An expression for multiple L-functions in terms of the confluent hypergeometric function

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Introduction

Throughout this talk, let $s = \sigma + it$ be a complex number.

The multiple zeta-function of depth r

$$\zeta_r(s_1, s_2, \dots, s_r) := \sum_{0<m_1<m_2<\cdots<m_r} \frac{1}{m_1^{s_1} m_2^{s_2} \dots m_r^{s_r}}.$$

- If $r = 1$, this is the Riemann zeta-function.
- The series converges absolutely when

$$\sigma_r > 1, \sigma_{r-1} + \sigma_r > 2, \dots, \sigma_1 + \cdots + \sigma_r > r.$$

- Zhao (2000), Akiyama, Egami and Tanigawa (2001) showed that $\zeta_r(s_1, s_2, \dots, s_r)$ can be continued meromorphically to $\mathbb{C}^r$-space.

The case $r = 2$.

- Let
$$g(s_1, s_2) := \zeta_2(s_1, s_2) - \frac{\Gamma(1-s_1)\Gamma(s_1+s_2-1)}{\Gamma(s_2)}\zeta(s_1+s_2-1).$$
- $\Psi(a, c; x)$ denotes the confluent hypergeometric function of the second kind defined by
$$\Psi(a, c; x) := \frac{\Gamma(1-c)}{\Gamma(a-c+1)}{}_1F_1(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}{}_1F_1(a-c+1, 2-c : x),$$
where ${}_1F_1$ is the Kummer's confluent hypergeometric function.

Ψ-expression for the double zeta-function (Matsumoto, 1998)

Let $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$. Then for $\sigma_1 < 0, \sigma_1 + \sigma_2 > 2$ we have
$$g(s_1, s_2) = (2\pi i)^{s_1+s_2-1}\Gamma(1-s_1)$$
$$\times \left\{ \sum_{n=1}^{\infty} \sigma_{s_1+s_2-1}(n)\Psi(s_2, s_1+s_2; 2\pi in) - \sum_{n=1}^{\infty} \sigma_{s_1+s_2-1}(n)\Psi(s_2, s_1+s_2; -2\pi in) \right\}.$$

Functional equation

the reflection formula

$$\Psi(a, c; x) = x^{1-c}\Psi(a - c + 1, 2 - c; x).$$

$$F_2^{\pm}(s_1, s_2) := \sum_{k=1}^{\infty} \sigma_{s_1+s_2-1}(k)\Psi(s_2, s_1 + s_2; \pm 2\pi ik)$$

can be continued meromorphically to $\mathbb{C}^2$.

It holds that

$$F_2^{\pm}(s_1, s_2) = (\pm 2\pi)^{1-s_1-s_2} F^{\pm}(1 - s_2, 1 - s_1).$$

Functional equation for the double zeta-function (Matsumoto, 2004)

For any $s_1, s_2 \in \mathbb{C}$, except for singularity points, we have

$$\frac{g(s_1, s_2)}{(2\pi)^{s_1+s_2-1}\Gamma(1 - s_1)} = \frac{g(1 - s_2, 1 - s_1)}{i^{s_1+s_2-1}\Gamma(s_2)} + 2i \sin\left(\frac{\pi}{2}(s_1 + s_2 - 1)\right) F_2^{+}(s_1, s_2).$$

Generalization of the Matsumoto's result

The Mordell-Tornheim multiple zeta-function

$$\zeta_{MT,r}(s_1,\ldots,s_r,s_{r+1}) := \sum_{m_1,\ldots,m_r\geq 1} \frac{1}{m_1^{s_1}\ldots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}}.$$

$$\begin{aligned} g_{MT,r}(s_1,\ldots,s_r,s_{r+1}) &:= \zeta_{MT,r}(s_1,\ldots,s_r,s_{r+1}) \\ &\quad - \frac{\Gamma(1-s_r)\Gamma(s_r+s_{r+1}-1)}{\Gamma(s_{r+1})}\zeta_{MT,r-1}(s_1,\ldots,s_r-1,s_r+s_{r+1}-1). \end{aligned}$$

Okamoto and Onozuka (2016)

$$\begin{aligned} &g_{MT,r}(s_1,\ldots,s_r,s_{r+1}) = (2\pi i)^{s_r+s_{r+1}-1}\Gamma(1-s_r) \\ &\times \left\{ \sum_{\ell_1,\ldots,\ell_{r-1}=1}^{\infty} \frac{\sigma_{s_1+\cdots+s_r+s_{r+1}-1}(\gcd(\ell_1,\ldots,\ell_{r-1}))}{\ell_1^{s_1}\ldots\ell_{r-1}^{s_{r-1}}}\Psi(s_{r+1},s_r+s_{r+1};2\pi i(\ell_1+\cdots+\ell_{r-1})) \right. \\ &\left. - \sum_{\ell_1,\ldots,\ell_{r-1}=1}^{\infty} \frac{\sigma_{s_1+\cdots+s_r+s_{r+1}-1}(\gcd(\ell_1,\ldots,\ell_{r-1}))}{\ell_1^{s_1}\ldots\ell_{r-1}^{s_{r-1}}}\Psi(s_{r+1},s_r+s_{r+1};-2\pi i(\ell_1+\cdots+\ell_{r-1})) \right\}. \end{aligned}$$

Mordell-Tornheim multiple L-function

Definition(Mordell-Tornheim multiple L-function)

$$\mathcal{L}_{MT,r}(s_1, \ldots, s_r, s_{r+1}; a_1, \ldots, a_r) := \sum_{m_1,\ldots,m_r \geq 1} \frac{a_1(m_1)\ldots a_r(m_r)}{m_1^{s_1}\ldots m_r^{s_r}(m_1+\cdots+m_r)^{s_{r+1}}},$$

where $\{a_1(n)\}_{n\geq 1}, \ldots, \{a_r(n)\}_{n\geq 1}$ satisfy that Dirichlet series $L(s, a_j) = \sum_{n=1}^{\infty} a_j(n)n^{-s}$ are belong to **the Selberg class** $\mathcal{S}$.

The Selberg class $\mathcal{S}$ is the class of L-functions $\mathcal{L}(s) = \sum_{n=1}^{\infty} a_{\mathcal{L}}(n) n^{-s}$ satisfying the following conditions (S1) - (S5):

- (S1) The series $\mathcal{L}(s) = \sum_{n=1}^{\infty} a_{\mathcal{L}}(n) n^{-s}$ is convergent absolutely for $\sigma > 1$.
- (S2) There exists $m_{\mathcal{L}} \in \mathbb{Z}_{\geq 0}$ such that $(s-1)^{m_{\mathcal{L}}} \mathcal{L}(s)$ is entire of finite order.
- (S3) For all $s \in \mathbb{C}$, $\mathcal{L}(s)$ satisfies the functional equation $Z_{\mathcal{L}}(s) = \omega_{\mathcal{L}} \overline{Z_{\mathcal{L}}(1-\overline{s})}$, where

$$Z_{\mathcal{L}}(s) = Q^s \prod_{j=1}^{k} \Gamma(\lambda_j s + \mu_j) \mathcal{L}(s) = \gamma(s) \mathcal{L}(s)$$

  with $\lambda_j > 0, Q > 0, \mathrm{Re}(\mu_j) \geq 0$, and $\mid \omega_{\mathcal{L}} \mid = 1$.
- (S4) For $\sigma > 1$, $\mathcal{L}(s)$ can be written as $\mathcal{L}(s) = \prod_p \exp\left(\sum_{k=1}^{\infty} \frac{b_{\mathcal{L}}(p^k)}{p^{ks}}\right)$, where $b_{\mathcal{L}}(n) = 0$ and $b_{\mathcal{L}}(n) \ll n^{\theta_{\mathcal{L}}}$ for $\theta_{\mathcal{L}} \in [0.\frac{1}{2})$.
- (S5) For any $n \in \mathbb{Z}_{\geq 1}$, $a_{\mathcal{L}}(n) \ll n^{\varepsilon}$.

The degree of $\mathcal{L}$

$$d_{\mathcal{L}} := 2 \sum_{j=1}^{k} \lambda_j.$$

Main result

Kaczorowski-Perelli (1999)

Let $\mathcal{L} \in \mathcal{S}$ with $d_{\mathcal{L}} = 1$. Then

$$\mathcal{L}(s) = \zeta(s), \quad \text{or} \quad L(s + i\theta, \chi^*)$$

where χ^* is a primitive Dirichlet character.

Theorem (T., 2023+)

If one of $L(s, a_1), \ldots, L(s, a_r)$ has the degree 1, then $\mathcal{L}_{MT,r}(s_1, \ldots, s_r, s_{r+1}; a_1, \ldots, a_r)$ has an expression in terms of the confluent hypergeometric function Ψ.

If $L(s, a_1) = \cdots = L(s, a_r) = \zeta(s)$, this recovers the result of Okamoto and Onozuka.

Sketch of Proof

the Mellin-Barnes formula

$$(1+\lambda)^{-s} = \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s+z)\Gamma(-z)}{\Gamma(s)}\lambda^z dz,$$

$$\begin{aligned}
&\mathcal{L}_{MT,r}(s_1,\ldots,s_r,s_{r+1};a_1,\ldots,a_r)\\
&=\sum_{m_1,\ldots,m_r\geq 1} \frac{a_1(m_1)\ldots a_r(m_r)}{m_1^{s_1}\ldots m_r^{s_r} m_r^{s_{r+1}}\left(1+\frac{m_1+\cdots+m_{r-1}}{m_r}\right)^{s_{r+1}}}\\
&=\frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\mathcal{L}_{MT,r-1}(s_1,\ldots,s_{r-1},-z;a_1,\ldots,a_{r-1})L(s_r+s_{r+1}+z,a_r)dz.
\end{aligned}$$

The Mellin-Barnes integral expression of Ψ:

$$\Psi(s_{r+1},s_r+s_{r+1};\pm 2\pi ik) = \frac{1}{2\pi i}\int_{(\gamma)} \frac{\Gamma(s_{r+1}+z)\Gamma(-z)\Gamma(1-s_r-s_{r+1}-z)}{\Gamma(s_{r+1})\Gamma(-s_r+1)}(\pm 2\pi ik)^z dz.$$

where $-\mathrm{Re}(s_{r+1}) < \gamma < \min\{0, 1-\mathrm{Re}(s_r+s_{r+1})\}$.

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$$\begin{aligned}
&\mathcal{L}_{MT,r}(s_1,\ldots,s_r,s_{r+1};a_1,\ldots,a_r)\\
&= \sum_{m_1,\ldots,m_r\geq 1} \frac{a_1(m_1)\ldots a_r(m_r)}{m_1^{s_1}\ldots m_r^{s_r} m_r^{s_{r+1}}\left(1+\frac{m_1+\cdots+m_{r-1}}{m_r}\right)^{s_{r+1}}}\\
&= \frac{1}{2\pi i}\int_{(c)} \frac{\Gamma(s_{r+1}+z)\Gamma(-z)}{\Gamma(s_{r+1})}\mathcal{L}_{MT,r-1}(s_1,\ldots,s_{r-1},-z;a_1,\ldots,a_{r-1})L(s_r+s_{r+1}+z,a_r)dz.
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where $-\mathrm{Re}(s_{r+1}) < \gamma < \min\{0, 1-\mathrm{Re}(s_r+s_{r+1})\}$.

- the functional equation for $\zeta(s)$:

$$\zeta(s_r + s_{r+1} + z) = 2^{s_r+s_{r+1}+z}\pi^{s_r+s_{r+1}+z-1}\cos\frac{\pi}{2}(1 - s_r - s_2 - z) \times \Gamma(1 - s_r - s_{r+1} - z)\zeta(1 - s_r - s_{r+1} - z).$$

- the functional equation for $L(s,\chi)$:

$$L(s_r + s_{r+1} + z, \chi) = \varepsilon(\chi)\frac{(2\pi)^{s_r+s_{r+1}+z}}{\pi}q^{\frac{1}{2}-s_r-s_{r+1}-z}\cos\frac{\pi}{2}(1 - s_r - s_{r+1} - z - \kappa) \times \Gamma(1 - s_r - s_{r+1} - z)L(1 - s_r - s_{r+1} - z, \overline{\chi}).$$

λ-conjecture

Every $\mathcal{L} \in \mathcal{S}$ has a γ-factor with $\lambda_j = \frac{1}{2}$ for $j = 1, 2, \ldots, k$.

$$d_{\mathcal{L}} = 2\sum_{j=1}^{k}\lambda_j.$$

If λ-conjecture is true, then there is no other $L(s,a) = \sum_{n=1}^{\infty} a(n)n^{-s} \in \mathcal{S}$ such that it has only one Γ-function in the functional equation.

Thank you for your attention!