# On ranks of quadratic twists of a Mordell curve

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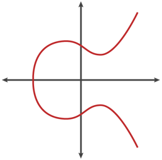
this is a joint work with A. Jyual and D. Moody

# Elliptic curve defined over number fields (Overview)



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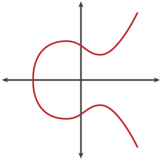

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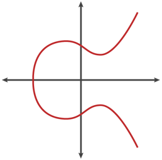

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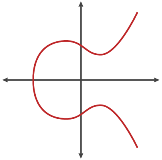
- We know $E(K) := \{P = (x, y) : x, y \in K \text{ and } E(x, y) = \mathcal{O}\} \cup \{\mathcal{O}\}$ is called the set of $K$-rational points of $E$.



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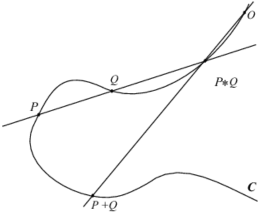


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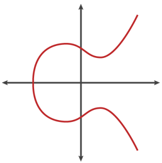

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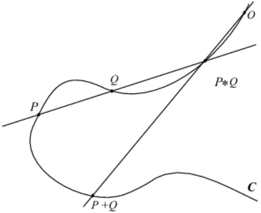


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• We also know that the group $E(K)$ is a finitely generated abelian group by Mordell-Weil theorem.

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**Known records:**

- There exist elliptic curves with rank as large as 28.

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A general elliptic curve $E/\mathbb{Q}$ given by the weierstrass equation

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$\rightsquigarrow$ Twists has importance in multiple directions like cryptography, understanding 'positive-ranks' etc .

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Understand Ranks of $E_d^0$, while varying $d$!

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- Kohen: 19.B of Cremona Level $\dashrightarrow$ Rank 1

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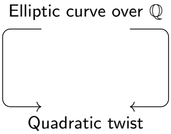
Application of decent methods:

⇢ Dabrowski: For any positive integer $k$, there exits pairwise non-isogenous elliptic curves $E^1, \ldots, E^k$ such that $rank(E_p^1) = \cdots = rank(E_p^k) = 0$, for positive proposition of primes.

Elliptic curve over $\mathbb{Q}$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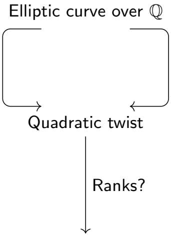
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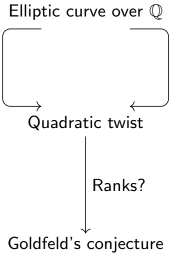
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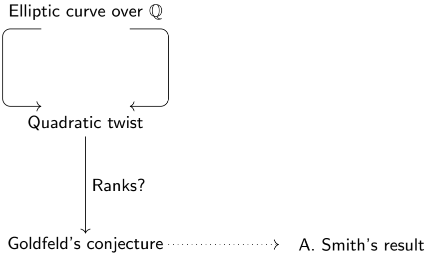
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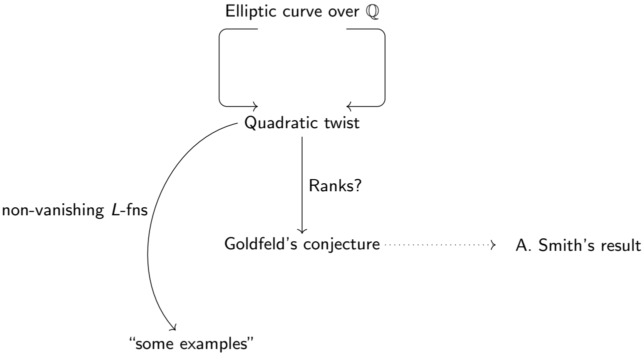
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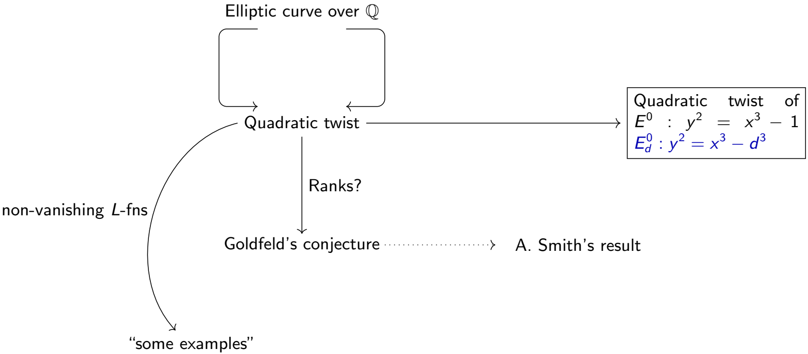
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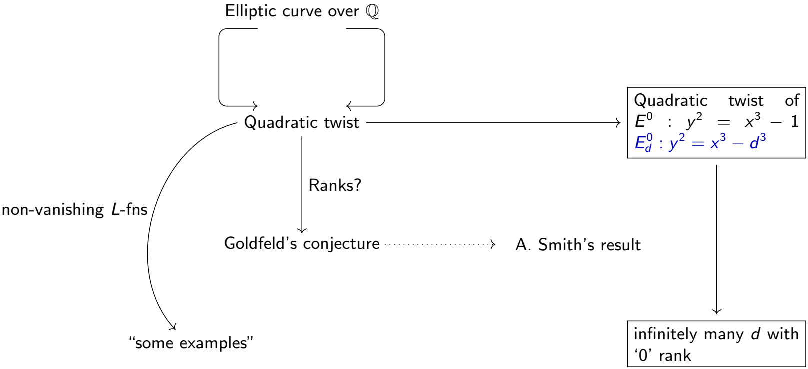
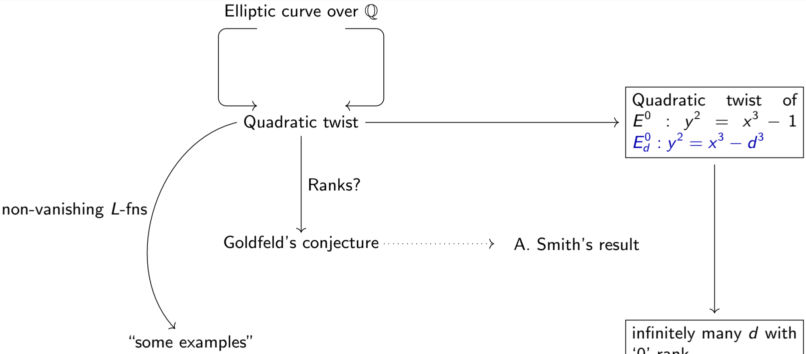



**Theorem: (2022)**

For any square-free integer $d$, we have the quadratic twist $E_d^0 : y^2 = x^3 - d^3$. Let $\omega(d)$ be the number of distinct prime divisors of $d$. Then there exist infinitely many square-free integers $d$, with $\omega(d) > 1$, such that $rank(E_d^0) = 0$.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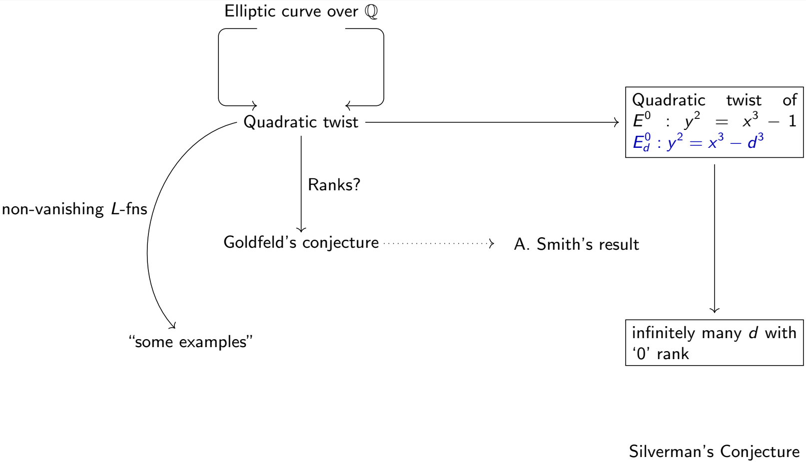
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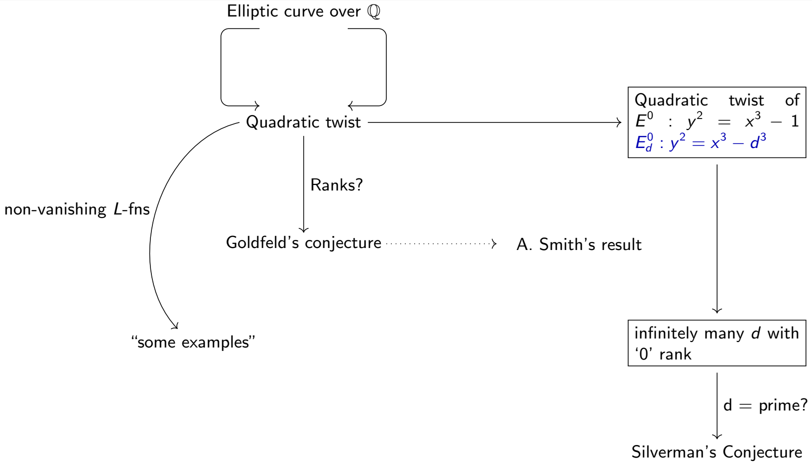

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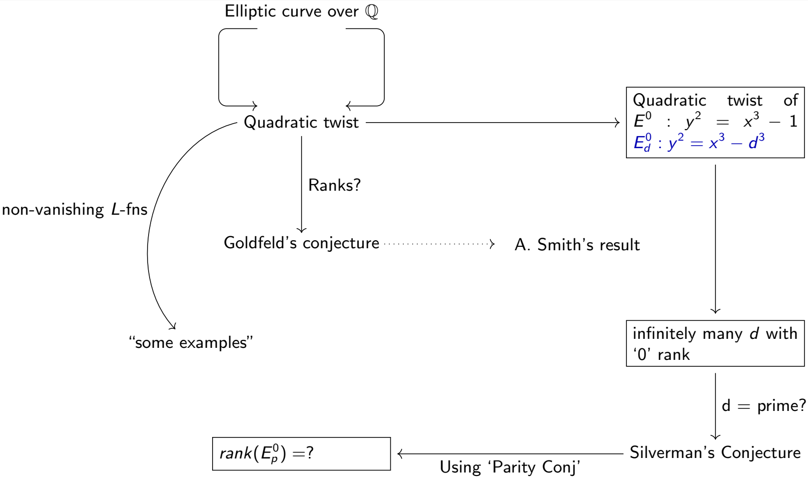


**Conj: Parity**

Let $E$ be an elliptic curve defined over a number field $K$. Then

$$(-1)^{rank(E/K)} = w(E/K)$$

where $w(E/K)$ is the *global root number*.

**Conj:(Silverman)**

If $E$ is an elliptic curve, then there are infinitely many primes $p$ for which $E_p(\mathbb{Q})$ has positive ranks, and there are infinitely many primes $q$ for which $E_q(\mathbb{Q})$ has rank 0.

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### Theorem: (2022)

For the elliptic curve $E_p^0 : y^2 = x^3 - p^3$, we have

$$rank(E_p^0) = \begin{cases} 0 & \text{if } p \equiv 5 \pmod{12}, \\ 1 & \text{if } p \equiv 11 \pmod{12} \text{ (assuming the Parity Conjecture).} \end{cases} \quad (1)$$

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♦ Corresponding to the 2-torsion point, there is a 2-isogeny $\phi : E_d^0 \longrightarrow E_d'$ which has the kernel $\{\mathcal{O}, (d, 0)\}$ with the image curve is

$$E_d' : y^2 = x^3 - 6dx^2 - 3d^2x,$$

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$$0 \to \frac{E'_d(\mathbb{Q})}{\phi(E^0_d(\mathbb{Q}))} \to Sel^{(\phi)}(E^0_d/\mathbb{Q}) \to Sha(E^0_d/\mathbb{Q})[\phi] \to 0,$$

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♦ The above is also true for the dual isogeny $\widehat{\phi}$

♦ For a square-free positive integer $d$, define a finite set $S$ of prime divisors of the rational numbers $\mathbb{Q}$ by

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♦ For each $k \in M$ we have the homogeneous spaces $C_k$ and $C'_k$ defined by

$$C_k : kw^2 = k^2t^4 - 6kdt^2z^2 - 3d^2z^4,$$
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♦ The Selmer group $Sel^{(\phi)}(E_d^0/\mathbb{Q})$ (respectively $Sel^{(\hat{\phi})}(E_d^{'}/\mathbb{Q})$) measures the possibility of $C_k$ (or $C'_k$) having non-trivial solutions in the local field $\mathbb{Q}_v$ for all $v \in S$.

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♦

$$Sel^{(\phi)}(E_d^0/\mathbb{Q}) = \{k \in M : C_k(\mathbb{Q}_v) \neq \emptyset \text{ for all } v \in S\},$$
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## Lemma

*Let $E/\mathbb{Q}$ be an elliptic curve with a rational point of order 2. Let $\phi : E \to E'$ be an isogeny of degree 2, with $\widehat{\phi} : E' \to E$ the dual of $\phi$. Then*

$$rank(E(\mathbb{Q})) \leq dim_{\mathbb{F}_2} Sel^{(\phi)}(E, \mathbb{Q}) + dim_{\mathbb{F}_2} Sel^{(\widehat{\phi})}(E', \mathbb{Q}) - 2.$$

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By calculating $C_k(\mathbb{Q}_v)$ and $C'_k(\mathbb{Q}_v)$ precisely,

$$\{1, -3\} = Sel^{(\phi)}(E^0_d/\mathbb{Q})$$

and

$$\{1, 3\} = Sel^{(\widehat{\phi})}(E'_d/\mathbb{Q})$$

□

# Positive rank evidence:

Both Goldfelds's Conjecture and Silverman's Conjecture are concerned with elliptic curves with positive rank.

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**Theorem: (2022)**

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Remark: There are infinitely many integers $n$ such that $E_n^0 : y^2 = x^3 - n^3$ has positive rank. There are infinitely many integers $m$ such that $E_m^0$ has rank zero.

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(a) There exist infinitely many integers $d$ such that $rank(E_d^0)$ is **positive**.

(b) There is an infinite family of curves $E_d^0$, over the number field $\mathbb{Q}(m)$, with rank **at least** 2. Here $d$ and $m$ are related by the following equation:

$$d = \left(\frac{1 - 2m^2}{m^2 + 1}\right)^3 - 1.$$

Remark: There are infinitely many integers $n$ such that $E_n^0 : y^2 = x^3 - n^3$ has positive rank. There are infinitely many integers $m$ such that $E_m^0$ has rank zero.

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Thank You for Your Attention!