Approximation of generalized Campana points

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Conventions

For simplicity, we work over $\mathbb{Q}$ and $\mathbb{Z}$, but the results work more generally over number fields and function fields of curves over any field. We denote $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$.

Many types of points

Let X be a compact variety over $\mathbb{Q}$ with integral model $\mathcal{X}$ over $\mathbb{Z}$ and let $D_1, \ldots, D_n$ be divisors on X with Zariski closure $\mathcal{D}_1, \ldots, \mathcal{D}_n$. There are a lot of special subsets of rational points defined relative to these, such as

- integral points,
- Campana points and weak Campana points,
- Darmon points etc.

We introduce W-points as a common framework for these points.

Multiplicity map

For a prime p and $P \in (X \setminus D_i)(\mathbb{Q})$ we define the **multiplicity** at a divisor $\mathcal{D}_i$ as the largest integer $N = n_p(P, \mathcal{D}_i)$ such that $P \bmod p^N$ lies in $\mathcal{D}_i(\mathbb{Z}/p^N\mathbb{Z})$. If $P \in D_i(\mathbb{Q})$ we set $n_p(P, \mathcal{D}_i) = \infty$. Using this we define the multiplicity map

$$\mathrm{mult}_p \colon X(\mathbb{Q}_p) \to \overline{\mathbb{N}}^n$$

$$P \mapsto (n_p(P, \mathcal{D}_1), \ldots, n_p(P, \mathcal{D}_n)).$$

$\mathcal{W}$-points

Given $\mathfrak{W} \subset \overline{\mathbb{N}}^n$ containing $\{0, \ldots, 0\}$ we set $\mathcal{W} = ((\mathcal{D}_1, \ldots, \mathcal{D}_n), \mathfrak{W})$ and we define the set of *p-adic $\mathcal{W}$-points* as

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}_p) = \{P \in X \mid \mathrm{mult}_p(P) \in W\},$$

and the set of *$\mathcal{W}$-points over $\mathbb{Z}$* as

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{P \in X \mid \mathrm{mult}_p(P) \in W \text{ for all primes } p\}.$$

Multiplicities on toric varieties

We take X to be a compact smooth split toric variety (such as $\mathbb{P}^{n-1}$), with integral model $\mathcal{X}$ induced by the fan. We let $D_1, \ldots, D_n$ be the torus-invariant prime divisors (on $\mathbb{P}^{n-1}$: coordinate hyperplanes) with ray generators $u_i \in \mathbb{Z}^d$, where $d = \dim X$.

Multiplicities on toric varieties

We can represent a point on $X(\mathbb{Q})$ by its Cox coordinates $P = (a_1 : \cdots : a_n)$, corresponding to the D_i, and by scaling we can assume that $a_i \in \mathbb{Z}$, and that for every prime p there exists a cone σ such that $p \nmid a_i$ for all i with $u_i \notin \sigma$.
(For $\mathbb{P}^{n-1}$ this is just $\gcd(a_1, \ldots, a_n) = 1$.)
Then we have

$$\mathrm{mult}_p(P) = (v_p(a_1), \ldots, v_p(a_n)).$$

Examples of $\mathcal{W}$-points

- $\mathfrak{W} = \{0\}^k \times \overline{\mathbb{N}}^{n-k}$ gives the integral points with respect to $D_1, \dots, D_k$:

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{(\pm 1 : \cdots : \pm 1 : a_{k+1} : \cdots : a_n)\}.$$

- $\mathfrak{W} = \{0, 1\}^n$ gives "squarefree" points $(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \colon a_i \text{ squarefree}\}$.

Let $m_1, \dots, m_n \in \mathbb{N} - \{0\}$.

- $\mathfrak{W} = \{(w_1, \dots, w_n) \colon m_i | w_i\}$ gives the **Darmon points**

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{(\pm a_1^{m_1} : \cdots : \pm a_n^{m_n})\}.$$

- $\mathfrak{W} = \{(w_1, \dots, w_n) \colon w_i = 0 \text{ or } w_i \geq m_i\}$ gives the **Campana points**

$$(\mathcal{X}, \mathcal{W})(\mathbb{Z}) = \{(a_1 : \cdots : a_n) \colon a_i \ m_i\text{-full}\}.$$

W-approximation

We say that $\mathcal{X}$ satisfies *(integral) $\mathcal{W}$-approximation* if the embedding

$$(\mathcal{X},\mathcal{W})(\mathbb{Z}) \to \prod_{p \text{ prime}} (\mathcal{X},\mathcal{W})(\mathbb{Z}_p) \times X(\mathbb{R})$$

has dense image,
and say it satisfies *(integral) $\mathcal{W}$-approximation* off ∞ if the embedding

$$(\mathcal{X},\mathcal{W})(\mathbb{Z}) \to \prod_{p \text{ prime}} (\mathcal{X},\mathcal{W})(\mathbb{Z}_p)$$

has dense image. This generalizes strong approximation, which is when $(\mathcal{X},\mathcal{W})(\mathbb{Z})$ are the integral points.

W-approximation for toric varieties

When is this satisfied? Consider the fan of X in $\mathbb{Z}^d$ ($d = \dim X$). Then we get a homomorphism

$$\mathbb{N}^n \to \mathbb{Z}^d$$

sending $e_i \mapsto u_i$, where u_i is the ray generator associated to D_i. (For $\mathbb{P}^{n-1}$ we take $u_i = e_i$ if $i \leq n-1 (= d)$ and $u_n = -\sum_{i=1}^{d} e_i$.) Using this map, W generates a submonoid

$$N_W^+ \subset \mathbb{Z}^d$$

and a subgroup

$$N_W \subset \mathbb{Z}^d.$$

W-approximation for toric varieties

Theorem (B.M.,2023)

1. $\mathcal{X}$ satisfies $\mathcal{W}$-approximation off ∞ if and only if $N_W = \mathbb{Z}^d$,
2. $\mathcal{X}$ satisfies $\mathcal{W}$-approximation if and only if $N_W^+ = \mathbb{Z}^d$.

As N_W and N_W^+ are easy to compute, it is easy to decide whether $\mathcal{W}$-approximation holds.

Implications of the theorem

Corollary (B.M.,2023)

$\mathcal{X}$ always satisfies $\mathcal{W}$-approximation for Campana points and for squarefree points.

Corollary

Strong approximation holds off ∞ with respect to $D_1, \ldots, D_k$ if and only if $X \setminus \cup_{i=1}^k D_i$ is simply connected as a complex manifold. This comes from the isomorphism

$$\mathbb{Z}^d / N_W \cong \pi_1(X \setminus \cup_{i=1}^k D_i).$$

Corollary (B.M.,2023)

$\mathcal{W}$-approximation holds for Darmon points if and only if there are no (nontrivial) finite covers $Y \to X$ ramified only over the D_i with ramification multiplicity $e_i | m_i$ at the D_i.
In particular: if $\gcd(m_i, m_j) = 1$ for all $i \neq j$ then $\mathcal{W}$-approximation holds for Darmon points, and if $X = \mathbb{P}^n$ then the converse also holds.

(The above condition is equivalent to the associated root stack being simply connected.)

Example: if $X = \mathbb{P}^1$ and $m_1, m_2 = 2$, then $\mathcal{X}$ does not satisfy $\mathcal{W}$-approximation, as 2 mod 5 is not of the form $\pm a^2$ mod 5, but $(2:1) \in (\mathcal{X}, \mathcal{W})(\mathbb{Z}_5)$ as $2 \in \mathbb{Z}_5^\times$.

In general

The results transfer verbatim to number fields, and after slight modification also for function fields of curves.