Some degree problems in number fields

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Sum-feasible and product-feasible triplets

Definition

A triplet $(a, b, c) \in \mathbb{N}^3$ is called sum-feasible (resp. product-feasible) if there exist algebraic numbers α and β such that $\deg \alpha = a$, $\deg \beta = b$ and $\deg(\alpha + \beta) = c$ (resp. $\deg(\alpha\beta) = c$).

For example, $(2, 2, 4)$ is sum-feasible:

$$\alpha = \sqrt{2},\ \beta = \sqrt{3},\ \alpha + \beta = \sqrt{2} + \sqrt{3}.$$

Also $(2, 2, 4)$ is product-feasible (e.g. $\alpha = \sqrt{2}$, $\beta = 1 + \sqrt{3}$).

In 2012 Drungilas, Dubickas and Smyth[1] proposed a problem to find all possible sum-feasible triplets.

Theorem (Isaacs, 1970)

If $\deg \alpha = a$, $\deg \beta = b$ *and* $\gcd(a, b) = 1$ *then* $\deg(\alpha + \beta) = ab$.

[1] P. Drungilas, A. Dubickas, C. J. Smyth, *A degree problem of two algebraic numbers and their sum*, Publ. Mat. Barc. **56** (2) (2012), 413-448.

Compositum-feasible triplets

Definition

A triplet $(a, b, c) \in \mathbb{N}^3$ is called compositum-feasible if there exist number fields K and L such that $[K : \mathbb{Q}] = a$, $[L : \mathbb{Q}] = b$ and $[KL : \mathbb{Q}] = c$ (here KL denotes the compositum of K and L).

Let $\mathcal{C}$, $\mathcal{S}$ and $\mathcal{P}$ denote sets of all possible compositum-feasible, sum-feasible and product-feasible triplets, respectively.

It is proved by Drungilas, Dubickas and Smyth that

$$\mathcal{C} \subsetneq \mathcal{S} \subsetneq \mathcal{P}.$$

Both inclusions are indeed strict:

- $(n, n, 1) \in \mathcal{S}\ \forall n \in \mathbb{N}$, but $(n, n, 1) \notin \mathcal{C}$ for $n > 1$.
- $(2, 3, 3) \in \mathcal{P}$ (e.g. $\alpha = e^{\frac{2\pi i}{3}}$, $\beta = \sqrt[3]{2}$), but $(2, 3, 3) \notin \mathcal{S}$ by the result of Isaacs.

Related results

Obvious necessary conditions:

- if $(a, b, c) \in \mathbb{N}^3$, $a \leqslant b \leqslant c$, is compositum-feasible, sum-feasible or product-feasible then $c \leqslant ab$.
- if $(a, b, c) \in \mathbb{N}^3$, $a \leqslant b \leqslant c$, is compositum-feasible then $a|c$ and $b|c$.

In 2012-2013 Drungilas, Dubickas, Luca and Smyth described all sum-feasible triplets $(a, b, c) \in \mathbb{N}^3$, $a \leqslant b \leqslant c$, $b \leqslant 7$, and also all possible compositum-feasible triplets under the same restrictions.

We say that a triplet $(a, b, c) \in \mathbb{N}^3$ satisfies the exponent triangle inequality with respect to a prime number p if

$$\operatorname{ord}_p a + \operatorname{ord}_p b \geqslant \operatorname{ord}_p c,\ \operatorname{ord}_p b + \operatorname{ord}_p c \geqslant \operatorname{ord}_p a \text{ and } \operatorname{ord}_p a + \operatorname{ord}_p c \geqslant \operatorname{ord}_p b. \tag{1}$$

Theorem (Drungilas, Dubickas, Smyth, 2012)

If (a, b, c) satisfies (1) with respect to every prime number then $(a, b, c) \in \mathcal{S}$.

Main results

Theorem (Drungilas, L.M., 2019[2])

1. *Let* $a \leqslant 8 \leqslant c$. *Then* $(a, 8, c) \in \mathcal{C}$ *if and only if* $c \leqslant 8a$, $a|c$ *and* $b|c$ *with a single exceptional triplet* $(8, 8, 40)$.
2. *Let* $a \leqslant 9 \leqslant c$. *Then* $(a, 9, c) \in \mathcal{C}$ *if and only if* $c \leqslant 9a$, $a|c$ *and* $b|c$ *with two exceptional triplets* $(9, 9, 45)$ *and* $(9, 9, 63)$.

Theorem (Drungilas, L.M.)

Suppose $n \in \mathbb{N}$ *and a prime* p *satisfy* $\frac{n}{2} < p < n - 2$. *Then* $(n, n, np) \notin \mathcal{P}$, *and therefore* $(n, n, np) \notin \mathcal{C}$, $(n, n, np) \notin \mathcal{S}$.

Theorem (Drungilas, L.M.)

Suppose $n \geqslant 4$. *Then* $(n, n, n(n-2)) \in \mathcal{C}$ *for even* n *and* $(n, n, n(n-2)) \notin \mathcal{P}$ *for odd* n.

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[2] P. Drungilas, L. Maciulevičius, *A degree problem for the compositum of two number fields*, Publ. Lith. Math. J. **59** (1) (2019), 39-47.

Main results

Theorem (L.M., 2023[3])

All the triplets $(a, b, c) \in \mathcal{P}$ with $a \leqslant b \leqslant c$, $b \leqslant 7$ are given in the following table with five exceptions that are circled.

$b \backslash a$	1	2	3	4	5	6	7
1	1						
2	2	2, 4					
3	3	3, 6	3, 6, 9				
4	4	4, 8	6, 12	4, 6, 8, 12, 16			
5	5	10	15	5, 10, 20	5, 10, 20, 25		
6	6	6, 12	6, 9, 12, 18	6, (8), 12, 24	(10), (15), 30	6, 8, 9, 12, 15, 18, 24, 30, 36	
7	7	14	21	(7), (14), 28	35	7, 14, 21, 42	7, 14, 21, 28, 42, 49

[3] L. Maciulevičius, *On the degree of product of two algebraic numbers*, Publ. Mathematics, **11** (9), Paper No. 2131

Supplementary results

Theorem (Virbalas, 2023)

Let α and β be algebraic numbers, $\deg\alpha = p$, $\deg\beta = m$, *where* $p > 2$ *is a prime,* $p \nmid m$ *and* $p - 1 \nmid m$. *Then* $\deg(\alpha\beta) = mp$.

E.g., $\deg\alpha = 5$, $\deg\beta = 6 \Rightarrow \deg(\alpha\beta) = 5 \cdot 6 = 30$. Hence $(5, 6, 10), (5, 6, 15) \notin \mathcal{P}$. Analogously $(4, 7, 7), (4, 7, 14) \notin \mathcal{P}$.

Recently we have showed with Dubickas that $(4, 6, 8) \notin \mathcal{P}$.

Conjecture for compositum-feasible triplets

In 2012 Drungilas, Dubickas and Smyth proposed the following conjecture:

Conjecture

If $(a, b, c), (a', b', c') \in \mathcal{C}$ *then* $(aa', bb', cc') \in \mathcal{C}$.

In 2016 Drungilas and Dubickas proved that this conjecture is true if the answer to the *inverse Galois problem* is positive. Recall that the inverse Galois problem asks whether every finite group occurs as a Galois group of some Galois extension K over $\mathbb{Q}$.

Theorem

If every finite group occurs as a Galois group of some Galois extension $K/\mathbb{Q}$ *then the Conjecture is true.*

Irreducible compositum-feasible trilpets

In other words, assuming affirmative answer to the inverse Galois problem, the set $\mathcal{C}$ forms a semigroup with respect to the multiplication defined by

$$(a, b, c) \cdot (a', b', c') := (aa', bb', cc'). \tag{2}$$

It is natural to ask which elements of $\mathcal{C}$ are irreducible.

Definition

A triplet $(A, B, C) \in \mathcal{C}$ is called irreducible if it cannot be written as

$$(A, B, C) = (a, b, c) \cdot (a', b', c'),$$

where $(a, b, c), (a', b', c') \in \mathcal{C}$, $(a, b, c) \neq (1, 1, 1)$ and $(a', b', c') \neq (1, 1, 1)$.

Main results

Proposition (L.M., 2023)

For any integer $n \geqslant 2$ the compositum-feasible triplet $(n, n, n(n-1))$ is irreducible.[4]

Among the compositum-feasible triplets (a, b, c), $a \leqslant b \leqslant c$, $b \leqslant 9$, the only irreducible triplets are of the form

$$(1, p, p), \ (p, p, pd), \ (n, n, n(n-1)),$$

where p is prime, $1 \leqslant d < p$ and $n \geqslant 2$.

Problem

Find all irreducible compositum-feasible triplets.

[4] It is proved by Drungilas, Dubickas and Smyth that $(n, n, n(n-1)) \in \mathcal{C}$ for any $n \geqslant 2$.

Thank you!