On Ergodic Theorems and The Riemann Hypothesis

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Riemann Zeta function Basics

Let

$$\zeta(s) = \sum_{n \geq 1} n^{-s}$$

for $s = \sigma + it \in \mathbb{C}$ where $\sigma > 1$. Then $g(s) := \zeta(s) - \frac{1}{s-1}$ can be analytically continued to $\mathbb{C}$. With this extended definition, ζ can be shown to satisfy the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s),$$

where $\Gamma(s)$ denotes Euler's gamma function.

This tells us ζ has zeros called trivial zeros, at negative even integers. All others, called non-trivial are confined to the "critical strip" $0 < \sigma < 1$ in $\mathbb{C}$. The Riemann Hypothesis is that they are on the line $\sigma = \frac{1}{2}$.

Some statements equivalent to the Riemann Hypothesis

(i) For a real number $x \geq 1$, let $\pi(x)$ denote the number prime numbers in the interval $[1, x]$.
Let

$$Li(x) = \int_0^x \frac{dt}{\log t}.$$

The Riemann Hypothesis is equivalent to the statement: Given $\epsilon > 0$

$$\pi(x) = Li(x) + o(x^{\frac{1}{2}+\epsilon}).$$

The Prime Number Theorem says, that as x tends to infinity, $\pi(x) \sim Li(x)$.

The Lindelhöf Hypothesis

The Lindelhöf Hypothesis, which is implied by the Riemann Hypothesis says that given $\epsilon > 0$ we have $\zeta\left(\frac{1}{2} + it\right) = o(t^{\epsilon})$.

The Riemann Hypothesis implies there exists a constant $A > 0$ such that

$$\zeta\left(\frac{1}{2} + it\right) = O\left(\exp A\left\{\frac{\log t}{\log\log t}\right\}\right),$$

which is stronger than the Lindelhöf Hypothesis.

Random Sampling

Let $(X_i)_{i\geq 1}$ be a sequence of independent Cauchy random variables, with characteristic function $\phi(t) = e^{|t|}$ and consider the partial sums $S_n = X_1 + \ldots, X_n$ $(n = 1, 2, \ldots)$.

M. Lifshits and M. Weber studied the value distribution of the Riemann zeta funtion $\zeta(s)$ sampled along the Cauchy random walk $(S_n)_{n\geq 1}$ showing, for $b > 2$, that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \zeta\left(\frac{1}{2} + iS_n\right) = 1 + o\left(\frac{(\log N)^b}{N^{\frac{1}{2}}}\right).$$

The Boole Dynamical System

For (X, β, μ) a measure space, usually a probability space, let $T : X \to X$ be a map of X preserving μ i.e. if $T^{-1}A = \{Tx : x \in A\}$, then $\mu(T^{-1}A) = \mu(A)$ for all $A \in \beta$.

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We say the dynamical system (X, β, μ, T) is ergodic if $T^{-1}A = A$ $\mu(A)$ is either 1 or 0.

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Observation of G. Boole :

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx.$$

But dx is not a probability measure on $\mathbb{R}$ so we use Cauchy probability $\frac{dx}{x^2+1}$ which is and is also preserved by the map $Tx = x - \frac{1}{x}$.

The Lee Surjiajaya Dynamical System

Lee and Suriajaya show, for $x \in \mathbb{R}$ that the maps

$$T_{\alpha,\beta}(x) \;=\; \begin{cases} \frac{\alpha}{2}\left(\frac{x+\beta}{\alpha} - \frac{\alpha}{x-\beta}\right), & \text{if } x \;\neq \beta; \\ \beta, & \text{if } x \;= \beta, \end{cases}$$

for $\alpha > 0$ and real β are measure preserving and ergodic with respect to the probability measure

$$\mu_{\alpha,\beta}(A) = \frac{\alpha}{\pi}\int_A \frac{dt}{\alpha^2 + (t-\beta)^2},$$

for any Lebesgue measurable subset A of the real numbers.

Birkhoff's Ergodic Theorem

Given a measure preserving dynamical system (X, β, μ, T), for $x \in X$ its orbit is $x, Tx, T^2x, T^3x, \ldots..$

Birkhoff's ergodic theorem : Suppose $f \in L^1(X, \beta, \mu)$. Then there exist $\overline{f} \in L^1(X, \beta, \mu)$ with $\overline{f}(Tx) = \overline{f}(x)$ μ almost everywhere such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \overline{f}(x),$$

μ almost everywhere.

If (X, β, μ, T) is ergodic then $\overline{f} = \int_X f d\mu$ μ almost everywhere.

J Steuding's work

Using Birkhoff's ergodic theorem, applied to the Boole dynamical system :

(I) For s with $\Re(s) > -\frac{1}{2}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \zeta(s + iT^n x) = \ell(s) := \int_{\mathbb{R}} \frac{\zeta(s+it)}{1+t^2} dt$$

for almost all x. If $\Re(s) > 1$ then $\ell(s) = \zeta(s+1) - \frac{2}{s(2-s)}$ and if $\Re(s) < 1$ then $\ell(s) = \zeta(s+1)$.

J Steuding's work II

(II) The Lindelhöf being true is equivalent for any $k \in \mathbb{N}$ to the existence of either side of the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\zeta(s + iT^n x)|^k = \int_{\mathbb{R}} \frac{|\zeta(s+it)|^k}{1+t^2} dt$$

for almost all x.

J Steuding's work III

(III) The Riemann Hypothsis being true is equivalent to the existence of either side of the limit and it being zero

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log |\zeta(s + iT^n x)| = \sum_{\Re(\rho)>\frac{1}{2}} \log \left| \frac{\rho}{1-\rho} \right|$$

for almost all x.

Good Universality

We say $(a_n)_{n\geq 1}$ is L^p good universal if given any dynamical system (X, β, μ, T) and $f \in L^p(X, \beta, \mu)$ the limit

$$\ell_{f,T}(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n}x),$$

exists μ almost everywhere.

Is $\ell_{f,T}(x) = \int_X d\mu$ almost everywhere if (X, β, μ, T) is ergodic. Not always. This is crutial for our considerations.

Polynomial like sequences

1. *The natural numbers:* The sequence $(n)_{n=1}^{\infty}$ is L^1-good universal. This is Birkhoff's pointwise ergodic theorem.

2. *Polynomial like sequences:* Note if $\phi(x)$ is a polynomial such that $\phi(\mathbb{N}) \subseteq \mathbb{N}$ (Bourgain 88, Nair 96) and $p > 1$ then $(\phi(n))_{n=1}^{\infty}$ and $(\phi(p_n))_{n=1}^{\infty}$ (Nair 90) where p_n is n^{th} prime are L^p good universal sequences. There are lots of other derminisitic and random constructions, now.

Special Cases

A) Set $X = \mathbb{Z}_p$ (p-adic integers), $Tx = x + 1$ (an ergodic map on $X = \mathbb{Z}_p$) and $\phi(n) = n^2$, $f \in L^p(\mathbb{Z}_p)$ gives

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x + n^2) = \sum_{\chi\in\hat{\mathbb{Z}}_p} G(\chi)\hat{f}_\chi\overline{\chi}(x)$$

with Gauss Sums

$$G(\chi) = \frac{1}{p^r} \sum_{n=1}^{p^r} e^{2\pi i \frac{n^2}{p^r}}.$$

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Special Cases

B) (Density version of Dirichlet Theorem on arithmetic progressions):

Suppose $a, d \in \mathbb{N}$ with $(a, d) = 1$. Let $X = \mathbb{Z}/d\mathbb{Z}$, $Tx = x + 1$, $f = \chi_{a,d}$ (the characteristic function of the residue class a modulo d).

$$\frac{1}{N}\sum_{n=1}^{N}\chi_{a,d}(x + p_n) \to \frac{1}{\phi(a)} \qquad (x = 0)$$

Contained in work of Bourgain 88, Wierdl 89, Nair 91, Mirek 2014, Trojan 2019

Uniform distribution on a group

Any compact abelian topological group G supports a unique translation invariant measure λ called Haar measure. We say a sequence $(x_n)_{n\geq 1}$ is uniform distributed if for each continuous $f : G \to \mathbb{C}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(y_n) = \int_G f(t) d\lambda.$$

We say $\chi : G \to \mathbb{T}$ is a character if $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$.
A sequence $(x_n)_{n\geq 1}$ is uniformly distributed on G if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi(y_n) = 0,$$

for χ other than χ_0 (the identity character) . This is Weyl's criterion.

Two important special cases

1) A sequence of real numbers $(x_n)_{n\geq 1}$ is uniformly distributed modulo one (u.d. mod1) if given any interval $I \subseteq [0,1)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_I(\{y_n\}) = |I|.$$

Here for a real number y we have used $\{y\}$ to denote its fractional part, χ_I denote the characteristic function of the interval I and $|I|$ denotes its length.

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Here for a real number y we have used y to denote its fractional part, χ_I denote the characteristic function of the interval I and $|I|$ denotes its length.

2) We say a sequence of integers $(k_n)_{n\geq 1}$ is uniformly distributed on $\mathbb{Z}$ if for ever natural number $m \geq 2$ and every residue class a modulo m we have

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \leq N : k_n \equiv a \bmod m\} = \frac{1}{m}.$$

Hartman uniform distribution

A sequence of integers $(k_n)_{n\geq 1}$ is Hartman uniformly distributed on $\mathbb{Z}$ if for any irrational number α we have $(\{k_n\alpha\})_{n\geq 1}$ u.d. mod1 and $(k_n)_{n\geq 1}$ is uniformly distributed on $\mathbb{Z}$.

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Note that if $(k_n)_{n\geq 0}$ is Hartman u.d. on $\mathbb{Z}$ and if and only if letting

$$F(N,z) := \frac{1}{N}\sum_{n=0}^{N-1} z^{k_n}, \qquad (N = 1, 2, \cdots)$$

we have $F(N,1) = 1$ for all $N \geq 1$ and if $|z| = 1, z \neq 1$ we have $\lim_{N\to\infty} F(N,z) = 0$.

So $(n)_{n\geq 1}$ is Hartman u.d. on $\mathbb{Z}$;

Hartman and Good Universal Sequences

1. *The natural numbers:* The sequence $(n)_{n=1}^{\infty}$ is L^1-good universal. This is Birkhoff's pointwise ergodic theorem.
2. *Polynomial like sequences:* Note if $\phi(x)$ is a polynomial such that $\phi(\mathbb{N}) \subseteq \mathbb{N}$ (Bourgain, Nair) and $p > 1$ then $(\phi(n))_{n=1}^{\infty}$ and $(\phi(p_n))_{n=1}^{\infty}$ (Nair) where p_n is n^{th} prime are L^p good universal sequences.
3 Specific sequences of integers that satisfy conditions H include $k_n = [g(n)]$ $(n = 1, 2, \ldots)$ where
I. $g(n) = n^{\omega}$ if $\omega > 1$ and $\omega \notin \mathbb{N}$.
II. $g(n) = e^{\log^{\gamma} n}$ for $\gamma \in (1, \frac{3}{2})$.
III. $g(n) = P(n) = b_k n^k + \ldots + b_1 n + b_0$ for $b_k, \ldots, b_1$ not all rational multiples of the same real number.
4. Many other families of sequences, random and deterministic.

Good Universality

We say $(a_n)_{n\geq 1} \subseteq \mathbb{N}$ is L^p good universal if given $f \in L^p(X, \beta, \mu)$ for any (X, β, μ) the limit

$$\overline{f}(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{a_n}x),$$

exists μ almost everywhere.

When is

$$\overline{f}(x) = \int_X f d\mu,$$

μ almost everywhere? Its crutial for our considerations.

Good Universality

We say $(a_n)_{n\geq 1}$ is L^p good universal if given any dynamical system (X, β, μ, T) and $f \in L^p(X, \beta, \mu)$ the limit

$$\ell_{f,T}(x) = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} f(T^{a_n}x),$$

exists μ almost everywhere.

Suppose $(k_i)_{i=1}^{\infty}$ is Hartman uniformly distributed, and L^p-good universal for $p \in [1, 2]$ and that the dynamical system $(X, \mathcal{B}, \mu, T)$ is ergodic. Then the limit $\ell_{T,f}(x)$, defined in the introduction, exists and equals $\int_X f d\mu$ for μ almost all x.

New Work

Suppose $(k_n)_{\geq 1}$ in Hartman uniformly distributed and L^p good univeral. Then

For s with $\Re(s) > -\frac{1}{2}$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \zeta(s + iT^{k_n}x) = \ell(s) := \int_{\mathbb{R}} \frac{\zeta(s+it)}{1+t^2} dt$$

for almost all x. If $\Re(s) > 1$ then $\ell(s) = \zeta(s+1) - \frac{2}{s(2-s)}$ and if $\Re(s) < 1$ then $\ell(s) = \zeta(s+1)$.

New Work II

Suppose $(k_n)_{\geq 1}$ in Hartman uniformly distributed and L^p good univeral. Then the Lindelhöf being true is equivalent for any $k \in \mathbb{N}$ to the existence of either side of the limit

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} |\zeta(s + iT^{k_n}x)|^k = \int_{\mathbb{R}} \frac{|\zeta(s+it)|^k}{1+t^2} dt$$

for almost all x.

New Work III

Suppose $(k_n)_{\geq 1}$ in Hartman uniformly distributed and L^p good univeral. Then the Riemann Hypothsis being true is equivalent for to the either side of the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \log|\zeta(s+iT^{k_n}x)| = \sum_{\Re(\rho)>\frac{1}{2}} \log\left|\frac{\rho}{1-\rho}\right|$$

being zero for almost all x.

Separable Measure Spaces

Suppose (X, β, μ) is a measure space. Given $A, B \in \beta$, we call $d(A, B) := \mu(A \Delta B)$ the Hausdorff metric on β. Here of course $A \Delta B$ denotes the symmetric difference $A \backslash B \cup B \backslash A$. We call (X, β, μ) separable if the metric space (β, d) is separable.

Dynamical and Stochastic results are related

Consider two ergodic separable dynamical systems $(X_1, \beta_1, \mu_1, T_1)$ and $(X_2, \beta_2, \mu_2, T_2)$. Suppose also that μ_1 and μ_2 are non-atomic. Then if for a particular sequence of integers $(k_n)_{n\geq 1}$ for each $f_1 \in L^p(X_1, \beta, \mu_1)$ for all $p > 1$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_1(T_1^{k_n} x_1) = \int_{X_1} f_1(x_1) d\mu_1,$$

μ_1 almost everywhere, then the same is true with 1 replaced by 2.

Now suppose that $(k_n)_{n\geq 1}$ is Hartman uniform distributed and L^p good universal for fixed $p \in [1,\infty)$ and μ is the Cauchy distribution $\mu_{\alpha,\beta}$ then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} f(s+iX_{k_n}(\omega)) = \frac{\alpha}{\pi}\int_{\mathbb{R}} \frac{f(s+i\tau)}{\alpha^2+(\tau-\beta)^2}d\tau,$$

for almost all ω in $\mathbb{R}$.
We can specialise this to

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \log\left|\zeta\left(\frac{1}{2}+\frac{1}{2}iX_{k_n}(\omega)\right)\right| = \sum_{Re(\rho)>\frac{1}{2}} \log\left|\frac{\rho}{1-\rho}\right|,$$

for almost all ω in $\mathbb{R}$.
Again, the Riemann Hypothesis follows if either side is zero

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Thank You For Listening