Log-behaviour of quasi-polynomial-like functions

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Introduction

Quasi-polynomials & Quasi-polynomial-like functions

Definition: Quasi-polynomial
Let $k \in \mathbb{N}$ and $M_1 \in \mathbb{N}_+$. A quasi-polynomial $f(n)$ of degree k is an expression of the form

$$f(n) = t_k(n)n^k + t_{k-1}(n)n^{k-1} + \cdots + t_0(n),$$

where the coefficients $t_0(n), \ldots, t_k(n)$ depend on the residue class of n mod M_1.

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Definition: Quasi-polynomial-like function
Let $d, l \in \mathbb{N}$, $M_2 \in \mathbb{N}_+$ and $d \leqslant l$. We say that a function $g(n)$ is a quasi-polynomial-like function if $g(n)$ might be written as

$$g(n) = \tilde{t}_l(n)n^l + \tilde{t}_{l-1}(n)n^{l-1} + \cdots + \tilde{t}_d(n)n^d + o(n^d),$$

where the coefficients $\tilde{t}_d(n), \ldots, \tilde{t}_l(n)$ depend on the residue class of n mod M_2.

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Definition: The restricted partition function

The restricted partition function $p_{\mathcal{A},k}(n)$ counts restricted partitions of n.

An example of the restricted partition function

Example: Restricted Plane Partitions

Let $\mathcal{A} = (1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots)$

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Let $\mathcal{A} = (1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots)$ and $k = 8$.

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Let $\mathcal{A} = (1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots)$ and $k = 8$. For $n = 4$, we have $p_{\mathcal{A},8}(4) = 11$:

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Remark: Connection between $p_{\mathcal{A},k}(n)$ and $p(n)$
If "$k = \infty$" and $\mathcal{A} = (1, 2, 3, \ldots)$, then $p_{\mathcal{A},\infty}(n) = p(n)$.

Bell's Theorem

Theorem (Bell 1943)
The function $p_{\mathcal{A},k}(n)$ is a quasi-polynomial — it takes the form

$$p_{\mathcal{A},k}(n) = t_{k-1}(n)n^{k-1} + t_{k-2}(n)n^{k-2} + \cdots + t_0(n),$$

where each $t_j(n)$ depends on $n \mod lcm(a_1, a_2, \ldots, a_k)$ for $0 \leqslant j \leqslant k-1$.

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Remark
We can say something more about these coefficients $t_j(n)$:

- Almkvist
- Beck, Gessel and Komatsu
- Israilov

The Motivation

Definition: A log-concave sequence

A sequence $(c_i)_{i=0}^{\infty} \in \mathbb{R}^{\infty}$ is log-concave if $c_n^2 > c_{n-1}c_{n+1}$ for $n \geqslant 1$.

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Theorem (Nicolas 1978, DeSalvo-Pak 2015)
Sequence $p(n)$ is log-concave for all $n > 25$. In other words, we have

$$p^2(n) > p(n+1)p(n-1)$$

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Remark
There are a lot of similar results for other variations of the partition function (e.g. the k-regular partition function $p_k(n)$, the k-colored partition function $p_{-k}(n)$, the plane partition function $pp(n), \ldots$).

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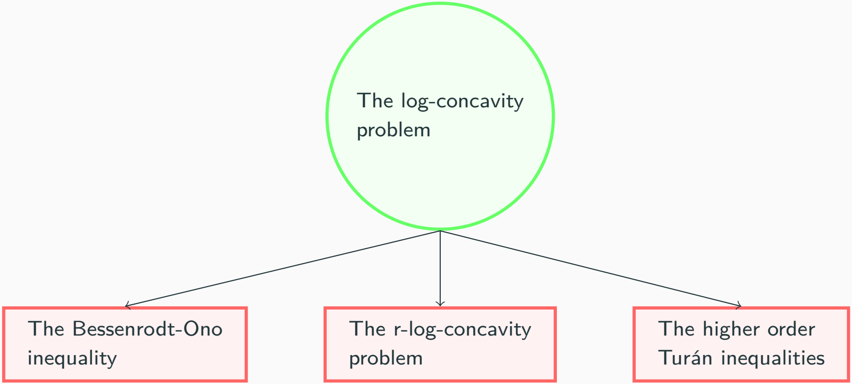

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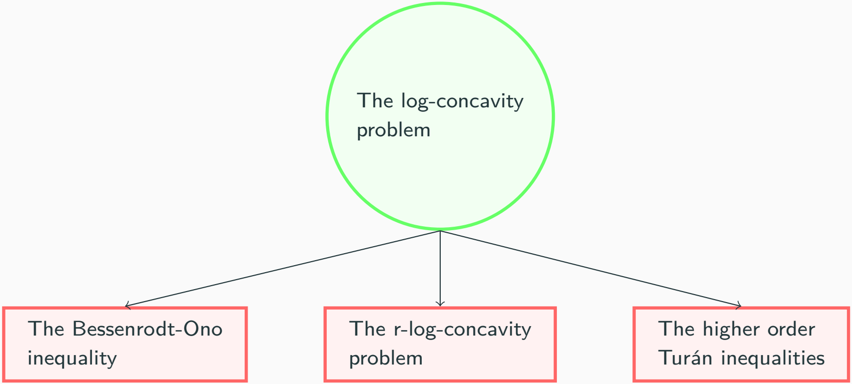

Theorem (Bessenrodt-Ono 2016)
If a, b are integers such that $a, b \geqslant 2$ and $a + b > 9$, then

$$p(a)p(b) > p(a + b).$$

The r-log-concavity problem

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- If $\widehat{\mathcal{L}}_c(n), \widehat{\mathcal{L}}_c^2(n), \ldots, \widehat{\mathcal{L}}_c^r(n) > 0$ for all sufficiently large values of n, then c_n is said to be asymptotically r-log-concave.

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Theorem (Hou-Zhang 2018)
For every positive integer r, the sequence $(p(n))_{n\geqslant 0}$ is asymptotically r-log-concave.

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Theorem (Hou-Zhang 2018)
Let $(b_n)_{n\geqslant 0}$ be a positive sequence such that the Puiseux-type approximation of $b_{n+2}b_n/b_{n+1}^2$ takes the form

$$\frac{b_{n+2}b_n}{b_{n+1}^2} = 1 + \frac{c_1}{n^{\alpha_1}} + \cdots + \frac{c_m}{n^{\alpha_m}} + o\left(\frac{1}{n^{\alpha_m}}\right).$$

If $c_1 < 0$ and $\alpha_1 < 2$, then $(b_n)_{n\geqslant 0}$ is asymptotically $\lfloor \alpha_m/\alpha_1 \rfloor$-log-concave.

Theorem (G. 202?)
Let $l, r \in \mathbb{N}_+$ be such that $l \geqslant 2r$. Suppose further that we have

$$f(n) = a_l(n)n^l + a_{l-1}(n)n^{l-1} + \cdots + a_{l-2r}(n)n^{l-2r} + o\left(n^{l-2r}\right),$$

where the coefficients $a_{l-2r}(n), \ldots, a_l(n)$ might depend on the residue class of $n \pmod M$ for some positive integer $M \geqslant 2$. Then the sequence $(f(n))_{n=0}^{\infty}$ is asymptotically r-log-concave if and only if all the numbers $a_{l-2r}(n), \ldots, a_l(n)$ are independent of the residue class of $n \pmod M$.

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Theorem (G. 202?)
Let $\mathcal{A} = (a_i)_{i=1}^{\infty}$, $r \in \mathbb{N}_+$ and $k > 2r$ be fixed. Then the sequence $(p_{\mathcal{A},k}(n))_{n=0}^{\infty}$ is asymptotically r-log-concave if and only if we have that $\gcd A = 1$ for all $(k-2r)$-multisubsets A of $\{a_1, a_2, \ldots, a_k\}$.

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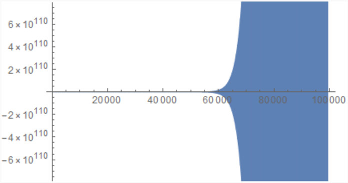

Figure 1: Values of $\widehat{\mathcal{L}^2_{p_{\mathcal{A},10}}}(n)$ for $n \leqslant 10^5$

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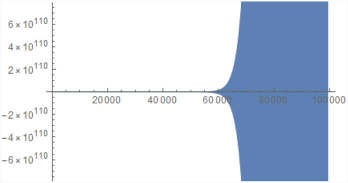

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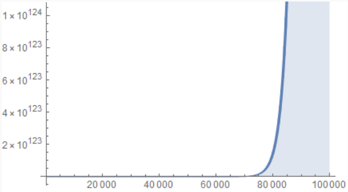

Figure 2: Values of $\widehat{\mathcal{L}^2}_{p_{\mathcal{A},11}}(n)$ for $n \leqslant 10^5$

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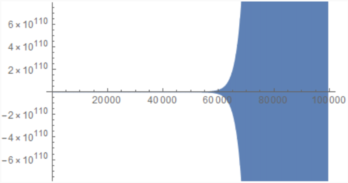

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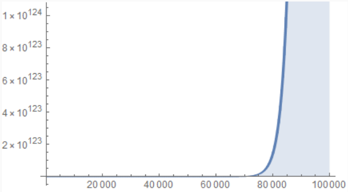

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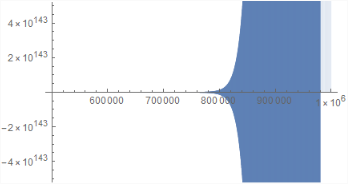

Figure 3: Values of $\widehat{\mathcal{L}^3_{p_{\mathcal{A},12}}}(n)$ for $n \leqslant 10^6$

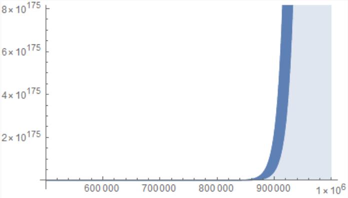

Figure 4: Values of $\widehat{\mathcal{L}^3_{p_{\mathcal{A},13}}}(n)$ for $n \leqslant 10^6$

The higher order Turán inequalities

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Theorem (Griffin-Ono-Rolen-Zagier 2019)
Suppose that $\alpha(n)$, $(E(n))$ and $(\delta(n))$ are positive real sequences with $\lim_{n\to\infty}\delta(n) = 0$, and that $F(t) = \sum_{i=0}^{\infty} c_i t^i$ is a formal power series with complex coefficients. For a fixed $d \geqslant 1$, suppose that there are real numbers $(C_0(n)), \ldots, (C_d(n))$, with $\lim_{n\to\infty} C_i(n) = c_i$ for $0 \leqslant i \leqslant d$, such that for $0 \leqslant j \leqslant d$, we have

$$\frac{\alpha(n+j)}{\alpha(n)} E(n)^{-j} = \sum_{i=0}^{d} C_i(n)\delta(n)^i j^i + o(\delta(n)^d) \qquad \text{as } n \to \infty.$$

Then, we have

$$\lim_{n\to\infty}\left(\frac{\delta(n)^{-d}}{\alpha(n)} J_{\alpha}^{d,n}\left(\frac{\delta(n)X-1}{E(n)}\right)\right) = d!\sum_{k=0}^{d}(-1)^{d-k} c_{d-k} X^k/k!,$$

uniformly for X in any compact subset of $\mathbb{R}$.

Theorem (in preparation)
Let $k, s \in \mathbb{N}$. Suppose further that $g(n)$ is a quasi-polynomial-like function of the form

$$g(n) = t_k(n)n^k + t_{k-1}(n)n^{k-1} + \cdots + t_s(n)n^s + o(n^s),$$

where the coefficients $t_s(n), \ldots, t_k(n)$ might depend on the residue class of $n \mod M$ for some $M \geqslant 2$. If $t_k(n), t_{k-1}(n), \ldots, t_{k-d}(n)$ are independent of the residue class of $n \mod M$, then $g(n)$ satisfies the order j Turán inequality for all sufficiently large values of n and $1 \leqslant j \leqslant d$.

Theorem (in preparation)

Let $\mathcal{A} = (a_i)_{i=1}^{\infty}$ be a weakly increasing sequence of positive integers, and let $k > d$. If $\gcd A = 1$ for every $(k-d)$-multisubset $A \subset \{a_1, \dots, a_k\}$, then the sequence $(p_{\mathcal{A},k}(n))_{n=0}^{\infty}$ satisfies the order j Turán inequality for all but finitely many values of n and $1 \leqslant j \leqslant d$.

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- $\mathcal{A} = (1, 2, 2, 3, 3, 3, \ldots)$ & $c_n = p_{\mathcal{A},k}(n)$
- $f_k(n) = 4(c_n^2 - c_{n+1}c_{n-1})(c_{n+1}^2 - c_n c_{n+2}) - (c_n c_{n+1} - c_{n-1}c_{n+2})^2$

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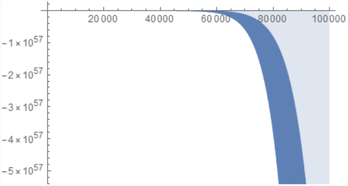

Figure 5: Values of $f_6(n)$ for $n \leqslant 10^5$

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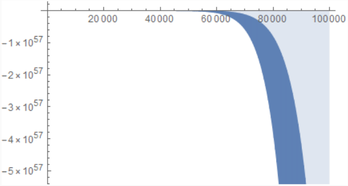

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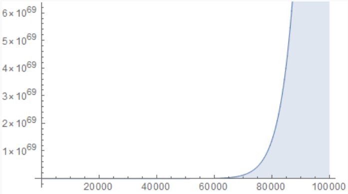

Figure 6: Values of $f_7(n)$ for $n \leqslant 10^5$

Research plan for the future

Polynomization

Let $\mathcal{A} = (a_i)_{i=1}^{\infty}$ and $k \in \mathbb{N}_+$ be fixed, and let $\sigma_A(j) = \sum_{\substack{i=1 \\ a_i | j}}^{k} a_i$

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Proposition

We have that $f_{A,0}(x) = 1$. Moreover, if $n \geqslant 1$, then

$$f_{A,n}(x) = \frac{x}{n} \sum_{j=1}^{n} \sigma_A(j) f_{A,n-j}(x) \quad \text{and} \quad f'_{A,n}(x) = \sum_{j=1}^{n} \frac{\sigma_A(j)}{j} f_{A,n-j}(x),$$

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Problem
Investigate the log-behaviour of $f_{A,n}(x)$.

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