FACULTY OF SCIENCES

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Zero-density estimates for Beurling generalized numbers

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$$\psi_{\mathcal{P}}(x) = \sum_{p_j^{\nu} \leq x} \log p_j.$$

Examples

- $(\mathcal{P}, \mathcal{N}) = (\mathbb{P}, \mathbb{N}_{>0})$, the classical primes and integers.

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- $\mathcal{O}_K$ the ring of integers of a number field K.

$$\mathcal{P} = \left(|P|, P \trianglelefteq \mathcal{O}_K, P \text{ prime ideal}\right),$$
$$\mathcal{N} = \left(|I|, I \trianglelefteq \mathcal{O}_K, I \text{ integral ideal}\right).$$

$$\pi_{\mathcal{O}_K}(x) \sim \frac{x}{\log x}, \quad N_{\mathcal{O}_K}(x) = \rho_K x + O\left(x^{1-\frac{2}{d+1}}\right).$$

THE LANDAU-DLVP PNT

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Theorem (Landau prime ideal theorem)

Under the above assumption, there is a constant $c = c(\theta) > 0$ such that

$$\pi_{\mathcal{P}}(x) = \mathrm{Li}(x) + O\big(x \exp(-c\sqrt{\log x})\big).$$

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Theorem (Diamond, Montgomery, Vorhauer, 2006)

For $\theta \in (1/2, 1)$, there exist systems $(\mathcal{P}, \mathcal{N})$ with $N_{\mathcal{P}}(x) = Ax + O(x^{\theta})$ and

$$\pi_{\mathcal{P}}(x) = \mathrm{Li}(x) + \Omega\big(x \exp(-c'\sqrt{\log x})\big)$$

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Beurling zeta function associated with $(\mathcal{P}, \mathcal{N})$:

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How many zeros can $\zeta_{\mathcal{P}}(s)$ have?

Number of zeros of $\zeta_{\mathcal{P}}(s)$

Lemma (Révész, '21)

Suppose (1) *holds. For each* $T > 2$, $\alpha > \theta$, *the number of zeros* $\rho = \beta + \mathrm{i}\gamma$ *of* $\zeta_{\mathcal{P}}(s)$ *satisfying* $\beta \in [\alpha, 1]$, $|\gamma| \in [T, T+1]$, *is* $O_\alpha(\log T)$.

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For $\alpha \in (\theta, 1]$, $T > 2$ denote

$$N(\alpha, T) := \#\{\rho = \beta + \mathrm{i}\gamma : \zeta_{\mathcal{P}}(\rho) = 0, \beta \geq \alpha, |\gamma| \leq T\}.$$

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For $\alpha \in (\theta, 1]$, $T > 2$ denote

$$N(\alpha, T) := \#\{\rho = \beta + \mathrm{i}\gamma : \zeta_{\mathcal{P}}(\rho) = 0, \beta \geq \alpha, |\gamma| \leq T\}.$$

Theorem (Révész, '22)

Suppose (1) *holds. Suppose additionally that $n_k \in \mathbb{N}$ for every $n_k \in \mathcal{N}$. Then for every $\varepsilon > 0$,*

$$N(\alpha, T) \ll T^{(6-2\theta)\frac{1-\alpha}{1-\theta}+\varepsilon}.$$

Zero-density estimate

Express α as convex combination of 1 and θ: $\alpha = (1-\mu)\theta + \mu$.
We also set

$$c(\mu) := \frac{4\mu}{2\mu^2 - 3\mu + 2}.$$

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Theorem (B., Debruyne, '23)

Suppose (1) *holds. Then for every* $\varepsilon > 0$ *and* $\delta > 0$ *there exists a* $T_0 = T_0(\varepsilon, \delta)$ *such that, uniformly for* $T \geq T_0$, $\mu \geq 2/3 + \delta$,

$$N(\alpha, T) = N((1-\mu)\theta + \mu, T) \ll T^{(c(\mu)+\varepsilon)(1-\mu)}(\log T)^9.$$

PNT IN SHORT INTERVALS

PNT in short intervals holds if there is a $\lambda < 1$ such that

$$\psi_{\mathcal{P}}(x+h) - \psi_{\mathcal{P}}(x) \sim h, \quad \text{for } h \gg x^{\lambda}, \quad x \to \infty.$$

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For classical primes, based on Ingham–Huxley zero density estimate:

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whenever $\lambda > 7/12$.
Can we obtain PNT in short intervals for any system $(\mathcal{P}, \mathcal{N})$ satisfying (1)?

PNT IN SHORT INTERVALS

Theorem (B., Debruyne, '23)

Suppose (1) *holds. If for some* $d > 0$ $\zeta_{\mathcal{P}}(s)$ *has no zeros for*

$$\sigma > 1 - d\frac{\log_2 |t|}{\log |t|}, \quad t > t_0,$$

and if for some $c > 0$*,* $L > 0$*,* $b \in (\theta, 1)$ *we have*

$$N(\alpha, T) \ll T^{c(1-\alpha)}(\log T)^L, \quad \text{for } b \leq \alpha \leq 1,$$

then $\psi_{\mathcal{P}}(x+h) - \psi_{\mathcal{P}}(x) \sim h$ *for* $h \gg x^{\lambda}$ *whenever*

$$\lambda > \max\left\{b, 1 - \frac{d}{cd + L}\right\}.$$

Counterexample

Proposition (B., Debruyne, '23)

For every $\theta \in (1/2, 1)$, *there exists a system satisfying* (1) *such that for every* $\lambda \in (0,1)$, *there exist sequences* $(x_K)_K$, $(h_K)_K$, $h_K \asymp x_K^{\lambda}$ *such that*

$$\limsup_{K\to\infty} \frac{\psi_{\mathcal{P}}(x_K + h_K) - \psi_{\mathcal{P}}(x_K)}{h_K} > 1,$$

$$\liminf_{K\to\infty} \frac{\psi_{\mathcal{P}}(x_K + h_K) - \psi_{\mathcal{P}}(x_K)}{h_K} < 1.$$

Chebyshev bounds in short intervals

Suppose (1) holds. If for some $d > 0$ $\zeta_{\mathcal{P}}(s)$ has no zeros for

$$\sigma > 1 - \frac{d}{\log|t|}, \quad t > t_0,$$

and if for some $c > 0$, $K > 0$, $b \in (\theta, 1)$ we have

$$N(\alpha, T) \leq KT^{c(1-\alpha)}, \quad \text{for } b \leq \alpha \leq 1,$$

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Questions:

Does a log-free zero-density estimate always holds under (1)?

Do Chebyshev bounds in short intervals always hold under (1)?

References

- H. G. Diamond, W.-B. Zhang, *Beurling generalized numbers*, Mathematical Surveys and Monographs series, AMS, Providence, RI, 2016.
- Sz. Gy. Révész, *Density theorems for the Beurling zeta function*, Mathematika **68** (2022), 1045–1072.
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QUESTIONS?

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