Hankel determinants associated with weighted binary sum of digits

Bartosz Sobolewski
(joint with Maciej Ulas)

Jagiellonian University

Journées Arithmétiques 2023
July 3, 2023

Hankel determinants

Let $(a_n)_{n\in\mathbb{N}}$ – a number sequence.

Definition

We define the Hankel determinant of order n as

$$H(n) = \det \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{bmatrix}.$$

Hankel determinants

Let $(a_n)_{n\in\mathbb{N}}$ – a number sequence.

Definition

We define the Hankel determinant of order n as

$$H(n) = \det \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_{n-1} & a_n & \cdots & a_{2n-2} \end{bmatrix}.$$

Applications:

- Padé approximation and irrationality exponents
- Orthogonal polynomials

Selected results

- Thue–Morse sequence:

$$t_n = (-1)^{s_2(n)},$$

where $s_2(n)$ – sum of binary digits of n.

Selected results

- Thue–Morse sequence:
$$t_n = (-1)^{s_2(n)},$$
where $s_2(n)$ – sum of binary digits of n.

Theorem (Allouche, Peyrière, Wen, Wen (1999))

For all $n \in \mathbb{N}$ we have $H(n) \neq 0$.

Selected results

- Thue–Morse sequence:

$$t_n = (-1)^{s_2(n)},$$

where $s_2(n)$ – sum of binary digits of n.

Theorem (Allouche, Peyrière, Wen, Wen (1999))

For all $n \in \mathbb{N}$ we have $H(n) \neq 0$.

Theorem (Bugeaud (2011))

For any base $b \geq 2$ the number

$$\sum_{n=0}^{\infty} \frac{t_n}{b^n}$$

has irrationality exponent 2.

Selected results

- Paperfolding sequence:

$$f_{2n} = (n+1) \bmod 2, \qquad f_{2n+1} = f_n.$$

Selected results

- Paperfolding sequence:

$$f_{2n} = (n+1) \bmod 2, \qquad f_{2n+1} = f_n.$$

Theorem (Guo, Wu, Wen (2014))

$$H(n) \equiv \begin{cases} 1 \pmod 2 & \textit{if } n \equiv 0,1,2,5,8,9 \pmod{10} \\ 0 \pmod 2 & \textit{if } n \equiv 3,4,6,7 \pmod{10}. \end{cases}$$

Selected results

- Paperfolding sequence:

$$f_{2n} = (n+1) \bmod 2, \qquad f_{2n+1} = f_n.$$

Theorem (Guo, Wu, Wen (2014))

$$H(n) \equiv \begin{cases} 1 \pmod 2 & \textit{if } n \equiv 0,1,2,5,8,9 \pmod{10} \\ 0 \pmod 2 & \textit{if } n \equiv 3,4,6,7 \pmod{10}. \end{cases}$$

Theorem (Guo, Wu, Wen (2014))

For any base $b \geq 2$ the number

$$\sum_{n=0}^{\infty} \frac{f_n}{b^n}$$

has irrationality exponent 2.

Selected results

- A general result concerning irrationality exponent of

$$\sum_{n=0}^{\infty} \frac{a_n}{b^n}$$

for a broader class of sequences (Bugeaud, Han, Wen, Yao (2016))

Selected results

- A general result concerning irrationality exponent of

$$\sum_{n=0}^{\infty} \frac{a_n}{b^n}$$

for a broader class of sequences (Bugeaud, Han, Wen, Yao (2016))

- Period-doubling sequence:

$$d_n = (s_2(n+1) - s_2(n)) \bmod 2.$$

Theorem (Fokkink, Kraaikamp, Shallit (2017))

$H(n)$ is [up to sign] a product of n Jacobsthal numbers.

Jacobsthal numbers:

$$J_0 = 0, \quad J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2}.$$

Selected results

- A general result concerning irrationality exponent of

$$\sum_{n=0}^{\infty} \frac{a_n}{b^n}$$

for a broader class of sequences (Bugeaud, Han, Wen, Yao (2016))

- Period-doubling sequence:

$$d_n = (s_2(n+1) - s_2(n)) \bmod 2.$$

Theorem (Fokkink, Kraaikamp, Shallit (2017))

$H(n)$ is [up to sign] a product of n Jacobsthal numbers.

Jacobsthal numbers:

$$J_0 = 0, \quad J_1 = 1, \quad J_n = J_{n-1} + 2J_{n-2}.$$

- Catalan, Euler, Bernoulli numbers, etc.

Hankel determinants for the binary sum of digits

Problem (Allouche, Shallit)

Find a simple closed form for the Hankel determinants for the binary sum of digits $(s_2(n))_{n\in\mathbb{N}}$.

Hankel determinants for the binary sum of digits

Problem (Allouche, Shallit)

Find a simple closed form for the Hankel determinants for the binary sum of digits $(s_2(n))_{n \in \mathbb{N}}$.

One can observe the following:

n	2	3	6	11	22	43	86	171	$\cdots$
$H(n)$	-1	2	3	-4	-5	6	7	-8	$\cdots$

Hankel determinants for the binary sum of digits

Problem (Allouche, Shallit)

Find a simple closed form for the Hankel determinants for the binary sum of digits $(s_2(n))_{n\in\mathbb{N}}$.

One can observe the following:

n	2	3	6	11	22	43	86	171	$\cdots$
$H(n)$	-1	2	3	-4	-5	6	7	-8	$\cdots$

More precisely, for $n_0 = 2, n_k = 2^{k+1} - n_{k-1} + 1$ we have

$$H(n_k) = (-1)^{\frac{(k+2)(k+3)}{2}}(k+1).$$

Weighted sum of binary digits

Let $\mathbf{w} = (w_j)_{j\in\mathbb{N}}$ – a sequence of weights.

Definition

Let $n \in \mathbb{N}$ have binary expansion

$$n = 2^m\varepsilon_m + \cdots + 2\varepsilon_1 + \varepsilon_0, \quad \varepsilon_j \in \{0,1\}.$$

We define weighted sum of binary digits n as

$$s_{\mathbf{w}}(n) = \varepsilon_m w_m + \cdots + \varepsilon_1 w_1 + \varepsilon_0 w_0.$$

Hankel determinants associated with $s_{\mathbf{w}}$

We define

$$H_{\mathbf{w}}(n) = \det \begin{bmatrix} s_{\mathbf{w}}(0) & s_{\mathbf{w}}(1) & \cdots & s_{\mathbf{w}}(n-1) \\ s_{\mathbf{w}}(1) & s_{\mathbf{w}}(2) & \cdots & s_{\mathbf{w}}(n) \\ \vdots & \vdots & & \vdots \\ s_{\mathbf{w}}(n-1) & s_{\mathbf{w}}(n) & \cdots & s_{\mathbf{w}}(2n-2) \end{bmatrix},$$

Hankel determinants associated with $s_{\mathbf{w}}$

We define

$$H_{\mathbf{w}}(n) = \det \begin{bmatrix} s_{\mathbf{w}}(0) & s_{\mathbf{w}}(1) & \cdots & s_{\mathbf{w}}(n-1) \\ s_{\mathbf{w}}(1) & s_{\mathbf{w}}(2) & \cdots & s_{\mathbf{w}}(n) \\ \vdots & \vdots & & \vdots \\ s_{\mathbf{w}}(n-1) & s_{\mathbf{w}}(n) & \cdots & s_{\mathbf{w}}(2n-2) \end{bmatrix},$$

$$G_{\mathbf{w}}(n) = \det \begin{bmatrix} d_{\mathbf{w}}(0) & d_{\mathbf{w}}(1) & \cdots & d_{\mathbf{w}}(n-1) \\ d_{\mathbf{w}}(1) & d_{\mathbf{w}}(2) & \cdots & d_{\mathbf{w}}(n) \\ \vdots & \vdots & & \vdots \\ d_{\mathbf{w}}(n-1) & d_{\mathbf{w}}(n) & \cdots & d_{\mathbf{w}}(2n-2) \end{bmatrix},$$

where $d_{\mathbf{w}}(n) = s_w(n+1) - s_{\mathbf{w}}(n)$.

Hankel determinants associated with $s_{\mathbf{w}}$

We define

$$H_{\mathbf{w}}(n) = \det \begin{bmatrix} s_{\mathbf{w}}(0) & s_{\mathbf{w}}(1) & \cdots & s_{\mathbf{w}}(n-1) \\ s_{\mathbf{w}}(1) & s_{\mathbf{w}}(2) & \cdots & s_{\mathbf{w}}(n) \\ \vdots & \vdots & & \vdots \\ s_{\mathbf{w}}(n-1) & s_{\mathbf{w}}(n) & \cdots & s_{\mathbf{w}}(2n-2) \end{bmatrix},$$

$$G_{\mathbf{w}}(n) = \det \begin{bmatrix} d_{\mathbf{w}}(0) & d_{\mathbf{w}}(1) & \cdots & d_{\mathbf{w}}(n-1) \\ d_{\mathbf{w}}(1) & d_{\mathbf{w}}(2) & \cdots & d_{\mathbf{w}}(n) \\ \vdots & \vdots & & \vdots \\ d_{\mathbf{w}}(n-1) & d_{\mathbf{w}}(n) & \cdots & d_{\mathbf{w}}(2n-2) \end{bmatrix},$$

where $d_{\mathbf{w}}(n) = s_w(n+1) - s_{\mathbf{w}}(n)$.

Goal

Compute $H_{\mathbf{w}}(n)$ and $G_{\mathbf{w}}(n)$.

A formula for $G_{\mathbf{w}}(n)$

Theorem

Let $n \in \mathbb{N}_+$ *have binary expansion*

$$1^{\ell_s} 0^{\ell_{s-1}} 1^{\ell_{s-2}} \cdots (1-a)^{\ell_3} a^{\ell_2} 1 0^{\ell_1 - 1},$$

where $s \geq 1$, $\ell_i \geq 1$ *for* $i = 1, \ldots, s$ *and* $a \in \{0, 1\}$. *Put*

$$k_i = \begin{cases} \ell_1 - 1 & \text{if } i = 1, \\ \sum_{j=1}^{i} \ell_i & \text{if } i \geq 2. \end{cases}$$

Then

$$G_{\mathbf{w}}(n) = \varepsilon_n \prod_{i=1}^{s} \prod_{j=1}^{\ell_i} (w_{k_i} - 2^j w_{k_i - j})^{\alpha_{i,j}},$$

where $\varepsilon_n \in \{1, -1\}$ *and the exponents* $\alpha_{i,j} \geq 1$ *sum up to* n.

A corollary for the period-doubling sequence

Corollary

If $w_j = J_{j+1}$, *then* $\mathbf{d_w}$ *is the period-doubling sequence and for all* $n \in \mathbb{N}_+$ *we have*

$$G_{\mathbf{w}}(n) = \gamma_n \prod_{i=1}^{s} \prod_{j=1}^{\ell_i} J_j^{\alpha_{i,j}},$$

where $\gamma_n \in \{1, -1\}$.

This generalizes the result of Fokkink, Kraaikamp and Shallit.

Formulas for $H_{\mathbf{w}}(n)$

Let $\tilde{G}_{\mathbf{w}}(n)$ denote $G_{\mathbf{w}}(n)$ after replacing the last column with 1's.

Theorem

Let $n \geq 5$ and let k be such that $2^k < n \leq 2^{k+1}$.

1. *If $2^k < n \leq 3 \cdot 2^{k-1}$, then*

$$\begin{bmatrix} H_{\mathbf{w}}(n) \\ \tilde{G}_{\mathbf{w}}(n) \end{bmatrix} = A_n \begin{bmatrix} -1 & C_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} H_{\mathbf{w}}(2^{k+1} - n + 1) \\ \tilde{G}_{\mathbf{w}}(2^{k+1} - n + 1) \end{bmatrix}$$

2. *If $3 \cdot 2^{k-1} < n \leq 2^{k+1}$, then*

$$\begin{bmatrix} H_{\mathbf{w}}(n) \\ \tilde{G}_{\mathbf{w}}(n) \end{bmatrix} = B_n \begin{bmatrix} 1 & C_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} H_{\mathbf{w}'}(n - 2^k) \\ \tilde{G}_{\mathbf{w}'}(n - 2^k) \end{bmatrix},$$

for some constants A_n, B_n, C_n depending on $n, \mathbf{w}$, where

$$\mathbf{w}' = (w_0, w_1, \ldots, w_{k-1}, \frac{1}{2} w_{k+1}, \ldots).$$

Corollaries

For $w_j = t^j$ write

$$H_{\mathbf{w}}(n) = H(n, t).$$

In particular $H(n, 1)$ is the Hankel determinant for the usual binary sum of digits.

Corollaries

For $w_j = t^j$ write

$$H_{\mathbf{w}}(n) = H(n, t).$$

In particular $H(n, 1)$ is the Hankel determinant for the usual binary sum of digits.

Corollary

Let $n_0 = 2$ *and*

$$n_k = 2^{k+1} - n_{k-1} + 1, \quad k \geq 1.$$

Then

$$H(n_k, t) = (-1)^{\frac{(k+2)(k+3)}{2}} t^{\alpha_k} (t-2)^{n_k - 2} (1 + t + \cdots + t^k)$$

for some $\alpha_k \in \mathbb{N}_+$. *In particular,*

$$H(n_k, 1) = (-1)^{\frac{(k+2)(k+3)}{2}} (k+1).$$

Corollaries

Corollary

For all $m \geq 2$ we have

$$H(2^m, 1) = \sum_{j=1}^{m} \frac{2^{j-1}}{2^j - 1} \prod_{j=1}^{m} (2^j - 1)^{2^{m-j}}.$$

and

$$H(2^m - 1, 1) = (2^{m-1} - 1) \left(1 + \sum_{j=1}^{m-1} \frac{2^{j-1}}{2^j - 1}\right) \prod_{j=1}^{m-2} (2^j - 1)^{2^{m-j}}.$$

Corollaries

Corollary

For all $n \geq 1$ we have

$$H(n,1) \equiv (n+1) \pmod 2.$$

Corollaries

Corollary

For all $n \geq 1$ we have

$$H(n,1) \equiv (n+1) \pmod{2}.$$

Corollary

For $n \in \mathbb{N}$ let

$$s(n) = \begin{cases} \operatorname{sgn} H(n,1) & \text{if } n \neq 1, \\ 1 & \text{if } n = 1. \end{cases}$$

Then for all $n \in \mathbb{N}$ we have

$$\begin{aligned} s(2n) &= (-1)^{\frac{n(n+1)}{2}} s(n), \\ s(4n+1) &= (-1)^n s(2n+1), \\ s(4n+3) &= -s(2n+1). \end{aligned}$$

In particular, $H(n,1) \neq 0$ for all $n \geq 2$.

(Non)vanishing of $H(n, t)$

For $w_j = t^j$ we write

$$s_{\mathbf{w}}(n) = S(n, t),$$
$$H_{\mathbf{w}}(n) = H(n, t).$$

(Non)vanishing of $H(n, t)$

For $w_j = t^j$ we write

$$s_{\mathbf{w}}(n) = S(n, t),$$
$$H_{\mathbf{w}}(n) = H(n, t).$$

Questions

- Which algebraic numbers $t \in \mathbb{C}$ can(not) be roots of $H(n, t)$?
- If $H(n, t) = 0$ for some $n \geq 2$, do there exist infinitely many such n?

(Non)vanishing of $H(n, t)$

For $w_j = t^j$ we write

$$s_{\mathbf{w}}(n) = S(n, t),$$
$$H_{\mathbf{w}}(n) = H(n, t).$$

Questions

- Which algebraic numbers $t \in \mathbb{C}$ can(not) be roots of $H(n, t)$?
- If $H(n, t) = 0$ for some $n \geq 2$, do there exist infinitely many such n?

What we know so far:

- $S(n, 0) = n \bmod 2 \implies H(n, 0) = 0$ for all $n \geq 2$,
- $S(n, 2) = n \implies H(n, 2) = 0$ for all $n \geq 2$,
- $\zeta^d = 1, \zeta \neq 1 \implies H(n, \zeta) = 0$ for infinitely many n.

(Non)vanishing of $H(n, 2\zeta)$

Theorem

For $x \in \mathbb{R}_+$ we have

$$\#\{n \leq x : H(n, -2) \neq 0\} = \Theta(\log x).$$

If $d \geq 3$ and $\zeta^d = 1$, then

$$\log x \ll \#\{n \leq x : H(n, 2\zeta) \neq 0\} \ll x^{\delta_d},$$

where $\delta_d < 1 + \log_2(1 - 2^{-d}) < 1$.

(Non)vanishing of $H(n, 2\zeta)$

Theorem

For $x \in \mathbb{R}_+$ we have

$$\#\{n \leq x : H(n, -2) \neq 0\} = \Theta(\log x).$$

If $d \geq 3$ and $\zeta^d = 1$, then

$$\log x \ll \#\{n \leq x : H(n, 2\zeta) \neq 0\} \ll x^{\delta_d},$$

where $\delta_d < 1 + \log_2(1 - 2^{-d}) < 1$.

Corollary

For any $d \geq 1$ we have

$$\mathrm{dens}\left\{n : (t^d - 2^d) \mid H(n, t)\right\} = 1.$$

Consequently, if g is odd, then

$$\mathrm{dens}\{n : g \mid H(n, 1)\} = 1.$$

(Non)vanishing of $H(n, t)$ for general t

Let $t \neq 0, 2\zeta$, where ζ is a root of unity.

Theorem

Assume there exists odd $n \geq 3$ such that $H(n, t) = 0$ and let m be such that $2^{m-1} < n < 2^m$. If we put

$$n_0 = n, \qquad n_\ell = 2^m(n_{\ell-1} - 1) + n_0, \quad \ell \geq 1,$$

then for all ℓ we have

$$H(n_i, t) = 0.$$

(Non)vanishing of $H(n, t)$ for general t

Let $t \neq 0, 2\zeta$, where ζ is a root of unity.

Theorem

Assume there exists odd $n \geq 3$ such that $H(n, t) = 0$ and let m be such that $2^{m-1} < n < 2^m$. If we put

$$n_0 = n, \qquad n_\ell = 2^m(n_{\ell-1} - 1) + n_0, \quad \ell \geq 1,$$

then for all ℓ we have

$$H(n_i, t) = 0.$$

Proposition

For $x \in \mathbb{R}_+$ we have

$$\#\{n \leq x : H(n, t) \neq 0\} \gg \log x.$$

Further problems

- Describe $H(n,1) \mod 2^k$.

Further problems

- Describe $H(n,1) \mod 2^k$.
- For $\zeta^d = 1$, give better bounds on

$$\#\{n \le x : H(n, 2\zeta) \ne 0\}.$$

Further problems

- Describe $H(n,1) \mod 2^k$.
- For $\zeta^d = 1$, give better bounds on
$$\#\{n \leq x : H(n, 2\zeta) \neq 0\}.$$
- Do there exist roots of $H(n,t)$ of multiplicity > 1 other than $t = 0, 2\zeta$?

Further problems

- Describe $H(n,1) \mod 2^k$.
- For $\zeta^d = 1$, give better bounds on
$$\#\{n \le x : H(n, 2\zeta) \neq 0\}.$$
- Do there exist roots of $H(n,t)$ of multiplicity > 1 other than $t = 0, 2\zeta$?
- Do there exist $n \geq 2$ such that $H(n, \sqrt{2}) = 0$ or $H(n,3) = 0$?

Thank you for your attention!