The distribution of partial quotients with fixed denominator

Manuel Hauke

Graz University of Technology

July 7, 2023

Introduction - Continued fractions

- Every $\alpha \in \mathbb{R}$ has a (unique) continued fraction expansion $\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, \dots]$, $a_i \in \mathbb{N}$.

Introduction - Continued fractions

- Every $\alpha \in \mathbb{R}$ has a (unique) continued fraction expansion $\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, \dots],\ a_i \in \mathbb{N}$.
- $\alpha \in \mathbb{Q}$, if and only if the expansion is finite, that is, $\alpha = [a_0; a_1, \dots, a_r]$ for some $r \in \mathbb{N}$.

Introduction - Continued fractions

- Every $\alpha \in \mathbb{R}$ has a (unique) continued fraction expansion $\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, \dots], \ a_i \in \mathbb{N}$.
- $\alpha \in \mathbb{Q}$, if and only if the expansion is finite, that is, $\alpha = [a_0; a_1, \dots, a_r]$ for some $r \in \mathbb{N}$.
- $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \dots], e = [2; 1, 2, 1, 1, 4, 1, 1, \dots], \Phi = \frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, 1, \dots]$.

Introduction - Continued fractions

- Every $\alpha \in \mathbb{R}$ has a (unique) continued fraction expansion $\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, \dots], \ a_i \in \mathbb{N}.$
- $\alpha \in \mathbb{Q}$, if and only if the expansion is finite, that is, $\alpha = [a_0; a_1, \dots, a_r]$ for some $r \in \mathbb{N}$.
- $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \dots], e = [2; 1, 2, 1, 1, 4, 1, 1, \dots], \Phi = \frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, 1, \dots].$
- The convergents $\frac{p_i}{q_i} = [a_0; a_1, a_2, \dots, a_i]$ approximate α by

$$\frac{1}{(a_{i+1} + 2)q_i^2} \leq (-1)^i \left(\alpha - \frac{p_i}{q_i} \right) \leq \frac{1}{a_{i+1} q_i^2}.$$

Introduction - Continued fractions

- Every $\alpha \in \mathbb{R}$ has a (unique) continued fraction expansion $\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}} = [a_0; a_1, a_2, \dots], \ a_i \in \mathbb{N}$.
- $\alpha \in \mathbb{Q}$, if and only if the expansion is finite, that is, $\alpha = [a_0; a_1, \dots, a_r]$ for some $r \in \mathbb{N}$.
- $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \dots], e = [2; 1, 2, 1, 1, 4, 1, 1, \dots], \Phi = \frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, 1, \dots]$.
- The convergents $\frac{p_i}{q_i} = [a_0; a_1, a_2, \dots, a_i]$ approximate α by

$$\frac{1}{(a_{i+1}+2)q_i^2} \leq (-1)^i \left(\alpha - \frac{p_i}{q_i}\right) \leq \frac{1}{a_{i+1}q_i^2}.$$

- Theorem of Legendre: if $|\alpha - \frac{p}{q}| \leq \frac{1}{2q^2}$, then $\frac{p}{q}$ is a convergent of α.

Distribution of a_k, q_k for random irrationals

- Drawing an irrational uniformly at random from $[0, 1)$, what is the probability that $a_i = m$?

Distribution of a_k, q_k for random irrationals

- Drawing an irrational uniformly at random from $[0, 1)$, what is the probability that $a_i = m$?
- $\mathbb{P}[q_k \geq f(k)], \mathbb{P}[a_{i+n} = j \mid a_i = k], \mathbb{P}[\exists\infty \text{ many } k : a_k > f(k)], \mathbb{P}[\sum_{i=1}^{k} a_k > f(k)], \dots$

Distribution of a_k, q_k for random irrationals

- Drawing an irrational uniformly at random from $[0,1)$, what is the probability that $a_i = m$?
- $\mathbb{P}[q_k \geq f(k)], \mathbb{P}[a_{i+n} = j \mid a_i = k], \mathbb{P}[\exists \infty \text{ many } k : a_k > f(k)], \mathbb{P}[\sum_{i=1}^{k} a_k > f(k)], \ldots$
- $\mathbb{P} =$ Lebesgue measure on $[0,1)$. The statements above (and many more) can be solved with measure-theoretic/probabilistic methods.

Distribution for random irrationals

- Gauss-Kuzmin theorem:

$$\mathbb{P}[a_n = m] = \log_2\left(1 + \frac{1}{m(m+2)}\right) + O(e^{-cn}).$$

Distribution for random irrationals

- Gauss-Kuzmin theorem:

$$\mathbb{P}[a_n = m] = \log_2\left(1 + \frac{1}{m(m+2)}\right) + O(e^{-cn}).$$

- Mixing property:

$$\mathbb{P}[a_i = j, a_{i+n} = k] = \mathbb{P}[a_i = j] \cdot \mathbb{P}[a_{i+n} = k] + O(e^{-cn}).$$

Distribution for random irrationals

- Gauss-Kuzmin theorem:

$$\mathbb{P}[a_n = m] = \log_2\left(1 + \frac{1}{m(m+2)}\right) + O(e^{-cn}).$$

- Mixing property:

$$\mathbb{P}[a_i = j, a_{i+n} = k] = \mathbb{P}[a_i = j] \cdot \mathbb{P}[a_{i+n} = k] + O(e^{-cn}).$$

- The distribution of $(a_i)_{i\in\mathbb{N}}$ is very close to i.i.d. variables $(X_i)_{i\in\mathbb{N}}$ where $\mathbb{P}[X_i = m] = \log_2\left(1 + \frac{1}{m(m+2)}\right) \Rightarrow$ many theorems from classical probability hold (laws of large numbers, central limit theorems, LDPs, ...).

Diophantine behaviour of random rationals

- Question: What can be transferred to a random rational? What is meant by random rational? Two natural candidates:

Diophantine behaviour of random rationals

- Question: What can be transferred to a random rational? What is meant by random rational? Two natural candidates:
- The Farey fractions: For a fixed large integer N, we pick a fraction uniformly at random from $\mathcal{F}_N := \{\frac{a}{b} : a \leq b \leq N, (a,b) = 1\}$ (Dynamical methods can be applied)

Diophantine behaviour of random rationals

- Question: What can be transferred to a random rational? What is meant by random rational? Two natural candidates:
- The Farey fractions: For a fixed large integer N, we pick a fraction uniformly at random from $\mathcal{F}_N := \{\frac{a}{b} : a \leq b \leq N, (a,b) = 1\}$ (Dynamical methods can be applied)
- Reduced fractions with fixed denominator: For a fixed large integer N, we pick a fraction uniformly at random from $\{\frac{a}{N} : 1 \leq a \leq N : (a,N) = 1\}$ (we worked with this one - equidistribution and sieve theory).

Considered objects

- Distribution of $\sum_i f(a_i(\alpha)), \sum_i (-1)^i f(a_i(\alpha))$ where f is a well-behaving function and $\alpha = [0; a_1, a_2, \ldots]$.

Considered objects

- Distribution of $\sum_i f(a_i(\alpha)), \sum_i (-1)^i f(a_i(\alpha))$ where f is a well-behaving function and $\alpha = [0; a_1, a_2, \ldots]$.
- Prominent entities within this framework:
  - Gauss-Kuzmin statistics ($f = \mathbb{1}_{[a_i = m]}$)
  - Sum of partial quotients ($f(x) = x$), related to the Discrepancy of $(n\alpha)_{n \in \mathbb{N}}$.
  - Alternating sum $\sum_i (-1)^i a_i(\alpha)$, closely related to Dedekind sums.
  - Maximal partial quotient $f = \mathbb{1}_{[a_i \geq m]}$, related to Zaremba's conjecture.

Gauss-Kuzmin statistics

Theorem (Gauss/Kuzmin, 1800/1929)

Irrational case: $\lim_{i\to\infty} \mathbb{P}[a_i = m] = \log_2\left(1 + \frac{1}{m(m+2)}\right).$

Gauss-Kuzmin statistics

Theorem (Gauss/Kuzmin, 1800/1929)

Irrational case: $\lim_{i\to\infty} \mathbb{P}[a_i = m] = \log_2\left(1 + \frac{1}{m(m+2)}\right).$

Theorem (Balladi,Vallée, 2015)

Farey: $\frac{\sum_{i=1}^{r} \mathbb{1}_{[a_i=m]} - \log_2\left(1+\frac{1}{m(m+2)}\right)\log N}{\sigma_m\sqrt{\log N}} \xrightarrow{d} \mathcal{N}(0,1).$

Gauss-Kuzmin statistics

Theorem (Gauss/Kuzmin, 1800/1929)

Irrational case: $$\lim_{i\to\infty} \mathbb{P}[a_i = m] = \log_2\left(1 + \frac{1}{m(m+2)}\right).$$

Theorem (Balladi,Vallée, 2015)

Farey: $$\frac{\sum_{i=1}^{r} \mathbb{1}_{[a_i=m]} - \log_2\left(1+\frac{1}{m(m+2)}\right)\log N}{\sigma_m\sqrt{\log N}} \xrightarrow{d} \mathcal{N}(0,1).$$

Theorem (Aistleitner, Borda, H., 2023+)

Reduced fractions with fixed denominator:

$$\lim_{N\to\infty} \frac{1}{\varphi(N)} \frac{\pi^2}{12\log 2 \log N} \sum_{a\in\mathbb{Z}_N^*} \sum_{i=1}^{r} \mathbb{1}_{[a_i=m]} = \log_2\left(1+\frac{1}{m(m+2)}\right).$$

Dedekind sums

Theorem (Vardi, 1993)

$$\textit{Farey: } \frac{2\pi \sum_i^r (-1)^i a_i}{\log N} \xrightarrow{d} \textit{Cauchy } (0,1).$$

Dedekind sums

Theorem (Vardi, 1993)

$$\textit{Farey: } \frac{2\pi \sum_i^r (-1)^i a_i}{\log N} \xrightarrow{d} \textit{Cauchy } (0,1).$$

Theorem (Aistleitner, Borda, H., 2023+)

Reduced fractions with fixed denominator: For any $0 < t \leq (\log N)^C$,

$$\mathbb{P}\left[\left|\sum_{i=1}^r (-1)^i a_i\right| \geq t \log N\right] \ll \frac{1}{t}.$$

- Same asymptotic tail estimate as in the Farey case.

Sum of partial quotients

Theorem (Bettin, Drappeau, 2022)

$$\textit{Farey:} \quad \frac{\sum_{i=1}^{r} a_i - \frac{12}{\pi^2}\log N \log\log N - \gamma \log N}{\log N} \xrightarrow{d} S_1\left(\frac{\pi}{6}, 1, 0\right).$$

Sum of partial quotients

Theorem (Bettin, Drappeau, 2022)

Farey: $$\frac{\sum_{i=1}^r a_i - \frac{12}{\pi^2}\log N \log\log N - \gamma \log N}{\log N} \xrightarrow{d} S_1\left(\frac{\pi}{6}, 1, 0\right).$$

Theorem (Aistleitner, Borda, H., 2023+)

Reduced fractions with fixed denominator: For any $0 < t \leq (\log N)^C$,

$$\mathbb{P}\left[\left|\sum_{i=1}^r a_i - \frac{12}{\pi^2}\log N \log\log N\right| \geq t \log N\right] \ll \frac{1}{t}.$$

- Same asymptotic tail estimate as in the Farey case.

$$\mathbb{P}\left[\left|\sum_{i=1}^{r} a_i - \frac{12}{\pi^2}\log N \log\log N\right| \geq t\log N\right] \ll \frac{1}{t}.$$

Corollary (Aistleitner, Borda, H., 2023+)

$$\forall N \in \mathbb{N}\ \exists a \in \mathbb{Z}_N^* : \sum_{i}^{r} a_i(a/N) \leq \frac{12}{\pi^2}\log N \log\log N + O(\log N).$$

- Improves upon the (implicit) constants found by Larcher (1986)/Rukavishnikova(2006).

Maximal partial quotient

Theorem (Hensley, 1991)

$$\textit{Farey:} \quad \lim_{N\to\infty} \mathbb{P}\left[\max_{i\leq r} a_i \geq t\log N\right] = 1 - e^{-\frac{12}{\pi^2 t}}.$$

Maximal partial quotient

Theorem (Hensley, 1991)

$$\textit{Farey:} \quad \lim_{N\to\infty} \mathbb{P}\left[\max_{i\le r} a_i \ge t \log N\right] = 1 - e^{-\frac{12}{\pi^2 t}}.$$

Theorem (Aistleitner, Borda, H., 2023+)

Reduced fractions with fixed denominator: For any $0 < t \le (\log N)^C$,

$$\mathbb{P}\left[\max_{i\le r} a_i \ge t \log N\right] \le \frac{12}{\pi^2 t} + O\left(\frac{(\log\log N)^3}{t \log N}\right).$$

- By $1 - e^{-x} = x + O(x^2)$, same tail behaviour.

Zaremba's conjecture

Conjecture (Zaremba, 1972 (still open))

$$\forall N \in \mathbb{N}\ \exists a \in \mathbb{Z}_N^* : \max_{i \leq r} a_i(a/N) \leq 5.$$

Zaremba's conjecture

Conjecture (Zaremba, 1972 (still open))

$$\forall N \in \mathbb{N}\ \exists a \in \mathbb{Z}_N^* : \max_{i \leq r} a_i(a/N) \leq 5.$$

Theorem (Aistleitner, Borda, H., 2023+)

$$\forall N \in \mathbb{N}\ \exists a \in \mathbb{Z}_N^* : \max_{i \leq r} a_i(a/N) \leq \frac{12}{\pi^2} \log N + O\left((\log \log N)^3\right).$$

- Best bound known so far for general N.

The method - expected value

Let $S_f\left(\frac{a}{N}\right) = \sum_{i=1}^{r} f(a_i)$ where $a/N = [0; a_1, \ldots, a_r]$.

The method - expected value

Let $S_f\left(\frac{a}{N}\right) = \sum_{i=1}^{r} f(a_i)$ where $a/N = [0; a_1, \ldots, a_r]$. For given $b/k = [0; a_1, a_2, \ldots, a_j], k < N$, let $I_m(b/k) = ([0; a_1, a_2, \ldots, a_j, m], [0; a_1, a_2, \ldots, a_j, m+1])$. Then $a/N \in I_m(b/k) \Leftrightarrow a/N = [0; a_1, a_2, \ldots, a_j, m, \ldots]$.

The method - expected value

Let $S_f\left(\frac{a}{N}\right) = \sum_{i=1}^{r} f(a_i)$ where $a/N = [0; a_1, \dots, a_r]$. For given $b/k = [0; a_1, a_2, \dots, a_j], k < N$, let $I_m(b/k) = ([0; a_1, a_2, \dots, a_j, m], [0; a_1, a_2, \dots, a_j, m+1])$. Then $a/N \in I_m(b/k) \Leftrightarrow a/N = [0; a_1, a_2, \dots, a_j, m, \dots]$.

$$\Rightarrow \sum_{a \in \mathbb{Z}_N^*} S_f\left(\frac{a}{N}\right) = \sum_{k \leq N} \sum_{b \in \mathbb{Z}_k^*} \sum_{m=1}^{\infty} f(m) \sum_{a \in \mathbb{Z}_N^*} \mathbb{1}_{I_m(b/k)}\left(\frac{a}{N}\right).$$

The method - expected value

Let $S_f\left(\frac{a}{N}\right) = \sum_{i=1}^{r} f(a_i)$ where $a/N = [0; a_1, \ldots, a_r]$. For given $b/k = [0; a_1, a_2, \ldots, a_j], k < N$, let $I_m(b/k) = ([0; a_1, a_2, \ldots, a_j, m], [0; a_1, a_2, \ldots, a_j, m+1])$. Then $a/N \in I_m(b/k) \Leftrightarrow a/N = [0; a_1, a_2, \ldots, a_j, m, \ldots]$.

$$\Rightarrow \sum_{a \in \mathbb{Z}_N^*} S_f\left(\frac{a}{N}\right) = \sum_{k \leq N} \sum_{b \in \mathbb{Z}_k^*} \sum_{m=1}^{\infty} f(m) \sum_{a \in \mathbb{Z}_N^*} \mathbb{1}_{I_m(b/k)}\left(\frac{a}{N}\right).$$

We have (on average) $\lambda(I_m(b/k)) \approx \frac{1}{k^2} \log_2\left(1 + \frac{1}{m(m+2)}\right)$ so if $\{a/N : (a, N) = 1\}$ is well uniformly distributed, everything is fine. Problem: For $k > \sqrt{N}$, interval length $\leq \frac{1}{k^2} < \frac{1}{N}$.

$k > \sqrt{N}$ - reflection by modular inverse

Define a^* by $aa^* = (-1)^r \mod N$ ($a \mapsto a^*$ is a bijection), If $a/N = [0; a_1, \dots, a_r]$, then $a^*/N = [0; a_r, \dots, a_1]$.

$k > \sqrt{N}$ - reflection by modular inverse

Define a^* by $aa^* = (-1)^r \mod N$ ($a \mapsto a^*$ is a bijection), If $a/N = [0; a_1, \ldots, a_r]$, then $a^*/N = [0; a_r, \ldots, a_1]$.

$$\Rightarrow \sum_{a \in \mathbb{Z}_N^*} S_f\left(\frac{a}{N}\right) \approx 2 \sum_{k \leq \sqrt{N}} \sum_{b \in \mathbb{Z}_k^*} \sum_{m=1}^{\infty} f(m) \sum_{a \in \mathbb{Z}_N^*} \mathbb{1}_{I_m(b/k)}\left(\frac{a}{N}\right)$$

$k > \sqrt{N}$ - reflection by modular inverse

Define a^* by $aa^* = (-1)^r \mod N$ ($a \mapsto a^*$ is a bijection), If $a/N = [0; a_1, \dots, a_r]$, then $a^*/N = [0; a_r, \dots, a_1]$.

$$\Rightarrow \sum_{a \in \mathbb{Z}_N^*} S_f\left(\frac{a}{N}\right) \approx 2 \sum_{k \leq \sqrt{N}} \sum_{b \in \mathbb{Z}_k^*} \sum_{m=1}^{\infty} f(m) \sum_{a \in \mathbb{Z}_N^*} \mathbb{1}_{I_m(b/k)}\left(\frac{a}{N}\right)$$

For $1 \leq k < \sqrt{N}$, we have

$$\frac{1}{\varphi(N)} \# \left\{a \in \mathbb{Z}_N^* : \frac{a}{N} \in I_m(b/k)\right\} \approx \lambda(I_m(b/k)) \approx \frac{1}{k^2} \log_2\left(1 + \frac{1}{m(m+2)}\right)$$

by sieve methods/discrepancy estimates.

Variance estimate

$$\mathbb{E}\left[S_f\left(\frac{a}{N}\right)^2\right] \approx \frac{1}{\varphi(N)}\frac{1}{8}\sum_{a\in\mathbb{Z}_N^*}\sum_{i:q_{i-1}<\sqrt{N}} f(a_i(a/N)) \underbrace{\sum_{j\leq i} f(a_j(a/N))}_{=S_f\left(\frac{b}{k}\right),\frac{b}{k}=[0;a_1,\dots,a_i]}$$

$$\approx \frac{1}{\varphi(N)}\frac{1}{8}\sum_{1\leq k<\sqrt{N}}\sum_{b\in\mathbb{Z}_k^*}\sum_{a\in\mathbb{Z}_N^*} w_f\left(\frac{b}{k}-\frac{a}{N}\right) S_f\left(\frac{b}{k}\right)$$

Variance estimate

$$\mathbb{E}\left[S_f\left(\frac{a}{N}\right)^2\right] \approx \frac{1}{\varphi(N)}\frac{1}{8}\sum_{a\in\mathbb{Z}_N^*}\sum_{i:q_{i-1}<\sqrt{N}} f(a_i(a/N)) \underbrace{\sum_{j\leq i} f(a_j(a/N))}_{=S_f\left(\frac{b}{k}\right), \frac{b}{k}=[0;a_1,\ldots,a_i]}$$

$$\approx \frac{1}{\varphi(N)}\frac{1}{8}\sum_{1\leq k<\sqrt{N}}\sum_{b\in\mathbb{Z}_k^*}\sum_{a\in\mathbb{Z}_N^*} w_f\left(\frac{b}{k}-\frac{a}{N}\right) S_f\left(\frac{b}{k}\right)$$

$$\sum_{a\in\mathbb{Z}_N^*} w_f\left(\frac{b}{k}-\frac{a}{N}\right) \approx \frac{\varphi(N)}{k^2}\int_0^\infty \frac{f(x)}{x^2}\mathrm{d}x,$$ almost independent of b.

$$\Rightarrow \approx \frac{1}{8}\int_0^\infty \frac{f(x)}{x^2}\mathrm{d}x \sum_{1\leq k<\sqrt{N}} \frac{1}{k^2} \underbrace{\sum_{b\in\mathbb{Z}_k^*} S_f\left(\frac{b}{k}\right)}_{\text{Expected value w.r.t. } k}.$$

Concentration inequalities

For any $0 < t \leq (\log N)^C$,

$$\mathbb{P}\left[\left|\sum_{i=1}^{r} a_i - \frac{12}{\pi^2}\log N \log\log N\right| \geq t \log N\right] \ll \frac{1}{t}.$$

- Heavy-tailed distribution: First, remove those a/N where $\max_i a_i(a/N) \geq (\log N)^C$ by Markov. Then apply mean/variance + Chebyshev on $f(x) = x\mathbb{1}_{[x \leq (\log N)^c]}$.

Concentration inequalities

For any $0 < t \leq (\log N)^C$,

$$\mathbb{P}\left[\left|\sum_{i=1}^{r} a_i - \frac{12}{\pi^2}\log N \log\log N\right| \geq t\log N\right] \ll \frac{1}{t}.$$

- Heavy-tailed distribution: First, remove those a/N where $\max_i a_i(a/N) \geq (\log N)^C$ by Markov. Then apply mean/variance + Chebyshev on $f(x) = x\mathbb{1}_{[x \leq (\log N)^c]}$.
- "Typical behaviour deviates from average behaviour": $\mathbb{E}[\sum_{i=1}^{r} a_i] \sim \frac{6}{\pi^2}(\log N)^2$ (Panov/Liehl, 1982/1983), but concentration around $\frac{12}{\pi^2}\log N \log\log N$ (median is much smaller than the mean).

Open questions

- Is it possible to obtain those estimates on (short) intervals or other measures than $Unif(\mathbb{Z}_N^*)$? Say, given $(X, Y) \subset [0, 1]$, what statistics hold for a/N such that $a/N \in (X, Y)$ or $\sum_{a \in \mathbb{Z}_N^*} S_f(a/N) g(a/N)$ where g is a smooth function?

Open questions

- Is it possible to obtain those estimates on (short) intervals or other measures than $Unif(\mathbb{Z}_N^*)$? Say, given $(X, Y) \subset [0, 1]$, what statistics hold for a/N such that $a/N \in (X, Y)$ or $\sum_{a \in \mathbb{Z}_N^*} S_f(a/N) g(a/N)$ where g is a smooth function?
- What about the mixing property?
  $\mathbb{P}[a_{i+n} = m \mid a_i = j] \approx \mathbb{P}[a_{i+n} = m] \cdot \mathbb{P}[a_i = j]$ for $n \to \infty$?

Open questions

- Is it possible to obtain those estimates on (short) intervals or other measures than $Unif(\mathbb{Z}_N^*)$? Say, given $(X, Y) \subset [0, 1]$, what statistics hold for a/N such that $a/N \in (X, Y)$ or $\sum_{a \in \mathbb{Z}_N^*} S_f(a/N) g(a/N)$ where g is a smooth function?
- What about the mixing property? $\mathbb{P}[a_{i+n} = m \mid a_i = j] \approx \mathbb{P}[a_{i+n} = m] \cdot \mathbb{P}[a_i = j]$ for $n \to \infty$?
- Do the same limit laws as in the Farey setting hold without the double-average? If so, do the center/scaling terms depend on the arithmetic structure of N?

Thanks for your attention!