Investigating divisibility properties of quotient sequences derived from Lucas and elliptic divisibility sequences

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Lucas Sequences

Definition 1 (Lucas Sequence)

Let P and Q be relatively prime integers. The Lucas sequence is defined by

- $U_0 = 0$, $U_1 = 1$, and
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- $P = 1$, $Q = -1 \implies$ the sequence of the Fibonacci numbers $(F_n)_{n\geq 0}$:

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- $P = 3$, $Q = 2 \implies$ the sequence of the Mersenne numbers $(M_n)_{n\geq 0}$:

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- In general,

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where α and β are the zeroes of the characteristic polynomial $p(x) = x^2 - Px + Q$.

Elliptic Divisibility Sequences - Recurrence Definition

Definition 2 (Elliptic Divisibility Sequence (EDS))

A sequence $(h_n)_{n\geq 0}$ is said to be an elliptic divisibility sequence if

- $h_{m+n}h_{m-n} = h_{m+1}h_{m-1}h_n^2 - h_{n+1}h_{n-1}h_m^2$ for all $m \geq n \geq 0$, and
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For example,

- The sequence $(n)_{n\geq 0}$ of nonnegative integers:

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- The sequence $\left((-1)^{(n-1)(n-2)/2}F_n\right)$ where F_n is the nth Fibonacci number:

$$0, \quad 1, \quad 1, \quad -2, \quad -3, \quad 5, \quad 8, \quad -13, \quad -21, \quad 34, \quad \ldots$$

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(F_n) satisfies the following identity:

$$F_{m+n}F_{m-n} = (-1)^{n+1}(F_{m+1}F_{m-1}F_n^2 - F_{n+1}F_{n-1}F_m^2).$$

Elliptic Divisibility Sequences - Elliptic Curve Based Definition

- Weierstrass equation: $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ with integer coefficients.

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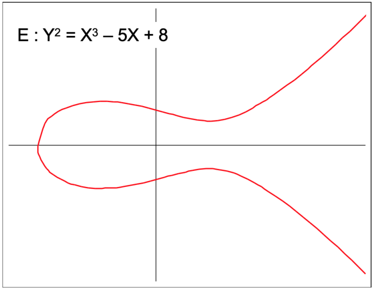
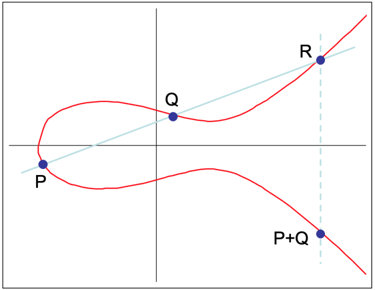
- Rational points on this curves form a group $E(\mathbb{Q})$.



credit: J. Silverman, K. Stange



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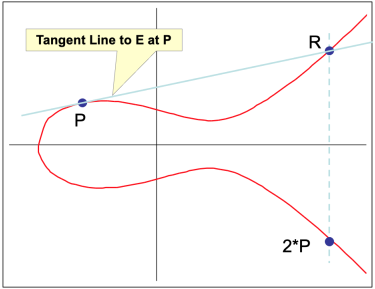
Adding Points on Elliptic Curves (cont.)



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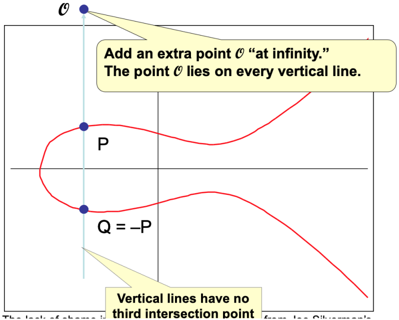

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Lemma 3 (Sanna)

Let p be a prime such that $p \nmid Q$. Then, for each positive integer n,

$$\nu_p(U_n) = \begin{cases} \nu_p(n) + \nu_p(U_p) - 1, & p \mid D \text{ and } p \mid n; \\ 0, & p \mid D \text{ and } p \nmid n; \\ \nu_p(n) + \nu_p(U_{p\tau(p)}) - 1, & p \nmid D, \tau(p) \mid n, \text{ and } p \mid n; \\ \nu_p(U_{\tau(p)}), & p \nmid D, \tau(p) \mid n, \text{ and } p \nmid n; \\ 0, & p \nmid D \text{ and } \tau(p) \nmid n, \end{cases}$$

where $\tau(p) =$ least positive integer such that $p \mid U_{\tau(p)}$.

Lemma 4 (Panraksa, T)

Let $n, k \geq 1$ and p a prime factor of U_k such that $p \nmid Q$. Then

- *if (i) p is odd, or (ii) $p = 2$ and k is even, or (iii) $p = 2$ and n is odd, we have*

$$\nu_p(U_{kn}) = \nu_p(n) + \nu_p(U_k);$$

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$$\nu_p(U_{kn}) = \nu_p(n) + \nu_p(U_k);$$

- *if* k *and* D *are odd and* n *is even, we have*

$$\nu_2(U_{kn}) = \nu_2(n) + \nu_2(U_k) + \nu_2(U_{2\tau(2)}) - \nu_2(U_{\tau(2)}) - 1,$$

where $D = P^2 - 4Q$*, the discriminant of the characteristic polynomial of the sequence* (U_n)*.*

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Lemma 5

Let $(B_n)_{n\geq 1}$ be an elliptic divisibility sequence corresponding to an elliptic curve E with the Weierstrass equation: $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ and a non-torsion point P in $E(\mathbb{Q})$.

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- *For every pair of positive integers m, n, if $\nu_2(B_n) > 0$ then $\nu_2(B_{mn}) = \nu_2(B_n) + \nu_2(m)$ if the coefficient a_1 is even and $|\nu_2(B_{mn}) - (\nu_2(B_n) + \nu_2(m))| \leq \epsilon$ otherwise, where the constant ϵ depends only on E and P.*

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- *For all positive integers m, n,*

$$\gcd(B_m, B_n) = B_{\gcd(m,n)},$$

  i.e., EDS is a strong divisibility sequence.

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$$T_n = \left| \frac{U_{n\Delta}}{U_n U_\Delta} \right|,$$

where $\Delta = |D|$ and D is the discriminant of the characteristic polynomial $x^2 - Px + Q$ associated with the Lucas sequence $(U_n)_{n\geq 0}$.

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For example, for the sequence $U(4, -7)$, we have $\Delta = 12$ and the first five terms of the sequence (T_n) are

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Definition 6

Let N be a positive integer. A sequence (u_n) of rational numbers is said to be an N-almost strong divisibility sequence if for all m and n where u_m and u_n are integers we have

$$\gcd(u_m, u_n) = u_{\gcd(m,n)}$$

whenever $\gcd(mn, N) = 1$.

Theorem 7 (Panraksa, T)

The sequence $(T_n)_{n\geq 1}$ is a Δ-almost strong divisibility sequence.

Let n be a positive integer. Define the sequence $(H_k(n))_{k\geq 1}$ by $H_1(n) = T_n$ and $H_k(n) = T_{nH_{k-1}(n)}$ for $k \geq 2$. The first few terms of the sequence $(H_k(n))_{k\geq 1}$ are

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$$T_n, \quad T_{nT_n}, \quad T_{nT_{nT_n}}, \quad T_{nT_{nT_{nT_n}}}.$$

Theorem 8 (Panraksa, T)

Suppose $\gcd(n, \Delta) = 1$ *and* $T_n \neq 1$. *Then, for each positive integer* k,

$$T_n^k \mid\mid H_k(n).$$

Let τ be a positive integer and $(B_n)_{n\geq 1}$ an elliptic divisibility sequence corresponding to an elliptic curve with the Weierstrass equation: $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ and a non-torsion point P. Define the sequence $(K_n)_{n\geq 1}$ by

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Theorem 9 (Panraksa, T)

If the coefficient a_1 is even and $\tau \mid B_\tau$, then the sequence $(K_n)_{n\geq 1}$ is a τ-almost strong divisibility sequence. That is, for all positive integers m,n, if $\gcd(mn, \tau) = 1$, then

$$\gcd(K_m, K_n) = K_{\gcd(m,n)}.$$

For example, the elliptic divisibility sequence $(B_n)_{n\geq 1}$ corresponding to the elliptic curve $E:\ y^2+y=x^3-x$ and the point $P=(0,0)$ is

$$1,\quad 1,\quad 1,\quad 1,\quad 2,\quad 1,\quad 3,\quad 5,\quad 7,\quad 4,\quad 23,\quad 29,\quad 59,\quad 129,\quad 314,\quad 65,\quad 1529,\quad \ldots$$

For example, the elliptic divisibility sequence $(B_n)_{n\geq 1}$ corresponding to the elliptic curve $E: \; y^2 + y = x^3 - x$ and the point $P = (0,0)$ is

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One can check that $40 \mid B_{40}$. Then the sequence $(K_n)_{n\geq 1}$ defined by

$$K_n = \frac{B_{40n}}{B_{40}B_n} = \frac{B_{40n}}{(40 \cdot 13526278251270010)B_n}$$

for all $n \geq 1$ satisfies

$$\gcd(K_m, K_n) = K_{\gcd(m,n)}$$

whenever $\gcd(mn, 40) = 1$.

Theorem 10 (Panraksa, T)

Let $(B_n)_{n\geq 1}$ be an elliptic divisibility sequence corresponding to an elliptic curve whose Weierstrass equation: $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ has a_1 even. Let n be a positive integer. Define a sequence $(G_k(n))_{k\geq 1}$ as follows: $G_1(n) = B_n$ and $G_k(n) = B(nG_{k-1}(n))$ for $k \geq 2$. Then, if $B_n \neq 1$, we have

$$B_n^k \,||\, G_k(n)$$

for all positive integers k.

Lemma 11 (Matijasevich)

For $n > 2$, we have

$$F_n^2 \mid F_m \quad \text{if and only if} \quad nF_n \mid m.$$

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Hilbert's 10th Problem

Is there a general algorithm to determine whether a given Diophantine equation (a polynomial equation with integer coefficients and a finite number of unknowns) has a solution in integers?

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Theorem 12 (Panraksa, T)

Let $(B_n)_{n\geq 1}$ be an elliptic divisibility sequence corresponding to an elliptic curve whose Weierstrass equation: $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ has a_1 even. Moreover, suppose that there exists a positive integer N such that all terms of the sequence $(B_n)_{n\geq N}$ are distinct and none of the terms $B_1, \ldots, B_{N-1}$ appears in $(B_n)_{n\geq N}$. Then, for all integers $n, r \geq N$ and for all positive integers k, we have

$$B_n^k \mid B_r \quad \textit{if and only if} \quad nB_n^{k-1} \mid r.$$

Table of Contents

Thank You!

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