{(n-1)^2} - 1 + \frac{\gamma^2}{4} - \frac{\gamma}{2} - \frac{\pi^2}{24} = -1.5171198\cdots,$$

$$\gamma_1^{[3]}(0) = \sum_{n=2}^{\infty} \frac{|b_n|}{(n-1)^3} + 1 - \frac{\gamma^2}{4} - \frac{\gamma\pi^2}{24} + \frac{\pi^2}{24} + \frac{\gamma^3}{12} + \frac{\zeta(3)}{6} = 1.3969896\cdots,$$

  and we have $(-1)^j \gamma_1^{[j]}(0) < 0$ for all j, with

$$\lim_{j\to\infty} \left(\gamma_1^{[j]}(0) + (-1)^j \right) = |b_2| = \frac{1}{12}.$$

At the negative integers

Corollary (2022)

For each positive integer k the function $\zeta(s,\{1\}^{j-1})$ has a Laurent series at $s=-k$ whose singular part (degree $i\le 0$) is described by

$$s^{j-1}\zeta(s-k,\{1\}^{j-1}) \equiv (-1)^{k-1}\binom{s-1}{k}\frac{B_{k+1}^{(s+1)}(1)}{(k+1)\Gamma(s+1)} \pmod{s^j\mathbb{C}[[s]]}.$$

Consequently the Laurent coefficient in degree $1-j$ is

$$\gamma_{1-j}^{[j]}(-k) = -\frac{B_{k+1}(1)}{k+1} = \zeta(-k).$$

If k is odd, then $\gamma_{1-j}^{[j]}(-k)\neq 0$ and thus $\zeta(s,\{1\}^{j-1})$ has a pole of order $j-1$ at $s=-k$. If k is even, then $\gamma_{1-j}^{[j]}(-k)=0$, and for $j>1$, $\zeta(s,\{1\}^{j-1})$ has a pole of order $j-2$ at $s=-k$, with $\gamma_{2-j}^{[j]}(-k)=(k+1)B_k/(2k)\neq 0$ in this case. For any positive integer k, the linear coefficient is given by the series

$$\frac{(-1)^{k+1}}{k!}\gamma_1^{[j]}(-k) = \sum_{n\neq k+1}^{\infty}\frac{(-1)^n b_n^{(k+1)}}{(n-k-1)^j} - [s^j]\left(\frac{\binom{s-1}{k}B_{k+1}^{(s+1)}}{(k+1)!\Gamma(s+1)}\right).$$

A special constant

The constant

$$\tau_1 = \sum_{n=1}^{\infty} \frac{\log(n+1)}{n(n+1)} \approx 1.2577468869\cdots$$

(decimal expansion A131688 in *OEIS*) appears in the asymptotic formula for $\log d(n!)$, and is also intimately related to series for the Stieltjes constants γ_i, having alternate expressions

$$\tau_1 = \int_0^1 \frac{\psi(t+1)+\gamma}{t}\,dt = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\zeta(n+1) = -\sum_{n=2}^{\infty} \zeta'(n).$$

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- Our series for γ_1 implies the additional series representation

$$\sum_{n=1}^{\infty} |b_n| H_n^{(2)} = \tau_1.$$

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- This constant and its representations have very natural analogues for height 1 zeta functions.

Generalized τ_1

Theorem

For all positive integers j, the constant $\tau_1^{[j]} := \sum_{n=1}^{\infty} |b_n| H_n^{(j+1)}$ is given by the series expressions

$$\begin{aligned}
\tau_1^{[j]} &= \sum_{n=1}^{\infty} |b_n| H_n^{(j+1)} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \zeta(n+1, \{1\}^{j-1}) \\
&= \sum_{m=1}^{\infty} \frac{1}{m!} \left[\begin{matrix} m \\ j \end{matrix} \right] \log\left(1 + \frac{1}{m}\right) \\
&= -\sum_{k=0}^{j-1} (-1)^k \sum_{n}^{*} \binom{n+k}{n} \zeta'(k+n+1, \{1\}^{j-k-1}),
\end{aligned}$$

where the sum $\sum_n^*$ denotes that the $n = 0$ term is omitted when $k = 0$.

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where the sum $\sum_n^*$ denotes that the $n = 0$ term is omitted when $k = 0$.

- In a forthcoming paper I give additional representations of $\tau_1^{[j]}$, such as

$$\tau_1^{[j]} = \int_0^1 \zeta_r(j+1)\, dr \qquad \text{and} \qquad \tau_1 = -\sum_{k=3}^{\infty} \sum_{j=0}^{\infty} \zeta'(k, \{1\}^j).$$

Generalized Euler - Mascheroni series

Corollary

For any nonnegative integer j we have

$$\sum_{n=1}^{\infty} \frac{|b_n|}{n^{j+1}} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n, \{1\}^j) = \gamma_1^{[j]}(1) + \tau_1^{[j]} + [s^{j+1}](\Gamma(s+1)^{-1}).$$

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$$F_j(s) := \sum_{m=j}^{\infty} \frac{1}{m!} \begin{bmatrix} m \\ j \end{bmatrix} (m+1)^{-s}$$

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- With this convention, for $j = 0$ we recover the classical series

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- For $j = 1$ we recover a recent evaluation due to Coppo,

$$\sum_{n=1}^{\infty} \frac{|b_n|}{n^2} = \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \zeta(n, 1) = \gamma_1 + \tau_1 + \frac{\gamma^2}{2} - \frac{\pi^2}{12}.$$

Ramanujan summation

Given a (divergent) series, Ramanujan assigned it an "algebraic constant" which is *"the constant obtained by completing the remaining part in the* [Euler-MacLaurin] *theorem. We can substitute this constant which is like the centre of gravity of a body instead of its divergent infinite series."*

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- Given a sequence $(f(n))_{n=1}^{\infty}$, and supposing there exists $f \in \mathcal{O}^{\pi}$ such that $f(z) = f(n)$ for $z = n \in \mathbb{N}$, the *Ramanujan summation* or *Ramanujan constant* $\sum_{n \geq 1}^{\mathcal{R}} f(n)$ is defined to be the value $R_f(1)$, where R_f is the unique solution in $\mathcal{O}^{\pi}$ to the difference equation

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- If in addition we have $f \in \mathcal{O}^{\log 2}$, then $\sum_{n\geq 1}^{\mathcal{R}} f(n)$ may also be given by the convergent series

$$\sum_{n\geq 1}^{\mathcal{R}} f(n) = \sum_{n=1}^{\infty} |b_n|(Df)(n),$$

where the operator D is defined on the space of sequences $(f(n))_{n=1}^{\infty}$ by

$$(Df)(n+1) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(j+1).$$

For a convergent series, the Ramanujan constant need not equal the sum of the series. However, one important property of Ramanujan summation is that *analyticity of the terms implies analyticity of the "sum"*.

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- For example,

$$\sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n^s} = \zeta(s) - \frac{1}{s-1} \quad (s \in \mathbb{C}), \qquad \sum_{n\geq 1}^{\mathcal{R}} \frac{1}{n} = \gamma.$$

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- For harmonic numbers H_n, we have (for example) the Ramanujan constants

$$\sum_{n\geq 1}^{\mathcal{R}} H_n = \zeta'(0) + \frac{3\gamma}{2} + \frac{1}{2}, \qquad \sum_{n\geq 1}^{\mathcal{R}} \frac{H_n}{n} = \gamma_1 + \frac{\gamma^2}{2} - \frac{\pi^2}{12} + \tau_1$$

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- The height 1 Stieltjes constants likewise appear in the Ramanujan constants of the *multiple harmonic star sums*

$$\zeta_n^\star(\{1\}^j) = \sum_{n\geq n_1\geq n_2\geq\cdots\geq n_j\geq 1} \frac{1}{n_1 n_2 \cdots n_j} = H_{n,j},$$

also known as *Roman harmonic numbers*. (Note $\zeta_n^\star(1) = H_{n,1} = H_n$).

Theorem

For all positive integers j,

$$\begin{aligned}
\sum_{n\geq 1}^{\mathcal{R}} \frac{\zeta^\star_{n-1}(\{1\}^j)}{n-1} &= \gamma_1^{[j]}(1) + \zeta(j+1) + [s^{j+1}](\Gamma(s+1)^{-1});\\
\sum_{n\geq 1}^{\mathcal{R}} \zeta^\star_{n-1}(\{1\}^j) &= -\gamma_1^{[j]}(0) - (-1)^j - [s^j]\left(\frac{s-1}{2\Gamma(s+1)}\right);\\
\sum_{n\geq 1}^{\mathcal{R}} \frac{\zeta^\star_{n}(\{1\}^{j-1}) - \zeta^\star_{n}(\{1\}^j)}{n(n-1)} &= \tau_1^{[j]} - \zeta(j+1);\\
\sum_{n\geq 1}^{\mathcal{R}} \frac{\zeta^\star_{n}(\{1\}^j)}{n} &= \gamma_1^{[j]}(1) + \tau_1^{[j]} + [s^{j+1}](\Gamma(s+1)^{-1});\\
\sum_{n\geq 1}^{\mathcal{R}} \zeta^\star_{n}(\{1\}^j) &= \gamma_1^{[j-1]}(1) + \tau_1^{[j-1]} - \gamma_1^{[j]}(0) - (-1)^j - [s^j]\left(\frac{s-3}{2\Gamma(s+1)}\right).
\end{aligned}$$

Ramanujan summation of multiple harmonic star sums

Theorem

For all positive integers j, using $\sim$ to denote congruence modulo $\mathbb{Q}[\gamma, \zeta(2), \ldots, \zeta(j+1)]$,

$$\begin{aligned}
\sum_{n\geq 1}^{\mathcal{R}} \frac{\zeta_{n-1}^{\star}(\{1\}^j)}{n-1} &\sim \gamma_1^{[j]}(1); \\
\sum_{n\geq 1}^{\mathcal{R}} \zeta_{n-1}^{\star}(\{1\}^j) &\sim -\gamma_1^{[j]}(0); \\
\sum_{n\geq 1}^{\mathcal{R}} \frac{\zeta_{n}^{\star}(\{1\}^{j-1}) - \zeta_{n}^{\star}(\{1\}^j)}{n(n-1)} &\sim \tau_1^{[j]} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \zeta(n+1, \{1\}^{j-1}); \\
\sum_{n\geq 1}^{\mathcal{R}} \frac{\zeta_{n}^{\star}(\{1\}^j)}{n} &\sim \gamma_1^{[j]}(1) + \tau_1^{[j]} \sim \sum_{n=2}^{\infty} \frac{(-1)^{n}}{n} \zeta(n, \{1\}^j); \\
\sum_{n\geq 1}^{\mathcal{R}} \zeta_{n}^{\star}(\{1\}^j) &\sim \gamma_1^{[j-1]}(1) + \tau_1^{[j-1]} - \gamma_1^{[j]}(0).
\end{aligned}$$

Curiosities

The classical series of Euler and of Mascheroni for γ, and the relations of the constant τ_1 to $\zeta(s)$, generalize very naturally to *height 1* multiple zeta functions. So do the Ramanujan constants of harmonic number series.

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