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Abstract

The fundamental theorem of arithmetic gives an important property of natural numbers: every natural number can be uniquely expressed as a product of primes. A similar decomposition can be defined in the set of meromorphic functions, except this time we use the composition of functions.

I will study the conditions under which functions belonging to the extended Selberg class are prime. In addition, I will give an analogous result about the Hurwitz zeta function.

Prime function

Definition
Let f be a meromorphic function satisfying

$$f = g \circ h \tag{1}$$

with g meromorphic and h entire or h meromorphic and g rational. Then the expression (1) is called a *decomposition* of f.

Definition
If for all decompositions (1) of f, g or h is a linear function, then f is *prime*.

Prime function

Definition
If for all decompositions (1), either g is rational or h is a polynomial, then f is *pseudo prime*. If g is linear whenever h is transcendental, then f is *left prime*. If h is linear whenever g is transcendental, then f is *right prime*

Definition
The *Selberg class* $\mathcal{S}$ consists of functions satisfying these axioms

- (simple Dirichlet series) $F(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ which converges absolutely in the half plane $\sigma > 1$.
- (analytic continuation) there exists a non-negative integer k such that $(s-1)^k F(s)$ is entire of finite order.
- (functional equation) $F(s)$ satisfies

$$\Lambda_F(s) = \omega \overline{\Lambda_F(1-\overline{s})}.$$

Selberg class

Definition

- (Ramanujan hypothesis) $a(n) \ll n^{\epsilon}$ for all ϵ, where the implicit constant does not depend on ϵ.
- (Euler product) $\log F(s) = \sum_{n=}^{\infty} b_F(n) n^{-s}$, where $b_F(n) = 0$ except when $n = p^m$ for some prime p with $m \geq 1$ and $b_F(n) \ll n^{\theta}$ for some $\theta < 1/2$.

Definition
The *extended Selberg class* $\mathcal{S}^\#$ consists of the Dirichlet series satisfying the first three axioms of the Selberg class.

Definition
The *degree* of $F \in \mathcal{S}^\#$ is $d_F = 2\sum_{j=1}^{N} \lambda_j$ where λ_j come from the functional equation. The degree is an invariant of F.

The zeros of F coming from the poles of the Gamma function in the functional equation are called *trivial*. Their distribution is known.

Theorem
A function $F \in \mathcal{S}^{\#}$, $d_F > 1$ is pseudo prime and right prime

Definition
Let $E \subset \mathbb{C}$. If $\theta \in [0, 2\pi)$ is an accumulation point of the set $S = \{\arg s : s \in E\}$, then the set $\{s : \arg s = \theta\}$ is an *accumulation line* of E.

Lemma (Monakhov)
Let f be an entire function and let there exist a sequence (ω_n) satisfying $\lim_{n\to\infty} |\omega_n| = \infty$ and

$$\bigcup_{n=1}^{\infty} \{s : f(s) = \omega_n\}$$

has q $(< \infty)$ accumulation lines (except possibly for a finite number of ω_n). Then f is at most $2q$ degree polynomial.

Lemma
Let $ad - bc \neq 0$, $f : \mathbb{C} \to \mathbb{C}$ is a meromorphic function and $h(z) = (af(z) + b)/(cf(z) + d)$. Then $h(z)$ is transcendental if and only if $f(z)$ is transcendental. In addition, $h(z)$ and $f(z)$ are of the same order.

Lemma (Polya)
If $f(s)$ and $h(s)$ are entire functions and $f(h(s))$ is an entire function of finite order, then there are two possibilities:

1. the inner function $h(s)$ is a polynomial and the outer function $f(s)$ is of finite order; or
2. the inner function $h(s)$ is not a polynomial but of finite order, and the outer function $f(s)$ is of zero order.

Lemma (Edrei and Fuchs)
Let $f(s)$ be a meromorphic function of non-zero order and $h(s)$ be an entire function that is non-polynomial. Then $f(h(s))$ is of infinite order.

Lemma
Let a_1, a_2 be any two distinct complex numbers or infinity. Let f be a meromorphic function of finite order. Let the number of accumulation lines of the set $E = \{s : f(s) = a_j\}$ be finite. Then f is pseudo prime.

Proof.
If f is not prime, then we have a decomposition $f(s) = g(h(s))$ where $g(z)$ is transcendental meromorphic and $h(z)$ is transcendental entire. According to the Edrei and Fuchs lemma, $g(z)$ is of order zero. According to the Monakhov lemma, $g(z) = a_j$, $j = 1, 2$ has a finite number of roots; otherwise, $h(z)$ would be a polynomial. Therefore

$$G(s) := \frac{g(s) - a_1}{g(s) - a_2} = R(s)e^{L(s)},$$

where $R(z)$ is rational and $L(z)$ is an entire function that is non-constant.

Proof (continued).
According to one of the lemmas and the property $T(r, G(s)/R(s)) \leq T(r, G(s)) + T(r, 1/R(s))$ (T is the Nevanlinna characteristic), we have that $G(s)/R(s)$ is an entire function of zero order. The order n of the function $e^{L(s)}$ is positive since $L(s)$ is non-constant. The contradiction proves the lemma.

Proof.
Take $a_1 = 0$ and $a_2 = \infty$. Then the function $F(s) \in \mathcal{S}^{\#}$ is pseudo prime by the above lemma.

We will prove that F is right prime. Let $F(s) = f(h(s))$ be a decomposition. Since F is pseudo prime, f is rational, or h is a polynomial. We are only interested in the case when f is transcendental. Thus, without loss of generality, suppose that h is a polynomial. The set $\mathcal{S}^{\#}$ consists of the Dirichlet series. Therefore $F(s) = O(1)$, $\sigma \to \infty$ uniformly. We have $\lim_{|s|\to\infty} F(s) = \infty$ uniformly in the set

$$A := \{s : \pi/2 + \epsilon \leq \arg s \leq \pi - \epsilon \text{ or } \pi + \epsilon \leq \arg s \leq 3\pi/2 - \epsilon\}.$$

Proof (continued).
Let $h(s) = a_d s^d + a_{d-1}s^{d-1} + \ldots + a_0$, $a_d \neq 0$. Let $\ell_1, \ldots, \ell_d$ be the pre-images of the half-lines $\ell = \{s : \arg a = \pi_2 - \arg a_d\}$ under the action of the polynomial h. Let us enumerate the preimages in such a way that as we approach infinity, the curve ℓ_j is close to the half-line
$L_j := \{s : \arg s = \pi/2d - \arg a_d/d + 2j\pi/2\}$. Thus taking $d \geq 2$, there exist indices p and q such that L_p (except for a finite portion) are in the half-plane $\sigma > 2$ and L_q is in the set A. Thus we have

$$\lim_{\substack{|s|\to\infty \\ s\in\ell_p}} = 1 \text{ and } \lim_{\substack{|s|\to\infty \\ s\in\ell_q}} = \infty.$$

On the other side, $F(s) = f(h(s))$ and $h(\ell_p) = \ell = h(\ell_q)$, thus $F(\ell_p) = F(\ell_q)$, a contradiction. Therefore $d = 1$.

Hurwitz zeta function

Definition
The *Hurwitz zeta function* is defined

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

Here $\Re s > 1$ and $0 < a \leq 1$. It is continued meromorphically to the rest of the complex plane except for the simple pole at $s = 1$ with residue 1. Observe that $\zeta(s,1) = \zeta(s)$.

Hurwitz zeta function is prime

Theorem
Hurwitz zeta function is prime.

Proof.
Pseudo-primeness is proved, as in the case of the extended Selberg class. To prove primeness, we use the Nevanlinna theory. □

References

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