Exposé court

141 *Series representations and asymptotically finite representations for the numbers* $\zeta(2m+1)$

Weba, Michael (Goethe University Frankfurt)

Consider the problem of finding representations for the numbers $\zeta(2m+1)$ where ζ denotes the Riemann zeta function and m is a prescribed positive integer. In the literature, special emphasis has been laid on series representations involving the numbers $\zeta(2n), n \geq 1$. In the specific case $m = 1$, a classical result due to Euler is given by

$$\zeta(3) = \frac{\pi^2}{7}\left(1 - 2\sum_{n=1}^{\infty} \frac{\zeta(2n)}{(n+1)(2n+1)\,4^n}\right). \tag{1}$$

Recently, Lupu and Orr established the formula

$$\zeta(3) = \frac{4\pi^2}{35}\left(\frac{3}{2} - \ln\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n\,(n+1)(2n+1)\,16^n}\right). \tag{2}$$

This talk presents a parameterized family of series representations for the numbers $\zeta(2m+1)$. Based on Clausen functions of higher order, several conventional results are improved and generalized; in particular, the rate of convergence can considerably be improved. Regarding $m = 1$, for instance, one obtains

$$\zeta(3) = \frac{\pi^2}{12}\left(\frac{3}{2} - \ln\frac{\pi}{3} + \sum_{n=1}^{\infty} \frac{\zeta(2n)}{n(n+1)(2n+1)36^n}\right), \tag{3}$$

i.e., the power 16^n in the infinite series (2) can be replaced be 36^n.
The parameterized family is also applicable to represent the values $\beta(2m)$ of the Dirichlet beta function and to derive asymptotically finite representations for both $\zeta(2m+1)$ and $\beta(2m)$.