Arithmetic Nature of q-Euler-Stieltjes Constants

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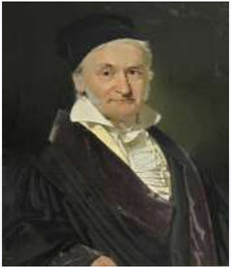
Outline

Leonhard Euler developed the theory of partitions, commonly known as additive analytic number theory in the 1740s, marking the beginning of the field of q-analysis.

Introduction

Leonhard Euler developed the theory of partitions, commonly known as additive analytic number theory in the 1740s, marking the beginning of the field of q-analysis.

Further, it was studied by Carl Friedrich Gauss in early 1800s. The q-binomial coefficients and their identities, which serve as the foundation for q-analysis, were created by him.



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In 1869, Carl Johannes Thomae (1840-1921) together with Jackson introduced q-integral. He also proved that the Heine transformation for ${}_2\phi_1(a,b;c|q,z)$ was a q-analogue of the Euler beta integral, which can be expressed as a quotient of q-Gamma functions.

Frank Hilton Jackson [1870-1960]

The real q-derivative was invented by Jackson in 1908. Further in 1910, he defined the general q-integral which occured in his most important papers on q-series which are “On q-definite integrals” (1910) [9], “On basic double hypergeometric functions” (1942) and “Basic double hypergeometric functions” (1944). He was the first person to work explicitly with the expression $\Gamma_q(x)\Gamma_q(1-x)$.

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Srinivasa Ramanujan was the first one to state the bilateral summation formula - an extension of q-binomial theorem.

Ramanujan's ${}_1\psi_1$ Summation:

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n = \frac{(az;q)_\infty (q/(az);q)_\infty (q;q)_\infty (b/a;q)_\infty}{(z;q)_\infty (b/(az);q)_\infty (b;q)_\infty (q/a;q)_\infty}$$

Richard Askey (1933-2019) discussed the various connections of the q-binomial theorem and the Ramanujan's ${}_1\psi_1$ Summation theorem with q-analogues of the gamma and beta functions. In 1978, he proved the q-analogue of the Bohr-Mollerup theorem.

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In 2003, Kurokawa and Wakayama studied the q-analogue of the Riemann zeta function and introduced a q-analogue of the Euler constant $\gamma_0(q)$. Further, they proved the irrationality of numbers involving $\gamma_0(q)$. In 2007, they gave a Jackson q-integral analogue of Euler's logarithmic sine integral.

Notations & Terminologies

q-analogue of a ($a \in \mathbb{C}$)

$$[a]_q = \frac{q^a - 1}{q - 1}, \qquad q \neq 1.$$

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q-factorial

$$\begin{aligned} [n]_q! &= [1]_q \cdot [2]_q \cdots [n-1]_q \cdot [n]_q \\ &= \frac{q-1}{q-1} \cdot \frac{q^2-1}{q-1} \cdots \frac{q^n-1}{q-1} \\ &= 1 \cdot (1+q) \cdots (1+q+q^2+\cdots+q^{(n-1)}). \end{aligned}$$

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q-shifted factorial of a

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{m=0}^{n-1} (1 - aq^m), \qquad n \geq 1,$$

$$(a;q)_\infty = \lim_{n \to \infty} (a;q)_n = \prod_{n=0}^{\infty} (1 - aq^n).$$

Contd...

q-analogue of the gamma function by Jackson [9]

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty} \qquad \text{for} \qquad 0 < q < 1$$

$$\Gamma_q(x) = \frac{q^{\binom{x}{2}} (q^{-1};q^{-1})_\infty (q-1)^{1-x}}{(q^{-x};q^{-1})_\infty} \qquad \text{for} \qquad q > 1.$$

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q-analogue of the digamma function

$$\psi_q(x) = -\log(1-q) + \log q \sum_{n\geq 0} \frac{q^{n+x}}{1-q^{n+x}}, \qquad 0 < q < 1$$

$$\begin{aligned} \psi_q(x) &= -\log(q-1) + \log q \left(x - \frac{1}{2} - \sum_{n\geq 0} \frac{q^{-n-x}}{1-q^{-n-x}} \right) \\ &= -\log(q-1) + \log q \left(x - \frac{1}{2} - \sum_{n\geq 1} \frac{q^{-nx}}{1-q^{-n}} \right), \qquad q > 1. \end{aligned}$$

Preliminaries

Kurokawa and Wakayama studied the following q-analogue of the Riemann zeta function in 2003 ([10])

$$\zeta_q(s) = \sum_{n=1}^{\infty} \frac{q^n}{[n]_q^s}, \qquad \mathrm{Re}(s) > 1.$$

[1] N. Kurokawa and M. Wakayama, *On q-Analogues of the Euler Constant and Lerch's Limit Formula*, Proceedings of the American Mathematical Society, **132** (2003), 935 – 943.

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Theorem (Kurokawa and Wakayama[1])

Suppose $q > 1$. Then the following statements hold:

1. $\zeta_q(s)$ *is meromorphic for* $s \in \mathbb{C}$.
2. *Around $s = 1$, we have the Laurent series*

$$\zeta_q(s) = \frac{q-1}{\log q} . \frac{1}{s-1} + \gamma(q) + c_1(q)(s-1) + \cdots$$

with

$$\gamma(q) = \sum_{n=1}^{\infty} \frac{1}{[n]_q} + \frac{(q-1)\log(q-1)}{\log q} - \frac{q-1}{2}.$$

[1] N. Kurokawa and M. Wakayama, *On q-Analogues of the Euler Constant and Lerch's Limit Formula*, Proceedings of the American Mathematical Society, **132** (2003), 935 – 943.

Theorem

Let $q \geq 2$ be an integer. Then

$$\gamma(q) - \frac{(q-1)\log(q-1)}{\log q}$$

is an irrational number. In particular, $\gamma(2)$ is irrational.

[2] Yu. V. Nesternko, *Modular functions and transcendence problems*, C. R. Acad. Sci. Paris Sér. I Math., **322**, (1996), 909 – 914.

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In 1916, Ramanujan defined the three functions:

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n,\ E_4(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n,\ E_6(q) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n.$$

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Theorem (Nesterenko's Theorem[2])

For any q with $\mid q \mid < 1$, the transcendence degree of the field

$$\mathbb{Q}(q, E_2(q), E_4(q), E_6(q))$$

is at least 3. Thus, for q algebraic, $E_2(q)$, $E_4(q)$ and $E_6(q)$ are algebraically independent and hence transcendental.

[2] Yu. V. Nesternko, *Modular functions and transcendence problems*, C. R. Acad. Sci. Paris Sér. I Math., **322**, (1996), 909 – 914.

Theorem (Duverney and Tachiya[3])

Let $d(n)$ *be the divisor function and let* $\{a_n\}_{n\geq 1}$ *be a sequence of nonzero integers satisfying* $\log|a_n| = O(\log\log n)$*. Then for every integer* $h \geq 1$*, the numbers*

$$1, \sum_{n=1}^{\infty} \frac{d(n)a_n}{q^n}, \sum_{n=1}^{\infty} \frac{d(n)a_n}{q^{2n}}, \ldots, \sum_{n=1}^{\infty} \frac{d(n)a_n}{q^{hn}}$$

are linearly independent over $\mathbb{Q}$*.*

[3] D. Duverney and Y. Tachiya, *Refinement of the Chowla-Erdős method and linear independence of certain Lambert series*, Forum Math., **31** (2019), 1557 – 1566

Our Results [4]

Theorem

The q-analogue of the Riemann zeta function is meromorphic for $s \in \mathbb{C}$ and its Laurent series expansion around $s = 1$ is given by

$$\zeta_q(s) = \frac{q-1}{\log q} \cdot \frac{1}{s-1} + \gamma_0(q) + \gamma_1(q)(s-1) + \gamma_2(q)(s-1)^2 + \gamma_3(q)(s-1)^3 + \cdots$$

$$\gamma_k(q) = \sum_{i=1}^{k+1} \left(\left(\sum_{n=1}^{\infty} \frac{s(n+1,i)}{[n]_q n!} \right) \frac{\log^{k+1-i}(q-1)}{(k+1-i)!} \right)$$

$$+ \sum_{j=1}^{k} (-1)^j \left(\sum_{i=1}^{k-(j-1)} \left(\sum_{n=1}^{\infty} \frac{s(n+1,i) q^n \mathcal{A}_{q^n}(j-1,j)}{n! [n]_q (q^n-1)^j} \frac{\log^j q}{j!} \right) \frac{\log^{(k-(j-1)-i)}(q-1)}{(k-(j-1)-i)!} \right)$$

$$- \frac{(q-1)\log^k(q-1)}{2(k!)} + \frac{(q-1)\log^{k+1}(q-1)}{(k+1)!\log q} + \sum_{i=1}^{\lceil \frac{k}{2} \rceil} (-1)^{i+1} \frac{(q-1)\log^{2i-1} q \log^{k-(2i-1)}(q-1)}{\mathcal{B}(i)(k-(2i-1))!}$$

where, $s(n+1,i)$ are the unsigned Stirling numbers of the first kind, $\mathcal{A}_{q^n}(j-1,j)$ is the polynomial in q^n of degree $(j-1)$ and coefficients from the j^{th} row in Eulerian numbers triangle, $\mathcal{B}(i)$ is the denominator of non-zero coefficients in the series expansion around zero of $\frac{1}{2}\cot(x/2)$ disregarding the first term and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

[4] T. Chatterjee and S. Garg, *On q-analogue of Euler-Stieltjes Constant*, Proc. Amer. Math. Soc., **151** (2023), 2011–2022.

Stirling numbers of the first kind

The Stirling numbers of the first kind count permutations according to their number of cycles (counting fixed points as cycles of length one). The number of permutations on n elements with k cycles is denoted by $s(n, k)$.

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Example: $s(4, 2) = 11$.
The symmetric group on 4 objects has 3 permutations of the form
$(\bullet\bullet)(\bullet\bullet)$ (having 2 orbits, each of size 2),
and 8 permutations of the form
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n \ k	0	1	2	3	4	5	6	7
0	1							
1	0	1						
2	0	1	1					
3	0	2	3	1				
4	0	6	11	6	1			
5	0	24	50	35	10	1		
6	0	120	274	225	85	15	1	
7	0	720	1764	1624	735	175	21	1

Recurrence relation: $s(n+1, k) = ns(n, k) + s(n, k-1)$

Eulerian numbers triangle

The classical **Eulerian number** $A(n, m)$ is the number of permutations of the set of numbers $\{1, \cdots, n\}$ in which exactly m elements are greater than the previous element.

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n	m	Permutations	A(n,m)
1	0	id	A(1,0) = 1
2	0	id	A(2,0)= 1
	1	(1 2)	A(2,1) = 1
3	0	id	A(3,0)= 1
	1	(1 2), (1 3), (2 3), (1 3 2)	A(3 1) = 4
	2	(1 2 3)	A(3,2)= 1

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	2	(1 2 3)	A(3,2)= 1

n \ m	0	1	2	3	4	5
1	1					
2	1	1				
3	1	4	1			
4	1	11	11	1		
5	1	26	66	26	1	
6	1	57	302	302	57	1

Recurrence relation:$A(n, m) = (n - m)A(n - 1, m - 1) + (m + 1)A(n - 1, m)$

Another representation

$$\gamma_k(q) = \frac{(q-1)\log^{k+1}(q-1)}{(k+1)!\log q} + \sum_{i=0}^{k-1}\left(\frac{a_{k-i}\log^{k-i}(q-1)}{(k-i)!} + \frac{(-1)^i b_{k-i}\log^{k-i} q}{(k-i)!}\right)$$
$$+ \sum_{n=k}^{\infty}\frac{s(n+1,k+1)}{n![n]_q} + \sum_{i=1}^{\lceil\frac{k}{2}\rceil}(-1)^{i+1}\frac{(q-1)\log^{2i-1} q\log^{k-(2i-1)}(q-1)}{\mathcal{B}(i)(k-(2i-1))!},$$

where a_{k-i} and b_{k-i} are the coefficients of $\log^{k-i}(q-1)$ and $\log^{k-i} q$ respectively.

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where a_{k-i} and b_{k-i} are the coefficients of $\log^{k-i}(q-1)$ and $\log^{k-i} q$ respectively.

Example

$$\gamma_1(q) = \frac{(q-1)\log^2(q-1)}{2\log q} + \left(\sum_{n=1}^{\infty}\frac{1}{[n]_q} - \frac{q-1}{2}\right)\log(q-1)$$
$$-\left(\sum_{n=1}^{\infty}\frac{q^n}{[n]_q(q^n-1)}\right)\log q + \sum_{n=1}^{\infty}\frac{s(n+1,2)}{n![n]_q} + \frac{(q-1)\log q}{12}.$$

So here, $a_1 = \sum_{n=1}^{\infty}\frac{1}{[n]_q} - \frac{q-1}{2}$ and $b_1 = -\left(\sum_{n=1}^{\infty}\frac{q^n}{[n]_q(q^n-1)}\right)$.

Here we obtain $\gamma_2(q)$ as follows:

Example

$$\gamma_2(q) = \frac{(q-1)\log^3(q-1)}{3!\log q} + \left(\sum_{n=1}^{\infty}\frac{1}{[n]_q} - \frac{q-1}{2}\right)\frac{\log^2(q-1)}{2!}$$
$$+ \left(-\sum_{n=1}^{\infty}\frac{(q^n)\log q}{[n]_q(q^n-1)} + \sum_{n=1}^{\infty}\frac{s(n+1,2)}{n![n]_q}\right)\log(q-1)$$
$$+ \left(\sum_{n=1}^{\infty}\frac{q^n(q^n+1)}{[n]_q(q^n-1)^2}\right)\frac{\log^2 q}{2} - \left(\sum_{n=1}^{\infty}\frac{s(n+1,2)q^n}{n![n]_q(q^n-1)}\right)\log q$$
$$+ \sum_{n=2}^{\infty}\frac{s(n+1,3)}{n![n]_q} + \frac{(q-1)\log q\log(q-1)}{12},$$

which again can be rewritten in the same form with

$$a_2 = \sum_{n=1}^{\infty}\frac{1}{[n]_q} - \frac{q-1}{2} \quad \text{and} \quad a_1 = -\sum_{n=1}^{\infty}\frac{q^n\log q}{[n]_q(q^n-1)} + \sum_{n=1}^{\infty}\frac{s(n+1,2)}{n![n]_q},$$
$$b_2 = \sum_{n=1}^{\infty}\frac{q^n(q^n+1)}{[n]_q(q^n-1)} \quad \text{and} \quad b_1 = -\left(\sum_{n=1}^{\infty}\frac{s(n+1,2)q^n}{n![n]_q(q^n-1)}\right).$$

Contd...

Define the normalized q-analogue of the Euler constant by

$$\gamma_0^*(q) = \gamma_0(q) - \frac{(q-1)\log(q-1)}{\log q}.$$

Theorem

For integers $r \geq 1$ and $q > 1$, the set of numbers

$$\{1, \gamma_0^*(q), \gamma_0^*(q^2), \gamma_0^*(q^3), \dots, \gamma_0^*(q^r)\}$$

is linearly independent over $\mathbb{Q}$.

Contd…

Lemma

[5] *For every integer* $q > 1$, $\sum_{n=1}^{\infty} \frac{\sigma_1(n)}{q^n}$ *is a transcendental number, where* $\sigma_1(n)$ *is the sum of the divisors of* n.

[5] P. Erdős, *On arithmetical properties of Lambert series*, J. Indian Math. Soc., **12** (1948), 63 – 66

Contd...

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Lemma

For every integer $t > 1$, $\sum_{n=1}^{\infty} \frac{t^n}{(t^n - 1)^2} = \sum_{n=1}^{\infty} \frac{\sigma_1(n)}{t^n}$.

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Lemma

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Theorem

Let $k = 1$ *and* $q = 2$. *Then*

$$\frac{1}{\log 2}\left(\gamma_1(2) - \sum_{n=1}^{\infty} \frac{H_n}{2^n - 1}\right)$$

is a transcendental number, where H_n *is the nth harmonic number.*

5 P. Erdős, *On arithmetical properties of Lambert series*, J. Indian Math. Soc., **12** (1948), 63 – 66

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Thank You!