On a conjecture of Levesque and Waldschmidt

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Overview

1. Motivation
2. Statement and Discussion

What's in a Thue equation?

- Diophantine equation $f(x,y) = m$
  - $f \in \mathbb{Z}[x,y]$
  - f irreducible, homogenous
  - $\deg f \geq 3$

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Theorem (A. Baker; 1968)

Let $\kappa > \deg f + 1$. All solutions of $f(x, y) = m$ in integers x, y satisfy

$$\max\left(|x|, |y|\right) < Ce^{(\log m)^{\kappa}},$$

where C is an effectively computable number depending only on $\deg f$, κ, and the coefficients of f.

What's in a family of Thue equations?

- Family of (parametrised) Thue equations $\{f_n : n \in \mathbb{N}\}$
  - f_n Thue equation
  - $f_n \in \mathbb{Z}[n][x, y]$

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Theorem (E. Thomas; 1990)

Let n be an integer with $n \geq 1.365 \times 10^7$. Then the equation

$$x^3 - (n-1)\,x^2 y - (n+2)\,x y^2 - y^3 = \pm 1, \quad n \geq 0,$$

has only the trivial solutions $(0, \pm 1)$, $(\pm 1, 0)$, $(\pm 1, \mp 1)$.

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Theorem (C. Levesque and M. Waldschmidt; 2015)

Let $f_{n,a}(x,y) = \pm 1$ with $a \neq 0$ and $\max(|x|,|y|) \geq 2$. Then there exists an effectively computable constant κ_2 such that

$$\max(|n|,|a|,|x|,|y|) \leq \kappa_2.$$

What's in a conjecture?

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Conjecture (C. Levesque and M. Waldschmidt; 2015)

For s, t and n in $\mathbb{Z}$ define

$$f_{n,s,t}(x,y) = \left(x - \lambda_0^s \lambda_1^t y\right)\left(x - \lambda_1^s \lambda_2^t y\right)\left(x - \lambda_2^s \lambda_0^t y\right).$$

There exists a positive absolute constant κ_3 with the following property: If n, s, t, x, y are integers satisfying

$$\max(|x|, |y|) \geq 2, \quad (s,t) \neq (0,0), \quad \text{and } f_{n,s,t}(x,y) = \pm 1,$$

then

$$\max(\log|n|, |s|, |t|, \log|x|, \log|y|) \leq \kappa_3.$$

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Theorem (T. Hilgart and V. Ziegler; 2023)

For s, t and n in $\mathbb{Z}$ define

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Let $\varepsilon > 0$. There exists an effectively computable constant $\kappa > 0$ with the following property: If n, s, t, x, y are integers satisfying

$$|y| \geq 2, \quad n \geq 3, \quad st \neq 0, \quad \text{and} \quad f_{n,s,t}(x,y) = \pm 1,$$

as well as

$$\min\left(|2s - t|, |2t - s|, |s + t|\right) > \varepsilon \cdot \max\left(|s|, |t|\right) > 2,$$

then

$$\max\left(\log|n|, |s|, |t|, \log|x|, \log|y|\right) \leq \kappa$$

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$$\frac{\lambda_0^s \, \lambda_1^t}{\lambda_1^s \, \lambda_2^t} = \lambda_0^{2s-t} \, \lambda_2^{2t-s}$$

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$$\frac{\lambda_0^s\,\lambda_1^t}{\lambda_1^s\,\lambda_2^t} = \lambda_0^{2s-t}\,\lambda_2^{2t-s}$$

- only λ_2^{2t-s} remains!

What's in a future paper?

- If we can solve the case $2s = t$, i.e.

$$\left(x - \lambda_0^s \lambda_1^{2s} y\right)\left(x - \lambda_1^s \lambda_2^{2s} y\right)\left(x - \lambda_2^s \lambda_0^{2s} y\right) = \pm 1,$$

we can solve the Conjecture.

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