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Unified treatment of Artin-type problems

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Introduction

Artin's Conjecture

Let $\alpha \in \mathbb{Z} \setminus \{0, \pm 1\}$.
Consider the index of $(\alpha \bmod p) \in (\mathbb{Z}/p\mathbb{Z})^\times$.

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[Hooley, 1967], under GRH:

- The density of primes for which we have **index** 1 exists and equals

$$\sum_{n \geq 1} \frac{\mu(n)}{[\mathbb{Q}(\zeta_n, \sqrt[n]{\alpha}) : \mathbb{Q}]}$$

The density is strictly positive if α is not a square.
It is a rational multiple of Artin's constant

$$\prod_{\ell \text{ prime}} \left(1 - \frac{1}{\ell(\ell - 1)}\right)$$

Variants of Artin's Conjecture for number fields

Let K be a number field and $\alpha \in K^\times \setminus \mu_K$.

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[Cooke and Weinberger, Lenstra, Ziegler, JPS,...], under GRH:

- **index** 1

$$\sum_{n \geqslant 1} \frac{\mu(n)}{[K(\zeta_n, \alpha^{1/n}) : K]}$$

- **index** t

$$\sum_{n \geqslant 1} \frac{\mu(n)}{[K(\zeta_{nt}, \alpha^{1/nt}) : K]}$$

- **index square-free**

$$\sum_{n \geqslant 1} \frac{\mu(n)}{[K(\zeta_{n^2}, \alpha^{1/n^2}) : K]}$$

Index Map

Results on the Index Map [JP]

$$f : \{\text{primes of } K\} \mapsto \mathbb{Z}_{\geqslant 1}$$

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$$\operatorname{dens}(f^{-1}S) = \sum_{s \in S} \operatorname{dens}(f^{-1}s)$$

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- Preimages are either finite or have a positive density. Moreover,

$$\text{dens}(f^{-1}S) = \sum_{s \in S} \text{dens}(f^{-1}s)$$

- The image of f is computable. There is some n_0 such that

$$n \in \text{Im}(f) \Leftrightarrow \gcd(n, n_0) \in \text{Im}(f)$$

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- $$\mathrm{dens}(f^{-1}S) = \mathrm{dens}(f^{-1}S_B) \cdot \prod_{\ell \nmid B} \mathrm{dens}(f^{-1}S_\ell)$$

  $$= \underbrace{F_B}_{\mathbb{Q}} \cdot \underbrace{\mathcal{A}_S}_{\text{Artin type constant}}$$

Special case

Index Map for $\mathbb{Q}$

Notation

$$\alpha = (-1)^{\epsilon} \left(b^2 \cdot 2^{\delta} \cdot T\right)^{2^d}$$

$\epsilon \in \{0, 1\}$ and $d \geqslant 0$ maximal

$b \in \mathbb{Q}^{\times}$ and $\delta \in \{0, 1\}$ and $T \equiv 1 \bmod 4$ squarefree

- **Excluded values for $\mathrm{ind}_p(\alpha)$** (where $p \neq 2$, $v_p(\alpha) = 0$)

$\{2n+1\}$	if $d \geqslant 1$ and $\epsilon = 0$
$\{(2n+1)\|T\|\}$	if $d = \delta = \epsilon = 0$
$\{(2n+1)2\|T\|\}$	if $d = \delta = \epsilon = 1$
$\{n2^m\|T\|/3 : 3 \nmid n\}$	if α is a cube and $3 \mid T$

Example $\mathrm{ind}_p(2)$ takes all possible values