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Arithmetic dynamics of unicritical polynomials: a study on rational periodic points

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Introduction

Some Definitions

- *A* ***(discrete) dynamical system*** *is a pair* (f, S) *of a set* S *and a map* $f : S \longrightarrow S$.

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- *The (forward) **orbit** of $z \in S$ by f is defined as*

$$\mathscr{O}_f(z) := \{z, f(z), f^{(2)}(z), f^{(3)}(z), \ldots\}.$$

Some Definitions (continued)

- *A point $z \in S$ is called* ***periodic*** *with respect to f if $f^{(k)}(z) = z$ for some $k \in \mathbb{N}$ and we call the smallest such k the* ***minimal period*** *of z.*

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- *A point $z \in S$ is called* ***preperiodic*** *with respect to f if $f^{(k)}(z)$ is periodic for some $k \in N$, which is equivalent to that $\mathscr{O}_f(z)$ is finite. The set of preperiodic points of f (in S) is denoted by* $\mathrm{PrePer}(f, S)$.

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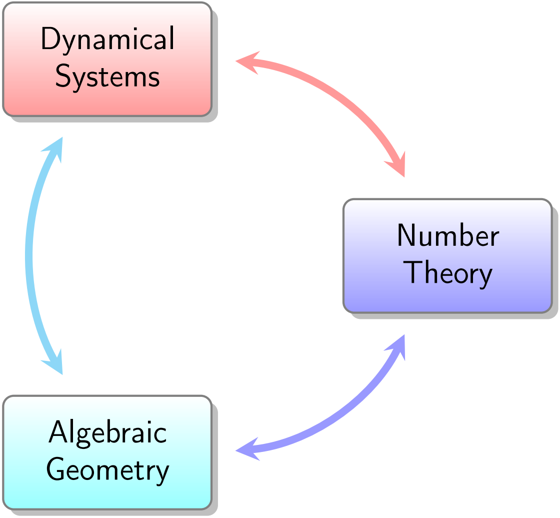
- *We say that a point $z \in S$ is **wandering** if z is not preperiodic.*

For arithmetic interests, we can let S be $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, $\mathbb{K}, \ldots$


Dynamical Systems
Number Theory
Algebraic Geometry

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No, it's not preperiodic.
Is $z = \frac{1}{4}$ preperiodic?

$$\frac{1}{4} \mapsto -\frac{7}{4} \mapsto -\frac{5}{4} \mapsto -\frac{1}{4} \mapsto -\frac{7}{4}$$

Conjecture (Morton-Silverman Uniform Boundedness, 1994)

There exists a bound $B = B(D, N, d)$ such that if K is a number field of degree D, and $\phi : \mathbb{P}^N(K) \longrightarrow \mathbb{P}^N(K)$ is a morphism of degree $d \geq 2$ defined over K, then the number of preperiodic points of ϕ is bounded by B.

This conjecture is remarkably strong. For $(D, N, d) = (1, 1, 4)$, the conjecture implies that the size of the torsion subgroup of an elliptic curve is uniformly bounded. This can be done via the associated Lattès map. Similarly, for $(D, N, d) = (D, 1, 4)$, the conjecture implies the Merel's theorem.

Theorem (Northcott, 1950)

Let $\phi \in K(z)$ of degree $d \geq 2$. Then

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Conjecture (Morton-Silverman, $\mathbb{Q}$ version)

For any integer $d \geq 2$, there is a constant $C(d)$ such that for any $\phi \in \mathbb{Q}(z)$ of degree d,

$$\mathrm{PrePer}(\phi) \leq C(d).$$

Rational periodic points

Theorem

Let $\phi_c(z) = z^2 + c$.

1. *There are infinitely many* $c \in \mathbb{Q}$ *such that* $\phi_c(z)$ *has a* $\mathbb{Q}$*-rational point of period 1, 2 or 3.*

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2. *(Morton, 1998, Flynn-Poonen-Schaefer, 1997) There is no* $c \in \mathbb{Q}$ *such that* $\phi_c(z)$ *has a* $\mathbb{Q}$*-rational point of period 4 or 5.*

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1. *There are infinitely many $c \in \mathbb{Q}$ such that $\phi_c(z)$ has a $\mathbb{Q}$-rational point of period 1, 2 or 3.*
2. *(Morton, 1998, Flynn-Poonen-Schaefer, 1997) There is no $c \in \mathbb{Q}$ such that $\phi_c(z)$ has a $\mathbb{Q}$-rational point of period 4 or 5.*
3. *(Stoll, 2008) There is no $c \in \mathbb{Q}$ such that $\phi_c(z)$ has a $\mathbb{Q}$-rational point of period 6 (based on Birch and Swinnerton-Dyer Conjecture).*

How about $f_c(x) = x^d + c$ over $\mathbb{Q}$?

- For every $d \in \mathbb{N}$, there are infinitely many $c \in \mathbb{Q}$ such that $f_c(x) = x^d + c$ has a fixed point in $\mathbb{Q}$, i.e., $c = t - t^d, t \in \mathbb{Q}$.

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- For every odd $d \geq 3$ and $n \geq 2$, $f_c(x)$ has no n-periodic point in $\mathbb{Q}$.
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  Ex: $f_c(x) = x^4 - \dfrac{19561}{10000}, \dfrac{9}{10} \longleftrightarrow \dfrac{-13}{10}$

2-periodic points of $x^4 + c$

Theorem

There are infinitely many $c \in \mathbb{Q}$ such that $f_{4,c}(x) = x^4 + c$ has rational periodic points of exact period 2.

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$$c = \frac{t^6 + 4t^3 - 1}{4t^2}$$

and the 2-periodic points are

$$x_1, x_2 = \frac{t^2 \pm \sqrt{-t^4 - 2t}}{2t},$$

where $-t^4 - 2t = y^2$ for some $y \in \mathbb{Q}$.

Lemma (Hutz, 2015)

Let $f_{d,c}(x) = x^d + c \in K[x]$ *and let* $\alpha \in \mathbb{A}(K)$ *be a preperiodic point of* $f_{d,c}$*. Then for each nonarchimedean place* v *such that* $v(c) < 0$*, we have* $v(c) = dv(\alpha)$*. For each nonarchimedean place* v *such that* $v(c) \geq 0$*, we have* $v(\alpha) \geq 0$*.*

Corollary

If $\dfrac{X}{Z}$ *is a rational preperiodic point of* $f_{d,c}(x) = x^d + c$ *with* $c = \dfrac{M}{N}$ *and both* $\dfrac{X}{Z}$ *and* $\dfrac{M}{N}$ *are rational numbers expressed in the lowest terms with* Z *and* N *positive integers, then* $N = Z^d$*.*

2-periodic points and Fermat-Catalan Equations

Theorem

Let k,n be positive integers such that $n>1$, and $f_{d,c}(x)=x^d+c\in\mathbb{Q}[x]$, where $d=2k$. If X_1/Z and $X_2/Z=f_{d,c}(X_1/Z)$ are rational periodic points of exact period n of $f_{d,c}$ with $c=C/Z^d$, and $X_1/Z, X_2/Z$ and C/Z^d are rational numbers expressed in the lowest terms with integers X_1,X_2 and Z, then $\gcd(X_1,X_2)=1$. Moreover, for $n=2$,

a) $X_1^k+X_2^k=\delta Z_1^{2k-1}$ *for some* $Z_1\in\mathbb{Z}$ *and* $\delta\in\{1,2\}$,

b) *if k is odd, then* $X_1^k+X_2^k=Z_1^{2k-1}$ *for some* $Z_1\in\mathbb{Z}$.

$f_{d,c}(x) = x^d + c$

Conjecture (Generalized Poonen, Hutz 2015)

For $n > 3$, there is no integer $d \geq 2$ and $c \in \mathbb{Q}$ such that $f_{d,c}(x) = x^d + c$ has a rational periodic point of exact period n. For maps of the form $f_{d,c}$, we have

$$\#\,\mathrm{PrePer}(f_{d,c}, \mathbb{P}^1(\mathbb{Q})) \leq 9.$$

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Theorem (Narkierwicz, 2013)

For $n > 1$ and $d > 2$ odd, there is no $c \in \mathbb{Q}$ such that $f_{d,c}$ has a $\mathbb{Q}$-rational periodic point of minimal period n. Furthermore,

$$\#PrePer(f_{d,c}, \mathbb{P}^1(\mathbb{Q})) \leq 4.$$

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Actually, Narkierwicz's statement is for number fields.

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Conjecture (Hutz, 2015)

For $n > 2$ there is no even $d > 2$ and $c \in \mathbb{Q}$ such that $f_{d,c}$ has a $\mathbb{Q}$-rational periodic point of minimal period n. Furthermore,

$$\#PrePer(f_{d,c}, \mathbb{P}^1(\mathbb{Q})) \leq 4.$$

2-Periodic Points: Modular Method

There are many useful results of **Fermat Catalan equation** $x^p + y^q = z^r$ of signature $(p,q,r) = (p,p,r)$ (**Darmon's Program**) such as works of Bennett, Ellenberg, Ng, Darmon, Merel, Freitas, etc.

Theorem (Bennett-Vatsal-Yazdani, 2004)

Let $C \in \{1,2,3,5,7,11,13,15,17,19\}$ and prime number $n > \max\{C,4\}$, then the Diophantine equation $x^n + y^n = Cz^3$ has no solutions in coprime nonzero integers x, y and z with $|xy| > 1$.

Theorem

Let k be an integer with a prime factor $p \geq 5$ and $f_{2k,c}(x) = x^{2k} + c$ where $c \in \mathbb{Q} \backslash \{-1\}$. If $3 \mid 2k-1$, then $f_{2k,c}$ has no rational periodic point of exact period 2.

Definition

Let $n = \prod_{i=1}^{s} p^{k_i} \in \mathbb{N}$. The **radical** of n is defined as

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Conjecture (Masser and Oestelé 1988))

For every $\varepsilon > 0$, there exists a constant $K_\varepsilon > 0$ such that for all triples (a, b, c) of coprime positve integers, with $a + b = c$ such that

$$c < K_\varepsilon rad(abc)^{1+\varepsilon}.$$

Theorem (Uniform Boundedness for $x^d + c$, Looper, 2021)

Let $d \geq 2$ and K be a number field. Let $f_{d,c}(x) = x^d + c \in K[x]$. If $d \geq 5$, assume the abc-conjecture. If $2 \leq d \leq 4$, assume the abcd-conjecture. There is a $B = B(d, K)$ such that $f_{d,c}$ has at most B preperiodic points.

Theorem (UB for Polynomials, Looper, 2021+)

Let K be a number field and let $f(z) \in K[z]$ be a polynomial of degree $d \geq 2$. Assume the abcd-conjecture. Then there is a constant $B = B(d, K)$ such that f has at most B preperiodic points contained in K.

Explicit $abc-$Conjecture

Conjecture (Explicit $abc-$conjecture, Baker, 2004)

There exists an absolute constant K such that for all triples (a,b,c) of coprime integers with $a+b=c$, $abc \neq 0$ and $N = \mathrm{rad}(|abc|)$,

$$\max(|a|,|b|,|c|) < KN\frac{(\log(N))^{\omega}}{\omega!}$$

where ω is the total number of distinct primes dividing a, b and c.

Explicit $abc-$Conjecture

Conjecture (Explicit $abc-$conjecture, Laishram and Shorey 2012 ($\varepsilon = 3/4$))

Assume the explicit abc-conjecture. For all triples (a, b, c) of coprime positive integers with $a + b = c$, we have

$$c < (rad(abc))^{1+\frac{3}{4}}.$$

Theorem (Chim-Shorey-Sinha, 2019)

Assume the explicit abc-conjecture. For all triples (a, b, c) of coprime integers with $a + b = c$, $abc \neq 0$ and $N = \mathrm{rad}(|abc|)$, we have

$$\max(|a|, |b|, |c|) < \min(N^{1.72}, 10N^{1.62991}, 32N^{1.6}).$$

Lemma ($x^d + y^d = \delta z^{d-1}$)

Assume the abc conjecture. Then the system

$$\begin{cases} X_2^d - X_1^d = (X_3 - X_2)Z^{d-1}, Z \neq 0 \\ \gcd(X_1, X_2) = 1 \\ \max(|X_1|, |X_2|, |X_3|) = |X_2| \\ \max(|X_2|, |Z|) > 1 \end{cases}$$

has no integer solutions for sufficiently large positive integers k, m. Moreover, if we assume the explicit abc−conjecture, then the result holds for $d \geq 10$.

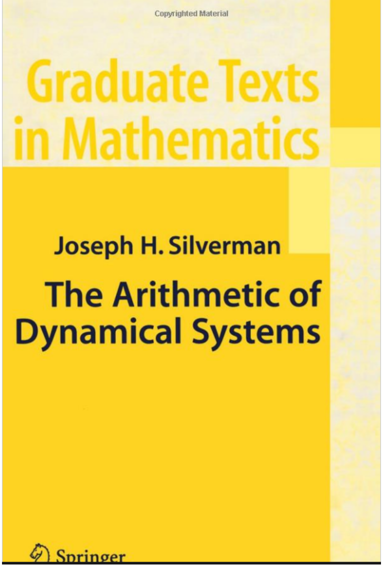
Theorem (P., 2022)

Let $f_{d,c}(x) = x^d + c \in \mathbb{Q}[x]$. If the abc-conjecture is valid, then for sufficiently large degree d,

1. *$f_{d,c}$ has no point of type 1_2 (i.e., $x_{-2} \to x_{-1} \to x_0 \circlearrowleft$),*
2. *$f_{d,c}$ has no rational periodic points of exact period greater than 1 except when $c = -1$ ($x^d - 1: \ 1 \to 0 \longleftrightarrow -1, \ \infty \circlearrowleft$),*
3. $\#\mathrm{PrePer}(f_{d,c}, \mathbb{P}^1(\mathbb{Q})) \leq 4$.

Moreover, if we assume the explicit abc-conjecture, then the result holds for $d \geq 7$.

The ADS by Joseph H. Silverman



CURRENT TRENDS AND OPEN PROBLEMS IN ARITHMETIC DYNAMICS, 2019

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CURRENT TRENDS AND OPEN PROBLEMS IN ARITHMETIC DYNAMICS

ROBERT BENEDETTO, PATRICK INGRAM, RAFE JONES, MICHELLE MANES,
JOSEPH H. SILVERMAN, AND THOMAS J. TUCKER

Abstract. Arithmetic dynamics is the study of number theoretic properties of dynamical systems. A relatively new field, it draws inspiration partly from dynamical analogues of theorems and conjectures in classical arithmetic geometry and partly from p-adic analogues of theorems and conjectures in classical complex dynamics. In this article we survey some of the motivating problems and some of the recent progress in the field of arithmetic dynamics.

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THANK YOU!