An extension of Euclid-Euler Theorem to certain α-perfect numbers

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Concepts and notation

1. We define $\sigma(N)$ as the sum of the positive divisors of N. It is a multiplicative function;
2. We say N is a perfect number if $\sigma(N) = 2N$; we say N is an α-perfect number if $\sigma(N) = \alpha N$;

Concepts and notation

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2. We say N is a perfect number if $\sigma(N) = 2N$; we say N is an α-perfect number if $\sigma(N) = \alpha N$;
3. We write $a \mid b$ if a divides b; we write $a^n \parallel b$ if a^n divides b exactly, i.e., $a^n \mid b$ and a^{n+1} does not divide b.
4. We say α is a p-abundancy outlaw if there is no positive integer N such that $\sigma(N) = \alpha N$ and $p \mid N$, where p is a prime number.

Motivation

Euclid-Euler Theorem

N is an even perfect number if and only if $N = 2^{p-1}(2^p - 1)$, where $2^p - 1$ is a Mersenne prime number.

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Since k and m divide m, and $\sigma(m) = k + m$, then $2^{a+1} - 1$ is a prime number and $k = 1$.

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Since k and m divide m, and $\sigma(m) = k + m$, then $2^{a+1} - 1$ is a prime number and $k = 1$.

Since $2^{a+1} - 1$ is prime number then $a + 1$ is a prime number p.

Therefore, $N = 2^{p-1}(2^p - 1)$.

Generalizing Euler's method

Let α be a rational number and $N > 1$ be an α-perfect number.

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As $\frac{\sigma(N)}{N}$ is multiplicative and $\frac{p+1}{p} \leq \frac{\sigma(p^a)}{p^a} < \frac{p}{p-1}$, then there exist positive integers r and m, prime numbers p_i, and positive integers a_i, with $1 \leq i \leq r$, such that

$$N = m \prod_{i=1}^{r} p_i^{a_i}, \tag{1}$$

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$$N = m \prod_{i=1}^{r} p_i^{a_i}, \tag{1}$$

$$\beta = \alpha \prod_{i=1}^{r} \frac{p_i - 1}{p_i} \leq 1, \tag{2}$$

and

$$\gcd\left(m, \prod_{i=1}^{r} p_i^{a_i}\right) = 1.$$

Generalizing Euler's method

Therefore, and by definition,

$$\alpha m \prod_{i=1}^{r} p_i^{a_i} = \sigma(N) = \sigma(m) \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

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Consider

$$d = \gcd\left(\beta \prod_{i=1}^{r} p_i^{a_i+1}, \prod_{i=1}^{r} \left(p_i^{a_i+1} - 1\right)\right).$$

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$$d = \gcd\left(\beta \prod_{i=1}^{r} p_i^{a_i+1}, \prod_{i=1}^{r}\left(p_i^{a_i+1} - 1\right)\right).$$

Hence, for some integers k and d, we have

$$\sigma(m) = \frac{\beta k}{d} \prod_{i=1}^{r} p_i^{a_i+1} \tag{3}$$

and

$$m = \frac{k}{d} \prod_{i=1}^{r}\left(p_i^{a_i+1} - 1\right), \tag{4}$$

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We find a lower bound for $\sigma(m)$, by summing divisors of m that are explicitly indicated in

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From now on, we will consider $\beta = 1$ (like Euler).

Differences between powers of 2 and 3

Lemma

The only solutions of the diophantine equation

$$2^a - 3^b = -1 \tag{7}$$

are $(1,1)$ and $(3,2)$. Also, the only solutions of the diophantine equation

$$2^a - 3^b = 2^c - 1, \tag{8}$$

are $(2,1,1)$, $(4,2,3)$, and $(a,0,a)$, $\forall a \in \mathbb{N} \cup \{0\}$.

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The first equation was solved by Mihăilescu (Catalan's conjecture).

For the second equation, we take $c \geq 2$.

If c is even then $3 \mid 2^c - 1$. Hence $3 \mid 2^a - 3^b$. Contradiction.

If c is odd then... $2^3 \parallel 3^b - 1$. We conclude $c = 3$...then $(a,b) = (4,2)$.

3-perfect numbers divisible by 6

Theorem

Suppose N is a 3-perfect number and $6 \mid N$. Hence $2 \parallel N$ and $3 \nparallel N$, or $2 \nparallel N$ and $3 \parallel N$.

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Then

$$3N = 2^a 3^{b+1} m = \sigma(N) = (2^{a+1} - 1) \frac{3^{b+1} - 1}{2} \sigma(m).$$

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Then
$$3N = 2^a 3^{b+1} m = \sigma(N) = (2^{a+1} - 1)\,\frac{3^{b+1} - 1}{2}\,\sigma(m).$$
Therefore,
$$\frac{\sigma(m)}{m} = \frac{2^{a+1} 3^{b+1}}{(2^{a+1} - 1)(3^{b+1} - 1)}.$$

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Therefore,

$$\frac{\sigma(m)}{m} = \frac{2^{a+1}\, 3^{b+1}}{(2^{a+1} - 1)(3^{b+1} - 1)}.$$

Let $d = \gcd\left(2^{a+1}\, 3^{b+1}, (2^{a+1} - 1)(3^{b+1} - 1)\right)$. It is easy to see that $d = 2^s\, 3^t$, where $1 \leq s \leq a + 1$ and $0 \leq t \leq b + 1$.

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Then, we have that

$$\sigma(m) = \frac{2^{a+1}\,3^{b+1}}{2^s\,3^t}k \text{ and } m = \frac{2^{a+1}-1}{3^t}\,\frac{3^{b+1}-1}{2^s}k,$$

for some positive integer k.

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Let us consider the following three cases, which will establish the claim.

Case A: Suppose that $t \neq 0$.

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for some positive integer k.

Let us consider the following three cases, which will establish the claim.

Case A: Suppose that $t \neq 0$.

Case B: Suppose that $t = 0$ and

$$2^{a+1} - 1 \neq \frac{3^{b+1}-1}{2^s}.$$

Case C: Suppose that $t = 0$ and

$$2^{a+1} - 1 = \frac{3^{b+1}-1}{2^s}.$$

3-perfect numbers divisible by 6

Case A: Suppose that $t \neq 0$ and let

$$M = \max\left(\frac{2^{a+1}-1}{3^t}, \frac{3^{b+1}-1}{2^s}\right).$$

Then we have

$$\frac{\sigma(m)}{k} = \frac{2^{a+1}3^{b+1}}{2^s\,3^t} = \ldots < \frac{m}{k} + M + 1.$$

Therefore,

$$\sigma(m) < m + Mk + k. \tag{9}$$

3-perfect numbers divisible by 6

Case A1: $Mk \neq m$ and $M \neq 1$.

Case A2: $Mk = m$ or $M = 1$.

In Case A1, we have: m, Mk and k are different divisors of m. Thus,

$$\sigma(m) \geq m + Mk + k. \tag{10}$$

By combination of inequalities (9) and (10), we have a contradiction.

3-perfect numbers divisible by 6

In Case A2, we have:

$$\frac{2^{a+1}-1}{3^t} = 1 \text{ or } \frac{3^{b+1}-1}{2^s} = 1.$$

Therefore, $2^{a+1} - 3^t = 1$ or $3^{b+1} - 2^s = 1$. We have $a = 1$ or $b = 1$ (by the solutions of the diophantine equations presented before).

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Case B: Suppose that $t=0$ and

$$2^{a+1}-1 \neq \frac{3^{b+1}-1}{2^s}.$$

Let

$$M' = \min\left(2^{a+1}-1, \frac{3^{b+1}-1}{2^s}\right).$$

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Therefore, $2^{a+1} = 2$ or $3^{b+1} - 2^s = 1$. As $a, b \geq 1$, we conclude that $b = 1$ (Catalan's conjecture!).

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Case B2: Since $M' \neq 1$ then

$$m, \left(2^{a+1} - 1\right) k, \frac{3^{b+1} - 1}{2^s} k, \text{ and } k,$$

are different divisors of m.

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are different divisors of m. Therefore,

$$\sigma(m) \geq m + (2^{a+1} - 1)k + \frac{3^{b+1} - 1}{2^s} k + k > ... = \sigma(m).$$

We obtain a contradiction.

3-perfect numbers divisible by 6

Case C: Suppose that $t = 0$ and

$$2^{a+1} - 1 = \frac{3^{b+1} - 1}{2^s}.$$

Then $2^{a+1+s} - 3^{b+1} = 2^s - 1$.

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We have $(a, b) \in \{(0, 0), (0, 1)\}$ (by the solutions of the diophantine equations presented before).
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Hence, we must have $a = 1$ or $b = 1$.
But...just one of them. Why? Black board, please!

Differences between powers of 2 and F_n known prime

Lemma

Let F_n be the n-th Fermat number and consider the following exponential diophantine equations

$$2^a - F_n^b = -1 \tag{11}$$

and

$$2^a - F_n^b = 2^c - 1. \tag{12}$$

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Then

(a) when $n \in \{1, 2, 3, 4\}$, Equation (11) only holds for $(a, b) = (2^n, 1)$;

(b) when $n \in \{2, 3, 4\}$, Equation (12) only holds for

$$(a, b, c) \in \{(a, 0, a) \mid a \in \mathbb{Z}\} \cup \{(2^n + 1, 1, 2^n)\};$$

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(c) when $n = 1$, Equation (12) only holds for

$$(a, b, c) \in \{(a, 0, a) \mid a \in \mathbb{Z}\} \cup \{(3, 1, 2), (7, 3, 2), (5, 2, 3)\}.$$

$\frac{2F_n}{F_n-1}$-perfect numbers divisible by F_n

Theorem

Let F_n be the n-th Fermat number. Then

1. *if exists N such that $\frac{\sigma(N)}{N} = \frac{F_1}{2}$ and $F_1 \mid N$, then $2^4 \parallel N, F_1^2 \parallel N$, and $31^2 \mid N$.*
2. *if $n \in \{2,3,4\}$ then $\frac{2F_n}{F_n-1}$ is a F_n-abundancy outlaw.*

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2. *if $n \in \{2, 3, 4\}$ then $\frac{2F_n}{F_n-1}$ is a F_n-abundancy outlaw.*

Sketch of the proof: Let $n \in \{1, 2, 3, 4\}$ and $F_n = 2^{2^n} + 1$. We can write $N = 2^a F_n^b m$ such that $a, b \geq 1$, $\gcd(2F_n, m) = 1$, and

$$\sigma(N) = \frac{2F_n}{F_n - 1} N.$$

Then

$$\frac{\sigma(m)}{m} = \frac{2^{a+1}\, F_n^{b+1}}{(2^{a+1} - 1)(F_n^{b+1} - 1)}.$$

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Let

$$d = \gcd\left(2^{a+1} F_n^{b+1}, (2^{a+1}-1)(F_n^{b+1}-1)\right).$$

As $F_n - 1 \mid F_n^{b+1} - 1$, then $d = 2^s F_n^t$, where $2^n \leq s \leq a+1$ and $0 \leq t \leq b+1$...(similar to 3-perfect, but trickier)...

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We will have

$$(F_n, a, b, s) \in \{(5,-1,-1,1), (5,0,0,2), (5,1,1,3), (5,4,2,2), (17,0,0,4), (257,0,0,8), (65537,0,0,16)\}.$$

$\frac{2F_n}{F_n-1}$-perfect numbers divisible by F_n

Since $a, b \geq 1$, we only have solutions for $F_n = 5$ and then $(F_n, a, b) = (5, 1, 1)$ or $(F_n, a, b) = (5, 4, 2)$.

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$$\frac{\sigma(N)}{N} = \frac{5}{2} = \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{\sigma(m)}{m}.$$

Therefore, $9 \mid m$.

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But then,

$$\frac{\sigma(N)}{N} = \frac{5}{2} \geq \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{13}{9} > \frac{5}{2}.$$

We have a contradiction.

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Hence, $(F_n, a, b) = (5, 4, 2)$ and so $31^2 \mid N$.

References

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