Exposé court

21 *Zero-density estimates for Beurling generalized numbers*

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We discuss zero-density estimates for Beurling zeta functions $\zeta(s)$ associated to Beurling number systems $(\mathscr{P}, \mathscr{N})$, where $\mathscr{P} = (p_1, p_2, \dots)$ is a sequence of Beurling generalized primes, and $\mathscr{N} = (n_0 = 1, n_1, n_2, \dots)$ is the corresponding sequences of generalized integers generated by these primes. Assuming that the integers are "well-behaved", i.e. that their counting function $N(x)$ satisfies $N(x) = Ax + O(x^{\theta})$ for some $A > 0$ and $\theta \in [0, 1)$, the zeta function has analytic continuation to the half-plane $\Re s > \theta$.

In this talk, we present our recent result stating that the number of zeros of such zeta functions in rectangles $\alpha \le \Re s \le 1$, $|\Im s| \le T$ is bounded as

$$N(\alpha, T) \ll T^{\frac{c(1-\alpha)}{1-\theta}} \log^9 T,$$

for a constant c arbitrarily close to 4. We also investigate the consequences of the obtained zero-density estimates on the PNT in short intervals. Our proofs crucially rely on an extension of the classical mean-value theorem for Dirichlet polynomials to generalized Dirichlet polynomials. This talk is based on collaborative work with Gregory Debruyne (Ghent University).