The Hasse principle for homogeneous polynomials with random coefficients over thin sets

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Notation

Throughout this talk, we shall use the following notation.

- $e(z) = e^{2\pi iz}$.
- $\boldsymbol{x} = (x_1, \dots, x_s) \in \mathbb{Z}^s$.
- $0 \leq \boldsymbol{x} \leq X$ for $0 \leq x_j \leq X$ $(1 \leq j \leq s)$.
- $A \ll B$ iff $A = O(B)$, with $B > 0$.
- $A \asymp B$ when both $A \gg B$ and $A \ll B$ hold.
- $\langle \cdot, \cdot \rangle$ denotes the inner product.

History and motivation

- Hasse principle (local-global principle)
  We say that an algebraic variety over $\mathbb{Q}$ satisfies the Hasse principle when this variety admits a point over $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p, respectively, if and only if the variety admits a $\mathbb{Q}$-rational point.

Remark (The *smooth* Hasse principle)

We say that an algebraic variety over $\mathbb{Q}$ satisfies the smooth Hasse principle when the fact that this variety admits a smooth point over $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p, respectively, implies that the variety admits a $\mathbb{Q}$-rational point.

- Birch (1962)
  We can say that a variety V, defined by a system of R forms $F_1, \ldots, F_R$ with integer coefficients of degree d in n variables, satisfies the *smooth* Hasse principle whenever n is sufficiently large in terms of d, R and the dimension of a certain singular locus. Specifically, it suffices to take

$$n > R(R+1)(d-1)2^{d-1} + \dim W,$$

  where

$$W = \{\boldsymbol{x} \in \mathbb{A}^n \mid \operatorname{rank}(J(\boldsymbol{x})) < R\},$$

  in which $J(\boldsymbol{x})$ is the Jacobian matrix of size $R \times n$ formed from the gradient vectors $\nabla F_1(\boldsymbol{x}), \ldots, \nabla F_R(\boldsymbol{x})$.

History and motivation

- Rydin Myerson (2017, 2018, 2019)
  The factor $R(R+1)(d-1)2^{d-1}$ in Birch's work above (1962) can be replaced by a factor growing linearly in R, in particular $Rd2^d + R$.

- Browning and Heath-Brown (2017)
  Beyond a system of equations with the same degrees, Browning and Heath-Brown verified the Hasse principle for a system of forms with different degrees.

In particular, consider $R = 1$.

- The Hasse-Minkowski theorem shows that the Hasse principle holds for a variety V defined by a quadratic form.
- For $d = 3$ and $R = 1$, Heath-Brown (1983) proved that non-singular cubic forms with 10 variables have a nontrivial integer solution. Later on, Hooley (1988, 1991, 1994, 2013) verified the Hasse principle for non-singular cubic forms in 9 variables.
- For $d = 4$ and $R = 1$, Marmon and Vishe (2019) proved that a smooth variety defined by a quartic form satisfies the Hasse principle whenever n is greater or equal to 28.
- In general, for $d \geq 3$ and $R = 1$, Browning and Prendiville (2017) showed that a smooth variety defined by a form satisfies the Hasse principle whenever $n \geq (d - \frac{1}{2}\sqrt{d})2^d$.

The classical approach through the circle method in order to verify the Hasse principle

Let $f(\boldsymbol{x}) \in \mathbb{Z}[\boldsymbol{x}]$ be a non-singular form in n variables of degree d. On observing the following relation

$$\int_0^1 e(n\alpha)\mathrm{d}\alpha = \begin{cases} 1 & \text{when } n = 0 \\ 0 & \text{when } n \in \mathbb{Z} \setminus \{0\} \end{cases},$$

we see that

$$\int_0^1 \sum_{\substack{1 \le \boldsymbol{x} \le X \\ \boldsymbol{x} \in \mathbb{Z}^n}} e(f(\boldsymbol{x})\alpha)\mathrm{d}\alpha = \sum_{\substack{1 \le \boldsymbol{x} \le X \\ \boldsymbol{x} \in \mathbb{Z}^n}} \int_0^1 e(f(\boldsymbol{x})\alpha)\mathrm{d}\alpha$$
$$= \#\{\boldsymbol{x} \in [1, X]^n \cap \mathbb{Z}^n |\ f(\boldsymbol{x}) = 0\}.$$

The classical approach through the circle method in order to verify the Hasse principle

Define the major arcs $\mathfrak{M} := \mathfrak{M}(Q)$ with $Q > 0$ by

$$\mathfrak{M} = \bigcup_{\substack{0 \le a \le q \le Q \\ (q,a)=1}} \mathfrak{M}_{q,a}(Q),$$

where

$$\mathfrak{M}_{q,a}(Q) = \left\{ \alpha \in [0,1) \middle| \, |\alpha - a/q| \le q^{-1} Q X^{-d} \right\}.$$

Define the minor arcs $\mathfrak{m} := [0,1) \setminus \mathfrak{M}$.

The classical approach through the circle method in order to verify the Hasse principle

By using major and minor arcs dissections, one deduces that for sufficiently large n in terms of d

$$\begin{aligned}
&\int_0^1 \sum_{\substack{1\leq \boldsymbol{x}\leq X\\ \boldsymbol{x}\in\mathbb{Z}^n}} e(f(\boldsymbol{x})\alpha)\mathrm{d}\alpha\\
&= \int_{\mathfrak{M}} \sum_{\substack{1\leq \boldsymbol{x}\leq X\\ \boldsymbol{x}\in\mathbb{Z}^n}} e(f(\boldsymbol{x})\alpha)\mathrm{d}\alpha + \int_{\mathfrak{m}} \sum_{\substack{1\leq \boldsymbol{x}\leq X\\ \boldsymbol{x}\in\mathbb{Z}^n}} e(f(\boldsymbol{x})\alpha)\mathrm{d}\alpha\\
&= \left(\prod_{p\ \text{prime}} \sigma_p\right)\cdot \sigma_\infty \cdot X^{n-d} + o(X^{n-d}),
\end{aligned}$$

where σ_p (p primes) are p-adic densities and σ_∞ is the Siegel volume which can be interpreted in terms of the surface measure of $\{\boldsymbol{x}\in[0,1]^n\cap\mathbb{R}^n \mid f(\boldsymbol{x})=0\}$.

The classical approach through the circle method in order to verify the Hasse principle

Furthermore, one can show that the existence of $\boldsymbol{x}$ over $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p, respectively, satisfying $f(\boldsymbol{x}) = 0$ implies that

$$\prod_{p \ prime} \sigma_p > 0 \text{ and } \sigma_\infty > 0,$$

by making use of Hensel's lemma and the implicit function theorem. Therefore, we conclude from the asymptotic formula above that

$$\int_0^1 \sum_{\substack{1 \leq \boldsymbol{x} \leq X \\ \boldsymbol{x} \in \mathbb{Z}^n}} e(f(\boldsymbol{x})\alpha) \mathrm{d}\alpha = \left(\prod_{p \ prime} \sigma_p \right) \cdot \sigma_\infty \cdot X^{n-d} + o(X^{n-d})$$

$$\asymp X^{n-d},$$

for sufficiently large $X > 0$.

The classical approach through the circle method in order to verify the Hasse principle

On recalling that

$$\#\{\mathbf{x} \in [1, X]^n \cap \mathbb{Z}^n \mid f(\mathbf{x}) = 0\} = \int_0^1 \sum_{\substack{1 \leq \mathbf{x} \leq X \\ \mathbf{x} \in \mathbb{Z}^n}} e(f(\mathbf{x})\alpha)\mathrm{d}\alpha,$$

we find that

$$\#\{\mathbf{x} \in [1, X]^n \cap \mathbb{Z}^n \mid f(\mathbf{x}) = 0\} \asymp X^{n-d}.$$

Therefore, when we write $V = \{\mathbf{x} \in \mathbb{P}^{n-1} \mid f(\mathbf{x}) = 0\}$, we conclude thus far that the variety V admits a point $\mathbf{x}$ over $\mathbb{R}$ and $\mathbb{Q}_p$ for all primes p, respectively, if and only if the variety V admits a $\mathbb{Q}$-rational point. This means that the variety V satisfies the Hasse principle.

- (Conjecture 1) Whenever $n \geq d+1$ with $d \geq 4$, a smooth variety V, defined by a homogeneous polynomial in n variables of degree d, satisfy the Hasse principle

Remark

We notice here that the current records described in the previous slides are very far from this conjecture.

Poonen and Voloch (2003) suggested a probabilistic point of view about this conjecture 1.

To be specific, denote a homogeneous polynomial in n variables of degree d with integer coefficients, by $\langle \boldsymbol{a}, \nu_{d,n}(\boldsymbol{x})\rangle$ with integer vectors $\boldsymbol{a}$ where $\langle \cdot, \cdot \rangle$ is the inner product and $\nu_{d,n}(\boldsymbol{x})$ is the Veronese embedding. For simplicity, we write

$$f_{\boldsymbol{a}}(\boldsymbol{x}) = \langle \boldsymbol{a}, \nu_{d,n}(\boldsymbol{x})\rangle.$$

Remark

We note that $\boldsymbol{a}$ *is an integer vector in* $\mathbb{Z}^N$ *with*

$$N := N_{d,n} = \binom{n+d-1}{d}.$$

History and motivation

Consider a variety $V_{\boldsymbol{a}}$ by

$$V_{\boldsymbol{a}} := \{\boldsymbol{x} \in \mathbb{P}^{n-1} |\ f_{\boldsymbol{a}}(\boldsymbol{x}) = 0\},$$

and define a set of these varieties $V_{\boldsymbol{a}}$ by

$$\mathbb{V}_{d,n}(A) := \{V_{\boldsymbol{a}} |\ \|\boldsymbol{a}\|_\infty \le A\}.$$

- Poonen and Voloch (2003) conjectured that the proportion of varieties in $\mathbb{V}_{d,n}(A)$, which satisfy the Hasse principle, converges to 1 as A goes to infinity, provided that $n \ge d+1$ and $d \ge 3$. (conjecture 2)
- Browning, Le Boudec, and Sawin (2023) confirmed this conjecture 2 of Poonen and Voloch except for the case $d = 3$ and $n = 4$, by using the geometry of numbers. (not the circle method!)

Recall the definition of $V_{\boldsymbol{a}}$, that is

$$V_{\boldsymbol{a}} := \{\boldsymbol{x} \in \mathbb{P}^{n-1} |\ f_{\boldsymbol{a}}(\boldsymbol{x}) = 0\}.$$

Our goal is to investigate the distribution of $\boldsymbol{a}$, with $\|\boldsymbol{a}\|_\infty \leq A$, whose associated varieties $V_{\boldsymbol{a}}$ do not satisfy the Hasse principle. In order to do so, for a given set $\mathcal{C} \subseteq \mathbb{Z}^N$, it is natural to consider a set

$$\mathbb{V}_{d,n}^{\mathcal{C}}(A) := \{V_{\boldsymbol{a}} |\ \|\boldsymbol{a}\|_\infty \leq A,\ \boldsymbol{a} \in \mathcal{C}\},$$

and ask a question that how many varieties $V_{\boldsymbol{a}}$ in $\mathbb{V}_{d,n}^{\mathcal{C}}(A)$ do not satisfy the Hasse principle.

Remark

With more answers to these questions for various sets $\mathcal{C}$, we have better information about the distribution of a set of $\boldsymbol{a}$ whose associated varieties $V_{\boldsymbol{a}}$ do not satisfy the Hasse principle.

Bridge between conjecture 1 and conjecture 2

Recall

$$\mathbb{V}^{\mathcal{C}}_{d,n}(A) := \{V_{\boldsymbol{a}} |\ \|\boldsymbol{a}\|_\infty \le A,\ \boldsymbol{a} \in \mathcal{C}\}.$$

- Brüdern and Dietmann (2017)
  Let $\mathcal{C}_1 \subseteq \mathbb{Z}^N$ be such that for each $\boldsymbol{a} \in \mathcal{C}_1$, the polynomial $f_{\boldsymbol{a}}(\boldsymbol{x})$ becomes a diagonal form, that is

$$f_{\boldsymbol{a}}(\boldsymbol{x}) = a_1x_1^d + a_2x_2^d + \cdots + a_nx_n^d \quad (a_i \in \mathbb{Z}).$$

  Brüdern and Dietmann (2017) showed that the proportion of varieties $V_{\boldsymbol{a}}$ in this particular set $\mathbb{V}^{\mathcal{C}_1}_{d,n}(A)$, which do not satisfy the Hasse principle, converges to 0 as $A \to \infty$, provided that $n \ge 3d + 2$.

Main result

For $\boldsymbol{a} \in \mathbb{Z}^N$, we let $P(\boldsymbol{a}) \in \mathbb{Z}[\boldsymbol{x}]$ be a non-singular form of degree $k \in \mathbb{N}$. We define

$$\mathcal{C}(P) := \{\boldsymbol{a} \in \mathbb{Z}^N |\ P(\boldsymbol{a}) = 0\}.$$

We examine the set of varieties

$$\mathbb{V}_{d,n}^{\mathcal{C}(P)}(A) = \{V_{\boldsymbol{a}} |\ \|\boldsymbol{a}\|_\infty \leq A,\ \boldsymbol{a} \in \mathcal{C}(P)\},$$

and investigate how many varieties $V_{\boldsymbol{a}}$ in $\mathbb{V}_{d,n}^{\mathcal{C}(P)}(A)$ do not satisfy the Hasse principle.

Recall that

$$\mathbb{V}_{d,n}^{\mathcal{C}(P)}(A) = \{V_{\boldsymbol{a}} |\ \|\boldsymbol{a}\|_\infty \leq A,\ \boldsymbol{a} \in \mathcal{C}(P)\}.$$

In advance of the statement of the main theorem, we define the following quantities

$$\rho_{d,n}^{P}(A) = \frac{\#\{V_{\boldsymbol{a}} \in \mathbb{V}_{d,n}^{\mathcal{C}(P)}(A) |\ V_{\boldsymbol{a}} \text{ admits a } \mathbb{Q}\text{-rational point}\}}{\#\mathbb{V}_{d,n}^{\mathcal{C}(P)}(A)}$$

and

$$\rho_{d,n}^{P,loc}(A) = \frac{\#\{V_{\boldsymbol{a}} \in \mathbb{V}_{d,n}^{\mathcal{C}(P)}(A) |\ V_{\boldsymbol{a}} \text{ admits a point over } \mathbb{R} \text{ and } \mathbb{Q}_p,\ \forall p\}}{\#\mathbb{V}_{d,n}^{\mathcal{C}(P)}(A)}.$$

The quantity $\rho_{d,n}^{P,loc}(A) - \rho_{d,n}^{P}(A)$ denotes the proportion of varieties $V_{\boldsymbol{a}}$ which do not satisfy the Hasse principle, in $\mathbb{V}_{d,n}^{\mathcal{C}(P)}(A)$.

Main result

Theorem (Y. 2023+)

Suppose that $P \in \mathbb{Z}[\boldsymbol{a}]$ is a non-singular homogeneous polynomial of degree $k \geq 2$, and that $P(\boldsymbol{a}) = 0$ has a nontrivial integer solution. Then, whenever $d \geq 14$, $k \leq d$, and $n \geq 32d + 17$, we have

$$\lim_{A\to\infty} \left(\rho_{d,n}^{P,loc}(A) - \rho_{d,n}^{P}(A)\right) = 0.$$

We can say that the proportion of varieties $V_{\boldsymbol{a}}$ in the set $\mathbb{V}_{d,n}^{\mathcal{C}(P)}(A)$, which do not satisfy the Hasse principle, converges to 0 as $A \to \infty$.

Main result

Remark (Joint work with H.Lee and S.Lee (2023+))

If we add an assumption that there exists $\boldsymbol{b} \in \mathbb{Z}^N$ *such that* $P(\boldsymbol{b}) = 0$ *and the variety* $V_{\boldsymbol{b}}$ *admits a smooth* $\mathbb{Q}$*-rational point, we have*

$$\liminf_{A\to\infty} \rho_{d,n}^{P,loc}(A) > 0.$$

Combining the result in the previous slide, we conclude that

$$\liminf_{A\to\infty} \rho_{d,n}^{P}(A) > 0.$$

Thank you!