PERTURBED BESSEL OPERATORS

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Abstract. We study perturbed Bessel operators $L_{m^2} = -\partial_x^2 + (m^2 - \frac{1}{4})\frac{1}{x^2} + Q(x)$ on $L^2]0,\infty[$, where $m \in \mathbb{C}$ and Q is a complex locally integrable potential. Assuming that Q is integrable near ∞ and $x \mapsto x^{1-\varepsilon}Q(x)$ is integrable near 0, with $\varepsilon \geq 0$, we construct solutions to $L_{m^2}f = -k^2 f$ with prescribed behaviors near 0. The special cases $m = 0$ and $k = 0$ are included in our analysis. Our proof relies on mapping properties of various Green's operators of the unperturbed Bessel operator. Then we determine all closed realizations of L_{m^2} and show that they can be organized as holomorphic families of closed operators.

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1. Introduction

One of the most important families of exactly solvable 1-dimensional Schrödinger operators is the family of *Bessel operators*

$$-\partial_x^2 + \frac{\mathrm{c}}{x^2}. \tag{1.1}$$

As is well-known, it is convenient to set $\mathrm{c} = m^2 - 1/4$, so that (1.1) is rewritten as

$$L_{m^2}^0 := -\partial_x^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{x^2}. \tag{1.2}$$

There exists large literature devoted to Bessel operators, mostly restricted to the case $m^2 \in \mathbb{R}$ (see e.g. [11, 20, 21, 26, 35] and references therein). They are also interesting for complex m^2. Their closed realizations on $L^2]0,\infty[$ were studied in [7, 16].

Many operators in mathematics and physics can be reduced to Bessel operators. Here are a few examples:

(1) the usual *Laplacian in dimension* $d \geq 3$, $m = \frac{d}{2} - 1 + \ell$, $\ell = 0, 1, 2, \dots$, see e.g. [16, Section 3];
(2) the *2d Aharonov-Bohm Hamiltonian with magnetic flux* θ, $m = \frac{\theta}{2\pi} + n$, $n \in \mathbb{Z}$, see e.g. [3, 9, 34] and [7, Appendix B];
(3) the *Laplacian on a conical surface of angle* α, $m = \frac{2\pi n}{\alpha}$, $n \in \mathbb{Z}$;
(4) the *Laplacian on a wedge of angle* α *with Dirichlet or Neumann boundary conditions*, $m = \frac{\pi n}{\alpha}$, $n \in \mathbb{Z}$;
(5) perturbed Bessel operators with m complex are used to define *Regge poles*, e.g. [5, 11];
(6) three-body systems with contact interactions.

In this paper we would like to investigate Bessel operators with complex m perturbed by complex valued locally integrable potentials $Q(x)$. Our goal is to show that under some assumptions on Q boundary conditions for perturbed Bessel operators can be described in a similar way as for unperturbed ones.

Before describing our results, let us review general Schrödinger operators on the half-line, and then unperturbed Bessel operators.

1.1. Basic facts about Schrödinger operators on the half-line.

We follow mostly [13]. Suppose that $]0,\infty[\ni x\mapsto V(x)$ is a function in $L^1_{\rm loc}]0,\infty[$, possibly complex valued. Consider the expression

$$L := -\partial_x^2 + V(x). \tag{1.3}$$

The basic meaning of L used in our paper will be that of a linear map from $AC^1]0,\infty[$ to $L^1_{\rm loc}]0,\infty[$. Recall that $AC]0,\infty[$ denotes the set of absolutely continuous functions from $]0,\infty[$ to $\mathbb{C}$, that is, functions whose distributional derivative belong to $L^1_{\rm loc}]0,\infty[$, and $AC^1]0,\infty[$ is the set of functions from $]0,\infty[$ to $\mathbb{C}$ whose distributional derivative belongs to $AC]0,\infty[$.

Let $\mathcal{N}(L)$ denote all functions in $AC^1[0,\infty[$ annihilated by L. The space $\mathcal{N}(L)$ is always 2-dimensional.

Let h_1, h_2 be two linearly independent elements of $\mathcal{N}(L)$. The *canonical bisolution* of L, denoted $G_\leftrightarrow$, is defined by the integral kernel

$$G_\leftrightarrow(x,y) = \frac{1}{\mathscr{W}(h_1,h_2)}\big(h_1(x)h_2(y) - h_2(x)h_1(y)\big), \tag{1.4}$$

where $\mathscr{W}(h_1,h_2)$ denotes the *Wronskian* of h_1 and h_2. Note that (1.4) does not depend on the choice of h_1, h_2. The operator (1.4) is usually unbounded on $L^2]0,\infty[$, however it is very useful in the study of L.

We will use the term *Green's operator* as a synonym for a right inverse of L. In other words, the integral kernel $G_\bullet(x,y)$ of Green's operator $G_\bullet$ satisfies

$$\big(-\partial_x^2 + V(x)\big)G_\bullet(x,y) = \delta(x-y). \tag{1.5}$$

Again, we do not insist on the boundedness of $G_\bullet$ on $L^2]0,\infty[$.

We have various types of Green's operators:

(1) the *forward Green's operator* $G_\rightarrow$:

$$G_\rightarrow(x,y) := \theta(x-y)G_\leftrightarrow(x,y), \tag{1.6}$$

(2) the *backward Green's operator* $G_\leftarrow$:

$$G_\leftarrow(x,y) := -\theta(y-x)G_\leftrightarrow(x,y). \tag{1.7}$$

These two Green's operators are forward, resp. backward Volterra operators: when they act on a function, they do not extend its support to the left, resp. to the right. They are uniquely defined given L: they do not depend on the choice of h_1, h_2.

We also have

(3) the *two-sided Green's operator corresponding to the boundary condition given by* h_1 *near* 0 *and* h_2 *near* ∞:

$$G_\bullet(x,y) := \frac{1}{\mathscr{W}(h_1,h_2)}\begin{cases} h_1(x)h_2(y), & x<y,\\ h_2(x)h_1(y), & y<x.\end{cases} \tag{1.8}$$

The expression (1.8) depends only on the choice of the 1-dimensional subspaces $\mathbb{C}h_1$, $\mathbb{C}h_2$.

Let us now discuss realizations of L as closed densely defined operators on $L^2]0,\infty[$. There are two obvious choices: the minimal realization $L^{\min}$ and the maximal realization $L^{\max}$. Their domains are given by

$$\mathcal{D}(L^{\max}) := \big\{ f \in L^2]0,\infty[\cap AC^1]0,\infty[\quad | \quad Lf \in L^2]0,\infty[\big\},$$
$$\mathcal{D}(L^{\min}) := \text{the closure of } \{ f \in \mathcal{D}(L^{\max}) \ | \ f = 0 \text{ near } 0 \text{ and } \infty \},$$

the closure being taken with respect to the graph norm of $L^{\max}$. In general, there may exist other operators $L^\bullet$ such that $L^{\min} \subset L^\bullet \subset L^{\max}$ defined by boundary conditions at 0 and ∞.

Potentials V considered in this paper usually vanish at ∞. In this case we do not need to specify boundary conditions for L near ∞. This implies that either

$$L^{\min} = L^{\max}, \tag{1.9}$$

$$\text{or} \qquad \dim \mathcal{D}(L^{\max})/\mathcal{D}(L^{\min}) = 2. \tag{1.10}$$

If (1.10) is true, there exists a one-parameter family of operators $L^\bullet$ that satisfy

$$L^{\min} \subsetneq L^\bullet \subsetneq L^{\max}. \tag{1.11}$$

Suppose that $L^\bullet$ satisfies (1.11) or coincides with (1.9). Let λ belong to the resolvent set of $L^\bullet$. Then the integral kernel of $(L^\bullet - \lambda)^{-1}$ has the form of a two-sided Green's operator (1.8) with appropriate h_1 and h_2.

1.2. **Basic facts about unperturbed Bessel operators.** We mostly follow [7, 16]. Let $m \in \mathbb{C}$ and $k \in \mathbb{C}$. Consider the space $\mathcal{N}(L^0_{m^2} + k^2)$, that is, solutions to the eigenvalue equation

$$L^0_{m^2} f = -k^2 f. \tag{1.12}$$

Solving (1.12) for $k = 0$ is easy: $\mathcal{N}(L^0_{m^2})$ is spanned by

$$x^{\frac12 + m}, \quad x^{\frac12 - m}, \quad m \neq 0; \tag{1.13}$$

$$x^{\frac12}, \quad x^{\frac12}\ln(x), \quad m = 0. \tag{1.14}$$

For $k \neq 0$, (1.12) can be reduced to the Bessel equation. This justifies the name *Bessel operator* for (1.2). Here is a pair of solutions of (1.12) for $k \neq 0$:

$$u^0_m(x,k) := \Big(\frac{2}{k}\Big)^m \sqrt{x} I_m(kx), \tag{1.15}$$

$$v^0_m(x,k) := \Big(\frac{k}{2}\Big)^m \sqrt{x} K_m(kx), \tag{1.16}$$

where I_m is the modified Bessel function and K_m the Macdonald function (see Subsection 2.1 below). Note that $u^0_m(\cdot,k)$ behaves as $\frac{x^{\frac12+m}}{\Gamma(m+1)}$ near zero and v^0_m decays exponentially at infinity. Their normalization is chosen in such a way that their Wronskians are 1 and they have a limit at $k = 0$:

$$u^0_m(x,0) = \frac{x^{\frac12+m}}{\Gamma(m+1)}, \tag{1.17}$$

$$v^0_m(x,0) = \frac{\Gamma(m) x^{\frac12 - m}}{2}, \quad \mathrm{Re}(m) \geq 0, \, m \neq 0. \tag{1.18}$$

It is convenient to introduce another solution of the unperturbed eigenequation (1.12), which differs from $v_m^0(x,k)$ only by a different normalization:

$$w_m^0(x,k) = \sqrt{\frac{2k}{\pi}}\Big(\frac{2}{k}\Big)^m v_m^0(x,k) = \sqrt{\frac{2xk}{\pi}}K_m(kx). \tag{1.19}$$

Note that $w_m^0(x,k) = w_{-m}^0(x,k) \sim \mathrm{e}^{-kx}$ for $x \to \infty$.

The Bessel operator for $m=0$ often needs a separate treatment. Note, for instance, that $v_0^0(\cdot,k)$ does not have a limit at $k=0$. To treat the case $m=0$ in a satisfactory way it is useful to introduce a family of eigenfunctions of L_0^0:

$$p_0^0(x,k) := -v_0^0(x,k) - \Big(\ln\Big(\frac{k}{2}\Big) + \gamma\Big)u_0^0(x,k), \tag{1.20}$$

where γ denotes Euler's constant. At $k=0$ it coincides with the logarithmic solution:

$$p_0^0(x,0) = x^{\frac{1}{2}}\ln(x).$$

We will often assume that $\mathrm{Re}(m) \geq 0$, because $L_{m^2}^0$ depends only on m^2. Based on the behavior near zero of its eigenfunctions, one can distinguish 3 regimes:

(1) $\mathrm{Re}(m) > 0$. Eigensolutions of $L_{m^2}^0$ can be divided into *principal*, that means proportional to u_m^0, and *non-principal*, all the others. Principal solutions behave as $x^{\frac{1}{2}+m}$ and are more regular than non-principal ones, which behave as $x^{\frac{1}{2}-m}$.
(2) $\mathrm{Re}(m) = 0$, $m \neq 0$. Eigensolutions of $L_{m^2}^0$ are spanned by u_m^0 and u_{-m}^0, with a comparable behavior $x^{\frac{1}{2}+m}$ and $x^{\frac{1}{2}-m}$ near zero.
(3) $m=0$. Eigensolutions of $L_{m^2}^0$ are spanned by u_0^0 and p_0^0. Those proportional to u_0^0 are again called *principal*, the remaining ones are called *non-principal*. Principal solutions behave as $x^{\frac{1}{2}}$ and are more regular than non-principal ones behaving as $x^{\frac{1}{2}}\ln(x)$.

As explained in the previous subsection, with $L_{m^2}^0 + k^2$ one can associate various Green's operators and the canonical bisolution. The most important are

(1) The canonical bisolution $G_{m^2\leftrightarrow}^0(-k^2)$;
(2) the *forward Green's operator* $G_{m^2,\rightarrow}^0(-k^2)$;
(3) the *backward Green's operator* $G_{m^2,\leftarrow}^0(-k^2)$;
(4) the *two-sided Green's operator with homogeneous boundary conditions* $G_{m^2,\bowtie}^0(-k^2)$.

Additionally, for $m=0$ we will use

(5) the *two-sided Green's operator logarithmic near zero* $G_{0,\triangle}^0(-k^2)$.

Green's operators $G_{m^2\leftrightarrow}^0(-k^2)$, $G_{m^2,\rightarrow}^0(-k^2)$, $G_{m^2,\leftarrow}^0(-k^2)$ and $G_{m^2,\bowtie}^0(-k^2)$ are defined as in (1.4), (1.6), (1.7), resp. (1.8) where we put $h_1(x) = u_m^0(x,k)$, $h_2(x) = v_m^0(x,k)$, and use the fact that their Wronskian is 1.

$G_{0,\triangle}^0(-k^2)$ is defined as in (1.8) where we put $h_1(x) = p_0^0(x)$, $h_2(x) = u_0^0(x)$ and again replace the Wronskian by 1.

For the sake of brevity, we will often abuse terminology, calling $G_{m^2,\bowtie}^0(-k^2)$ "two-sided" and $G_{0,\triangle}^0(-k^2)$ "logarithmic". However, both are two kinds of two-sided Green's operators according to the terminology of Subsection 1.1.

Let us now sketch the theory of closed realizations $L_{m^2}^0$ on $L^2]0,\infty[$. First of all, we can define the minimal and maximal realization of $L_{m^2}^0$ denoted by $L_{m^2}^{0,\min}$ and $L_{m^2}^{0,\max}$, respectively.

They satisfy

$$|\mathrm{Re}(m)| \geq 1 \text{ implies} \quad L_{m^2}^{0,\min} = L_{m^2}^{0,\max}, \tag{1.21}$$

$$|\mathrm{Re}(m)| < 1 \text{ implies} \quad \dim \mathcal{D}(L_{m^2}^{0,\max})/\mathcal{D}(L_{m^2}^{0,\min}) = 2. \tag{1.22}$$

Thus for $|\mathrm{Re}(m)| < 1$ there exists a 1-parameter family of closed realisations of $L_{m^2}^0$ between $L_{m^2}^{0,\min}$ and $L_{m^2}^{0,\max}$ defined by boundary conditions at zero. To describe these realizations one can introduce the following three families of Bessel operators

$$\{-1 < \mathrm{Re}(m)\} \ni m \mapsto H_m^0, \tag{1.23}$$

$$\{-1 < \mathrm{Re}(m) < 1\} \times \big(\mathbb{C} \cup \{\infty\}\big) \ni (m,\kappa) \mapsto H_{m,\kappa}^0, \tag{1.24}$$

$$\big(\mathbb{C} \cup \{\infty\}\big) \ni \nu \mapsto H_0^{0,\nu}. \tag{1.25}$$

The family H_m^0 is the most basic one. It is holomorphic on $\{-1 < \mathrm{Re}(m)\}$ (see Appendix B for the definition of holomorphic families of closed operators). For $1 \leq \mathrm{Re}(m)$ it is the unique closed realization of $L_{m^2}^0$. Then it is extended to the strip $-1 < \mathrm{Re}(m) < 1$ by analytic continuation. Its domain is defined by the boundary condition $\sim x^{\frac12+m}$ at zero, called *homogeneous* or *pure*. In other words, functions in the domain of H_m^0 belong to the domain of the maximal operator $L_{m^2}^{0,\max}$ and behave as $x^{\frac12+m}$ near 0.

The operator $H_{m,\kappa}^0$ is defined by the boundary condition $\sim x^{\frac12+m} + \kappa x^{\frac12-m}$ at zero. For $m=0$ and all κ we simply have $H_{0,\kappa}^0 = H_0^0$. The map (1.24) is holomorphic except for a singularity at $(m,\kappa) = (0,-1)$ (see Proposition 3.11(ii) in [12]).

Finally, for the special case $m=0$, $H_0^{0,\nu}$ is defined by the boundary condition $\sim x^{\frac12}\ln(x) + \nu x^{\frac12}$ at zero. The map (1.25) is holomorphic.

For $\mathrm{Re}(m) > -1$ and $\mathrm{Re}(k) > 0$ the two-sided Green's operator (with pure boundary conditions) is bounded on $L^2]0,\infty[$ and coincides with the resolvent of H_m^0:

$$G_{m^2,\bowtie}^0(-k^2) = \big(H_m^0 + k^2\big)^{-1}.$$

It should, however, be remarked that the integral kernel $G_{m^2,\bowtie}^0(-k^2,x,y)$ is well defined and useful also for other values of k and m, when it does not define a bounded operator.

1.3. **Overview of main results.** Our paper is devoted to *perturbed Bessel operators*, that is, to operators of the form

$$L_{m^2} := -\partial_x^2 + \Big(m^2 - \frac14\Big)\frac{1}{x^2} + Q(x).$$

We allow m to be complex and Q to be complex-valued. Throughout the paper, we will assume that $Q \in L^1_{\mathrm{loc}}]0,\infty[$.

Note that the condition $\mathrm{Re}(m) > -1$ which we saw e.g in (1.23) appears in several places in our analysis. One can argue that the case $\mathrm{Re}(m) \leq -1$ is less important for applications, because then $x^{\frac12+m}$ is not square integrable at zero. Nevertheless, if possible we keep m arbitrary, without restricting it to $\mathrm{Re}(m) > -1$.

Our first concern in this paper is the construction of solutions in $AC^1]0,\infty[$ to the equation

$$L_{m^2} f = -k^2 f \tag{1.26}$$

with a prescribed behavior near 0 or near ∞. We will show that under some conditions on perturbations these solutions are quite similar to solutions of the unperturbed eigenequation (1.12).

First of all we show that if the perturbation is slightly weaker than $1/x^2$ near zero, then there exists a solution of the perturbed equation approximating the principal solution, as described in the following proposition:

Proposition 1.1. *Let* $\mathrm{Re}(m) \geq 0$*,* $k \in \mathbb{C}$ *and suppose that*

$$\int_0^1 x|Q(x)|\mathrm{d}x < \infty, \quad \text{if } m \neq 0; \tag{1.27}$$

$$\int_0^1 x(1 + |\ln(x)|)|Q(x)|\mathrm{d}x < \infty, \quad \text{if } m = 0. \tag{1.28}$$

Suppose that g^0 *is a solution of* (1.12) *such that* $g^0(x) = \mathcal{O}(x^{\frac{1}{2}+\mathrm{Re}(m)})$ *near* 0*. Then, there exists a unique solution* $g \in AC^1]0,\infty[$ *to* (1.26) *such that,*

$$g(x) - g^0(x) = o(x^{\frac{1}{2}+\mathrm{Re}(m)}),$$
$$\partial_x g(x) - \partial_x g^0(x) = o(x^{-\frac{1}{2}+\mathrm{Re}(m)}), \quad x \to 0.$$

In order to be able to well approximate all unperturbed solutions, including the more singular ones, we need to strengthen the assumption on the perturbation.

Proposition 1.2. *Let* $\mathrm{Re}(m) \geq 0$*,* $k \in \mathbb{C}$ *and suppose that*

$$\int_0^1 x^{1-2\mathrm{Re}(m)}|Q(x)|\mathrm{d}x < \infty, \quad \text{if } m \neq 0; \tag{1.29}$$

$$\int_0^1 x\big(1 + (\ln(x))^2\big)|Q(x)|\mathrm{d}x < \infty, \quad \text{if } m = 0. \tag{1.30}$$

Then for any $g^0 \in AC^1]0,\infty[$ *solving* (1.12) *there exists a unique* $g \in AC^1]0,\infty[$ *solving* (1.26) *such that*

$$g(x) - g^0(x) = o(x^{\frac{1}{2}+\mathrm{Re}(m)}),$$
$$\partial_x g(x) - \partial_x g^0(x) = o(x^{-\frac{1}{2}+\mathrm{Re}(m)}), \quad x \to 0.$$

Here are consequences of Propositions 1.1 and 1.2:

Corollary 1.3. *Let* $m \in \mathbb{C}$*,* $k \in \mathbb{C}$ *and suppose that*

$$\int_0^1 x^{1-\varepsilon}|Q(x)|\mathrm{d}x < \infty, \quad \varepsilon \geq 0, \quad \mathrm{Re}(m) \geq -\frac{\varepsilon}{2}, \quad m \neq 0; \tag{1.31}$$

$$\int_0^1 x(1 + |\ln(x)|)|Q(x)|\mathrm{d}x < \infty, \quad m = 0. \tag{1.32}$$

Then there exists a unique $u_m(\cdot,k) \in AC^1]0,\infty[$ *that solves* (1.26) *and satisfies*

$$u_m(x,k) - u_m^0(x,k) = o(x^{\frac{1}{2}+|\mathrm{Re}(m)|}),$$
$$\partial_x u_m(x,k) - \partial_x u_m^0(x,k) = o(x^{-\frac{1}{2}+|\mathrm{Re}(m)|}), \quad x \to 0.$$

Corollary 1.4. *Let $k \in \mathbb{C}$ and suppose that*

$$\int_0^1 x(1+(\ln(x))^2)|Q(x)|\mathrm{d}x < \infty, \quad m=0. \tag{1.33}$$

Then there exists a unique $p_0(\cdot,k) \in AC^1]0,\infty[$ that solves (1.26) and satisfies

$$p_0(x,k) - p_0^0(x,k) = o(x^{\frac12}),$$
$$\partial_x p_0(x,k) - \partial_x p_0^0(x,k) = o(x^{-\frac12}), \quad x \to 0.$$

Note that if $|Q(x)| \lesssim |x|^\alpha$ near 0, then Condition (1.31) is satisfied for $\alpha > -2+\varepsilon$.

Conditions (1.27) and (1.28) are the minimal assumptions near zero in the context of our paper. They are enough to guarantee that the behavior near zero of non-principal solutions is roughly as in the unperturbed case:

Proposition 1.5. *Let $\mathrm{Re}(m) \geq 0$, $\mathrm{Re}(k) \geq 0$. Under the assumptions (1.27) and (1.28), for all $g \in \mathcal{N}(L_{m^2}+k^2)$, we have*

$$g(x) = \mathcal{O}(x^{\frac12 - \mathrm{Re}(m)}), \qquad \partial_x g(x) = \mathcal{O}(x^{-\frac12-\mathrm{Re}(m)}), \tag{1.34}$$

$$g(x) = \mathcal{O}(x^{\frac12}\ln(x)), \qquad \partial_x g(x) = \mathcal{O}(x^{-\frac12}\ln(x)), \qquad x \to 0. \tag{1.35}$$

As described in Proposition 1.1, the above assumptions are enough for the existence of u_m with $\mathrm{Re}(m) \geq 0$. However, it seems that to have *distinguished* non-principal solutions one needs to impose stronger conditions on Q, as described in Corollaries 1.3 and 1.4: In particular, if $\varepsilon \geq 0$ and Condition (1.31) holds, then u_m is constructed in the region $\mathrm{Re}(m) \geq -\frac{\varepsilon}{2}$. This suggests the following question, which we believe is open and interesting:

Open Problem 1.6. *Let Q satisfy condition (1.31) with $\varepsilon > 0$. Does it imply that the function $m \mapsto u_m(x,k)$ extends holomorphically (or at least meromorphically) to the whole $\mathbb{C}$? (This is true for the Coulomb potential [17].).*

Let us now consider the behavior near infinity. To prove the existence of solutions well approximating exponentially decaying solutions, called *Jost solutions*, we need the so-called short-range condition on the potential.

Proposition 1.7. *Let $m \in \mathbb{C}$. Suppose that*

$$\int_1^\infty |Q(x)|\mathrm{d}x < \infty.$$

Let $k \neq 0$ be such that $\mathrm{Re}(k) \geq 0$. Then there exists a unique solution $w_m(\cdot,k) = w_{-m}(\cdot,k) \in AC^1]0,\infty[$ to (1.26) such that

$$w_m(x,k) - w_m^0(x,k) = o(\mathrm{e}^{-x\mathrm{Re}(k)}),$$
$$\partial_x w_m(x,k) - \partial_x w_m^0(x,k) = o(\mathrm{e}^{-x\mathrm{Re}(k)}), \qquad x \to \infty.$$

Similarly as in unperturbed case, it is often convenient to use differently normalized Jost solutions $v_m(x,k) := \sqrt{\frac{\pi}{2k}}\left(\frac{k}{2}\right)^m w_m(x,k)$.

Proposition 1.7 does not cover the zero energy, that is, $k=0$. To handle this case we need to impose stronger conditions on the decay of perturbations, as described in the following two propositions.

Proposition 1.8. *Let $m \in \mathbb{C}$, $k = 0$. Suppose that*

$$\int_1^\infty x^\delta |Q(x)|\mathrm{d}x < \infty, \qquad \text{if } m \neq 0, \quad \text{with } \delta = 1 + 2\max\big(\mathrm{Re}(m), 0\big);$$

$$\int_1^\infty x(1+\ln(x))|Q(x)|\mathrm{d}x < \infty, \qquad \text{if } m = 0.$$

Then there exists a unique $q_m \in AC^1]0,\infty[$ solving (1.26) at $k=0$ such that,

$$q_m(x) - x^{\frac{1}{2}+m} = o(x^{\frac{1}{2}-\mathrm{Re}(m)}),$$
$$\partial_x q_m(x) - \partial_x x^{\frac{1}{2}+m} = o(x^{-\frac{1}{2}-\mathrm{Re}(m)}), \quad x \to \infty.$$

Proposition 1.9. *Let $m = 0$, $k = 0$. Suppose that*

$$\int_1^\infty x(1+(\ln(x))^2)|Q(x)|\mathrm{d}x < \infty.$$

Then there exists a unique $q_{0,\ln} \in AC^1]0,\infty[$ solving (1.26) for $k=0$ such that,

$$q_{0,\ln}(x) - x^{\frac{1}{2}}\ln(x) = o(x^{\frac{1}{2}}),$$
$$\partial_x q_{0,\ln}(x) - \partial_x x^{\frac{1}{2}}\ln(x) = o(x^{-\frac{1}{2}}), \quad x \to \infty.$$

The zero energy eigenequation near infinity is equivalent to the zero energy eigenequation near zero. This follows from the identity

$$-\partial_x^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{x^2} + Q(x) = y^3\Big(-\partial_y^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{y^2} + \tilde{Q}(y)\Big)y, \tag{1.36}$$

$$y = \frac{1}{x}, \qquad \tilde{Q}(y) := y^{-4}Q(y^{-1}). \tag{1.37}$$

Note also a simple relationship between the integral conditions near zero on Q and near infinity on $\tilde{Q}$:

$$\int_0^1 x^{1-\varepsilon}|Q(x)|\mathrm{d}x = \int_1^\infty y^{1+\varepsilon}|\tilde{Q}(y)|\mathrm{d}y, \tag{1.38}$$

$$\int_0^1 x(1+|\ln(x)|^\alpha)|Q(x)|\mathrm{d}x = \int_1^\infty y(1+\ln(y)^\alpha)|\tilde{Q}(y)|\mathrm{d}y. \tag{1.39}$$

Thus one can derive Propositions 1.8 and 1.9 from the $k=0$ case of Corollaries 1.3 and 1.4.

The main tools used in the construction of eigenfunctions are various Green's operators for the unperturbed Bessel operator. The forward Green's operator is used in Propositions 1.1, 1.2 and their corollaries. For instance,

$$u_m(\cdot,k) = \big(\mathbb{1} + G^0_{m^2,\to}(-k^2)Q\big)^{-1}u^0_m(\cdot,k), \tag{1.40}$$

$$p_0(\cdot,k) = \big(\mathbb{1} + G^0_{0,\to}(-k^2)Q\big)^{-1}p^0_0(\cdot,k). \tag{1.41}$$

The backward Green's operator is used in Propositions 1.7, 1.8 and 1.9:

$$w_m(\cdot,k) = \big(\mathbb{1} + G^0_{m^2,\leftarrow}(-k^2)Q\big)^{-1}w^0_m(\cdot,k), \tag{1.42}$$

$$q_m = \big(\mathbb{1} + G^0_{m^2,\leftarrow}(0)Q\big)^{-1}u^0_m(\cdot,0), \tag{1.43}$$

$$q_{0,\ln} = \big(\mathbb{1} + G^0_{0,\leftarrow}(0)Q\big)^{-1}p^0_0(\cdot,0). \tag{1.44}$$

In quantum physics the equation for the Jost solution (1.42) is called the *Lippmann–Schwinger Equation.*

If (1.31) holds and $\frac{\varepsilon}{2} < \mathrm{Re}(m)$, then Corollary 1.32 guarantees the existence only of $u_m(\cdot,k)$, but not of $u_{-m}(\cdot,k)$. Therefore, in this case it is more complicated to describe non-principal solutions. One way to do this is to use the two-sided Green's operator with pure boundary conditions $G^0_{m,\bowtie}$ (where we assume that $\mathrm{Re}(m) \geq 0$). Unfortunately, $\mathbb{1} + G^0_{m,\bowtie}(-k^2)Q$ may be not invertible. In order to guarantee the invertibility we can *compress* $G^0_{m,\bowtie}(-k^2)$ to a sufficiently small interval $]0,a[$. The corresponding compressed Green's operator is denoted $G^{0(a)}_{m,\bowtie}(-k^2)$ (see Appendix A).

In the case $m = 0$ one may prefer to use the logarithmic Green's operator $G^0_{\triangle}(-k^2)$, or actually its compressed version $G^{0(a)}_{\triangle}(-k^2)$.

Proposition 1.10. *Suppose the assumptions of Proposition 1.1 hold. If a is small enough, the following functions are well defined and solve* (1.26) *on* $]0,a[$*:*

$$u^{\bowtie(a)}_{-m}(\cdot,k) := \big(\mathbb{1} + G^{0(a)}_{m,\bowtie}(-k^2)Q\big)^{-1}u^0_{-m}(\cdot,k), \tag{1.45}$$

$$p^{\triangle(a)}_0(\cdot,k) := \big(\mathbb{1} + G^{0(a)}_{0,\triangle}(-k^2)Q\big)^{-1}p^0_0(\cdot,k). \tag{1.46}$$

In the case $m = 0$ the function $p^{\triangle(a)}_0$ is well defined by (1.46) under slightly less restrictive condition on Q than p_0 defined in Corollary 1.4: the difference is just one power of the logarithm less in (1.28) than in (1.29), which is not much. However the difference between the assumptions for $u^{\bowtie(a)}_{-m}(\cdot,k)$ defined in (1.45) and $u_m(\cdot,k)$ defined in Corollary 1.3 is quite substantial.

Unfortunately, the construction (1.45) and (1.46) has obvious drawbacks. It is not very explicit: it involves inverting a complicated integral operator. It also depends on an arbitrary parameter a even if the dependence on a is actually quite weak – if we change a, (1.45) and (1.46) are shifted by a multiple of the corresponding principal solution. Note that we cannot fix the value of a once for all, because the invertibility of $\mathbb{1}+G^{0(a)}_{m,\bowtie}(-k^2)Q$ and $\mathbb{1}+G^{0(a)}_{0,\triangle}(-k^2)Q$ depends on Q and other parameters.

We will describe below an alternative approach, which leads to a simpler description of non-principal solutions for $\mathrm{Re}(m) > 0$. We choose a non-negative integer n. We expand the denominator (1.45) into a formal power series, retaining n first terms. For definiteness, we fix $a = 1$ (quite arbitrarily) and set

$$u^{0[n]}_{-m}(x,k) = \sum_{j=0}^{n}(-G^{0(1)}_{\bowtie}(0)Q)^j u^0_{-m}(x,k). \tag{1.47}$$

Proposition 1.11. *Let* $\mathrm{Re}(k) \geq 0$. *Let n be a nonnegative integer such that Condition* (1.31) *is satisfied for* $-\frac{\varepsilon}{2}(n+1) \leq \mathrm{Re}(-m) \leq 0$. *Then there exists a unique solution* $u^{[n]}_{-m}(\cdot,k)$ *in* $AC^1]0,\infty[$ *of* (1.26) *such that*

$$u^{[n]}_{-m}(x,k) - u^{0[n]}_{-m}(x,k) = o(x^{\frac12+\mathrm{Re}(m)}), \tag{1.48}$$

$$\partial_x u^{[n]}_{-m}(x,k) - \partial_x u^{0[n]}_{-m}(x,k) = o(x^{-\frac12+\mathrm{Re}(m)}), \quad x \to 0. \tag{1.49}$$

Thus for sufficiently large n the function $u^{[n]}_{-m}(\cdot,k)$ determines uniquely a non-principal element of $\mathcal{N}(L_{m^2} + k^2)$ under much weaker assumptions than before.

Boundary conditions determined by $u_{-m}^{0[n]}(\cdot,k)$ still have an unpleasant feature – they depend on k. If we want to have boundary conditions independent of k we need to assume that $|\mathrm{Re}(m)|<1$. Then it is reasonable to choose $k=0$, which we do setting

$$u_{-m}^{0[n]}(x) := u_{-m}^{0[n]}(x,0). \tag{1.50}$$

In particular, under the condition $|\mathrm{Re}(m)|<1$ in Proposition 1.11 we can replace $u_{-m}^{0[n]}(\cdot,k)$ with $u_{-m}^{0[n]}(\cdot)$. This condition is also important in the L^2 theory of perturbed Bessel operators as we explain below.

In concrete cases, the function $u_{-m}^{0[n]}$ can be easily computed. For instance, if Q has a Coulomb singularity at 0, such as $Q(x) = -\frac{\beta}{x}\mathbb{1}_{]0,1]}(x)$ with $\beta\in\mathbb{C}$, then we need to take $n=1$ to cover $|\mathrm{Re}(m)|<1$. Then in the generic case $m\neq\frac12$ we have

$$u_{-m}^{0[1]}(x) = \frac{j_{\beta,-m}(x)}{\Gamma(1-m)} + \mathcal{O}(x^{\frac12+\mathrm{Re}(m)}), \quad j_{\beta,-m}(x) := x^{\frac12-m}\big(1-\frac{\beta x}{1-2m}\big),$$

the function $j_{\beta,-m}$ being precisely the function used to define Whittaker operators in [12, 17].

An important object of our analysis is the *Jost function* $\mathscr{W}_m(k)$, that is the Wronskian of the two main solutions $u_m(\cdot,k)$ and $v_m(\cdot,k)$. We prove that it is well-behaved as a function of k:

Proposition 1.12. *Assume* $\mathrm{Re}(m)>-1$*, as well as* (1.31) *if* $m\neq0$*, or* (1.33) *if* $m=0$*. Then*

$$\lim_{|k|\to\infty}\mathscr{W}_m(k) = 1, \quad \mathrm{Re}(k)\geq 0. \tag{1.51}$$

Note the assumption $\mathrm{Re}(m)>-1$ that appears in the above proposition – which again anticipates the basic condition needed in the L^2 analysis.

The last section of our paper, Section 6, is devoted to closed realizations of L_{m^2} on the Hilbert space $L^2]0,\infty[$. First we prove that under the assumptions of Propositions 1.1 and 1.7, we have

$$|\mathrm{Re}(m)|\geq 1 \text{ implies} \quad L_{m^2}^{\min} = L_{m^2}^{\max}, \tag{1.52}$$

$$|\mathrm{Re}(m)|<1 \text{ implies} \quad \dim\mathcal{D}(L_{m^2}^{\max})/\mathcal{D}(L_{m^2}^{\min}) = 2. \tag{1.53}$$

Thus the basic picture is the same as in the unperturbed case described in (1.21) and (1.22)

In particular, for $|\mathrm{Re}(m)|<1$, beside the minimal and maximal realizations, there exists a 1-parameter family of closed realizations of L_{m^2} defined by boundary conditions at zero. Boundary conditions can be fixed by specifying continuous linear functionals on $\mathcal{D}(L_{m^2}^{\max})$ vanishing on $\mathcal{D}(L_{m^2}^{\min})$, called *boundary functionals*. The method to descibe boundary functionals which seems to work the best in our context uses the Wronskian at zero, that is $\mathscr{W}(f,\cdot;0) := \lim_{x\to0}\mathscr{W}(f,\cdot;x)$, for appropriately chosen functions f. In practice the most convenient f are approximate zero energy eigenfunctions of L_{m^2}.

One can ask about distinguished bases of the boundary space

$$\mathcal{B}_{m^2} := \big(\mathcal{D}(L_{m^2}^{\max})/\mathcal{D}(L_{m^2}^{\min})\big)',$$

where the prime denotes the dual. Under the assumptions of Proposition 1.1 one can always distinguish the *principal boundary functional*. For $0\leq\mathrm{Re}(m)<1$ it can be defined as $\mathscr{W}(x^{\frac12+m},\cdot;0)$. There are also "non-principal boundary functional", which lead to boundary conditions roughly of the type $x^{\frac12-m}$ for $m\neq0$, or $x^{\frac12}\ln(x)$ for $m=0$. In general their choice is less canonical: under the assumptions of Proposition 1.1, a basis of $\mathcal{B}_{m^2}$, $0\leq\mathrm{Re}(m)<1$, $m\neq0$, is given by

$$\big(\mathscr{W}(x^{\frac12+m},\cdot;0),\mathscr{W}(u_{-m}^{\bowtie(a)}(\cdot,k),\cdot;0)\big),$$

with a small enough as in Proposition 1.10. Likewise, if $m=0$, then

$$\big(\mathscr{W}(x^{\frac12},\cdot;0),\mathscr{W}(p_0^{\triangle(a)},\cdot;0)\big),$$

is a basis of $\mathcal{B}_0$.

Let us now impose the assumption

$$\int_0^1 x^{1-\varepsilon}|Q(x)|\mathrm{d}x<\infty. \tag{1.54}$$

If $2>\varepsilon>0$, then for $0\leq\mathrm{Re}(m)\leq\varepsilon/2$ we have a distinguished non-principal boundary functional given by $\mathscr{W}(x^{\frac12-m},\cdot;0)$ if $m\neq0$ and $\mathscr{W}(x^{\frac12}\ln(x),\cdot;0)$ if $m=0$. Thus we obtain three families of perturbed Bessel operators

$$\Big\{-\frac{\varepsilon}{2}<\mathrm{Re}(m)\Big\}\ni m\mapsto H_m, \tag{1.55}$$

$$\Big\{|\mathrm{Re}(m)|<\frac{\varepsilon}{2}\Big\}\times\big(\mathbb{C}\cup\{\infty\}\big)\ni(m,\kappa)\mapsto H_{m,\kappa}, \tag{1.56}$$

$$\big(\mathbb{C}\cup\{\infty\}\big)\ni\nu\mapsto H_0^\nu, \tag{1.57}$$

fully analogous to the families of the unperturbed case. All three families are holomorphic except for a singularity of (1.56) at $(m,\kappa)=(0,-1)$. They are defined as the restrictions of L_{m^2} to the domains:

$$\begin{aligned}
\mathcal{D}(H_m)&:=\Big\{f\in\mathcal{D}(L_{m^2}^{\max})\mid\mathscr{W}(x^{\frac12+m},f;0)=0\Big\},\\
\mathcal{D}(H_{m,\kappa})&:=\Big\{f\in\mathcal{D}(L_{m^2}^{\max})\mid\mathscr{W}\big(x^{\frac12+m}+\kappa x^{\frac12-m},f;0\big)=0\Big\},\quad\kappa\in\mathbb{C},\\
\mathcal{D}(H_{m,\infty})&:=\Big\{f\in\mathcal{D}(L_{m^2}^{\max})\mid\mathscr{W}\big(x^{\frac12-m},f;0\big)=0\Big\},\\
\mathcal{D}(H_0^\nu)&:=\Big\{f\in\mathcal{D}(L_0^{\max})\mid\mathscr{W}\big(\nu x^{\frac12}+x^{\frac12}\ln(x),f;0\big)=0\Big\},\quad\nu\in\mathbb{C},\\
\mathcal{D}(H_0^\infty)&:=\mathcal{D}(H_0).
\end{aligned}$$

The maps $m\mapsto H_m$ and $(m,\kappa)\mapsto H_{m,\kappa}$ are also continuous on $\{-\frac{\varepsilon}{2}\leq\mathrm{Re}(m)\}$, respectively $\{|\mathrm{Re}(m)|\leq\frac{\varepsilon}{2},\kappa\in\mathbb{C}\cup\{\infty\},(m,\kappa)\neq(0,-1)\}$ (continuous families of closed operators are defined similarly as holomorphic families of closed operators, see Appendix B).

The holomorphic family (1.55) for $\mathrm{Re}(m)\geq1$ coincides with $L_{m^2}^{\min}=L_{m^2}^{\max}$. It involves the boundary conditions that can be viewed as "the most natural", and which we call *pure*. Note that (1.55) is restricted to $\{\mathrm{Re}(m)>-\frac{\varepsilon}{2}\}$. This leaves the following open question.

Open Problem 1.13. *Under the minimal conditions of Proposition 1.1, does $m\mapsto H_m$ extend to $\{\mathrm{Re}(m)>-1\}$ holomorphically, or at least meromorphically?*

Let us now consider a nonnegative integer n. Under the assumption (1.54), $2(n+1)>\varepsilon>0$ and $0\leq\mathrm{Re}(m)\leq\frac{\varepsilon}{2}(n+1)$ we can use the function $u_{-m}^{0[n]}$ that was defined in (1.50). Then every non-principal boundary functional can be written as

$$\mathscr{W}(\Gamma(1-m)u_{-m}^{0[n]}+\kappa x^{\frac12+m},\cdot;0) \tag{1.58}$$

for some $\kappa \in \mathbb{C}$. Clearly, (1.58) is proportional to $\mathscr{W}(x^{\frac12-m}+\kappa x^{\frac12+m},\cdot;0)$ for $n=0$. For $n\geq 1$ (1.58) is less canonical. If n,n' are two integers and (1.58) are well defined for n and n', then their difference is proportional to the principal boundary functional $\mathscr{W}(x^{\frac12+m},\cdot;0)$. Thus the set of non-principal boundary conditions can be viewed as a 1-dimensional affine space, where we can use $\mathscr{W}(\Gamma(1-m)u_{-m}^{0[n]},\cdot;0)$ as a possible "reference point".

The boundary functional (1.58) can be used to define a family of perturbed Bessel operators which includes all possible boundary conditions at 0:

$$\Big\{|\mathrm{Re}(m)|<\frac{\varepsilon}{2}(n+1)\Big\}\times\big(\mathbb{C}\cup\{\infty\}\big)\ni(m,\kappa)\mapsto H_{m,\kappa}^{[n]}. \tag{1.59}$$

Note that (1.59) is less canonical than (1.56), however it is defined on a wider region.

The distinguished solutions to (1.26) can be used to write down the resolvent of H_m and its cousins with mixed boundary conditions. For instance, the integral kernel of $(H_m+k^2)^{-1}$ coincides with (1.8) with $h_1(x)=u_m(x,k)$ and $h_2(x)=v_m(x,k)$.

One of the main difficulties of the analysis comes from the need to consider separately the case $m=0$, because generic estimates are not true due to logarithmic terms. This case is actually very important – it corresponds to the 2-dimensional Laplacian in the *s-wave sector*.

The case $k=0$ also requires special care and is particularly important. One can argue that the most natural way to define boundary conditions at zero involves zero-energy eigenfunctions [12]. Moreover, the behavior of zero energy eigenfunctions at large distances described by the so-called *scattering length* is responsible for large scale properties of quantum systems, see Subsection 6.9 and [30].

1.4. **Comparison with the literature.** The present paper can be viewed as a continuation of a series of related papers devoted to 1d Schrödinger operators. This series includes [7, 14–16] about holomorphic families of Bessel operators, [12, 17] about holomorphic families of Whittaker operators and [13] devoted to the general theory.

Of course, the literature devoted to Schrödinger operators on the half-line is vast and goes back several decades. Here is a selection of classical sources: Edmunds-Evans [21], Reed-Simon vol. II [35], Titchmarsh [38], Coddington-Levinson [10], Dunford-Schwartz [20], Yafaev [41], Levitan-Sargsjan [29], Weidmann [40], Derkach-Malamud [19]. See also Gesztesy-Zinchenko [24].

Most of this literature is restricted to real potentials and to self-adjoint realizations. The theory of general closed realizations of Schrödinger 1d operators with complex potentials is actually a relatively straightforward extension of the real theory and also has a long tradition. (However it has a rather different terminology: e.g. "self-adjoint extensions of symmetric operators" and "limit point/limit circle case" replace "closed realizations of the formal operator", and "trivial/nontrivial boundary space"). Here the number of sources is much smaller, but includes some of the classics, such as Titchmarsh [38], Edmunds-Evans [21] and Dunford-Schwartz [20].

Most of these sources start from the 1-dimensional Laplacian on the half-line with Dirichlet or Neumann conditions. This corresponds to the Bessel operator $H_{\frac12}$ (Dirichlet) and $H_{-\frac12}$ (Neumann) in the terminology of our paper. Usually the potential is assumed to be integrable near zero. Note that this excludes not only the $1/x^2$ potential, but even the $1/x$ potential, and makes the theory of boundary conditions very straightforward.

Self-adjoint extensions for potentials $1/x^2$ and $1/x$ are of course also discussed in the literature by many authors, e.g. in [1, 2, 6, 22, 25, 28].

There are also many treatments of d-dimensional Schrödinger operators. They are closely related to the Bessel operators for $m = \frac{d}{2} - 1$, especially in the spherically symmetric case. We are convinced that for many readers our analysis of perturbed Bessel operators can serve as a good introduction to the subject of Schrödinger operators in various dimensions.

Perturbed Bessel operators with complex m were considered to be an important subject already in the 70's, especially in view of applications to the so-called Regge poles [11].

There exists large literature about defining boundary conditions with the help of the so-called *boundary triplets*, see e.g. [4]. In order to define a boundary triplet one needs to select a transversal pair of Lagrangian subspaces inside the boundary space. In the case of perturbed Bessel operators this amounts to selecting two complementary 1-dimensional subspaces, such as (if possible) those defined by $\mathscr{W}(x^{\frac12+m}, \cdot, 0)$ and $\mathscr{W}(x^{\frac12-m}, \cdot; 0)$. Thus the analysis of our paper can be treated as a preparation for an application of the boundary triplets formalism.

The concept of a holomorphic family of closed operators goes back to [26]. The usefulness of organizing perturbed Bessel operators in holomorphic families was recognized by Kato [26, 27].

The behavior of zero energy eigensolutions near infinity and the related concept of the scattering length is a standard tool of contemporary physics, at least in dimension 3 (sometimes also 2). Mathematical treatment of this concept for all dimensions can be found in [30].

Many elements and partial results of our paper can be found in the literature. We are not aware, however, of previous work about all closed realizations of L_{m^2}, their pure point spectra and their holomorphic properties under the general (and rather weak) assumptions on Q that we consider. In this respect, we believe that our results are not far from being sharp, at least concerning the behavior of Q near zero. As we see in our paper, the full picture is quite complex. Note in particular that the cases $m = 0$ and $k = 0$ are quite subtle, both near 0 and ∞. We have also never seen a systematic analysis involving the boundary conditions given by $u_m^{0[n]}$, see (1.59), which shows how to deal with a perturbation where the most straightforward approach fails.

The direction where our results could be somewhat strengthened is the regularity of Q. This can be done e.g. using the method of Shkalikov and Savchuk [36, 37], however it would introduce an extra layer of technical complication in our analysis.

1.5. **Notations.** On $L^2]0,\infty[$, the notation $\langle\cdot|\cdot\rangle$ stands for the bilinear form defined by

$$\langle f|g\rangle := \int_0^\infty f(x)g(x)\mathrm{d}x, \quad f,g \in L^2]0,\infty[. \tag{1.60}$$

More generally, we will use the notation

$$\langle f|g\rangle = \int_0^\infty f(x)g(x)\mathrm{d}x,$$

for any measurable functions f, g such that $fg \in L^1]0,\infty[$.

The *transpose* of an operator A, that is, the adjoint with respect to (1.60) will be denoted $A^\#$, as in [13].

The Wronskian of two differentiable functions f, g is denoted by

$$\mathscr{W}(f,g;x) := f(x)g'(x) - f'(x)g(x), \quad x \in]0,\infty[. \tag{1.61}$$

Moreover,

$$\mathscr{W}(f,g;0) := \lim_{x\to0}\mathscr{W}(f,g;x), \quad \mathscr{W}(f,g;\infty) := \lim_{x\to\infty}\mathscr{W}(f,g;x),$$

if these limits exist. If $f, g \in AC^1]0,\infty[$ are two solutions to the equation $(L^{\bullet}_{m^2}+k^2)u=0$, where $L^{\bullet}_{m^2}$ stands for $L^0_{m^2}$ or L_{m^2}, then their Wronskian is constant and is denoted by $\mathscr{W}(f,g)$.

To shorten notations below, for $b,c\in\mathbb{R}\cup\infty$, we use the shorthand

$$(m,k)\in\{\mathrm{Re}(m)>b,\mathrm{Re}(k)>c\},$$

with the obvious meaning that $(m,k)\in\mathbb{C}^2$ are such that $\mathrm{Re}(m)>b$, $\mathrm{Re}(k)>c$, and likewise if $\mathrm{Re}(m)>b$ is replaced by $\mathrm{Re}(m)\geq b$ and so on.

Let $\Omega\subset\mathbb{C}\times\mathbb{C}$. We will say that a function

$$\Omega\ni(m,k)\mapsto f(m,k)$$

is *regular* on Ω if it is continuous and for any $m_0,k_0\in\mathbb{C}$ it is analytic on

$$\{k\in\mathbb{C}\mid(m_0,k)\in\Omega\}^\circ\ni k\mapsto f(m_0,k),$$
$$\{m\in\mathbb{C}\mid(m,k_0)\in\Omega\}^\circ\ni m\mapsto f(m,k_0),$$

where K° denotes the interior of a set $K\subset\mathbb{C}$. Note that if Ω is open then by Hartog's theorem f is regular if and only if it is analytic.

In several proofs, C will stand for a positive constant depending on the parameters and which may vary from line to line. Moreover the notation $a\lesssim b$ stands for $a\leq Cb$ where C is a positive constant depending on the parameters.

If A is an operator, then $\mathcal{D}(A)$ will denote its domain and $\mathcal{N}(A)$ its kernel (nullspace).

1.6. **Hypotheses.** Recall that throughout the paper, we assume that $Q\in L^1_{\mathrm{loc}}]0,\infty[$. Depending on the results, we will require further integrability conditions near 0 and/or ∞. Our minimal conditions will be

$$Q\in\mathscr{L}^{(0)}_0:=\Big\{Q\in L^1_{\mathrm{loc}}]0,\infty[\,\Big|\int_0^1 x|Q(x)|\mathrm{d}x<\infty\Big\},\quad\text{if } m\neq0;$$
$$Q\in\mathscr{L}^{(0)}_{0,\ln}:=\Big\{Q\in L^1_{\mathrm{loc}}]0,\infty[\,\Big|\int_0^1 x(1+|\ln(x)|)|Q(x)|\mathrm{d}x<\infty\Big\},\quad\text{if } m=0,$$

near 0 and

$$Q\in\mathscr{L}^{(\infty)}_0:=\Big\{Q\in L^1_{\mathrm{loc}}]0,\infty[\,\Big|\int_1^\infty |Q(x)|\mathrm{d}x<\infty\Big\},$$

near ∞. We will sometimes strengthen these conditions to

$$Q\in\mathscr{L}^{(0)}_\varepsilon:=\Big\{Q\in L^1_{\mathrm{loc}}]0,\infty[\,\Big|\int_0^1 x^{1-\varepsilon}|Q(x)|\mathrm{d}x<\infty\Big\};$$
$$Q\in\mathscr{L}^{(0)}_{\varepsilon,\ln^\beta}:=\Big\{Q\in L^1_{\mathrm{loc}}]0,\infty[\,\Big|\int_0^1 x^{1-\varepsilon}(1+|\ln(x)|^\beta)|Q(x)|\mathrm{d}x<\infty\Big\}.$$

with $\varepsilon\geq0$, $\beta\geq0$ and

$$Q\in\mathscr{L}^{(\infty)}_\delta:=\Big\{Q\in L^1_{\mathrm{loc}}]0,\infty[\,\Big|\int_1^\infty x^\delta|Q(x)|\mathrm{d}x<\infty\Big\},$$
$$Q\in\mathscr{L}^{(\infty)}_{\delta,\ln}:=\Big\{Q\in L^1_{\mathrm{loc}}]0,\infty[\,\Big|\int_1^\infty x^\delta(1+\ln(x))|Q(x)|\mathrm{d}x<\infty\Big\}.$$

with $\delta\geq0$.

Obviously, if $0\leq\varepsilon<\varepsilon'$, $0\leq\beta<\beta'$, then

$$\mathscr{L}^{(0)}_{\varepsilon'}\subset\mathscr{L}^{(0)}_{\varepsilon,\ln^{\beta'}}\subset\mathscr{L}^{(0)}_{\varepsilon,\ln^\beta}\subset\mathscr{L}^{(0)}_\varepsilon.$$

Likewise, if $0 \le \delta < \delta'$ then $\mathscr{L}_{\delta'}^{(\infty)} \subset \mathscr{L}_{\delta,\ln}^{(\infty)} \subset \mathscr{L}_{\delta}^{(\infty)}$. Moreover, since $Q \in L^1_{\mathrm{loc}}]0,\infty[$, the integrability conditions on $]0,1[$ are equivalent to the same integrability conditions on $]0,a[$ for any $a>0$ and the integrability conditions on $]1,\infty[$ are equivalent to the same integrability conditions on $]a,\infty[$ for any $a>0$.

2. Solutions of the unperturbed eigenequation

In this section we describe solutions to the unperturbed eigenequation

$$L^0_{m^2} g = -k^2 g. \tag{2.1}$$

2.1. Bessel equation.

The eigenequation (2.1) with the eigenvalue $-k^2=-1$ has the form

$$\Big(-\partial_z^2 + \Big(m^2 - \frac14\Big)\frac{1}{z^2} + 1\Big) g = 0. \tag{2.2}$$

We will call (2.2) the *hyperbolic Bessel equation for dimension* 1, or the *hyperbolic 1d Bessel equation.*

Eq. (2.2) is fully equivalent to the usual modified Bessel equation, which corresponds to dimension 2,

$$\Big(-\partial_z^2 - \frac{1}{z}\partial_z + \frac{m^2}{z^2} + 1\Big) g = 0. \tag{2.3}$$

We use the name the *hyperbolic 2d Bessel equation* for (2.3). In general, we will prefer to use (2.2) as our standard form of the Bessel equation.

In this subsection we briefly describe solutions of the hyperbolic 1d Bessel equation, following mostly [12, 17]. There are two kinds of standard solutions to the hyperbolic 1d Bessel equation (2.2).

The *hyperbolic 1d Bessel function* $\mathcal{I}_m$ is defined by

$$\mathcal{I}_m(z) = \sum_{n=0}^{\infty} \frac{\sqrt{\pi}\,\left(\frac{z}{2}\right)^{2n+m+\frac12}}{n!\Gamma(m+n+1)} = \sqrt{\frac{\pi z}{2}} I_m(z) = \sqrt{\pi}\Big(\frac{z}{2}\Big)^{\frac12+m} \mathbf{F}_m\Big(\frac{z^2}{4}\Big).$$

Here I_m is the usual modified Bessel function, which solves the hyperbolic 2d Bessel equation and $\mathbf{F}_m$ is the appropriately normalized $(0,1)$-hypergeometric function

$$\mathbf{F}_m(w) := \frac{{}_0F_1(m+1;w)}{\Gamma(m+1)} = \sum_{n=0}^{\infty} \frac{w^n}{n!\Gamma(m+n+1)}.$$

Note that for $m \in \mathbb{Z}$

$$\mathcal{I}_m(z) = \mathcal{I}_{-m}(z), \qquad \mathbf{F}_{-m}(w) = w^{2m}\mathbf{F}_m(w). \tag{2.4}$$

The analytic continuation around 0 by the angle $\pm\pi$ multiplies $\mathcal{I}_m$ by a phase factor, namely

$$\mathcal{I}_m(\mathrm{e}^{\pm\mathrm{i}\pi} z) = \mathrm{e}^{\pm\mathrm{i}\pi(m+\frac12)} \mathcal{I}_m(z).$$

The *1d Macdonald function* $\mathcal{K}_m$ is defined by

$$\begin{aligned}\mathcal{K}_m(z) &= \frac{\sqrt{z}}{\sqrt{2\pi}} \int_0^\infty \exp\Big(-\frac{z}{2}(s+s^{-1})\Big)\, s^{-m-1}\mathrm{d}s = \sqrt{\frac{2z}{\pi}} K_m(z)\\ &= \frac{1}{\sin(\pi m)}\big(\mathcal{I}_{-m}(z) - \mathcal{I}_m(z)\big).\end{aligned}$$

Here K_m is the usual Macdonald function, which solves the hyperbolic 2d Bessel equation.

For any $m \in \mathbb{C}$ we have $\mathcal{K}_m(z) = \mathcal{K}_{-m}(z)$.

For any fixed $m \in \mathbb{C}$, the maps $z \mapsto \mathcal{I}_m(z)$ and $z \mapsto \mathcal{K}_m(z)$ are analytic except for a branch point at $z = 0$. Thus the natural domain for these solutions is the Riemann surface of the logarithm. One can parametrize this surface by $|z| \in]0, \infty[$ and $\arg(z) \in \mathbb{R}$. It is often convenient to restrict the domain to $\mathbb{C}\backslash]-\infty, 0]$, that is, to $|\arg(z)| < \pi$. One can also include two copies of $]-\infty, 0]$, from above and from below, that is $\arg(z) = \pm\pi$. For any fixed z in this domain, the maps $m \mapsto \mathcal{I}_m(z)$ and $m \mapsto \mathcal{K}_m(z)$ are analytic on $\mathbb{C}$.

The functions $z \mapsto \mathcal{K}_m(\mathrm{e}^{\pm \mathrm{i}\pi} z)$, obtained from $\mathcal{K}_m$ by analytic continuation, are also solutions of (2.2). Typically, it is natural to consider the pairs of functions

$$z \mapsto \mathcal{K}_m(z), \quad z \mapsto \mathcal{K}_m(\mathrm{e}^{\pm \mathrm{i}\pi} z),$$

on $0 \leq \mp\arg(k) \leq \pi$. Both pairs are linearly independent. In particular, $\mathcal{I}_m(z)$ can be expressed in terms of these functions:

$$\begin{aligned}
\mathcal{K}_m(\mathrm{e}^{\pm \mathrm{i}\pi m} z) &= \frac{\mp \mathrm{i}}{\sin(\pi m)} \left(\mathrm{e}^{\pm \mathrm{i}\pi m} \mathcal{I}_m(z) - \mathrm{e}^{\mp \mathrm{i}\pi m} \mathcal{I}_{-m}(z)\right), \\
\mathcal{I}_m(z) &= \frac{1}{2} \left(\mathcal{K}_m(\mathrm{e}^{\pm \mathrm{i}\pi} z) \mp \mathrm{i}\, \mathrm{e}^{\mp \mathrm{i}\pi m} \mathcal{K}_m(z)\right).
\end{aligned}$$

Here are the asymptotics of the solutions $\mathcal{I}_m$ and $\mathcal{K}_m$ near 0:

$$\mathcal{I}_m(z) = \frac{\sqrt{\pi}}{\Gamma(m+1)} \Big(\frac{z}{2}\Big)^{m+\frac{1}{2}} + \mathcal{O}(|z|^{\frac{5}{2}+\mathrm{Re}(m)}), \qquad m \neq -1, -2, \dots; \tag{2.5}$$

$$\mathcal{K}_m(z) = \frac{\Gamma(m)}{\sqrt{\pi}} \Big(\frac{z}{2}\Big)^{-m+\frac{1}{2}} + \mathcal{O}(|z|^{\frac{5}{2}-\mathrm{Re}(m)}), \qquad \mathrm{Re}(m) > 1; \tag{2.6}$$

$$\mathcal{K}_m(z) = \frac{\Gamma(m)}{\sqrt{\pi}} \Big(\frac{z}{2}\Big)^{-m+\frac{1}{2}} - \frac{\Gamma(-m)}{\sqrt{\pi}} \Big(\frac{z}{2}\Big)^{m+\frac{1}{2}} + \mathcal{O}(|z|^{\frac{5}{2}-\mathrm{Re}(m)}), \qquad |\mathrm{Re}(m)| < 1,\ m \neq 0; \tag{2.7}$$

$$\mathcal{K}_0(z) = -\frac{\sqrt{2z}}{\sqrt{\pi}} \Big(\ln\Big(\frac{z}{2}\Big) + \gamma\Big) + \mathcal{O}\big(|z|^{\frac{5}{2}} \ln|z|\big), \qquad m = 0; \tag{2.8}$$

$$\mathcal{K}_1(z) = \frac{1}{\sqrt{\pi}} \Big(\frac{z}{2}\Big)^{-\frac{1}{2}} + \mathcal{O}(|z|^{\frac{3}{2}} \ln|z|), \qquad m = \pm 1. \tag{2.9}$$

Recall that γ denotes Euler's constant.

Using the integral representation of $\mathcal{K}_m(z)$ one can prove the following asymptotics at infinity: for any $\epsilon > 0$,

$$\mathcal{K}_m(z) = \mathrm{e}^{-z} \left(1 + \mathcal{O}(z^{-1})\right), \quad |\arg(z)| \leq \frac{3}{2}\pi - \epsilon, \quad |z| \to \infty. \tag{2.10}$$

Note that the sector $|\arg(z)| < \frac{3}{2}\pi$ is maximal for the estimate (2.10). Beyond this sector the estimate no longer holds.

The following estimates near zero, say, for $|z| \leq 1$, follow from the series expansions:

$$\begin{aligned}
&|\mathcal{I}_m(z)| \lesssim |z|^{\frac{1}{2}+\mathrm{Re}(m)}; \\
&|\mathcal{K}_m(z)| \lesssim |z|^{\frac{1}{2}-|\mathrm{Re}(m)|}, \quad m \neq 0; \\
&|\mathcal{K}_0(z)| \lesssim |z|^{\frac{1}{2}}(1 + |\ln(z)|), \quad m = 0.
\end{aligned}$$

We also have the following estimates near ∞, say, for $|z| \geq 1$ (and any $\epsilon > 0$):

$$|\mathcal{K}_m(z)| \lesssim \mathrm{e}^{-\mathrm{Re}(z)} \quad |\arg(z)| \leq \frac{3}{2}\pi - \epsilon;$$
$$|\mathcal{K}_m(\mathrm{e}^{\pm \mathrm{i}\pi} z)| \lesssim \mathrm{e}^{\mathrm{Re}(z)}, \quad |\arg(z) \mp \pi| \leq \frac{3}{2}\pi - \epsilon;$$
$$|\mathcal{I}_m(z)| \lesssim \mathrm{e}^{\mathrm{Re}(z)} + \mathrm{e}^{-\mathrm{Re}(z)}.$$

Here are global estimates:

$$|\mathcal{K}_m(z)| \lesssim \min(1,|z|)^{\frac{1}{2}-|\mathrm{Re}(m)|}\,\mathrm{e}^{-\mathrm{Re}(z)}, \quad |\arg(z)| \leq \frac{3}{2}\pi - \epsilon, \quad m \neq 0; \tag{2.11}$$

$$|\mathcal{K}_m(\mathrm{e}^{\pm \mathrm{i}\pi} z)| \lesssim \min(1,|z|)^{\frac{1}{2}-|\mathrm{Re}(m)|}\,\mathrm{e}^{\mathrm{Re}(z)}, \quad |\arg(z) \mp \pi| \leq \frac{3}{2}\pi - \epsilon, \quad m \neq 0; \tag{2.12}$$

$$|\mathcal{K}_0(z)| \lesssim \min(1,|z|)^{\frac{1}{2}}(1 + |\ln \min(1,|z|)|)\,\mathrm{e}^{-\mathrm{Re}(z)}, \quad |\arg(z)| \leq \frac{3}{2}\pi - \epsilon, \quad m = 0; \tag{2.13}$$

$$|\mathcal{K}_0(\mathrm{e}^{\pm \mathrm{i}\pi} z)| \lesssim \min(1,|z|)^{\frac{1}{2}}(1 + |\ln \min(1,|z|)|)\,\mathrm{e}^{\mathrm{Re}(z)}, \quad |\arg(z) \mp \pi| \leq \frac{3}{2}\pi - \epsilon, \quad m = 0; \tag{2.14}$$

$$|\mathcal{I}_m(z)| \lesssim \min(1,|z|)^{\frac{1}{2}+\mathrm{Re}(m)}\big(\mathrm{e}^{-\mathrm{Re}(z)} + \mathrm{e}^{\mathrm{Re}(z)}\big). \tag{2.15}$$

Here are the Wronskians of various solutions of the hyperbolic 1d Bessel equation:

$$\mathscr{W}(\mathcal{I}_m, \mathcal{I}_{-m}) = -\sin(\pi m),$$
$$\mathscr{W}(\mathcal{K}_m, \mathcal{I}_m) = 1,$$
$$\mathscr{W}\big(\mathcal{K}_m, \mathcal{K}_m(\mathrm{e}^{\pm \mathrm{i}\pi}\,\cdot)\big) = 2,$$
$$\mathscr{W}\big(\mathcal{I}_m, \mathcal{K}_m(\mathrm{e}^{\pm \mathrm{i}\pi}\,\cdot)\big) = \mp \mathrm{i}\,\mathrm{e}^{\pm \mathrm{i}\pi m}.$$

2.2. **Equation $L_{m^2}^0 g = -k^2 g$.** Let us now analyze the eigenvalue equation for $L_{m^2}^0$ and an arbitrary eigenvalue $-k^2 \neq 0$:

$$\Big(-\partial_x^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{x^2}\Big)g = -k^2 g. \tag{2.16}$$

A direct computation shows that for $k \neq 0$ (2.16) is solved by the following functions

$$u_m^0(x,k) := \sqrt{\frac{2}{\pi k}}\Big(\frac{2}{k}\Big)^m \mathcal{I}_m(kx), \tag{2.17}$$

$$v_m^0(x,k) := \sqrt{\frac{\pi}{2k}}\Big(\frac{k}{2}\Big)^m \mathcal{K}_m(kx). \tag{2.18}$$

For $m = 0$ we also introduce the solution

$$\begin{aligned} p_0^0(x,k) &:= -\sqrt{\frac{\pi}{2k}}\mathcal{K}_0(kx) - \Big(\ln\Big(\frac{k}{2}\Big) + \gamma\Big)\sqrt{\frac{2}{\pi k}}\mathcal{I}_0(kx) \\ &= -v_0^0(x,k) - \Big(\ln\Big(\frac{k}{2}\Big) + \gamma\Big)u_0^0(x,k). \end{aligned} \tag{2.19}$$

Here are their Wronskians:

$$\mathscr{W}\big(u_m^0(\cdot,k), u_{-m}^0(\cdot,k)\big) = -\frac{2\sin(\pi m)}{\pi}, \tag{2.20}$$

$$\mathscr{W}\big(v_m^0(\cdot,k), u_m^0(\cdot,k)\big) = 1, \quad \mathscr{W}\big(u_0^0(\cdot,k), p_0^0(\cdot,k)\big) = 1. \tag{2.21}$$

We define these functions also for $k=0$:

$$u_m^0(x,0) = \frac{x^{\frac12+m}}{\Gamma(m+1)}, \tag{2.22}$$

$$v_m^0(x,0) = \frac{\Gamma(m)x^{\frac12-m}}{2}, \quad \mathrm{Re}(m)\geq 0, \quad m\neq 0, \tag{2.23}$$

$$p_0^0(x,0) = x^{\frac12}\ln(x). \tag{2.24}$$

Clearly (2.22), (2.23) and (2.24) are annihilated by $L_{m^2}^0$. Note that for any fixed $x>0$, $u_m^0(x,k)$ and $p_0^0(x,k)$ are continuous in k down to $k=0$. If $\mathrm{Re}(m)\geq 0$, $m\neq 0$, the same is true for $v_m^0(x,k)$.

Proposition 2.1. *Let $x>0$. Then*

(i) The function $(m,k)\mapsto u_m^0(x,k)$ is analytic on $\mathbb{C}\times\mathbb{C}$.
(ii) The function $(m,k)\mapsto v_m^0(x,k)$ is regular on

$$\mathbb{C}\times\{\mathrm{Re}(k)\geq 0\}\setminus\big(\{\mathrm{Re}(m)<0\}\times\{k=0\}\cup\{m=0\}\times\{k=0\}\big).$$

(iii) The function $k\mapsto p_0^0(x,k)$ is regular on

$$\{\mathrm{Re}(k)\geq 0\}.$$

Proof. We can rewrite the definitions (2.17) and (2.18) as

$$u_m^0(x,k) := x^{\frac12+m}\mathbf{F}_m\Big(\frac{k^2x^2}{4}\Big), \tag{2.25}$$

$$v_m^0(x,k) := \frac{\pi x^{\frac12-m}}{2\sin(\pi m)}\bigg(\mathbf{F}_{-m}\Big(\frac{k^2x^2}{4}\Big) - \Big(\frac{k^2x^2}{4}\Big)^m\mathbf{F}_m\Big(\frac{k^2x^2}{4}\Big)\bigg). \tag{2.26}$$

From (2.25) the reguarity of $u_m^0(x,k)$ is obvious. (2.26) directly shows the regularity of $v_m^0(x,k)$ on the considered domain at $m\notin\mathbb{Z}$. To see the regularity of $v_m^0(x,k)$ at $m\in\mathbb{Z}$ it suffices to use (2.4) and the de l'Hopital rule.

(iii) follows from (i) and (ii) at $k\neq 0$. At $k=0$, as mentioned above, a direct computation shows that $p_0^0(x,k)\to p_0^0(x,0)$, as $k\to 0$. □

Remark 2.2. *Note that $v_m^0(x,k)$ can be continued across the cut $\arg(k)=\pm\frac{\pi}{2}$, so that $k=0$ becomes its branch point. The restriction to $\{\mathrm{Re}(k)\geq 0\}=\{|\arg(k)|\leq\frac{\pi}{2}\}$ is convenient in view of applications. It is however interesting to note that for $m\in\mathbb{Z}+\frac12$, after this continuation, one obtains an univalent function of k.*

For $\mathrm{Re}(k)\geq 0$, we have the following estimates:

$$|u_m^0(x,k)| \lesssim \min(|k|^{-1},x)^{\frac12+\mathrm{Re}(m)}\,\mathrm{e}^{\mathrm{Re}(k)x}; \tag{2.27}$$

$$|v_m^0(x,k)| \lesssim \min(|k|^{-1},x)^{\frac12-|\mathrm{Re}(m)|}\,\mathrm{e}^{-\mathrm{Re}(k)x}, \quad \mathrm{Re}(m)\geq 0, \quad m\neq 0; \tag{2.28}$$

$$|v_0^0(x,k)| \lesssim \min(|k|^{-1},x)^{\frac12}(1-\ln\min(1,|k|x))\,\mathrm{e}^{-\mathrm{Re}(k)x}, \quad m=0; \tag{2.29}$$

$$|p_0^0(x,k)| \lesssim \min(|k|^{-1},x)^{\frac12}(1-\ln\min(|k|^{-1},x))\,\mathrm{e}^{\mathrm{Re}(k)x}, \quad m=0. \tag{2.30}$$

The reason for complicated prefactors in (2.17) and (2.18) is the good behavior near $k=0$. When we are interested in k large, we usually prefer to replace v_m^0 with a differently normalized solution

$$w_m^0(x,k) := \sqrt{\frac{2k}{\pi}}\Big(\frac{2}{k}\Big)^m v_m^0(x,k) = \mathcal{K}_m(kx) \tag{2.31}$$

behaving as e^{-kx} for $x \to \infty$.

2.3. **Canonical bisolution.** In this and the following subsection we introduce a few integral kernels naturally associated with $L_{m^2}^0 + k^2$. The corresponding operators are not always bounded on $L^2]0,\infty[$, however, they will play an important role in our paper.

Let h_1^0, h_2^0 be any pair of solutions to $L_{m^2}^0 f = -k^2 f$ satisfying $\mathscr{W}(h_1^0, h_2^0) \neq 0$. Then, following [13], we introduce the operator

$$G_{\leftrightarrow}^0 = G_{m^2,\leftrightarrow}^0(-k^2) := \frac{1}{\mathscr{W}(h_1^0,h_2^0)}\Big(h_1^0\langle h_2^0|\cdot\rangle - h_2^0\langle h_1^0|\cdot\rangle\Big).$$

It has the kernel

$$G_{\leftrightarrow}^0(x,y) = G_{m^2,\leftrightarrow}^0(-k^2;x,y) = \frac{1}{\mathscr{W}(h_1^0,h_2^0)}\big(h_1^0(x)h_2^0(y) - h_2^0(x)h_1^0(y)\big).$$

Note that $G_{\leftrightarrow}^0$ does not depend on the choice of the functions h_1^0, h_2^0, which justifies the adjective *canonical*. The operator $G_{\leftrightarrow}^0$ is called the *canonical bisolution of* $L_{m^2}^0 + k^2$. It satisfies

$$(L_{m^2}^0 + k^2)G_{\leftrightarrow}^0 = G_{\leftrightarrow}^0(L_{m^2}^0 + k^2) = 0,$$

which justifies calling it *bisolution*. In particular, since the Wronskian of v_m^0 and u_m^0 is 1, one has

$$G_{m^2,\leftrightarrow}^0(-k^2;x,y) = v_m^0(x,k)u_m^0(y,k) - u_m^0(x,k)v_m^0(y,k).$$

For $m=0$ we also have

$$G_{0,\leftrightarrow}^0(-k^2;x,y) = -p_0^0(x,k)u_0^0(y,k) + u_0^0(x,k)p_0^0(y,k).$$

The canonical bisolution is defined also for $k=0$:

$$G_{m^2,\leftrightarrow}^0(0;x,y) = \frac{1}{2m}\big(x^{\frac12-m}y^{\frac12+m} - x^{\frac12+m}y^{\frac12-m}\big), \quad m \neq 0;$$
$$G_{0,\leftrightarrow}^0(0;x,y) = x^{\frac12}y^{\frac12}\big(\ln(y) - \ln(x)\big), \quad m = 0.$$

Proposition 2.3. *Let $x,y>0$. Then the map*

$$(m^2,k^2) \mapsto G_{m^2,\leftrightarrow}^0(-k^2;x,y),$$

is analytic on $\mathbb{C}\times\mathbb{C}$.

Proof. We can write

$$G_{\leftrightarrow}^0(x,y) = \frac{\pi\sqrt{xy}}{\sin(\pi m)}x^m y^{-m}\Bigg(\mathbf{F}_m\Big(\frac{k^2x^2}{4}\Big)\mathbf{F}_{-m}\Big(\frac{k^2y^2}{4}\Big)$$
$$-x^{-2m}y^{2m}\mathbf{F}_{-m}\Big(\frac{k^2x^2}{4}\Big)\mathbf{F}_m\Big(\frac{k^2y^2}{4}\Big)\Bigg).$$

For $m \notin \mathbb{Z}$, the analyticity in m, k^2 is obvious from this expression. For $m \in \mathbb{Z}$, we apply first the de l'Hopital rule. Then we obtain a function analytic in m, k^2.

$G_{\leftrightarrow}^0$ is invariant with respect to the change $m \to -m$. Together with the analyticity in m, it implies the analyticity in m^2. □

2.4. **Green's operators.** We will need several kinds of Green's operators. The *forward Green's operator* $G^0_{m^2,\rightarrow}(-k^2)$ and the *backward Green's operator* $G^0_{m^2,\leftarrow}(-k^2)$ are defined by

$$\begin{aligned} G^0_{\rightarrow}(x,y) &= G^0_{m^2,\rightarrow}(-k^2;x,y) := \theta(x-y)G^0_{m^2,\leftrightarrow}(-k^2;x,y),\\ G^0_{\leftarrow}(x,y) &= G^0_{m^2,\leftarrow}(-k^2;x,y) := -\theta(y-x)G^0_{m^2,\leftrightarrow}(-k^2;x,y).\end{aligned}$$

Here θ is the Heaviside function:

$$\theta(x) := \begin{cases} 0 & \text{if } x < 0,\\ 1 & \text{if } x \geq 0.\end{cases}$$

Many properties of forward and backward Green's operators can be directly deduced from those of the canonical bisolution.

Proposition 2.4. *Let $x,y>0$. Then the maps*

$$(m^2,k^2) \mapsto G^0_{m^2,\rightarrow}(-k^2;x,y), \quad G^0_{m^2,\leftarrow}(-k^2;x,y)$$

are analytic on $\mathbb{C}\times\mathbb{C}$.

$L^0_{m^2}$ possesses various Green's operators defined by imposing boundary conditions at 0 and ∞, which can be called *two-sided.* Among them we should distinguish $G^0_{m,\bowtie}(-k^2)$ given for $\mathrm{Re}(k)\geq 0$, $k\neq 0$, by its integral kernel

$$\begin{aligned} G^0_{\bowtie}(x,y) &= G^0_{m,\bowtie}(-k^2;x,y)\\ &:= \theta(x-y)v^0_m(x,k)u^0_m(y,k) + \theta(y-x)u^0_m(x,k)v^0_m(y,k).\end{aligned}$$

We will call it the *two-sided Green's operator with pure boundary conditions*, often abusing the terminology and shortening the name to just the *two-sided Green's operator*. Note that it depends on m and not on m^2. Note also that at the moment we do not insist on the conditions $\mathrm{Re}(m) > -1$ and $\mathrm{Re}(k)>0$, which will be needed to make it a bounded operator.

Note the connection between the forward and two-sided Green's operators:

$$\begin{aligned} G^0_{m,\bowtie}(-k^2;x,y) &= u^0_m(x,k)v^0_m(y,k) + G^0_{m^2,\rightarrow}(-k^2;x,y), && (2.32)\\ &= v^0_m(x,k)u^0_m(y,k) + G^0_{m^2,\leftarrow}(-k^2;x,y). && (2.33)\end{aligned}$$

For $m\neq 0$, $G^0_{m,\bowtie}(-k^2)$ can be also defined for $k=0$, when its integral kernel is

$$G^0_{m,\bowtie}(0;x,y) := \frac{1}{2m}\big(x^{\frac12-m}y^{\frac12+m}\theta(x-y) + x^{\frac12+m}y^{\frac12-m}\theta(y-x)\big).$$

For $m=0$ and $k=0$ Green's operator $G^0_{\bowtie}$ is not well defined. This motivates us to introduce another kind of Green's operator at $m=0$, which will be called *Green's operator logarithmic near zero*:

$$G^0_{\triangle}(x,y) = G^0_{0,\triangle}(-k^2;x,y) := -u^0_0(x,k)p^0_0(y,k)\theta(x-y) - p^0_0(x,k)u^0_0(y,k)\theta(y-x).$$

It has a limit at $k=0$:

$$G^0_{0,\triangle}(0;x,y) := -x^{\frac12}y^{\frac12}\big(\ln(x)\theta(x-y) + \ln(y)\theta(y-x)\big).$$

Observe that $G^0_{\triangle}$ and $G^0_{\bowtie}$ differ by a term that diverges as $k\to 0$:

$$G^0_{0,\triangle}(-k^2;x,y) = G^0_{0,\bowtie}(-k^2;x,y) + \Big(\ln\Big(\frac{k}{2}\Big)+\gamma\Big)u^0_0(x,k)u^0_0(y,k).$$

For $m=0$ it is also natural to introduce *Green's operator logarithmic near infinity*:

$$G^0_{0,\triangledown}(x,y)=G^0_{0,\triangledown}(-k^2;x,y):=p^0_0(x,k)u^0_0(y,k)\theta(x-y)+u^0_0(x,k)p^0_0(y,k)\theta(y-x).$$

It also has a limit at $k=0$:

$$G^0_{0,\triangledown}(0;x,y):=x^{\frac12}y^{\frac12}\big(\ln(y)\theta(x-y)+\ln(x)\theta(y-x)\big).$$

One could compare $G^0_{0,\triangledown}$ and $G^0_{0,\leftarrow}$:

$$G^0_{0,\triangledown}(-k^2;x,y)=G^0_{0,\leftarrow}(-k^2;x,y)+p^0_0(x,k)u^0_0(y,k).$$

Proposition 2.5. *Let $x,y>0$.*

(i) The function $(m,k)\mapsto G^0_{m,\bowtie}(-k^2;x,y)$ is regular on

$$\mathbb{C}\times\{\mathrm{Re}(k)\geq 0\}\setminus\{\mathrm{Re}(m)\leq 0\}\times\{k=0\}.$$

(ii) The function $k\mapsto G^0_{0,\triangle}(-k^2;x,y)$ is regular on $\{\mathrm{Re}(k)\geq 0\}$.

Proof. This is a direct consequence of Proposition 2.1. □

Let $a>0$. If $G^0_\bullet$ is one of Green's operators, then we introduce the corresponding Green's operator *compressed to the interval* $]0,a[$ by

$$G^{0(a)}_\bullet(x,y):=G^0_\bullet(x,y)\theta(a-x)\theta(a-y). \tag{2.34}$$

For $m\in\mathbb{C},\kappa\in\mathbb{C}\cup\{\infty\}$ and $\nu\in\mathbb{C}\cup\{\infty\}$ one can also introduce Green's operators with mixed boundary conditions

$$G^0_{m,\kappa}(-k^2;x,y):=\frac{\frac{1}{\Gamma(-m)}(k/2)^{-m}G^0_{m,\bowtie}(-k^2)-\frac{\kappa}{\Gamma(m)}(k/2)^{m}G^0_{-m,\bowtie}(-k^2)}{\frac{1}{\Gamma(-m)}(k/2)^{-m}-\frac{\kappa}{\Gamma(m)}(k/2)^{m}},\quad m\neq 0; \tag{2.35}$$

$$G^0_{0,\kappa}(-k^2;x,y):=G^0_{0,\bowtie}(-k^2;x,y); \tag{2.36}$$

$$G^{0,\nu}_0(-k^2;x,y):=\frac{\big(\nu-\gamma-\ln(k/2)\big)G^0_{0,\bowtie}(-k^2)+G^{0\prime}_{0,\bowtie}(-k^2)}{\nu-\gamma-\ln(k/2)}; \tag{2.37}$$

where $G^{0\prime}_{0,\bowtie}(-k^2)$ denotes $\partial_m G^0_{m,\bowtie}(-k^2)\big|_{m=0}$. Eq. (2.35) is (6.3) of [16] (generalized to $m\in\mathbb{C}$). Eq. (2.37) follows from (2.35) by the de l'Hopital method if we set $\kappa=\frac{\nu m-1}{\nu m+1}$ as in Remark 2.5 of [16]. Note that (2.37) is consistent with (7.1) of [16].

3. Solutions of the perturbed eigenequation with prescribed behavior near origin

In this section we construct solutions to the equation

$$L_{m^2}g=-k^2g, \tag{3.1}$$

and study their properties. We will try to find solutions whose behavior near origin is similar to solutions of the unperturbed equation

$$L^0_{m^2}g^0=-k^2g^0. \tag{3.2}$$

To shorten notations below, we will often write

$$u^0(x)=u^0_m(x)=u^0_m(x,k),\qquad v^0(x)=v^0_m(x)=v^0_m(x,k),$$

where $u^0_m(x,\cdot)$, $v^0_m(x,\cdot)$ are the solutions of (3.2) introduced in (2.17) and (2.18). Recall that the space of solutions in $AC^1]0,\infty[$ to (3.1), respectively to (3.2) is denoted $\mathcal{N}(L_{m^2}+k^2)$, respectively $\mathcal{N}(L^0_{m^2}+k^2)$.

3.1. **Weights.** One of our main tools will be various weighted L^∞ spaces. Let us introduce notation that we will use to denote such spaces.

Let $]a,b[\subset]0,\infty[$. For any positive measurable function ϕ on $]a,b[$, we define the following Banach space of (equivalence classes of) measurable functions on $]a,b[$:

$$L^\infty(]a,b[,\phi) := \left\{ f:]a,b[\to\mathbb{C} \ \middle|\ \left\|\frac{f}{\phi}\right\|_\infty := \operatorname*{ess\,sup}_{x\in]a,b[}\left|\frac{f(x)}{\phi(x)}\right| < \infty \right\}, \tag{3.3}$$

$$L_0^\infty(]0,b[,\phi) := \left\{ f\in L^\infty(]0,b[,\phi) \ \middle|\ \lim_{x\to 0}\frac{f(x)}{\phi(x)} = 0 \right\}, \tag{3.4}$$

$$L_\infty^\infty(]a,\infty[,\phi) := \left\{ f\in L^\infty(]a,\infty[,\phi) \ \middle|\ \lim_{x\to \infty}\frac{f(x)}{\phi(x)} = 0 \right\}. \tag{3.5}$$

We will use the following convention. Suppose that the operator $\mathbb{1}+G_\bullet^0 Q$ is invertible on $L^\infty(]0,a[,\phi)$ for some $a>0$ and some positive measurable function ϕ on $]0,a[$, where $G_\bullet^0$ is a Green's operator. If $f:]0,\infty[\to\mathbb{C}$ is such that its restriction to $]0,a[$ belongs to $L^\infty(]0,a[,\phi)$, then $(\mathbb{1}+G_\bullet^0 Q)^{-1}f$ should be understood as $(\mathbb{1}+G_\bullet^0 Q)^{-1}$ applied to the restriction of f on $]0,a[$. Clearly, if in addition $f\in\mathcal{N}(L_{m^2}^0+k^2)$, then $(\mathbb{1}+G_\bullet^0 Q)^{-1}f$ is a solution to (3.1) on $]0,a[$. The unique solution on $]0,\infty[$ which coincides with $(\mathbb{1}+G_\bullet^0 Q)^{-1}f$ on $]0,a[$ will be denoted by the same symbol.

In order to make the notation more compact, we introduce the following k-dependent weights on $]0,\infty[$:

$$\mu_k(x) := \min(|k|^{-1},x), \quad \eta_{\pm k}(x) := \mathrm{e}^{\pm\mathrm{Re}(k)x}, \tag{3.6}$$

$$\lambda_k(x) := 1-\ln\big(|k|\mu_k(x)\big). \tag{3.7}$$

Note that for $k=0$ we have $\mu_k(x)=x$ and λ_k is ill defined. For $x>|k|^{-1}$, we have $\mu_k(x)=|k|^{-1}$ and $\lambda_k(x)=1$.

With these shorthands we can concisely rewrite our basic estimates on unperturbed eigenfunctions (2.27)–(2.30):

$$|u_m^0(x,k)| \lesssim \mu_k(x)^{\frac12+\mathrm{Re}(m)}\eta_k(x); \tag{3.8}$$

$$|v_m^0(x,k)| \lesssim \mu_k(x)^{\frac12-\mathrm{Re}(m)}\eta_{-k}(x), \quad \mathrm{Re}(m)\ge 0, \quad m\neq 0; \tag{3.9}$$

$$|v_0^0(x,k)| \lesssim \mu_k(x)^{\frac12}\lambda_k(x)\eta_{-k}(x), \quad m=0, \quad k\neq 0; \tag{3.10}$$

$$|p_0^0(x,k)| \lesssim \mu_k(x)^{\frac12}\big(1+|\ln\mu_k(x)|\big)\eta_k(x), \quad m=0. \tag{3.11}$$

Note that estimates (3.8), (3.9), (3.10) and (3.11) are uniform in $x\in]0,\infty[$ and $\mathrm{Re}(k)\ge 0$.

3.2. **The forward Green's operator.** In this subsection we study the boundedness of the operator $G_\to^0 Q$ between suitable weighted L^∞-spaces. The forward Green's operator is insensitive to the change of the sign at m. Therefore, without limiting generality, we can assume that $\mathrm{Re}(m)\ge 0$.

The first lemma is devoted to global properties of $G_\to^0 Q$ on the whole $]0,\infty[$. Note that, if $\varepsilon\ge 0$ and $k\neq 0$, the condition (3.12) is equivalent to $Q\in\mathscr{L}_\varepsilon^{(0)}\cap\mathscr{L}_0^{(\infty)}$, while (3.13) is equivalent to $Q\in\mathscr{L}_{\varepsilon,\ln^\beta}^{(0)}\cap\mathscr{L}_0^{(\infty)}$.

Lemma 3.1. *Let* $\mathrm{Re}(k)\ge 0$ *and* $Q\in L^1_{\mathrm{loc}}]0,\infty[$.

(i) Let $\mathrm{Re}(m) \geq 0$*,* $m \neq 0$ *and* $\varepsilon_1 + \varepsilon \geq \frac{1}{2} + \mathrm{Re}(m)$*. Suppose that*

$$\int_0^\infty \mu_k(y)^{1-\varepsilon} |Q(y)| \mathrm{d}y < \infty. \tag{3.12}$$

Then

$$G^0_{\to} Q : L^\infty\big(]0,\infty[, \mu_k^{\varepsilon_1} \eta_k\big) \to L^\infty_0\big(]0,\infty[, \mu_k^{\varepsilon_1+\varepsilon} \eta_k\big)$$

is bounded by $C \times$(3.12) *uniformly in* k*.*

(ii) Let $m = 0$*,* $k \neq 0$*,* $1 \leq \beta - \alpha$ *and* $\varepsilon_1 + \varepsilon \geq \frac{1}{2}$*. Suppose that*

$$\int_0^\infty \mu_k(y)^{1-\varepsilon} \lambda_k(y)^\beta |Q(y)| \mathrm{d}y < \infty. \tag{3.13}$$

Then

$$G^0_{\to} Q : L^\infty\big(]0,\infty[, \mu_k^{\varepsilon_1} \lambda_k^\alpha \eta_k\big) \to L^\infty_0\big(]0,\infty[, \mu_k^{\varepsilon_1+\varepsilon} \lambda_k^{\alpha+1-\beta} \eta_k\big)$$

is bounded by $C \times$(3.13) *uniformly in* k*.*

Proof. For $m \neq 0$ we use

$$(G^0_{\to} Q f)(x) = v^0(x) \int_0^x u^0(y) Q(y) f(y) \mathrm{d}y - u^0(x) \int_0^x v^0(y) Q(y) f(y) \mathrm{d}y.$$

By (3.8)–(3.9), we have

$$\begin{aligned}&\left| u^0(x) \int_0^x v^0(y) Q(y) f(y) \mathrm{d}y \right| \\ &\lesssim \mu_k(x)^{\frac{1}{2}+\mathrm{Re}(m)} \eta_k(x) \int_0^x \mu_k(y)^{\frac{1}{2}-\mathrm{Re}(m)+\varepsilon_1} \eta_{-k}(y) |Q(y)| \eta_k(y) \mathrm{d}y \left\| \frac{f}{\mu_k^{\varepsilon_1} \eta_k} \right\|_\infty,\end{aligned}$$

and

$$\begin{aligned}&\left| v^0(x) \int_0^x u^0(y) Q(y) f(y) \mathrm{d}y \right| \\ &\lesssim \mu_k(x)^{\frac{1}{2}-\mathrm{Re}(m)} \eta_{-k}(x) \int_0^x \mu_k(y)^{\frac{1}{2}+\mathrm{Re}(m)+\varepsilon_1} \eta_k(y) |Q(y)| \eta_k(y) \mathrm{d}y \left\| \frac{f}{\mu_k^{\varepsilon_1} \eta_k} \right\|_\infty.\end{aligned}$$

Using the fact that $y \mapsto \mu_k(y)$ and $y \mapsto \eta_k(y)$ are increasing together with $-\frac{1}{2} \mp \mathrm{Re}(m) + \varepsilon + \varepsilon_1 \geq 0$, we estimate both expressions by

$$\lesssim \mu_k(x)^{\varepsilon_1+\varepsilon} \eta_k(x) \int_0^x \mu_k(y)^{1-\varepsilon} |Q(y)| \mathrm{d}y \left\| \frac{f}{\mu_k^{\varepsilon_1} \eta_k} \right\|_\infty.$$

Since

$$\int_0^x \mu_k(y)^{1-\varepsilon} |Q(y)| \mathrm{d}y = o(x^0),$$

we obtain

$$\big|(G^0_{\to} Q f)(x)\big| \leq o(x^0) \mu_k(x)^{\varepsilon_1+\varepsilon} \eta_k(x) \left\| \frac{f}{\mu_k^{\varepsilon_1} \eta_k} \right\|_\infty.$$

This proves (i).

For $m = 0$, using (3.8) and (3.10), we have

$$\left|u^0(x)\int_0^x v^0(y)Q(y)f(y)\mathrm{d}y\right| \lesssim \mu_k(x)^{\frac{1}{2}}\eta_k(x)\int_0^x \mu_k(y)^{\frac{1}{2}+\varepsilon_1}\lambda_k(y)^{1+\alpha}\eta_{-k}(y)|Q(y)|\eta_k(y)\mathrm{d}y\left\|\frac{f}{\mu_k^{\varepsilon_1}\lambda_k^{\alpha}\eta_k}\right\|_\infty,$$

and

$$\left|v^0(x)\int_0^x u^0(y)Q(y)f(y)\mathrm{d}y\right| \lesssim \mu_k(x)^{\frac{1}{2}}\lambda_k(x)\eta_{-k}(x)\int_0^x \mu_k(y)^{\frac{1}{2}+\varepsilon_1}\lambda_k(y)^{\alpha}\eta_k(y)|Q(y)|\eta_k(y)\mathrm{d}y\left\|\frac{f}{\mu_k^{\varepsilon_1}\lambda_k^{\alpha}\eta_k}\right\|_\infty.$$

Besides the arguments used in (i), we need to notice that $y \mapsto \lambda_k(y)$ is decreasing and that $1+\alpha-\beta \le 0$. We then estimate both above expressions by

$$\lesssim \mu_k(x)^{\varepsilon_1+\varepsilon}\lambda_k(x)^{1+\alpha-\beta}\eta_k(x)\int_0^x \mu_k(y)^{1-\varepsilon}\lambda_k(y)^{\beta}|Q(y)|\mathrm{d}y\left\|\frac{f}{\mu_k^{\varepsilon_1}\lambda_k^{\alpha}\eta_k}\right\|_\infty.$$

Since

$$\int_0^x \mu_k(y)^{1-\varepsilon}\lambda_k(y)^{\beta}|Q(y)|\mathrm{d}y = o(x^0),$$

we obtain

$$\left|(G^0_{\to}Qf)(x)\right| \le o(x^0)\mu_k(x)^{\varepsilon_1+\varepsilon}\lambda_k(x)^{1+\alpha-\beta}\eta_k(x)\left\|\frac{f}{\mu_k^{\varepsilon_1}\lambda_k^{\alpha}\eta_k}\right\|_\infty.$$

This proves (ii). □

Corollary 3.2. *Let* $\mathrm{Re}(k) \ge 0$ *and* $n \in \mathbb{N}$.

(i) Let $\mathrm{Re}(m) \ge 0$, $m \neq 0$. *Suppose that* $Q \in \mathscr{L}_0^{(0)}$. *Then, for all* $a > 0$, *for all* $f \in L^\infty\big(]0,a[,\mu_k^{\frac{1}{2}+\mathrm{Re}(m)}\eta_k\big)$ *and* $0 < x < a$,

$$\frac{|(G^0_{\to}Q)^n f(x)|}{\mu_k(x)^{\frac{1}{2}+\mathrm{Re}(m)}\eta_k(x)} \le \frac{C^{n+1}}{n!}\Big(\int_0^x \mu_k(y)|Q(y)|\mathrm{d}y\Big)^n \sup_{y<x}\frac{|f(y)|}{\mu_k(y)^{\frac{1}{2}+\mathrm{Re}(m)}\eta_k(y)}.$$

(ii) Suppose $k \neq 0$. *Let* $m = 0$ *and* $\beta \ge 1$. *Suppose that* $Q \in \mathscr{L}^{(0)}_{0,\ln^\beta}$. *Then, for all* $a > 0$, $f \in L^\infty\big(]0,a[,\mu_k^{\frac{1}{2}}\lambda_k^{\beta-1}\eta_k\big)$ *and* $0 < x < a$,

$$\frac{|(G^0_{\to}Q)^n f(x)|}{\mu_k(x)^{\frac{1}{2}}\lambda_k(x)^{\beta-1}\eta_k(x)} \le \frac{C^{n+1}}{n!}\Big(\int_0^x \mu_k(y)\lambda_k(y)^{\beta}|Q(y)|\mathrm{d}y\Big)^n \sup_{y<x}\frac{|f(y)|}{\mu_k(y)^{\frac{1}{2}}\lambda_k(y)^{\beta-1}\eta_k(y)}.$$

(iii) Let $\mathrm{Re}(m) \ge 0$, $m \neq 0$. *Suppose that* $Q \in \mathscr{L}^{(0)}_{2\mathrm{Re}(m)}$. *Then, for all* $a > 0$, *for all* $f \in L^\infty\big(]0,a[,\mu_k^{\frac{1}{2}-\mathrm{Re}(m)}\eta_k\big)$ *and* $0 < x < a$,

$$\frac{|(G^0_{\to}Q)^n f(x)|}{\mu_k(x)^{\frac{1}{2}-\mathrm{Re}(m)}\eta_k(x)} \le \frac{C^{n+1}}{n!}\Big(\int_0^x \mu_k(y)^{1-2\mathrm{Re}(m)}|Q(y)|\mathrm{d}y\Big)^n \sup_{y<x}\frac{|f(y)|}{\mu_k(y)^{\frac{1}{2}-\mathrm{Re}(m)}\eta_k(y)}.$$

Above, C *is a constant independent of* n *and* k.

Proof. To prove (i) we first follow the proof of Lemma 3.1(i) with $\varepsilon_1 = \frac{1}{2} + \mathrm{Re}(m)$, $\varepsilon = 0$, obtaining

$$\left| \frac{\mu_k(y)^{\frac{1}{2}+\mathrm{Re}(m)}\eta_k(y)}{\mu_k(x)^{\frac{1}{2}+\mathrm{Re}(m)}\eta_k(x)} G^0_{\rightarrow}(x,y)Q(y) \right| \leq \mu_k(y)Q(y)\theta(x-y). \tag{3.14}$$

The operator $G^0_{\rightarrow}Q$ is clearly a forward Volterra operator (see Appendix A). Applying Proposition A.2 with $K(x,y)$ given by the integral kernel appearing in the left hand side of (3.14) then yields (i).

To prove (ii) we proceed analogously, using Lemma 3.1(ii) with $\varepsilon_1 = \frac{1}{2}$, $\varepsilon = 0$, $\alpha = \beta - 1$, obtaining

$$\left| \frac{\mu_k(y)^{\frac{1}{2}+\mathrm{Re}(m)}\lambda_k(y)^{\beta-1}\eta_k(y)}{\mu_k(x)^{\frac{1}{2}+\mathrm{Re}(m)}\eta_k(x)} G^0_{\rightarrow}(x,y)Q(y) \right| \leq \mu_k(y)\lambda_k(y)^{\beta}Q(y)\theta(x-y).$$

Assuming $\beta \geq 1$ and using that $\lambda_k(x) \geq 1$, this implies

$$\left| \frac{\mu_k(y)^{\frac{1}{2}+\mathrm{Re}(m)}\lambda_k(y)^{\beta-1}\eta_k(y)}{\mu_k(x)^{\frac{1}{2}+\mathrm{Re}(m)}\lambda_k(x)^{\beta-1}\eta_k(x)} G^0_{\rightarrow}(x,y)Q(y) \right| \leq \mu_k(y)\lambda_k(y)^{\beta}Q(y)\theta(x-y),$$

and hence we can conclude as in the case (i).

To prove (iii) we follow the proof of Lemma 3.1(i) with $\varepsilon_1 = \frac{1}{2} - \mathrm{Re}(m)$, $\varepsilon = 2\mathrm{Re}(m)$. □

Unfortunately, the case $m = 0$, $k = 0$ is not covered by Lemma 3.1 and Corollary 3.2, because then λ_k is ill defined. The following lemma and its corollary work for this case. Note however that Lemma 3.3 is not global in $x \in]0,\infty[$ and $\mathrm{Re}(k) \geq 0$ – we need to restrict the values of x and k.

Lemma 3.3. *Let* $\mathrm{Re}(k) \geq 0$, $m = 0$ *and* $Q \in L^1_{\mathrm{loc}}]0,\infty[$. *Let* $\varepsilon_1 + \varepsilon \geq \frac{1}{2}$, $1 \leq \beta - \alpha$. *Suppose that*

$$\int_0^1 y^{1-\varepsilon}(1 + |\ln(y)|^{\beta})|Q(y)|\mathrm{d}y < \infty. \tag{3.15}$$

Let $k_0 > 0$. *Then*

$$G^0_{\rightarrow}Q : L^{\infty}\big(]0,1[, x^{\varepsilon_1}(1 + |\ln(x)|^{\alpha})\big) \rightarrow L^{\infty}_0\big(]0,1[, x^{\varepsilon_1+\varepsilon}(1 + |\ln(x)|^{\alpha+1-\beta})\big) \tag{3.16}$$

is bounded by $C\times$(3.15) *uniformly in* $|k| \leq k_0$.

Proof. Suppose (3.15). Recall that the solution $p^0_0(\cdot,k)$ has been introduced in (2.19). We write $p^0_0(x) = p^0_0(x,k)$ and $u^0_0(x) = u^0_0(x,k)$ to shorten notations. We then have that

$$(G^0_{\rightarrow}Qf)(x) = -p^0_0(x)\int_0^x u^0_0(y)Q(y)f(y)\mathrm{d}y + u^0_0(x)\int_0^x p^0_0(y)Q(y)f(y)\mathrm{d}y.$$

Now, for $0 < x \leq 1$, by (3.8) and (3.11),

$$\left| u^0_0(x)\int_0^x p^0_0(y)Q(y)f(y)\mathrm{d}y \right| \lesssim x^{\frac{1}{2}}\int_0^x y^{\frac{1}{2}+\varepsilon_1}(1 + |\ln(y)|^{1+\alpha})|Q(y)|\mathrm{d}y \left\| \frac{f}{x^{\varepsilon_1}(1 + |\ln(x)|^{\alpha})} \right\|_{\infty},$$

and

$$\left| p^0_0(x)\int_0^x u^0_0(y)Q(y)f(y)\mathrm{d}y \right| \lesssim x^{\frac{1}{2}}(1 + |\ln(x)|)\int_0^x y^{\frac{1}{2}+\varepsilon_1}(1 + |\ln(y)|^{\alpha})|Q(y)|\mathrm{d}y \left\| \frac{f}{x^{\varepsilon_1}(1 + |\ln(x)|^{\alpha})} \right\|_{\infty}.$$

Using the fact that $y \mapsto |\ln(y)|$ is decreasing and $1+\alpha-\beta \le 0$, we estimate both above expressions by

$$\lesssim x^{\varepsilon_1+\varepsilon}(1+|\ln(x)|^{1+\alpha-\beta}) \int_0^x y^{1-\varepsilon}(1+|\ln(y)|^{\beta})|Q(y)|\mathrm{d}y \left\| \frac{f}{x^{\varepsilon_1}(1+|\ln(x)|^{\alpha})} \right\|_\infty .$$

Applying

$$\int_0^x y^{1-\varepsilon}(1+|\ln(y)|^{\beta})|Q(y)|\mathrm{d}y = o(x^0),$$

we obtain

$$\left|(G^0_{\to}Qf)(x)\right| \le o(x^0)x^{\varepsilon_1+\varepsilon}(1+|\ln(x)|^{1+\alpha-\beta}) \left\| \frac{f}{x^{\varepsilon_1}|(1+|\ln(x)|^{\alpha})} \right\|_\infty .$$

This concludes the proof of (3.16). □

Corollary 3.4. *Let $k=0$, $m=0$ and $n \in \mathbb{N}$.*

(i) Suppose that $Q \in \mathscr{L}^{(0)}_{0,\ln}$. Then, for all $0<a<1$, $f \in L^\infty\big(]0,a[, x^{\frac12}\big)$ and $0<x<a$,

$$\frac{|(G^0_{\to}Q)^n f(x)|}{x^{\frac12}} \le \frac{C^{n+1}}{n!}\Big(\int_0^x y\big(1+|\ln(y)|\big)|Q(y)|\mathrm{d}y\Big)^n \sup_{y<x}\frac{|f(y)|}{y^{\frac12}}.$$

(ii) Suppose that $Q \in \mathscr{L}^{(0)}_{0,\ln^2}$. Then, for all $0<a<1$, $f \in L^\infty\big(]0,a[, x^{\frac12}(|\ln(x)|+1)\big)$ and $0<x<a$,

$$\frac{|(G^0_{\to}Q)^n f(x)|}{x^{\frac12}(1+|\ln(x)|)} \le \frac{C^{n+1}}{n!}\Big(\int_0^x y\big(1+|\ln(y)|^2\big)|Q(y)|\mathrm{d}y\Big)^n \sup_{y<x}\frac{|f(y)|}{y^{\frac12}(1+|\ln(y)|)}.$$

Proof. The proof is the same as that of Corollary 3.2, applying Lemma 3.3. To prove (i), we use (3.16) with $\varepsilon_1=\frac12$, $\varepsilon=0$, $\beta=1$ and $\alpha=0$. To prove (ii), we use (3.16) with $\varepsilon_1=\frac12$, $\varepsilon=0$, $\beta=2$ and $\alpha=1$. □

3.3. **Solutions constructed with the help of the forward Green's operator.** In this subsection we construct solutions to (3.1) that approximate near 0 the solutions to the unperturbed equation using the forward Green's operator as the main tool. Here the behavior of the perturbation at infinity is irrelevant, therefore we need only assumptions on Q restricted to the interval $]0,a[$, where $a>0$ is arbitrary.

The following theorem implies Propositions 1.1 and 1.2 from the introduction. Note that (i) and (ii) concern principal solutions, (iii) and (iv) concern arbitrary solutions.

Theorem 3.5. *Let $\mathrm{Re}(k) \ge 0$.*

(i) Suppose that $\mathrm{Re}(m) \ge 0$, $m \ne 0$, $\varepsilon \ge 0$, $Q \in \mathscr{L}^{(0)}_{\varepsilon}$. Let $g^0 \in \mathcal{N}(L^0_{m^2}+k^2)$ and $g^0 = \mathcal{O}(x^{\frac12+\mathrm{Re}(m)})$. Then

$$g := (\mathbb{1}+G^0_{\to}Q)^{-1}g^0$$

is the unique solution in $AC^1]0,\infty[$ to (3.1) such that,

$$g(x)-g^0(x) = o(x^{\frac12+\mathrm{Re}(m)+\varepsilon}),$$
$$\partial_x g(x)-\partial_x g^0(x) = o(x^{-\frac12+\mathrm{Re}(m)+\varepsilon}), \quad x \to 0.$$

(ii) Suppose that $m=0$, $\varepsilon \geq 0$, $Q \in \mathscr{L}_{\varepsilon,\ln}^{(0)}$. *Let* $g^0 \in \mathcal{N}(L_0^0+k^2)$ *and* $g^0 = \mathcal{O}(x^{\frac12})$. *Then*

$$g := (\mathbb{1}+G^0_{\to}Q)^{-1}g^0$$

is the unique solution in $AC^1]0,\infty[$ *to* (3.1) *such that,*

$$g(x)-g^0(x) = o(x^{\frac12+\varepsilon}),$$
$$\partial_x g(x) - \partial_x g^0(x) = o(x^{-\frac12+\varepsilon}), \quad x\to 0.$$

(iii) Suppose that $\mathrm{Re}(m) \geq 0$, $m \neq 0$, $\varepsilon \geq 2\mathrm{Re}(m)$, $Q \in \mathscr{L}_{\varepsilon}^{(0)}$. *Let* $g^0 \in \mathcal{N}(L_{m^2}^0+k^2)$. *Then*

$$g := (\mathbb{1}+G^0_{\to}Q)^{-1}g^0$$

is the unique solution in $AC^1]0,\infty[$ *to* (3.1) *such that,*

$$g(x)-g^0(x) = o(x^{\frac12-\mathrm{Re}(m)+\varepsilon}),$$
$$\partial_x g(x) - \partial_x g^0(x) = o(x^{-\frac12-\mathrm{Re}(m)+\varepsilon}), \quad x\to 0.$$

(iv) Suppose that $m=0$, $\varepsilon \geq 0$, $Q \in \mathscr{L}_{\varepsilon,\ln^2}^{(0)}$. *Let* $g^0 \in \mathcal{N}(L_0^0+k^2)$. *Then*

$$g := (\mathbb{1}+G^0_{\to}Q)^{-1}g^0$$

is the unique solution in $AC^1]0,\infty[$ *to* (3.1) *such that,*

$$g(x)-g^0(x) = o(x^{\frac12+\varepsilon}),$$
$$\partial_x g(x) - \partial_x g^0(x) = o(x^{-\frac12+\varepsilon}), \quad x\to 0.$$

Proof. To prove (i), we use Corollary 3.2(i) which shows that, for any $a>0$, $\mathbb{1}+G^0_{\to}Q$ is invertible on $L^\infty(]0,a[,\mu_k^{\frac12+\mathrm{Re}(m)})$ with inverse given by

$$\big((\mathbb{1}+G^0_{\to}Q)^{-1}f\big)(x) = \sum_{n=0}^{\infty}\big((-G^0_{\to}Q)^n f\big)(x). \tag{3.17}$$

Hence, if $g^0 \in \mathcal{N}(L_{m^2}^0+k^2)$ satisfies $g^0 = \mathcal{O}(x^{\frac12+\mathrm{Re}(m)})$, $g=(\mathbb{1}+G^0_{\to}Q)^{-1}g^0$ is well defined in $L^\infty_{\mathrm{loc}}]0,\infty[$. Since $G^0_{\to}$ is a Green's operator, it then easily follows that g belongs to $AC^1]0,\infty[$ and is a solution to (3.1). The asymptotic behavior near 0 of g and $\partial_x g$ follow from the Neumann series expansion (3.17) and Lemma 3.1(i). Finally, uniqueness is a consequence of standard properties of the Wronskian of two solutions in $\mathcal{N}(L_{m^2}+k^2)$.

To prove (ii) we proceed analogously, using Corollary 3.2(ii) (with $\beta=1$) and Lemma 3.1(ii) in the case where $k\neq 0$. If $k=0$, we use Corollary 3.4(i) and Lemma 3.3.

To prove (iii) we use Corollary 3.2(iii) and Lemma 3.1(i).

To prove (iv), we use Corollary 3.2(ii) with $\beta=2$ and Lemma 3.1(ii) in the case where $k\neq 0$. If $k=0$, we use Corollary 3.4(ii) and Lemma 3.3. □

We can apply Theorem 3.5(i) and (ii) to $g^0(x) = u_m^0(x,k)$. We obtain the following result, which implies Corollary 1.3 from the introduction.

Proposition 3.6. *Let* $m\in\mathbb{C}$, $\varepsilon \geq 2\max\big(-\mathrm{Re}(m),0\big)$. *Suppose that*

$$Q \in \mathscr{L}_{\varepsilon}^{(0)}, \text{ if } m\neq 0, \qquad Q\in \mathscr{L}_{\varepsilon,\ln}^{(0)}, \text{ if } m=0.$$

Then

$$u_m(\cdot,k) := (\mathbb{1} + G^0_{\to}Q)^{-1}u^0_m(\cdot,k)$$

is the unique solution in $AC^1]0,\infty[$ *to* (3.1) *such that,*

$$u_m(x,k) - u^0_m(x,k) = o(x^{\frac{1}{2}+\mathrm{Re}(m)+\varepsilon}), \tag{3.18}$$

$$\partial_x u_m(x,k) - \partial_x u^0_m(x,k) = o(x^{-\frac{1}{2}+\mathrm{Re}(m)+\varepsilon}), \quad x \to 0. \tag{3.19}$$

Note that $\mathrm{Re}(m) + \varepsilon \geq |\mathrm{Re}(m)|$. Therefore, the error in (3.18) and (3.19) is always of a smaller order than the most regular solutions to (3.2). Let us stress that Proposition 3.6 includes the case $k = 0$, where $u^0_m(x,0) = x^{\frac{1}{2}+m}/\Gamma(m+1)$.

Recall that in (2.19) we have introduced the family of solutions to (3.2) for $m = 0$ with a logarithmic behavior near zero, denoted $p^0_0(x,k)$. This family includes the logarithmic case $m = 0$, $k = 0$, which does not belong to the family $u^0_0(x,k)$:

$$p^0_0(x,0) := x^{\frac{1}{2}}\ln(x).$$

We can apply Theorem 3.5(iv) to $g^0(x) = p^0_0(x,k)$, obtaining the following eigensolutions of the perturbed eigenequation. Note that Proposition 3.7 implies Corollary 1.4 from the introduction.

Proposition 3.7. *Let* $\mathrm{Re}(k) \geq 0$, $m = 0$, $\varepsilon \geq 0$ *and* $Q \in \mathscr{L}^{(0)}_{\varepsilon,\ln^2}$. *Then*

$$p_0(\cdot,k) := (\mathbb{1} + G^0_{\to}Q)^{-1}p^0_0(\cdot,k) \tag{3.20}$$

is the unique solution in $AC^1]0,\infty[$ *to* (3.1) *such that,*

$$p_0(x,k) - p^0_0(x,k) = o(x^{\frac{1}{2}+\varepsilon}),$$

$$\partial_x p_0(x,k) - \partial_x p^0_0(x,k) = o(x^{-\frac{1}{2}+\varepsilon}), \quad x \to 0.$$

In the following proposition we fix the perturbation Q and study the regularity of the solutions $u_m(\cdot,k)$ and $p_0(\cdot,k)$ with respect to m and k.

Proposition 3.8.

(i) Let $\varepsilon > 0$ *and suppose that* $Q \in \mathscr{L}^{(0)}_{\varepsilon}$. *Then for any* $x > 0$ *the maps*

$$\Big\{\mathrm{Re}(m) \geq -\frac{\varepsilon}{2}\Big\} \times \mathbb{C} \ni (m,k) \mapsto u_m(x,k), \partial_x u_m(x,k) \tag{3.21}$$

are regular.

(ii) Let $Q \in \mathscr{L}^{(0)}_0$. *Then for any* $x > 0$ *the maps*

$$\Big\{\mathrm{Re}(m) \geq 0,\ m \neq 0\Big\} \times \mathbb{C} \ni (m,k) \mapsto u_m(x,k),\ \partial_x u_m(x,k) \tag{3.22}$$

are regular. If we strengthen the assumption to $Q \in \mathscr{L}^{(0)}_{0,\ln}$, *then in* (3.22) *we can include* $m = 0$.

(iii) Let $Q \in \mathscr{L}^{(0)}_{0,\ln^2}$. *Then for any* $x > 0$ *the maps*

$$\Big\{\mathrm{Re}(k) \geq 0\Big\} \ni k \mapsto p_0(x,k),\ \partial_x p_0(x,k) \tag{3.23}$$

are regular.

Proof. We use the continuity and analyticity of the function u^0 and of the map $G^0_{\rightarrow}Q$ with respect to parameters. More precisely, for all fixed (x, y), the map $(m, k) \mapsto G^0_{\rightarrow}(x, y)$ is analytic. Lemma 3.2 and an induction argument then shows that, for all $x > 0$, $(G^0_{\rightarrow}Q)^n u^0(x)$ is analytic on $\{\mathrm{Re}(m) > -\varepsilon/2\}$. Since, by Lemma 3.2, the series

$$u_m(x,k) = \sum_{n=0}^{\infty} (-G^0_{\rightarrow}Q)^n u^0_m(x,k),$$

converges uniformly on every compact subset of $\{\mathrm{Re}(m) > -\varepsilon/2\}$. This proves the statement concerning the analyticity of u_m. Continuity is proven similarly.

The regularity of $\partial_x u_m$, p_0 and $\partial_x p_0$ follows in the same way. □

We conclude this subsection with the following more precise estimate (compared to Proposition 3.6) on the difference between u_0 and u_0^0 (assuming that the stronger condition $Q \in \mathscr{L}^{(0)}_{\varepsilon,\ln^2}$ holds). We will need this estimate to study closed realization of L_0 in Section 6.

Proposition 3.9. *Let* $\mathrm{Re}(k) \geq 0$*,* $m = 0$ *and suppose that* $Q \in \mathscr{L}^{(0)}_{0,\ln^2}$*. Then*

$$u_0(x,k) - u_0^0(x,k) = o(x^{\frac12}|\ln(x)|^{-1}),$$
$$\partial_x u_0(x,k) - \partial_x u_0^0(x,k) = o(x^{-\frac12}|\ln(x)|^{-1}), \quad x \to 0.$$

Proof. Recall from Proposition 3.6 that $u_0 - u_0^0 = -G^0_{\rightarrow}Qu_0$ and that $u_0(x,k) = \mathcal{O}(x^{\frac12})$. If $k = 0$, it then suffices to use Lemma 3.1(ii) with $\varepsilon_1 = \frac12$, $\varepsilon = 0$, $\beta = 2$, $\alpha = 0$. If $k = 0$, we use Lemma 3.3, also with $\varepsilon_1 = \frac12$, $\varepsilon = 0$, $\beta = 2$, $\alpha = 0$. □

3.4. **Asymptotics of non-principal solutions near 0.** In this subsection, under the minimal assumptions $Q \in \mathscr{L}^{(0)}_0$ if $m \neq 0$, $Q \in \mathscr{L}^{(0)}_{0,\ln}$ if $m = 0$, we show that any solution to (3.1) behaves like non-principal unperturbed solutions near 0.

The following proposition provides a rather rough estimate on all eigensolutions. Note that, for $m \neq 0$, if $Q \in \mathscr{L}^{(0)}_\varepsilon$ and $\varepsilon \geq 2\mathrm{Re}(m)$, then Proposition 3.10(i) is a consequence of Theorem 3.5(iii). Likewise, if $m = 0$ and $Q \in \mathscr{L}^{(0)}_{0,\ln^2}$, then Proposition 3.10(ii) is a consequence of Theorem 3.5(iv).

Proposition 3.10. *Let* $\mathrm{Re}(k) \geq 0$*.*

(i) Let $\mathrm{Re}(m) \geq 0$*,* $m \neq 0$*. Suppose that* $Q \in \mathscr{L}^{(0)}_0$*. Then, for all* $g \in \mathcal{N}(L_{m^2} + k^2)$*,*

$$g(x) = \mathcal{O}(x^{\frac12 - \mathrm{Re}(m)}), \qquad \partial_x g(x) = \mathcal{O}(x^{-\frac12 - \mathrm{Re}(m)}), \qquad x \to 0. \tag{3.24}$$

Moreover, if $\mathrm{Re}(m) > 0$ *and* g *is linearly independent of* $u_m(\cdot, k)$*, then*

$$\lim_{x\to 0} \frac{g(x)}{x^{\frac12 - m}} \quad \text{exists and does not vanish.} \tag{3.25}$$

(ii) Let $m = 0$ *and* $Q \in \mathscr{L}^{(0)}_{0,\ln}$*. Then, for all* $g \in \mathcal{N}(L_0 + k^2)$*,*

$$g(x) = \mathcal{O}(x^{\frac12}\ln(x)), \qquad \partial_x g(x) = \mathcal{O}(x^{-\frac12}\ln(x)), \qquad x \to 0. \tag{3.26}$$

Moreover, if g *is linearly independent of* $u_0(\cdot, k)$*, then*

$$\lim_{x\to 0} \frac{g(x)}{x^{\frac12}\ln(x)} \quad \text{exists and does not vanish.} \tag{3.27}$$

Proof. We prove (i), (ii) follows in the same way. It is well known that the Wronskian of two eigensolutions of a 1-dimensional Schrödinger equation is constant. Proposition 3.6 gives the solution $u = u_m \in \mathcal{N}(L_{m^2} + k^2)$. Assuming that u_m and W are known, we solve the ordinary differential equation

$$g(x)u'(x) - g'(x)u(x) = W, \tag{3.28}$$

for the unknown function g. Obviously the solutions to

$$g(x)u'(x) - g'(x)u(x) = 0,$$

are given by $g(x) = \lambda u(x)$, $\lambda \in \mathbb{C}$, and we seek a particular solution to (3.28) of the form $g(x) = \lambda(x)u(x)$, with $\lambda \in C^1]0,\infty[$. This gives

$$\lambda'(x)u(x)^2 = W.$$

By (2.17) we know that for some $C_0 \neq 0$

$$u(x) - C_0 x^{\frac{1}{2}+m} = o(x^{\frac{1}{2}+\mathrm{Re}(m)}).$$

This implies that there exists $\alpha > 0$ such that $u(x) \neq 0$ for $0 < x \leq \alpha$, and hence

$$\begin{aligned}\lambda(x) - \lambda(\alpha) = \int_\alpha^x \frac{W}{u(y)^2}\mathrm{d}y &= \int_\alpha^x W\Big(C_0 y^{-1-2m} + o(y^{-1-2\mathrm{Re}(m)})\Big)\mathrm{d}y \\ &= Cx^{-2m} + o(x^{-2\mathrm{Re}(m)}).\end{aligned}$$

Now

$$g(x) = \Big(\lambda(\alpha) + \int_\alpha^x \frac{W}{u(y)^2}\mathrm{d}y\Big)u(x),$$

implies (3.24) and (3.25). □

Note that Proposition 3.10 implies, under rather weak assumptions, that $u_m(\cdot,k)$ is the only solution square integrable near zero if $\mathrm{Re}(m) \geq 1$.

3.5. **The two-sided Green's operator.** Mapping properties of the two-sided Green's operator $G_{\bowtie}^0$ will be needed to construct solutions with a prescribed behavior near zero in situations where we cannot apply the forward Green's operator $G_{\rightarrow}^0$. Note that the two-sided Green's operator is not invariant with respect to the change $m \to -m$. The following lemma is meaningful only for $\mathrm{Re}(m) \geq 0$.

The operator $G_{\bowtie}^0 Q$ is not Volterra. In order to make $\mathbb{1} + G_{\bowtie}^0 Q$ invertible, we will compress it to a sufficiently small interval $]0,a[$. Recall from (2.34) that $G_{\bowtie}^0$ compressed to the interval $]0,a[$ is denoted $G_{\bowtie}^{0(a)}$.

Lemma 3.11. *Let* $\mathrm{Re}(k) \geq 0$, $0 < a \leq \infty$ *and* $Q \in L^1_{\mathrm{loc}}]0,\infty[$.

(i) Let $m \neq 0$, $\frac{1}{2} - \mathrm{Re}(m) \leq \varepsilon_1 + \varepsilon \leq \frac{1}{2} + \mathrm{Re}(m)$ *and*

$$\int_0^a \mu_k(y)^{1-\varepsilon}|Q(y)|\mathrm{d}y < \infty. \tag{3.29}$$

Then

$$G_{\bowtie}^{0(a)} Q : L^\infty\big(]0,a[, \mu_k^{\varepsilon_1}\eta_{-k}\big) \to L^\infty\big(]0,a[, \mu_k^{\varepsilon_1+\varepsilon}\eta_{-k}\big) \tag{3.30}$$

is bounded by $C\times$(3.29) *uniformly in* k *and* a. *Moreover, if* $\varepsilon_1 + \varepsilon < \frac{1}{2} + \mathrm{Re}(m)$, *then the image of* (3.30) *is in* $L_0^\infty\big(]0,a[, \mu_k^{\varepsilon_1+\varepsilon}\eta_{-k}\big)$.

(ii) Let $m=0$, $k\neq 0$, $\frac12=\varepsilon_1+\varepsilon$ and $0\leq\beta-\alpha\leq 1$. Let

$$\int_0^a \mu_k(y)^{1-\varepsilon}\lambda_k(y)^{\beta}|Q(y)|\mathrm{d}y<\infty. \tag{3.31}$$

Then

$$G_{\bowtie}^{0(a)}Q : L^\infty\big(]0,a[,\mu_k^{\varepsilon_1}\lambda_k^{\alpha}\eta_{-k}\big)\to L^\infty\big(]0,a[,\mu_k^{\varepsilon_1+\varepsilon}\lambda_k^{\alpha-\beta+1}\eta_{-k}\big) \tag{3.32}$$

is bounded by $C\times$(3.31) uniformly in k and a. Moreover, if $\beta-\alpha<1$, then the image of (3.32) is in $L_0^\infty\big(]0,a[,\mu_k^{\varepsilon_1+\varepsilon}\lambda_k^{\alpha-\beta+1}\eta_{-k}\big)$.

Proof. For simplicity, let $a=\infty$. We prove (i). We have

$$G_{\bowtie}^{0}Qf(x)=u^0(x)\int_x^\infty v^0(y)Q(y)f(y)\mathrm{d}y+v^0(x)\int_0^x u^0(y)Q(y)f(y)\mathrm{d}y.$$

The second term is treated as in the proof of Lemma 3.1, using $\frac12-\mathrm{Re}(m)\leq\varepsilon_1+\varepsilon$, namely

$$\begin{aligned}&\Big|v^0(x)\int_0^x u^0(y)Q(y)f(y)\mathrm{d}y\Big|\\ &\lesssim \mu_k(x)^{\varepsilon_1+\varepsilon}\eta_{-k}(x)\int_0^x\mu_k(y)^{1-\varepsilon}|Q(y)|\mathrm{d}y\,\Big\|\frac{f}{\mu_k^{\varepsilon_1}\eta_{-k}}\Big\|_\infty.\end{aligned}$$

Consider now the first term. We estimate

$$\begin{aligned}&\Big|u^0(x)\int_x^\infty v^0(y)Q(y)f(y)\mathrm{d}y\Big|\\ &\lesssim \mu_k(x)^{\frac12+\mathrm{Re}(m)}\eta_k(x)\int_x^\infty\mu_k(y)^{\frac12-\mathrm{Re}(m)+\varepsilon_1}\eta_{-k}(y)^2|Q(y)|\mathrm{d}y\,\Big\|\frac{f}{\mu_k^{\varepsilon_1}\eta_{-k}}\Big\|_\infty\\ &\lesssim \mu_k(x)^{\varepsilon_1+\varepsilon}\eta_{-k}(x)\int_x^\infty\mu_k(y)^{1-\varepsilon}|Q(y)|\mathrm{d}y\,\Big\|\frac{f}{\mu_k^{\varepsilon_1}\eta_{-k}}\Big\|_\infty,\end{aligned}$$

where we used $\frac12+\mathrm{Re}(m)\geq\varepsilon_1+\varepsilon$. This proves (3.30).

Suppose now that $\varepsilon_1+\varepsilon<\frac12+\mathrm{Re}(m)$. Since $y\mapsto\mu_k(y)^{1-\varepsilon}|Q(y)|$ is integrable on $]0,\infty[$, we can apply Lemma C.1 with $h(y)=\mu_k(y)^{\frac12+\mathrm{Re}(m)-\varepsilon_1-\varepsilon}\eta_k(x)$, which gives

$$\int_x^\infty\mu_k(y)^{\frac12-\mathrm{Re}(m)+\varepsilon_1}\eta_{-k}(y)^2|Q(y)|\mathrm{d}y=o\big(\mu_k(x)^{-\frac12-\mathrm{Re}(m)+\varepsilon_1+\varepsilon}\eta_{-k}(x)^2\big),\quad x\to 0.$$

This yields

$$G_{\bowtie}^{0}Qf(x)=o\big(\mu_k^{\varepsilon_1+\varepsilon}(x)\eta_{-k}(x)\big)\Big\|\frac{f}{\mu_k^{\varepsilon_1}\eta_{-k}}\Big\|_\infty,$$

and hence concludes the proof of (i).

To prove (ii), we proceed similarly, replacing the estimate (3.9) on v_m^0 by the estimate (3.10) for $m=0$. ☐

Remark 3.12. *Applying Lemma 3.11 with $\varepsilon=0$, it follows that, for $m\neq0$ and $\frac12-\mathrm{Re}(m)\leq\varepsilon_1\leq\frac12+\mathrm{Re}(m)$,*

$$\big\|G_{\bowtie}^{0(a)}Q\big\|\leq C\int_0^a\mu_k(y)^{1-\varepsilon}|Q(y)|\mathrm{d}y\ \text{ on } L^\infty\big(]0,a[,\mu_k^{\varepsilon_1}\eta_{-k}\big),$$

where the constant C is independent of k and a, but dependent on m. However, if the values of m are restricted to $|m| > m_0$, $0 \le \mathrm{Re}(m) \le M$ for some $m_0 > 0$, $M > 0$, then one infers from the proof that the constant C can be chosen uniformly.

In the next corollary we show the invertibility of $\mathbb{1} + G_{\bowtie}^{0(a)}Q$ for small enough $a > 0$.

Corollary 3.13. *Let* $\mathrm{Re}(k) \ge 0$.

(i) Let $m \ne 0$ and $\frac{1}{2} - \mathrm{Re}(m) \le \varepsilon_1 \le \frac{1}{2} + \mathrm{Re}(m)$. Suppose that $Q \in \mathscr{L}_0^{(0)}$. Then, for small enough $a > 0$, we have

$$\left\| G_{\bowtie}^{0(a)}Q \right\| < 1 \text{ on } L^\infty(]0,a[, \mu_k^{\varepsilon_1}\eta_{-k}),$$

so that $(\mathbb{1} + G_{\bowtie}^{0(a)}Q)^{-1}$ exists.

(ii) Let $m = 0$, $k \ne 0$. Suppose that $Q \in \mathscr{L}_{0,\ln}^{(0)}$ and $0 \le \alpha \le 1$. Then, for small enough $a > 0$, we have

$$\left\| G_{\bowtie}^{0(a)}Q \right\| < 1 \text{ on } L^\infty(]0,a[, \mu_k^{\varepsilon_1}\lambda_k^\alpha\eta_{-k}),$$

so that $(\mathbb{1} + G_{\bowtie}^{0,(a)}Q)^{-1}$ exists.

Proof. To prove (i), it suffices to apply Lemma 3.11(i) with $\varepsilon = 0$.
To prove (ii), we apply Lemma 3.11(ii) with $\varepsilon = 0$ and $\beta = 1$. □

3.6. **Solutions constructed with the help of the two-sided Green's operator.** The goal of this subsection is similar to that of Subsection 3.3: to construct solutions to (3.1) that approximate near 0 the solutions to the unperturbed equation (3.2). In this subsection we cover a different parameter range than in Subsection 3.3. This is accomplished by using a different tool. Instead of the forward Green's operator, we use the two-sided Green's operator compressed to a sufficiently small interval $]0, a[$, $G_{\bowtie}^{0(a)}$, which was studied in the previous subsection. The construction here will be less canonical than in Subsection 3.3 – it will depend on the parameter a.

Theorem 3.14. *Let* $\mathrm{Re}(k) \ge 0$.

(i) Let $m_0 > 0$, $M > 0$ and $Q \in \mathscr{L}_0^{(0)}$. Then for small enough $a > 0$, for all $m \in \mathbb{C}$ such that $|m| > m_0$, $0 \le \mathrm{Re}(m) \le M$, for all $g^0 \in \mathcal{N}(L_{m^2}^0 + k^2)$,

$$g^{\bowtie} := (\mathbb{1} + G_{\bowtie}^{0(a)}Q)^{-1}g^0$$

is a solution in $AC^1]0,\infty[$ to (3.1). If in addition $0 \le \varepsilon < 2\mathrm{Re}(m)$, then

$$g^{\bowtie}(x) - g^0(x) = o(x^{\frac{1}{2} - \mathrm{Re}(m) + \varepsilon}), \tag{3.33}$$

$$\partial_x g^{\bowtie}(x) - \partial_x g^0(x) = o(x^{-\frac{1}{2} - \mathrm{Re}(m) + \varepsilon}). \tag{3.34}$$

(ii) Let $m = 0$ and assume that $k \ne 0$. Suppose that $Q \in \mathscr{L}_{0,\ln}^{(0)}$. Then for small enough $a > 0$, for all $g^0 \in \mathcal{N}(L_0^0 + k^2)$,

$$g^{\bowtie} := (\mathbb{1} + G_{\bowtie}^{0(a)}Q)^{-1}g^0$$

is a solution in $AC^1]0,\infty[$ to (3.1) such that

$$g^{\bowtie}(x) - g^0(x) = o(x^{\frac{1}{2}}\ln(x)),$$

$$\partial_x g^{\bowtie}(x) - \partial_x g^0(x) = o(x^{-\frac{1}{2}}\ln(x)).$$

Proof. To prove (i), we apply Corollary 3.13(i) (and Remark 3.12) with $\varepsilon_1 = \frac{1}{2} - \mathrm{Re}(m)$: By making $a > 0$ small enough, we can thus make sure that for $|m| > m_0$, $0 \le \mathrm{Re}(m) \le M$, $m \neq 0$, the operator $G_{\bowtie}^{0(a)} Q$ has the norm < 1 on the space $L^\infty(]0,a[, \mu_k^{\frac{1}{2}-\mathrm{Re}(m)} \eta_{-k})$. Hence $g^{\bowtie}$ is well-defined and is a solution to (3.1).

We have $g^{\bowtie} - g^0 = (-G_{\bowtie}^{0(a)} Q) g^{\bowtie}$. Since $g^{\bowtie} = \mathcal{O}(x^{\frac{1}{2}-\mathrm{Re}(m)})$ by Proposition 3.10 and since $\varepsilon < 2\mathrm{Re}(m)$, we can apply Lemma 3.11(i) with $\varepsilon_1 = \frac{1}{2} - \mathrm{Re}(m)$. This yields (3.33)–(3.34).

To prove (ii) we proceed in the same way, using Corollary 3.13(ii) and Lemma 3.11(ii) with $\varepsilon = 0$ and $\alpha = \beta = 1$. □

We can apply Theorem 3.14(i) to the unperturbed solutions g^0 equal to $u_{-m}^0(\cdot,k)$ and $u_m^0(\cdot,k)$, obtaining solutions $u_{-m}^{\bowtie(a)}(\cdot,k)$ and $u_m^{\bowtie(a)}(\cdot,k)$. The solutions $u_m^{\bowtie(a)}(\cdot,k)$ for $\mathrm{Re}(m) \ge 0$ are not very useful, since we have then $u_m(\cdot,k)$ at our disposal. Therefore, in the following proposition we restrict ourselves to $u_{-m}^{\bowtie(a)}(\cdot,k)$. They can serve as a non-principal solution defined when u_{-m} is not available.

Proposition 3.15. *Let $\mathrm{Re}(k) \ge 0$. Suppose that $Q \in \mathscr{L}_0^{(0)}$. Let $m_0 > 0$, $M > 0$. Let $a > 0$ be small enough as in the previous theorem. Then for $|m| > m_0$, $0 \le \mathrm{Re}(m) \le M$, setting*

$$u_{-m}^{\bowtie(a)}(\cdot,k) := (\mathbb{1} + G_{\bowtie}^{0(a)} Q)^{-1} u_{-m}^0(\cdot,k)$$

we obtain a solution in $AC^1]0,\infty[$ to (3.1). If we impose the assumption $Q \in \mathscr{L}_\varepsilon^{(0)}$ for $0 \le \varepsilon < 2\mathrm{Re}(m)$, then

$$u_{-m}^{\bowtie(a)}(x,k) - u_{-m}^0(x,k) = o(x^{\frac{1}{2}-\mathrm{Re}(m)+\varepsilon}), \tag{3.35}$$

$$\partial_x u_{-m}^{\bowtie(a)}(x,k) - \partial_x u_{-m}^0(x,k) = o(x^{-\frac{1}{2}-\mathrm{Re}(m)+\varepsilon}). \tag{3.36}$$

If $Q \in \mathscr{L}_\varepsilon^{(0)}$ with $0 \le 2\mathrm{Re}(m) \le \varepsilon$ and $m \notin \mathbb{N}$, then there exists $c_m^{\bowtie(a)}(k) \in \mathbb{C}$ such that

$$u_{-m}^{\bowtie(a)}(x,k) = u_{-m}(x,k) + c_m^{\bowtie(a)}(k) u_m(x,k). \tag{3.37}$$

Proof. The first part of the proposition is a direct consequence of Theorem 3.14(i). To see (3.37) note that by Proposition 3.6, both $u_m(\cdot,k)$ and $u_{-m}(\cdot,k)$ are well-defined. Moreover, if $m \notin \mathbb{N}$, they are linearly independent, as follows from their asymptotics near 0. □

The eigensolutions $u_{-m}^{\bowtie(a)}$ constructed in Proposition 3.15 are given by convergent expansions

$$u_{-m}^{\bowtie(a)}(x,k) = \sum_{j=0}^{\infty} (-G_{\bowtie}^{0(a)} Q)^j u_{-m}^0(x,k). \tag{3.38}$$

Note that the individual terms on the right hand side of (3.38) are well defined under rather weak assumptions and their behavior near zero weakly depends on a.

Lemma 3.16. *Let $\mathrm{Re}(k) \ge 0$. Assume that $\mathrm{Re}(m) \ge 0$, $m \neq 0$ and $Q \in \mathscr{L}_0^{(0)}$, or $m = 0$, $Q \in \mathscr{L}_{0,\ln}^{(0)}$ and $k \neq 0$. Let $0 < a, b$. Then for any $j \in \mathbb{N}$ there exists $c_m^j(k)$ such that*

$$(G_{\bowtie}^{0(a)} Q)^j u_{-m}^0(x,k) - (G_{\bowtie}^{0(b)} Q)^j u_{-m}^0(x,k) = c_m^j(k) u_m^0(x,k) + o(x^{\frac{1}{2}+\mathrm{Re}(m)}). \tag{3.39}$$

Proof. Suppose that $m \neq 0$. We will prove (3.39) by induction with respect to j. Let us denote the left hand side of (3.39) by $z^j(x,k)$. Clearly, $z^0(x,k)=0$. Assume that (3.39) is true for a given j. Let $0 \leq x \leq a < b$. Now

$$\begin{aligned} z^{j+1}(x) &= (G_{\bowtie}^{0(a)}Q - G_{\bowtie}^{0(b)}Q)(G_{\bowtie}^{0(a)}Q)^j u_{-m}^0(x) + G_{\bowtie}^{0(b)}Qz^j(x) \\ &= u_m^0(x)\int_a^b v_m^0(y)Q(y)(G_{\bowtie}^{0(a)}Q)^j u_{-m}^0(y)\mathrm{d}y + u_m^0(x)\int_0^b v_m^0(y)Q(y)z^j(y)\mathrm{d}y \\ &\quad - u_m^0(x)\int_0^x v_m^0(y)Q(y)z^j(y)\mathrm{d}y + v_m^0(y)\int_0^x u_m^0(y)Q(y)z^j(y)\mathrm{d}y. \end{aligned}$$

The first term is clearly proportional to u_m^0. By the induction assumption $z_j = \mathcal{O}(x^{\frac12+\mathrm{Re}(m)})$. Therefore the integral in the second term is finite, and hence the second term is also proportional to u_m^0. By the same argument the third term is $o(x^{\frac12+\mathrm{Re}(m)})$. Finally, since $z_j = \mathcal{O}(x^{\frac12+\mathrm{Re}(m)})$, the integral in the fourth term is $o(x^{2\mathrm{Re}(m)})$. Hence the fourth term is also $o(x^{\frac12+\mathrm{Re}(m)})$.

The case $m=0$ with $Q \in \mathscr{L}_{0,\ln}^{(0)}$ and $k \neq 0$ can be treated in the same way. □

Under the assumptions of Lemma 3.16, we introduce the following notation for a partial sum of the series (3.38):

$$u_{-m}^{0(a)[n]}(x,k) := \sum_{j=0}^{n}(-G_{\bowtie}^{0(a)}Q)^j u_{-m}^0(x,k), \tag{3.40}$$

$$u_{-m}^{0[n]}(x,k) := u_{-m}^{0(1)[n]}(x,k). \tag{3.41}$$

Thus we choose (quite arbitrarily) $a=1$ as the "standard value" in (3.40).

The next proposition shows that the functions $u_{-m}^{0[n]}(\cdot,k)$ well approximate non-principal solutions under the assumption $0 \leq \mathrm{Re}(m) \leq \frac{\varepsilon}{2}(n+1)$, $m \neq 0$.

Proposition 3.17. *Let $\mathrm{Re}(k) \geq 0$ and $\mathrm{Re}(m) \geq 0$, $m \neq 0$. Assume that $\varepsilon \geq 0$ and $Q \in \mathscr{L}_{\varepsilon}^{(0)}$. Let n be a nonnegative integer such that $\varepsilon \geq \frac{2}{n+1}\mathrm{Re}(m)$. Suppose that $a>0$ is small enough, so that $u_{-m}^{\bowtie(a)}(\cdot,k)$ is well defined, as described in Proposition 3.15. Then there exists $c_m^{(a)[n]}(k) \in \mathbb{C}$ such that*

$$u_{-m}^{\bowtie(a)}(x,k) - u_{-m}^{0[n]}(x,k) - c_m^{(a)[n]}(k)u_m(x,k) = o(x^{\frac12+\mathrm{Re}(m)}),$$
$$\partial_x u_{-m}^{\bowtie(a)}(x,k) - \partial_x u_{-m}^{0[n]}(x,k) - c_m^{(a)[n]}(k)\partial_x u_m(x,k) = o(x^{-\frac12+\mathrm{Re}(m)}).$$

Proof. Applying

$$(G_{\bowtie}^{0(a)}Qf)(x) = u^0(x)\int_0^a v^0(y)Q(y)f(y)\mathrm{d}y + G_{\rightarrow}^0 Qf(x), \tag{3.42}$$

we can write

$$\begin{aligned} u_{-m}^{\bowtie(a)}(x,k) =& u_{-m}^{0(a)[n]}(x,k) + (-G_{\bowtie}^{0(a)}Q)^{n+1}u_{-m}^{\bowtie(a)}(x,k) \\ =& u_{-m}^{0(a)[n]}(x,k) - u_m^0(x,k)\int_0^a v_m^0(x,k)Q(y)(-G_{\bowtie}^{0(a)}Q)^n u_{-m}^{\bowtie(a)}(y,k)\mathrm{d}y \\ & - G_{\rightarrow}^0 Q(-G_{\bowtie}^{0(a)}Q)^n u_{-m}^{\bowtie(a)}(x,k). \end{aligned} \tag{3.43}$$

Suppose $n > 0$. Let $\tilde{n} \le n$ be a positive integer such that $\frac{2}{\tilde{n}}\mathrm{Re}(m) > \varepsilon \ge \frac{2}{\tilde{n}+1}\mathrm{Re}(m)$. We apply repeatedly Lemma 3.11(i) with $\varepsilon_1 = \frac{1}{2} - \mathrm{Re}(m) + j\varepsilon$ for $j = 0, \dots, \tilde{n}-1$, noting that $\frac{1}{2} - \mathrm{Re}(m) \le \varepsilon_1 + \varepsilon < \frac{1}{2} + \mathrm{Re}(m)$, to show that

$$(-G_{\bowtie}^{0(a)}Q)^{\tilde{n}} u_{-m}^{\bowtie(a)}(x,k) = o(x^{\frac{1}{2}-\mathrm{Re}(m)+\tilde{n}\varepsilon}). \tag{3.44}$$

Applying then again repeatedly Lemma 3.11(i) with $\varepsilon = 0$ we deduce that

$$(-G_{\bowtie}^{0(a)}Q)^{n} u_{-m}^{\bowtie(a)}(x,k) = o(x^{\frac{1}{2}-\mathrm{Re}(m)+\tilde{n}\varepsilon}). \tag{3.45}$$

Because of this, and since $\varepsilon \ge \frac{2}{\tilde{n}+1}\mathrm{Re}(m)$,

$$\int_0^a v_m^0(y,k)Q(x)(-G_{\bowtie}^{0(a)}Q)^n u_{-m}^{\bowtie(a)}(y,k)\mathrm{d}y \tag{3.46}$$

is finite. Now we apply Lemma 3.1(i) with $\varepsilon_1 = \frac{1}{2} - \mathrm{Re}(m) + \tilde{n}\varepsilon$, noting that $\varepsilon_1 + \varepsilon \ge \frac{1}{2} + \mathrm{Re}(m)$, to show that

$$G_{\rightarrow}^0 Q(-G_{\bowtie}^{0(a)}Q)^n u_{-m}^{\bowtie(a)}(x,k) = o(x^{\frac{1}{2}+\mathrm{Re}(m)}). \tag{3.47}$$

If $n = 0$, applying Lemma 3.1(i) with $\varepsilon_1 = \frac{1}{2} - \mathrm{Re}(m)$, we see that (3.47) still holds. Finally, by Lemma 3.16 we can replace $u_{-m}^{0(a)[n]}(x,k)$ with $u_{-m}^{0[n]}(x,k)$. □

We can use the functions $u_{-m}^{0[n]}(\cdot,k)$ to describe boundary conditions near zero of non-principal solutions.

Proposition 3.18. *Let* $\mathrm{Re}(k) \ge 0$ *and* $\mathrm{Re}(m) \ge 0$, $m \ne 0$. *Suppose that* $Q \in \mathscr{L}_\varepsilon^{(0)}$, $\varepsilon \ge 0$. *Let* n *be a nonnegative integer such that* $\frac{\varepsilon}{2}(n+1) \ge \mathrm{Re}(m)$. *Then*

$$u_{-m}^{[n]}(\cdot,k) := u_{-m}^{0[n]}(\cdot,k) + (-1)^{n+1}\big(\mathbb{1} + G_{\rightarrow}^0 Q\big)^{-1} G_{\rightarrow}^0 Q\big(G_{\bowtie}^{0(1)}Q\big)^n u_{-m}^0(\cdot,k) \tag{3.48}$$

is a solution in $AC^1]0,\infty[$ *to* (3.1) *such that*

$$u_{-m}^{[n]}(x,k) - u_{-m}^{0[n]}(x,k) = o(x^{\frac{1}{2}+\mathrm{Re}(m)}), \tag{3.49}$$

$$\partial_x u_{-m}^{[n]}(x,k) - \partial_x u_{-m}^{0[n]}(x,k) = o(x^{-\frac{1}{2}+\mathrm{Re}(m)}). \tag{3.50}$$

Proof. Note that $u_{-m}^{0[n]}(x,k) = \mathcal{O}(x^{\frac{1}{2}-\mathrm{Re}(m)})$. As in the proof of the previous proposition, applying repeatedly Lemma 3.11(i) and next Lemma 3.1(i), we obtain that

$$G_{\rightarrow}^0 Q\big(G_{\bowtie}^0 Q\big)^n u_{-m}^{0[n]}(\cdot,k) = o(x^{\frac{1}{2}+\mathrm{Re}(m)}).$$

Then we can use Corollary 3.2(i) which shows that, for any $a > 0$, $\mathbb{1} + G_{\rightarrow}^0 Q$ is invertible on $L^\infty(]0,a[,\mu_k^{\frac{1}{2}+\mathrm{Re}(m)})$. Applying $L_{m^2} + k^2 = (L_{m^2}^0 + k^2)(\mathbb{1} + G_{\rightarrow}^0 Q)$ to (3.48) and using the definition (3.40), we then obtain

$$\begin{aligned}&(L_{m^2} + k^2)(u_{-m}^{[n]}(\cdot,k)) \\ &= (L_{m^2} + k^2)\sum_{j=0}^{n}(-G_{\bowtie}^{0(1)}Q)^j u_{-m}^0(\cdot,k) - Q\big(-G_{\bowtie}^{0(1)}Q\big)^n u_{-m}^0(\cdot,k).\end{aligned}$$

Next, using that $L_{m^2}+k^2=L^0_{m^2}+k^2+Q$ together with the fact that $G^{0(1)}_{\bowtie}$ is a right inverse of $L^0_{m^2}+k^2$ gives

$$\begin{aligned}&(L_{m^2}+k^2)(u^{[n]}_{-m}(\cdot,k))\\&=-Q\sum_{j=1}^{n}(-G^{0(1)}_{\bowtie}Q)^{j-1}u^0_{-m}(\cdot,k)+Q\sum_{j=0}^{n}(-G^{0(1)}_{\bowtie}Q)^{j}u^0_{-m}(\cdot,k)-Q\big(-G^{0(1)}_{\bowtie}Q)^{n}u^0_{-m}(\cdot,k)\\&=0.\end{aligned}$$

Hence $u^{[n]}_{-m}(\cdot,k)$ belongs to $AC^1]0,\infty[$ and is a solution to (3.1) satisfying (3.49)–(3.50). □

Proposition 3.19. *Let $\varepsilon\geq 0$, n a nonnegative integer and suppose that $Q\in\mathscr{L}^{(0)}_{\varepsilon}$. Then for any $x>0$ the maps*

$$\Big\{\frac{\varepsilon}{2}(n+1)\geq\operatorname{Re}(m)\geq 0,\ m\neq 0\Big\}\times\Big\{\operatorname{Re}(k)\geq 0\Big\}\ni(m,k)\mapsto u^{[n]}_{-m}(x,k),\partial_x u^{[n]}_{-m}(x,k)$$

are regular.

Proof. The proof is similar to that of Proposition 3.8. □

Here is a drawback of Proposition 3.18: the boundary conditions are described by a function $u^{0[n]}_{-m}(\cdot,k)$ which depends on k. We already know that for principal solutions the boundary condition does not depend on k. One can ask whether one can use the same boundary conditions for all k in the non-principal case, e.g.

$$u^{0[n]}_{-m}(x):=u^{0[n]}_{-m}(x,0). \tag{3.51}$$

Thus we would like to use $k=0$ as the "standard value" in (3.51), which typically gives the simplest expressions.

Let us check what is the situation in the unperturbed case. Let $\operatorname{Re}(m)\geq 0$. We have

$$u^0_{-m}(x,k)=\frac{x^{\frac12-m}}{\Gamma(1-m)}+\mathcal{O}(x^{\frac52-\operatorname{Re}(m)}),\qquad u^0_{m}(x,k)=\mathcal{O}(x^{\frac12+\operatorname{Re}(m)}). \tag{3.52}$$

Hence we need the condition $\operatorname{Re}(m)<1$ to make sure that

$$u^0_{-m}(x,k)=\frac{x^{\frac12-m}}{\Gamma(1-m)}+o(x^{\frac12+\operatorname{Re}(m)}), \tag{3.53}$$

which guarantees that $u^0_{-m}(x,k)$ with distinct k give the same boundary condition.

Proposition 3.20. *In addition to the assumptions of Proposition 3.18 suppose that* $\operatorname{Re}(m)<1$. *Then in (3.49) and (3.50) we can replace $u^{0[n]}_{-m}(x,k)$ with $u^{0[n]}_{-m}(x)$ defined in (3.51), (or with $u^{0[n]}_{-m}(x,k')$ for any k').*

Proof. Proposition 3.20 easily follows from the definition (3.51) of $u^{0[n]}_{-m}(x,k)$ together with (3.53). □

In concrete cases, it is not difficult to compute $u^{0[n]}_{-m}$ explicitly. The following remark provides an example in the case where Q has a Coulomb singularity at 0.

Remark 3.21. *Suppose that* $Q(x) = -\frac{\beta}{x}1\!\!1_{]0,1]}(x)$ *with* $\beta \in \mathbb{C}$. *Then* $Q \in \mathscr{L}_\varepsilon^{(0)}$ *for* $\varepsilon < 1$. *Hence for* $0 \le \mathrm{Re}(m) < 1$, $m \neq 0$, *we can take* $n = 1$ *in Proposition 3.17 and we have that*

$$u_{-m}^{0[1]}(x) = u_{-m}^0(x,0) - G_{\bowtie}^{0(1)} Q u_{-m}^0(x,0).$$

Consider for simplicity the generic case $m \neq \frac{1}{2}$. *Since* $u_{\pm m}^0(x,0) = \frac{x^{\frac12 \pm m}}{\Gamma(1\pm m)}$ *and* $v_m^0(x,0) = \frac12 \Gamma(m) x^{\frac12 - m}$, *we can compute*

$$\begin{aligned} G_{\bowtie}^{0(1)} Q u_{-m}^0(x) &= \frac{1}{2m\Gamma(1-m)} \Big(x^{\frac12+m} \int_x^1 y^{1-2m} \frac{-\beta}{y} \mathrm{d}y + x^{\frac12 - m} \int_0^x y \frac{-\beta}{y} \mathrm{d}y \Big) \\ &= \frac{\beta}{2m\Gamma(1-m)} \Big(x^{\frac12+m} \frac{x^{1-2m}}{1-2m} - \frac{x^{\frac12+m}}{1-2m} - x^{\frac32 - m} \Big) \\ &= \frac{x^{\frac12-m}}{\Gamma(1-m)} \frac{\beta x}{1-2m} - \frac{\beta x^{\frac12+m}}{2m(1-2m)\Gamma(1-m)}. \end{aligned}$$

Hence

$$u_{-m}^{0[1]}(x) = \frac{x^{\frac12-m}}{\Gamma(1-m)} \Big(1 - \frac{\beta x}{1-2m} \Big) - \frac{\beta x^{\frac12+m}}{2m(1-2m)\Gamma(1-m)}.$$

We recover the function $j_{\beta,-m}$ *from (2.3) of [12], which was used to describe the boundary conditions of the Whittaker operator.*

3.7. **The logarithmic Green's operator.** For $m = 0$ we could use Lemma 3.11(ii), Corollary 3.13(ii) and Theorem 3.14(ii) to construct eigensolutions with the help of the two-sided Green's operator $G_{\bowtie}^0$. The drawback of this approach is the lack of the limit at $k = 0$. Therefore for $m = 0$ we prefer to use the logarithmic Green's operator $G_{\triangle}^0$, which is well defined for $k = 0$. More precisely, we will use the logarithmic Green's operator compressed to a finite interval, $G_{\triangle}^{0(a)}$.

Below we describe mapping properties of $G_{\triangle}^{0(a)}$. The result is analogous to Lemma 3.11(ii), however includes $k = 0$.

Lemma 3.22. *Let* $k_0 > 0$, $\mathrm{Re}(k) \ge 0$ *such that* $|k| \le k_0$, $0 < a < 1$, $\frac12 = \varepsilon_1 + \varepsilon$ *and* $0 \le \beta - \alpha \le 1$. *Suppose that* $Q \in L^1_{\mathrm{loc}}]0,\infty[$ *and*

$$\int_0^a y^{1-\varepsilon} \big(1 - \ln(y)\big)^\beta |Q(y)| \mathrm{d}y < \infty. \tag{3.54}$$

Then

$$G_{\triangle}^{0(a)} Q : L^\infty\big(]0,a[, x^{\varepsilon_1}(1-\ln(x))^\alpha\big) \to L^\infty\big(]0,a[, x^{\varepsilon_1+\varepsilon}(1-\ln(x))^{\alpha+1-\beta}\big) \tag{3.55}$$

is bounded by $C\times$(3.54) *uniformly in* $0 < a < 1$ *and* $|k| \le k_0$. *If in addition* $\beta - \alpha < 1$, *then the image of* (3.55) *is contained in* $L_0^\infty\big(]0,a[, x^{\varepsilon_1+\varepsilon}(1-\ln(x))^{\alpha+1-\beta}\big)$.

Proof. The proof is identical to that of Lemma 3.11(ii), using the solution p_0^0 instead of v_0^0 and (3.11) instead of (3.10). ☐

Corollary 3.23. *Let* $k_0 > 0$, $\mathrm{Re}(k) \ge 0$ *such that* $|k| \le k_0$. *Suppose that* $Q \in \mathscr{L}_{0,\ln}^{(0)}$. *Then for* $0 \le \alpha \le 1$ *and small enough* $a > 0$ *we have*

$$\|G_{\triangle}^{0(a)} Q\| < 1 \text{ on } L^\infty(]0,a[, x^{\frac12}(1-\ln(x))^\alpha),$$

so that $(\mathbb{1}+G_{\triangle}^{0(a)}Q)^{-1}$ *exists.*

Proof. It suffices to apply Lemma 3.22 with $\varepsilon = 0$ and $\beta = 1$. □

3.8. **Solutions constructed with help of the logarithmic Green's operator.** We continue with the case $m = 0$. The following theorem is the analog of Theorem 3.14(ii) in the context of the logarithmic Green's operator $G_{\triangle}^{0}$.

Theorem 3.24. *Let* $k_0 > 0$ *and* $\mathrm{Re}(k) \geq 0$ *such that* $|k| \leq k_0$. *Suppose that* $Q \in \mathscr{L}_{0,\ln}^{(0)}$. *Then for all* $g^0 \in \mathcal{N}(L_0^0 + k^2)$*, for small enough* a

$$g^{\triangle} := (\mathbb{1}+G_{\triangle}^{0(a)}Q)^{-1}g^0$$

exists and is a solution in $AC^1]0,\infty[$ *to* (3.1) *such that*

$$\begin{aligned} g^{\triangle}(x) - g^0(x) &= o(x^{\frac{1}{2}}\ln(x)),\\ \partial_x g^{\triangle}(x) - \partial_x g^0(x) &= o(x^{-\frac{1}{2}}\ln(x)). \end{aligned}$$

Proof. It suffices to proceed as in the proof of Proposition 3.14, using Corollary 3.23 and Lemma 3.22 with $\alpha = \beta = 1$. □

Applying Theorem 3.24 to p_0^0, we obtain the following result.

Proposition 3.25. *Let* $k_0 > 0$, $\mathrm{Re}(k) \geq 0$ *such that* $|k| \leq k_0$. *Suppose that* $Q \in \mathscr{L}_{0,\ln}^{(0)}$. *Then for* $a > 0$ *small enough,*

$$p_0^{\triangle(a)} := (\mathbb{1}+G_{\triangle}^{0(a)}Q)^{-1}p_0^0 \tag{3.56}$$

is a non-principal solution in $AC^1]0,\infty[$ *such that to* (3.1)

$$\begin{aligned} p_0^{\triangle(a)}(x,k) - p_0^0(x,k) &= o(x^{\frac{1}{2}}\ln(x)),\\ \partial_x p_0^{\triangle(a)}(x,k) - \partial_x p_0^0(x,k) &= o(x^{-\frac{1}{2}}\ln(x)). \end{aligned}$$

Proof. This is a direct consequence of Theorem 3.24. □

3.9. **Summary of distinguished solutions.** The next table summarizes the distinguish solutions of the perturbed eigenequation with a prescribed behavior near the origin constructed in this section.

Solution	Parameters	Conditions on Q	Green's operator
$u_m(\cdot,k)$	$\mathrm{Re}(m)\geq-\frac{\varepsilon}{2},\ m\neq 0$ $m=0$	$Q\in\mathscr{L}_{\varepsilon}^{(0)},\ \varepsilon\geq 0$ $Q\in\mathscr{L}_{0,\ln}^{(0)}$	Forward $G^0_{\rightarrow}$
$p_0(\cdot,k)$	$m=0$	$Q\in\mathscr{L}_{0,\ln^2}^{(0)}$	Forward $G^0_{\rightarrow}$
$u_m^{\bowtie(a)}(\cdot,k)$	$-M\leq\mathrm{Re}(m)<0,\ \lvert m\rvert>m_0>0$	$Q\in\mathscr{L}_{0}^{(0)}$	Two-sided $G^{0(a)}_{\bowtie}$ compressed to $]0,a[$
$u_{-m}^{[n]}(\cdot,k)$	$\frac{\varepsilon}{2}(n+1)\geq\mathrm{Re}(m)\geq 0,$	$Q\in\mathscr{L}_{\varepsilon}^{(0)},\ \varepsilon>0$	Forward $G^0_{\rightarrow}$ and two-sided $G^{0(1)}_{\bowtie}$
$p_0^{\triangle(a)}(\cdot,k)$	$m=0,\ \lvert k\rvert\leq k_0$	$Q\in\mathscr{L}_{0,\ln}^{(0)}$	Logarithmic $G^{0(a)}_{\triangle}$ compressed to $]0,a[$

TABLE 1. *Distinguished solutions of the perturbed eigenequation with a prescribed behavior near* 0. Our convention is that a solution $g_m(\cdot,k)$ of (1.26) (with $g=u,p,\dots$) has the same behavior near 0 as the unperturbed solution $g_m^0(\cdot,k)$. We everywhere assume that $\mathrm{Re}(k)\geq 0$. The second column recalls the range of parameters for which the solution $g_m(\cdot,k)$ is defined, the third column gives the conditions on Q that are required in order to define $g_m(\cdot,k)$ and the fourth column recalls the Green's operator used to construct $g_m(\cdot,k)$.

4. Solutions of the perturbed Bessel equation regular near infinity

Recall that $w_m^0(\cdot,k)$ is a solution of the unperturbed eigenequation which is proportional to $v_m^0(\cdot,k)$ and behaves as e^{-kx} at infinity. In this section we construct and study the solution to (3.1) with the same asymptotic behavior. In the literature, when $m=\pm 1/2$ and Q is real-valued, this solution is usually called the *Jost solution*. We will use the same name in our more general context.

We will assume that $m\in\mathbb{C}$ is arbirary and $\mathrm{Re}(k)\geq 0$, or equivalently, $\lvert\arg(k)\rvert\leq\frac{\pi}{2}$. The proofs of the results stated in this section are often similar to that of Section 3. We will focus on the differences.

Recall that $\mu_k,\lambda_k,\eta_{\pm k}$ are defined in (3.6)–(3.7) and that the spaces $L^\infty(]a,\infty[,\phi)$ and $L^\infty_\infty(]a,\infty[,\phi)$ are defined in (3.3)–(3.5). We use a similar convention as in the previous section: if the operator $\mathbb{1}+G^0_\bullet Q$ is invertible on $L^\infty(]a,\infty[,\phi)$ for some $a>0$ and some positive measurable function ϕ on $]a,\infty[$, where $G^0_\bullet$ is a Green's operator, and if $f:]0,\infty[\to\mathbb{C}$ is such that its restriction to $]a,\infty[$ belongs to $L^\infty(]a,\infty[,\phi)$, then $(\mathbb{1}+G^0_\bullet Q)^{-1}f$ should be understood as $(\mathbb{1}+G^0_\bullet Q)^{-1}$ applied to the restriction of f on $]a,\infty[$. Clearly, if in addition $f\in\mathcal{N}(L^0_{m^2}+k^2)$, then $(\mathbb{1}+G^0_\bullet Q)^{-1}f$ is a solution to (3.1) on $]a,\infty[$. The unique solution on $]0,\infty[$ which coincides with $(\mathbb{1}+G^0_\bullet Q)^{-1}f$ on $]a,\infty[$ will be denoted by the same symbol.

To simplify notations, we often write $w^0=w_m^0(\cdot,k)$.

4.1. **The backward Green's operator.** We consider the operator $G_{\leftarrow}^{0}Q$. The results proven here will be used to construct Jost solutions. Note that $G_{\leftarrow}^{0}$ is invariant with respect to the change of sign of m. Therefore, it is enough to assume that $\mathrm{Re}(m)\geq 0$.

Lemma 4.1. *Let* $\mathrm{Re}(k)\geq 0$ *and* $Q\in L^1_{\mathrm{loc}}]0,\infty[$.

(i) Let $\mathrm{Re}(m)\geq 0$, $m\neq 0$ *and* $\varepsilon+\varepsilon_1\leq\frac12-\mathrm{Re}(m)$. *Suppose that*

$$\int_0^\infty \mu_k(y)^{1-\varepsilon}|Q(y)|\mathrm{d}y<\infty. \tag{4.1}$$

Then

$$G_{\leftarrow}^{0}Q: L^\infty\big(]0,\infty[,\mu_k^{\varepsilon_1}\eta_{-k}\big)\to L^\infty_\infty\big(]0,\infty[,\mu_k^{\varepsilon_1+\varepsilon}\eta_{-k}\big)$$

is bounded by $C\times$(4.1) *uniformly in* k.

(ii) Let $m=0$, $k\neq 0$, $\varepsilon_1+\varepsilon\leq\frac12$, $\alpha\geq\beta$ *and*

$$\int_0^\infty \mu_k(y)^{1-\varepsilon}\lambda_k(y)^\beta|Q(y)|\mathrm{d}y<\infty. \tag{4.2}$$

Then

$$G_{\leftarrow}^{0}Q: L^\infty\big(]0,\infty[,\mu_k^{\varepsilon_1}\lambda_k^{\alpha}\eta_{-k}\big)\to L^\infty_\infty\big(]0,\infty[,\mu_k^{\varepsilon_1+\varepsilon}\lambda_k^{\alpha+1-\beta}\eta_{-k}\big)$$

is bounded by $C\times$(4.2) *uniformly in* k.

Proof. The proof is essentially the same as that of Lemma 3.1. □

Here is a corollary of the above lemma.

Corollary 4.2. *Let* $\mathrm{Re}(k)\geq 0$ *and* $n\in\mathbb{N}$.

(i) Let $\mathrm{Re}(m)\geq 0$, $m\neq 0$. *Suppose that* $Q\in\mathscr{L}_0^{(\infty)}$. *Then, for all* $a>0$, *for all* $f\in L^\infty\big(]a,\infty[,\mu_k^{\frac12-\mathrm{Re}(m)}\eta_{-k}\big)$ *and* $x>a$,

$$\frac{\big|(G_{\leftarrow}^{0}Q)^n f(x)\big|}{\mu_k(x)^{\frac12-\mathrm{Re}(m)}\eta_{-k}(x)}\leq\frac{C^{n+1}}{n!}\Big(\int_x^\infty\mu_k(y)|Q(y)|\mathrm{d}y\Big)^n\sup_{y>x}\frac{|f(y)|}{\mu_k(y)^{\frac12-\mathrm{Re}(m)}\eta_{-k}(y)}.$$

(ii) Suppose $k\neq 0$. *Let* $m=0$. *Suppose that* $Q\in\mathscr{L}_0^{(\infty)}$. *Then, for all* $a>0$, *for all* $f\in L^\infty\big(]a,\infty[,\mu_k^{\frac12}\lambda_k\eta_{-k}\big)$ *and* $x>a$,

$$\frac{\big|(G_{\leftarrow}^{0}Q)^n f(x)\big|}{\mu_k(x)^{\frac12}\lambda_k(x)\eta_{-k}(x)}\leq\frac{C^{n+1}}{n!}\Big(\int_x^\infty\mu_k(y)\lambda_k(y)|Q(y)|\mathrm{d}y\Big)^n\sup_{y>x}\frac{|f(y)|}{\mu_k(y)^{\frac12}\lambda_k(y)\eta_{-k}(y)}.$$

Above, C *is a constant independent of* n *and* k.

Proof. We proceed as in the proof of Corollary 3.2.

To prove (i) we use Lemma 4.1(i) with $\varepsilon=1$ and $\varepsilon_1=-\frac12+\mathrm{Re}(m)$.

To prove (ii) we use Lemma 4.1(ii) with $\varepsilon=1$, $\varepsilon_1=-\frac12$ and $\alpha=\beta=1$. □

The case $m=0$, $k=0$ is not covered by Lemma 4.1 and Corollary 4.2, because then λ_k is ill defined. The following lemma and its corollary work for this case.

Lemma 4.3. *Let $m = 0$, $k = 0$ and $Q \in L^1_{\mathrm{loc}}]0,\infty[$. Let $\varepsilon_1 + \varepsilon \le \frac{1}{2}$, $\beta - \alpha \ge 1$ and suppose*

$$\int_1^\infty y^{1-\varepsilon}(1+\ln(y))^\beta |Q(y)|\mathrm{d}y < \infty.$$

Then

$$G^0_{\leftarrow} Q : L^\infty(]1,\infty[, x^{\varepsilon_1}(1+\ln(x))^\alpha\big) \to L^\infty_\infty\big(]1,\infty[, x^{\varepsilon+\varepsilon_1}(1+\ln(x))^{\alpha+1-\beta}\big)$$

is bounded.

Proof. It suffices to proceed as in the proof of Lemma 3.3, using that $y \mapsto y^{-\frac{1}{2}+\varepsilon_1+\varepsilon}$ and $y \mapsto (\ln(y))^{1-\beta+\alpha}$ are decreasing on $]1,\infty[$. □

Corollary 4.4. *Let $k = 0$, $m = 0$ and $n \in \mathbb{N}$.*

(i) Suppose that $Q \in \mathscr{L}^{(\infty)}_{1,\ln}$. Then, for all $a > 1$, $f \in L^\infty\big(]a,\infty[, x^{\frac{1}{2}}\big)$ and $a < x$,

$$\frac{|(G^0_{\leftarrow} Q)^n f(x)|}{x^{\frac{1}{2}}} \le \frac{C^{n+1}}{n!}\Big(\int_x^\infty y\big(1+|\ln(y)|\big)|Q(y)|\mathrm{d}y\Big)^n \sup_{y>x}\frac{|f(y)|}{y^{\frac{1}{2}}}.$$

(ii) Suppose that $Q \in \mathscr{L}^{(\infty)}_{1,\ln^2}$. Then, for all $a > 1$, $f \in L^\infty\big(]a,\infty[, x^{\frac{1}{2}}|\ln(x)|\big)$ and $a < x$,

$$\frac{|(G^0_{\leftarrow} Q)^n f(x)|}{x^{\frac{1}{2}}(1+|\ln(x)|)} \le \frac{C^{n+1}}{n!}\Big(\int_x^\infty y\big(1+|\ln(y)|\big)^2|Q(y)|\mathrm{d}y\Big)^n \sup_{y>x}\frac{|f(y)|}{y^{\frac{1}{2}}(1+|\ln(y)|)}.$$

Proof. The proof is the same as that of Corollary 3.2, applying Lemma 4.3. To prove (i), we use Lemma 4.3 with $\varepsilon_1 = \frac{1}{2}$, $\varepsilon = 0$, $\beta = 1$ and $\alpha = 0$. To prove (ii), we use Lemma 4.3 with $\varepsilon_1 = \frac{1}{2}$, $\varepsilon = 0$, $\beta = 2$ and $\alpha = 1$. □

4.2. **Jost solutions constructed with the help of the backward Green's operator.** In this subsection, using the backward Green's operator, we construct the solution to (3.1) which behaves as e^{-kx} at infinity.

The next proposition implies Proposition 1.7 from the introduction.

Proposition 4.5. *Suppose that $Q \in \mathscr{L}^{(\infty)}_0$. Let $m \in \mathbb{C}$ and $\mathrm{Re}(k) \ge 0$, $k \ne 0$. Then*

$$w_m(\cdot,k) := (\mathbb{1} + G^0_{\leftarrow} Q)^{-1} w^0(\cdot,k)$$

is the unique solution in $AC^1]0,\infty[$ to (3.1) such that

$$w_m(x,k) - w^0_m(x,k) = o(\mathrm{e}^{-x\mathrm{Re}(k)}), \tag{4.3}$$

$$\partial_x w_m(x,k) - \partial_x w^0_m(x,k) = o(\mathrm{e}^{-x\mathrm{Re}(k)}), \qquad x \to \infty. \tag{4.4}$$

Moreover, for all $m \in \mathbb{C}$, we have

$$w_m(x,k) = w_{-m}(x,k). \tag{4.5}$$

Proof. Let $a > 0$. Clearly, $w^0 \in L^\infty(]a,\infty[, \mu_k\eta_{-k})$, Hence, by Corollary 4.2,

$$w = (\mathbb{1} + G^0_{\leftarrow} Q)^{-1} w^0 \in L^\infty(]a,\infty[, \mu_k\eta_{-k}) \tag{4.6}$$

is well defined. As in the proof of Proposition 3.7, this implies that $w \in AC^1]0,\infty[$ and, using in addition that $G^0_{\leftarrow}$ is a right inverse of L_{m^2}, that w is a solution to (3.1).

Next, since

$$w - w^0 = -G^0_{\leftarrow} Q w, \tag{4.7}$$

applying Lemma 4.1, we obtain that $w - w^0 \in L^\infty_\infty\big(]a,\infty[, \mathrm{e}^{-x\mathrm{Re}(k)}\big)$. This proves (4.3). Equation (4.4) is proven similarly.

Uniqueness follows exactly as in the proof of Proposition 3.7. This in turn implies (4.5) since $w_m^0 = w_{-m}^0$. □

Now, for a fixed Q, we can study the regularity of Jost solutions with respect to (m, k).

Proposition 4.6. *Suppose that $Q \in \mathscr{L}_0^{(\infty)}$. Then for all $x > 0$, the maps*

$$\left\{\mathrm{Re}(k) \geq 0,\ k \neq 0\right\} \ni (m, k) \mapsto w_m(x, k),\ \partial_x w_m(x, k)$$

are regular.

Proof. The analyticity and continuity follows as in the proof of Proposition 3.8, using the analyticity of w^0 and the map $G_{\leftarrow}^0 Q$ together with Lemma 4.2. □

Remark 4.7. *In general, $(m, k) \mapsto w_m(x, k)$ does not extend analytically to $\{\frac{\pi}{2} \leq |\arg(k)|\}$ in the same way as for $(m, k) \mapsto u_m(x, k)$. However, if the condition $Q \in \mathscr{L}_0^{(\infty)}$ is strengthened, assuming*

$$\int_1^\infty \mathrm{e}^{\Lambda y}\, |Q(y)| \mathrm{d}t < \infty,$$

for some $\Lambda > 0$, then one can verify that $(m, k) \mapsto w_m(x, k)$ extends analytically to

$$\{\mathrm{Re}(k) > -\Lambda/2, |\arg(k)| < \pi\}.$$

Proposition 4.5 is restricted to $k \neq 0$, because the usual short-range condition is insufficient to cover the zero energy case. Therefore, in our analysis most of the time we avoid considering Jost solutions for $k = 0$. In the remainder of this subsection, we describe a modification of Proposition 4.5 about the case $k = 0$.

It will be convenient to introduce notation for differently normalized Jost solutions, parallel to the unperturbed case:

$$v_m(x, k) := \sqrt{\frac{\pi}{2k}} \left(\frac{k}{2}\right)^m w_m(x, k). \tag{4.8}$$

Recall that the unperturbed eigenequation has the following solutions at $k = 0$:

$$u_m^0(x, 0) := \frac{x^{\frac{1}{2}+m}}{\Gamma(1+m)}, \quad m \in \mathbb{C}; \qquad p^0(x, 0) := x^{\frac{1}{2}} \ln(x), \quad m = 0.$$

The following proposition implies Proposition 1.8 of the introduction.

Proposition 4.8. *Let $m \in \mathbb{C}$, $k = 0$. Suppose that $\delta \geq 1 + 2\max(-\mathrm{Re}(m), 0)$ and*

$$Q \in \mathscr{L}_\delta^{(\infty)},\ \text{if } m \neq 0, \qquad Q \in \mathscr{L}_{\delta,\ln}^{(\infty)},\ \text{if } m = 0.$$

Then

$$q_{-m} := (\mathbb{1} + G_{\leftarrow}^0 Q)^{-1} u_{-m}^0(\cdot, 0)$$

is the unique solution in $AC^1]0, \infty[$ to (3.1) for $k = 0$ such that,

$$q_{-m}(x) - x^{\frac{1}{2}-m} = o(x^{\frac{3}{2}-\mathrm{Re}(m)-\delta}), \tag{4.9}$$

$$\partial_x q_{-m}(x) - \partial_x x^{\frac{1}{2}-m} = o(x^{\frac{1}{2}-\mathrm{Re}(m)-\delta}), \quad x \to \infty. \tag{4.10}$$

Besides, if $m \neq 0$, then

$$\lim_{k \to 0} v_m(x, k) = \frac{1}{2}\Gamma(m) q_{-m}(x). \tag{4.11}$$

Proof. We proceed as in the proof of Theorem 3.5. In particular, the fact that q_{-m} is well-defined follows from Corollary 4.2(i) in the case $m \neq 0$ and Corollary 4.4(i) if $m = 0$.

The limit (4.11) follows from (4.3), (4.9) and (2.31). □

Note that $\frac{3}{2} - \mathrm{Re}(m) - \delta \leq \frac{1}{2} - \mathrm{Re}(m)$. Therefore, the error in (4.9) is always of a smaller order than $x^{\frac{1}{2}-m}$.

Proposition 4.9.

(i) Let $Q \in \mathscr{L}_1^{(\infty)}$. Then for any $x > 0$ the maps

$$\Big\{\mathrm{Re}(m) \geq 0,\ m \neq 0\Big\} \ni m \mapsto q_{-m}(x),\, \partial_x q_{-m}(x) \tag{4.12}$$

are regular. If we strengthen the assumption to $Q \in \mathscr{L}_{1,\ln}^{(\infty)}$, then in (4.12) we can include $m = 0$.

(ii) Let $\delta > 1$ and $Q \in \mathscr{L}_\delta^{(\infty)}$. Then for any $x > 0$ the maps

$$\Big\{\mathrm{Re}(m) \geq \frac{1}{2}(1-\delta)\Big\} \ni m \mapsto q_{-m}(x),\, \partial_x q_{-m}(x) \tag{4.13}$$

are regular.

Proof. The proof is similar to that of Proposition 3.8. □

For $k = 0$, $m = 0$, we can also construct a solution with the same behavior as $x^{\frac{1}{2}}\ln(x)$ at ∞, but we need to strengthen the condition on Q. Note that Proposition 4.10 implies Proposition 1.9 from the introduction.

Proposition 4.10. *Let $m = 0$, $k = 0$, $\delta \geq 1$, $\beta \geq 2$. Suppose that $Q \in \mathscr{L}_{\delta,\ln^\beta}^{(\infty)}$. Then*

$$q_{0,\ln} := (1\!\!1 + G^0_{\leftarrow} Q)^{-1} p^0(\cdot, 0) \tag{4.14}$$

is the unique solution in $AC^1]0,\infty[$ to (3.1) for $k = 0$ such that

$$q_{0,\ln}(x) - x^{\frac{1}{2}}\ln(x) = o(x^{\frac{3}{2}-\delta}\ln(x)^{2-\beta}), \tag{4.15}$$

$$\partial_x q_{0,\ln}(x) - \partial_x x^{\frac{1}{2}}\ln(x) = o(x^{\frac{1}{2}-\delta}\ln(x)^{2-\beta})), \quad x \to \infty. \tag{4.16}$$

Proof. The proof is again similar to that of Theorem 3.5, using now Corollary 4.4(ii) and Lemma 4.3. □

4.3. **Asymptotics of non-principal solutions near ∞.** In this subsection, under the minimal assumptions $Q \in \mathscr{L}_0^{(\infty)}$, we show that all elements of $\mathcal{N}(L_{m^2} + k^2)$ not proportional to Jost solutions behave like non-principal unperturbed solutions near ∞. In particular, the following proposition shows that they are not square integrable.

Proposition 4.11. *Let $m \in \mathbb{C}$, $\mathrm{Re}(k) \geq 0$ and $k \neq 0$. Suppose that $Q \in \mathscr{L}_0^{(\infty)}$. Let g be a solution of (3.1) linearly independent with $w_m(\cdot, k)$ constructed in Proposition 4.5. Then there exists a constant $C \neq 0$ such that*

$$g(x) = C\,\mathrm{e}^{kx} + o(\mathrm{e}^{\mathrm{Re}(k)x}), \quad x \to \infty. \tag{4.17}$$

Proof. Similarly as in the proof of Proposition 3.10, assuming that $w \equiv w_m(\cdot, k)$ and W are known, we solve the ordinary differential equation

$$g(x)w'(x) - g'(x)w(x) = W.$$

The ansatz $g(x) = \lambda(x)w(x)$ yields

$$\lambda'(x)w(x)^2 = W.$$

By Proposition 4.5 and (2.10), we know that

$$w(x) = \mathrm{e}^{-kx} + o(\mathrm{e}^{-kx}), \quad x \to \infty.$$

This implies that there exists $\alpha > 0$ such that $w(x) \neq 0$ for $x \geq \alpha$, and hence

$$\lambda(x) - \lambda(\alpha) = \int_\alpha^x \frac{W}{w(y)^2}\mathrm{d}y = \int_\alpha^x W\Big(\mathrm{e}^{2ky} + o(\mathrm{e}^{2ky})\Big)\mathrm{d}y = \frac{W}{2k}\mathrm{e}^{2kx} + o(\mathrm{e}^{2kx}).$$

Now

$$g(x) = \Big(\lambda(\alpha) + \int_\alpha^x \frac{W}{w(y)^2}\mathrm{d}y\Big)w(x),$$

implies the estimate (4.17). □

4.4. **Global estimates on Jost solutions.** In the sequel we will need an estimate of the Jost solutions constructed in Proposition 4.5 global in x and k, which we prove in this subsection.

Since $w_m(\cdot, k) = w_{-m}(\cdot, k)$, we can assume in the next proposition that $\mathrm{Re}(m) \geq 0$.

Proposition 4.12. *Let $m \in \mathbb{C}$ be such that $\mathrm{Re}(m) \geq 0$. Suppose that $Q \in \mathscr{L}_0^{(\infty)}$. We have then the following estimates on the solution $w_m(\cdot, k)$ to (3.1), where the constant C is uniform in $\mathrm{Re}(k) \geq 0$, $k \neq 0$: for all $x > 0$,*

$$|w_m(x,k)| \leq \begin{cases} C|k|^{\frac12 - \mathrm{Re}(m)}\mu_k(x)^{\frac12 - \mathrm{Re}(m)}\eta_{-k}(x)\exp\Big(C\displaystyle\int_x^\infty \mu_k(y)|Q(y)|\mathrm{d}y\Big), & m \neq 0; \\ C|k|^{\frac12}\mu_k(x)^{\frac12}\lambda_k(x)\eta_{-k}(x)\exp\Big(C\displaystyle\int_x^\infty \mu_k(y)\lambda_k(y)|Q(y)|\mathrm{d}y\Big), & m = 0. \end{cases}$$

The same estimates hold for $|\partial_x w_m(x,k)|$, replacing $\mu_k(x)^{\frac12 - \mathrm{Re}(m)}$ by $\mu_k(x)^{-\frac12 - \mathrm{Re}(m)}$.

Proof. Note that by (2.31) and the estimates recalled in Subsection 2.1, we have

$$\begin{aligned} \big|w_m^0(x,k)\big| &\lesssim |k|^{\frac12 - \mathrm{Re}(m)}\mu_k(x)^{\frac12 - \mathrm{Re}(m)}\eta_{-k}(x), \\ \big|\partial_x w_m^0(x,k)\big| &\lesssim |k|^{\frac12 - \mathrm{Re}(m)}\mu_k(x)^{-\frac12 - \mathrm{Re}(m)}\eta_{-k}(x), \quad \text{if } m \neq 0, \\ \big|w_0^0(x,k)\big| &\lesssim |k|^{\frac12}\mu_k(x)^{\frac12}\lambda_k(x)\eta_{-k}(x), \\ \big|\partial_x w_0^0(x,k)\big| &\lesssim |k|^{\frac12}\mu_k(x)^{-\frac12}\lambda_k(x)\eta_{-k}(x), \quad \text{if } m = 0. \end{aligned}$$

Hence $w_m^0(\cdot,k) \in L^\infty\big(]0,\infty[, \mu_k^{\frac12 - \mathrm{Re}(m)}\eta_{-k}\big)$ for $m \neq 0$, $w_0^0(\cdot,k) \in L^\infty\big(]0,\infty[, \mu_k^{\frac12}\lambda_k\eta_{-k}\big)$ for $m = 0$, and therefore since $w_m(\cdot,k) = (\mathbb{1} + G_{\leftarrow}^0 Q)^{-1} w_m^0(\cdot,k)$, we can apply Corollary 4.2. □

4.5. **Asymptotics of Jost solutions near 0.** In the next section we will need the asymptotics near 0 of the Jost solution $w = w_m(\cdot, k)$. It is given by the following proposition.

Proposition 4.13. *Let $\mathrm{Re}(k) \geq 0$, $k \neq 0$. Suppose that $Q \in \mathscr{L}_0^{(\infty)}$.*

(i) If $m \neq 0$, $0 \leq \varepsilon < 2\mathrm{Re}(m)$ and $Q \in \mathscr{L}_\varepsilon^{(0)}$, then as $x \to 0$,

$$w_m(x) = w_m^0(x) + \langle u_m^0 | Q w_m\rangle v_m^0(x) + o(x^{\frac12 - \mathrm{Re}(m) + \varepsilon}), \tag{4.18}$$

$$\partial_x w_m(x) = \partial_x w_m^0(x) + \langle u_m^0 | Q w_m\rangle \partial_x v_m^0(x) + o(x^{-\frac12 - \mathrm{Re}(m) + \varepsilon}). \tag{4.19}$$

(ii) If $m \neq 0$, $0 \leq 2\mathrm{Re}(m) \leq \varepsilon$, $Q \in \mathscr{L}_\varepsilon^{(0)}$, then as $x \to 0$,

$$w_m(x) = w_m^0(x) + \langle u_m^0|Qw_m\rangle v_m^0(x) - \langle v_m^0|Qw_m\rangle u_m^0(x) + o(x^{\frac12-\mathrm{Re}(m)+\varepsilon}), \tag{4.20}$$

$$\partial_x w_m(x) = \partial_x w_m^0(x) + \langle u_m^0|Qw_m\rangle \partial_x v_m^0(x) - \langle v_m^0|Qw_m\rangle \partial_x u_m^0(x) + o(x^{-\frac12-\mathrm{Re}(m)+\varepsilon}). \tag{4.21}$$

(iii) If $0 \leq \varepsilon$, $Q \in \mathscr{L}_{\varepsilon,\ln^2}^{(0)}$, and $m = 0$, then as $x \to 0$,

$$w_0(x) = w_0^0(x) + \langle u_0^0|Qw_0\rangle v_0^0(x) - \langle v_0^0|Qw_0\rangle u_0^0(x) + o(x^{\frac12+\varepsilon}), \tag{4.22}$$

$$\partial_x w_0(x) = \partial_x w_0^0(x) + \langle u_0^0|Qw_0\rangle \partial_x v_0^0(x) - \langle v_0^0|Qw_0\rangle \partial_x u_0^0(x) + o(x^{-\frac12+\varepsilon}). \tag{4.23}$$

Proof. We only prove the statements for w_m. The statements for $\partial_x w_m$ are proven in the same way. We omit the index m and the argument k in this proof. Recall (4.7):

$$w = w^0 - G_{\leftarrow}^0 Qw.$$

Suppose first that $m \neq 0$. By Proposition 4.12, we have

$$w(x) = \mathcal{O}(x^{\frac12-\mathrm{Re}(m)}), \quad \partial_x w(x) = \mathcal{O}(x^{-\frac12-\mathrm{Re}(m)}), \quad x \to 0;$$
$$w(x) = \mathcal{O}(\mathrm{e}^{-\mathrm{Re}(k)x}) \quad \partial_x w(x) = \mathcal{O}(\mathrm{e}^{-\mathrm{Re}(k)x}), \quad x \to \infty.$$

First assume that $2\mathrm{Re}(m) \leq \varepsilon$. Then, since $Q \in \mathscr{L}_0^{(\infty)} \cap \mathscr{L}_\varepsilon^{(0)}$, it follows from Proposition 4.12 that $G_{\leftrightarrow}^0 Qw$ are well-defined. Thus we can write

$$G_{\leftarrow}^0 Qw = G_{\leftrightarrow}^0 Qw + G_{\rightarrow}^0 Qw.$$

We can use Lemma 3.1(i) with $\varepsilon_1 = \frac12 - \mathrm{Re}(m)$, which gives

$$G_{\rightarrow}^0 Qw = o(x^{\frac12-\mathrm{Re}(m)+\varepsilon}).$$

This proves (4.20).

Suppose now that $0 \leq \varepsilon < 2\mathrm{Re}(m)$. Then $\langle u^0|Qw\rangle$ is still well-defined but $\langle v^0|Qw\rangle$ is not any more. For $a > 0$ and $0 < x < a$, we write

$$G_{\leftarrow}^0 Qw(x) = -\langle u^0|Qw\rangle v^0(x) - G_{\bowtie}^{0(a)} Qw(x) + \mathcal{O}(x^{\frac12+\mathrm{Re}(m)}).$$

Applying Lemma 3.11(i) with $\varepsilon_1 = \frac12 - \mathrm{Re}(m)$, we obtain

$$G_{\bowtie}^{0(a)} Qw = o(x^{\frac12-\mathrm{Re}(m)+\varepsilon}),$$

which establishes (4.18).

The proof in the case $m = 0$ is similar, using Lemma 3.11(i) with $\varepsilon_1 = \frac12$, $\alpha = 1$, $\beta = 2$. □

4.6. **Summary of distinguished solutions.** In this subsection we recall the distinguished solutions of the perturbed eigenequation with a prescribed behavior near infinity constructed in this section. They are described in the following table, which has the same structure as that of Subsection 3.9.

Solution	Parameters	Conditions on Q	Green's operator
$w_m(\cdot,k)$ $v_m(\cdot,k)$	$m\in\mathbb{C}$, $\mathrm{Re}(k)\geq 0$, $k\neq 0$	$Q\in\mathscr{L}_0^{(\infty)}$	Backward $G_{\leftarrow}^0$
$q_{-m}(\cdot)$	$\mathrm{Re}(m)\geq\frac{1-\delta}{2}$, $k=0$	$Q\in\mathscr{L}_\delta^{(\infty)}$, $\delta\geq 1$, if $m\neq 0$ $Q\in\mathscr{L}_{1,\ln}^{(\infty)}$, if $m=0$	Backward $G_{\leftarrow}^0$
$q_{0,\ln}(\cdot)$	$m=0$, $k=0$	$Q\in\mathscr{L}_{1,\ln^2}^{(\infty)}$	Backward $G_{\leftarrow}^0$

TABLE 2. *Distinguished solutions of the perturbed eigenequation with a prescribed behavior near* ∞.

5. WRONSKIANS

In this section we study the wronskians of distinguished solutions constructed in the previous two sections.

5.1. **The Jost function.** Suppose that $Q\in\mathscr{L}_0^{(\infty)}$ and $Q\in\mathscr{L}_\varepsilon^{(0)}$ with $\varepsilon\geq\max\big(0,-2\mathrm{Re}(m)\big)$ if $m\neq 0$, $Q\in\mathscr{L}_{0,\ln}^{(0)}$ if $m=0$. Let $\mathrm{Re}(k)\geq 0$, $k\neq 0$. Then, by Propositions 3.6 and 4.5, both $u_m(\cdot,k)$ and $v_m(\cdot,k)$ are well defined. Their Wronskian

$$\mathscr{W}_m(k):=\mathscr{W}\big(v_m(\cdot,k),u_m(\cdot,k)\big), \tag{5.1}$$

will be called the *Jost function*. Assuming in addition that $Q\in\mathscr{L}_\delta^{(\infty)}$ for some $\delta\geq 1$, we also set for $k=0$ and $\mathrm{Re}(m)\geq\frac12(1-\delta)$, $m\neq 0$,

$$\mathscr{W}_m(0):=\frac{1}{2}\Gamma(m)\mathscr{W}\big(q_{-m}(\cdot),u_m(\cdot,0)\big). \tag{5.2}$$

Using the regularity properties of $u_m(x,k)$ and $v_m(x,k)$, we obtain the regularity of the map $(m,k)\mapsto\mathscr{W}_m(k)$ on suitable domains.

Proposition 5.1.

(i) If $Q\in\mathscr{L}_\varepsilon^{(0)}\cap\mathscr{L}_0^{(\infty)}$ for some $\varepsilon>0$, then the map $(m,k)\mapsto\mathscr{W}_m(k)$ is regular on

$$\Big\{\mathrm{Re}(m)\geq-\frac{\varepsilon}{2}\Big\}\times\Big\{\mathrm{Re}(k)\geq 0,\ k\neq 0\Big\}. \tag{5.3}$$

(ii) If $Q\in\mathscr{L}_0^{(0)}\cap\mathscr{L}_0^{(\infty)}$, then the map $(m,k)\mapsto\mathscr{W}_m(k)$ is regular on

$$\Big\{\mathrm{Re}(m)\geq 0,\ m\neq 0\Big\}\times\Big\{\mathrm{Re}(k)\geq 0,\ k\neq 0\Big\}. \tag{5.4}$$

(iii) If $Q\in\mathscr{L}_{0,\ln}^{(0)}\cap\mathscr{L}_1^{(\infty)}$, then the map $(m,k)\mapsto\mathscr{W}_m(k)$ is regular on

$$\Big\{\mathrm{Re}(m)\geq 0\Big\}\times\Big\{\mathrm{Re}(k)\geq 0\Big\}\setminus\big\{m=0\big\}\times\big\{k=0\big\}. \tag{5.5}$$

Proof. Analyticity and continuity are consequences of the analyticity and continuity of the maps $u_m(x,k)$, $w_m(x,k)$, $q_m(x)$ and their derivatives (see Propositions 3.8, 4.6 and 4.9).

Continuity at $k=0$ in (5.5) uses in addition the limit (4.11). ☐

The next proposition gives a convenient representation of the Jost function.

Proposition 5.2. *Suppose that* $Q \in \mathscr{L}_0^{(\infty)}$*. Let* $m \in \mathbb{C}$*. Suppose that* $Q \in \mathscr{L}_\varepsilon^{(0)}$ *with* $\varepsilon = \max(0, -2\mathrm{Re}(m))$ *if* $m \neq 0$*, or* $Q \in \mathscr{L}_{0,\ln^2}^{(0)}$ *if* $m = 0$*. Then for* $\mathrm{Re}(k) \geq 0$*,* $k \neq 0$

$$\mathscr{W}_m(k) = 1 + \langle u_m^0(\cdot, k) | Q v_m(\cdot, k)\rangle. \tag{5.6}$$

Proof. Consider first $m \neq 0$. By Proposition 3.6 with $\varepsilon = \max(0, -2\mathrm{Re}(m))$, for $x \to 0$ we have

$$u_m(x,k) = u_m^0(x,k) + o(x^{\frac{1}{2}+|\mathrm{Re}(m)|}) \tag{5.7}$$

$$\partial_x u_m(x,k) = \partial_x u_m^0(x,k) + o(x^{-\frac{1}{2}+|\mathrm{Re}(m)|}). \tag{5.8}$$

By Proposition 4.13(i) with $\varepsilon = 0$ we have

$$v_m(x,k) = v_m^0(x,k) + \langle u_m^0 | Q v_m\rangle v_m^0(x,k) + o(x^{\frac{1}{2}-|\mathrm{Re}(m)|}), \tag{5.9}$$

$$\partial_x v_m(x,k) = \partial_x v_m^0(x,k) + \langle u_m^0 | Q v_m\rangle \partial_x v_m^0(x,k) + o(x^{-\frac{1}{2}-|\mathrm{Re}(m)|}). \tag{5.10}$$

Now (2.31), (5.7)–(5.8) and (5.9)–(5.10) yield

$$\mathscr{W}(v_m, u_m; x) = \mathscr{W}(v_m^0, u_m; x)\Big(1 + \langle u_m^0 | Q v_m\rangle\Big) + o(x^0).$$

Moreover it follows from (5.7) and (5.8) that

$$\lim_{x\to 0} \mathscr{W}(v_m^0, u_m; x) = \mathscr{W}(v_m^0, u_m^0; x) + o(x^0) = 1 + o(x^0).$$

Then note that $\mathscr{W}(v_m, u_m; x)$ does not depend on x and use the definition (5.1) of $\mathscr{W}_m(k)$.

In the case $m = 0$, we use Proposition 4.13 (iii). Then we repeat the same arguments, using in addition

$$\mathscr{W}(u_0^0, u_0; x) = \mathscr{W}(u_0^0, u_0^0; x) + o(x^0), \quad \mathscr{W}(u_0^0, u_0^0; x) = 0.$$

This ends the proof of (5.6). □

The asymptotic behavior of the Jost function can be deduced from Proposition 5.2 together with the following lemma.

Lemma 5.3. *Let* $m \in \mathbb{C}$*. Suppose that* $Q \in \mathscr{L}_\varepsilon^{(0)} \cap \mathscr{L}_0^{(\infty)}$ *with* $\varepsilon = \max(0, -2\mathrm{Re}(m))$ *if* $m \neq 0$*, or* $Q \in \mathscr{L}_{0,\ln^2}^{(0)} \cap \mathscr{L}_0^{(\infty)}$ *if* $m = 0$*. Then*

$$\langle u_m^0 | Q v_m\rangle = o(|k|^0) + \mathcal{O}(|k|^{-1+\varepsilon}), \qquad |k| \to \infty, \quad \mathrm{Re}(k) \geq 0. \tag{5.11}$$

Proof. Assume that $m \neq 0$. We have the estimate

$$|u_m^0(x,k)| \lesssim \mu_k(x)^{\frac{1}{2}+\mathrm{Re}(m)} \eta_k(x), \tag{5.12}$$

uniformly in k. Next, in the estimate of Proposition 4.12, we first note that the big exponential on the right hand side is uniformly bounded for large k. Therefore, uniformly for large enough $|k|$,

$$|v_m(x,k)| \lesssim \mu_k(x)^{\frac{1}{2}-|\mathrm{Re}(m)|} \eta_{-k}(x). \tag{5.13}$$

Hence

$$\begin{aligned}
&\big|\langle u_m^0|Qv_m\rangle\big| \\
&= \Big|\int_0^\infty u_m^0(y)Q(y)v_m(y)\mathrm{d}y\Big| \\
&\lesssim \int_0^\infty \mu_k(x)^{1-\varepsilon}|Q(x)|\mathrm{d}x \\
&\lesssim \Big(\int_0^{\frac{1}{|k|}} x^{1-\varepsilon}|Q(x)|\mathrm{d}x + |k|^{-1+\varepsilon}\int_{\frac{1}{|k|}}^1 |Q(x)|\mathrm{d}x + |k|^{-1+\varepsilon}\int_1^\infty |Q(x)|\mathrm{d}x\Big).
\end{aligned}$$

First note that

$$\int_0^{\frac{1}{|k|}} x^{1-\varepsilon}|Q(x)|\mathrm{d}x = o(1) \quad \text{and} \quad \int_1^\infty |Q(x)|\mathrm{d}x = \mathcal{O}(1).$$

Therefore, the first term on the right is $o\big(|k|^0\big)$ and the third is $\mathcal{O}(|k|^{-1+\varepsilon})$.

If $1 > \varepsilon \geq 0$, then applying Lemma C.1 with $h(y) = y^{1-\varepsilon}$ and $x = \frac{1}{|k|}$, we obtain

$$\int_{\frac{1}{|k|}}^1 |Q(x)|\mathrm{d}x = o(|k|^{1-\varepsilon}).$$

So the middle term is $o(|k|^0)$.

If $\varepsilon \geq 1$, then

$$\int_{\frac{1}{|k|}}^1 |Q(x)|\mathrm{d}x = o(|k|^{1-\varepsilon}).$$

So the middle term is $\mathcal{O}(|k|^{-1+\varepsilon})$.

Finally, if $m = 0$, the proof is identical, the only difference being that

$$\begin{aligned}
&\big|\langle u_0^0|Qv_0\rangle\big| \\
&\lesssim \Big(\int_0^{\frac{1}{|k|}} x(1-|\ln(|k|x)|)|Q(x)|\mathrm{d}x + |k|^{-1}\int_{\frac{1}{|k|}}^1 |Q(x)|\mathrm{d}x + |k|^{-1}\int_1^\infty |Q(x)|\mathrm{d}x\Big).
\end{aligned}$$

□

We deduce from the previous lemma that, for $\mathrm{Re}(m) > -1$, $\mathscr{W}_m$ cannot constantly vanish except on a discrete set. Corollary 5.4 implies Proposition 1.12 from the introduction.

Corollary 5.4. *In addition to the assumptions of Proposition 5.2, suppose that* $\mathrm{Re}(m) > -1$. *Then*

$$\lim_{|k|\to\infty} \mathscr{W}_m(k) = 1, \quad \mathrm{Re}(k) \geq 0. \tag{5.14}$$

Therefore,

$$\big\{k \in \mathbb{C},\, \mathrm{Re}(k) > 0,\, \mathscr{W}_m(k) = 0\big\} \quad \textit{is discrete.}$$

Proof. Under the condition $\mathrm{Re}(m) > -1$ the right hand side of (5.11) becomes $o(|k|^0)$. Hence (5.14) follows by (5.6).

The fact that $\{k \in \mathbb{C}, \mathrm{Re}(k) > 0, \mathscr{W}_m(k) = 0\}$ is discrete is then a consequence of the analyticity of $\mathscr{W}_m$ stated in Proposition 5.1. □

Note that Corollary 5.4 is the second place in our paper where the condition $\mathrm{Re}(m) > -1$ appears (see also Theorem 3.20). This condition will play an important role in Section 6 about closed realizations.

5.2. **Wronskians – refined results.** If u_{-m} is ill-defined, we can often use $u_{-m}^{[n]}$ instead.

Proposition 5.5. *Let* $\mathrm{Re}(k) \geq 0$, $k \neq 0$. *Suppose that* $Q \in \mathscr{L}_{\varepsilon}^{(0)} \cap \mathscr{L}_0^{(\infty)}$, $\varepsilon \geq 0$. *Let n be a nonnegative integer,* $\frac{\varepsilon}{2}(n+1) \geq \mathrm{Re}(m) \geq 0$, $m \notin \mathbb{N}$. *Then*

$$\mathscr{W}\big(u_{-m}^{[n]}(\cdot,k), u_m(\cdot,k)\big) = \frac{2\sin(m\pi)}{\pi}. \tag{5.15}$$

Hence there exists a constant $C_m^{[n]}(k)$ *such that*

$$v_m(\cdot,k) = \frac{\mathscr{W}_m(k)\pi}{2\sin(m\pi)} u_{-m}^{[n]}(\cdot,k) + C_m^{[n]}(k) u_m(\cdot,k). \tag{5.16}$$

Proof. First we check (5.15), which follows from (2.20), using also Propositions 3.6 and 3.18. Then we write

$$v_m(\cdot,k) = B_m^{[n]}(k) u_{-m}^{[n]}(\cdot,k) + C_m^{[n]}(k) u_m(\cdot,k), \tag{5.17}$$

and take the Wronskian of both sides with $u_m(\cdot,k)$. This allows us to compute $B_m^{[n]}(k)$ and yields (5.16). □

Proposition 5.6. *Suppose that the assumptions of Proposition 5.5 are satisfied. Then the map*

$$\Big\{0 \leq \mathrm{Re}(m) \leq \frac{\varepsilon}{2}(n+1), \quad m \neq 0\Big\} \times \Big\{\mathrm{Re}(k) \geq 0,\, k \neq 0\Big\} \ni (m,k) \mapsto \mathscr{W}\big(v_m(\cdot,k), u_{-m}^{[n]}(\cdot,k)\big)$$

is regular.

Proof. This follows from Propositions 3.19 and 4.6. □

5.3. **Green's functions for perturbed Bessel operators.** As for every 1-dimensional Schrödinger operator, we can define the canonical bisolution and various Green's functions for perturbed Bessel operators. The solutions that we constructed allow us to give explicit expressions for these Green's functions.

As usual, when defining Green's operators we will always assume that $\mathrm{Re}(k) \geq 0$ (although sometimes an extension to a larger domain is possible). Let $Q \in \mathscr{L}_0^{(\infty)} \cap \mathscr{L}_0^{(0)}$ for $m \neq 0$, and $Q \in \mathscr{L}_0^{(\infty)} \cap \mathscr{L}_{0,\ln}^{(0)}$, for $m = 0$. The canonical bisolution associated with the operator $L_{m^2} + k^2$ is

$$G_{m^2,\leftrightarrow}(-k^2; x, y) = \frac{1}{\mathscr{W}_m(k)}\big(v_m(x,k)u_m(y,k) - u_m(x,k)v_m(y,k)\big), \tag{5.18}$$

where $v_m(\cdot,k)$, $u_m(\cdot,k)$ are the solutions to (1.26) constructed in Propositions 3.6 and 4.5, and $\mathscr{W}_m(k)$ is the Jost function defined in (5.1). The expression (5.18) is well-defined when $\mathscr{W}_m(k) \neq 0$ and has a limit at $k = 0$. For $m = 0$ we can use

$$G_{0,\leftrightarrow}(-k^2; x, y) = \frac{1}{\mathscr{W}_0(k)}\big(-p_0(x,k)u_0(y,k) + u_0(x,k)p_0(y,k)\big).$$

From the canonical bisolution, we can construct in the usual way the *forward Green's operator* $G_{m^2,\to}(-k^2)$ and the *backward Green's operator* $G_{m^2,\leftarrow}(-k^2)$ *of* $L_{m^2}+k^2$:

$$G_{m^2,\to}(-k^2;x,y) := \theta(x-y)G^0_{m^2,\leftrightarrow}(-k^2;x,y), \tag{5.19}$$

$$G_{m^2,\leftarrow}(-k^2;x,y) := -\theta(y-x)G^0_{m^2,\leftrightarrow}(-k^2;x,y). \tag{5.20}$$

Green's operators defined by specifying boundary conditions at zero and at infinity will be called *two-sided.* They will often be bounded on $L^2]0,\infty[$ and coincide with the resolvents of various closed realizations of L_{m^2}. However, they are of interest even when they do not define bounded operators and do not correspond to closed realizations of L_{m^2}.

The most natural two-sided Green's operator corresponds to *pure boundary conditions.* In the unperturturbed case they are usually called *homogeneous boundary conditions*, but in the perturbed case this name seems no longer appropriate. It can be defined for $0\leq\varepsilon$, $Q\in\mathscr{L}_0^{(\infty)}\cap\mathscr{L}_\varepsilon^{(0)}$, $m\neq 0$, $-\frac{\varepsilon}{2}\leq\mathrm{Re}(m)$, and $Q\in\mathscr{L}_0^{(\infty)}\cap\mathscr{L}_{0,\ln}^{(0)}$, $m=0$. Moreover, if $\mathrm{Re}(m)\leq 0$ we neeed to assume $k\neq 0$. Then if $\mathscr{W}_m(k)\neq 0$ we set

$$G_{m,\bowtie}(-k^2;x,y) := \frac{1}{\mathscr{W}_m(k)}\begin{cases} u_m(x,k)v_m(y,k), & x<y,\\ v_m(x,k)u_m(y,k), & y<x.\end{cases} \tag{5.21}$$

We also have Green's operators with mixed boundary conditions. The cleanest situation we have under the assumption $0\leq\varepsilon$, $Q\in\mathscr{L}_\varepsilon^{(\infty)}\cap\mathscr{L}_0^{(0)}$, $m\neq 0$, $|\mathrm{Re}(m)|\leq\frac{\varepsilon}{2}$, $k\neq 0$. Then if $k\neq 0$, $\kappa\in\mathbb{C}\cup\{\infty\}$ and $\mathscr{W}_m(k)+\kappa\frac{\Gamma(1-m)}{\Gamma(1+m)}\frac{k^{2m}}{2^{2m}}\mathscr{W}_{-m}(k)\neq 0$, we define

$$\begin{aligned} &G_{m,\bowtie,\kappa}(-k^2;x,y) \\ :=&\frac{1}{\big(\mathscr{W}_m(k)+\kappa\frac{\Gamma(1-m)}{\Gamma(1+m)}\frac{k^{2m}}{2^{2m}}\mathscr{W}_{-m}(k)\big)}\begin{cases} (u_m+\kappa\frac{\Gamma(1-m)}{\Gamma(1+m)}u_{-m})(x,k)v_m(y,k), & x<y,\\ v_m(x,k)(u_m+\kappa\frac{\Gamma(1-m)}{\Gamma(1+m)}u_{-m})(y,k), & y<x.\end{cases}\end{aligned} \tag{5.22}$$

Note that

$$G_{m,\bowtie,\kappa}(-k^2;x,y) = G_{-m,\bowtie,\kappa^{-1}}(-k^2;x,y). \tag{5.23}$$

Similarly, for $Q\in\mathscr{L}_0^{(\infty)}\cap\mathscr{L}_{0,\ln}^{(0)}$, $m=0$, if $\nu\mathscr{W}_0(k)+\mathscr{W}(v_0(\cdot,k),p_0(\cdot,k))\neq 0$, then we define

$$\begin{aligned} &G^\nu_{0,\bowtie}(-k^2;x,y) \\ :=&\frac{1}{\big(\nu\mathscr{W}_0(k)+\mathscr{W}(v_0(\cdot,k),p_0(\cdot,k))\big)}\begin{cases} (\nu u_0+p_0)(x,k)v_0(y,k), & x<y,\\ v_0(x,k)(\nu u_0+p_0)(y,k), & y<x.\end{cases}\end{aligned} \tag{5.24}$$

Without the assumption $\mathrm{Re}(m)\geq-\frac{\varepsilon}{2}$ we are not guaranteed the existence of u_m, and hence we are not sure whether Green's function with pure boundary conditions can be extended. However, we can use the boundary conditions given by $u_{-m}^{[n]}$. Choose $\varepsilon>0$, $Q\in\mathscr{L}_\varepsilon^{(\infty)}\cap\mathscr{L}_0^{(0)}$, $m\neq 0$, and a nonnegative integer n. Then for $-\frac{\varepsilon}{2}(n+1)\leq\mathrm{Re}(m)<0$ if $k\neq 0$, $\kappa\in\mathbb{C}\cup\{\infty\}$

and $\mathscr{W}\big(v_m(\cdot,k),u_m^{[n]}(\cdot,k)\big)+\kappa\frac{\Gamma(1-m)}{\Gamma(1+m)}\frac{k^{2m}}{2^{2m}}\mathscr{W}_{-m}(k)\neq 0$, we can define

$$G_{m,\bowtie,\kappa}^{[n]}(-k^2;x,y) \tag{5.25}$$
$$:=\frac{1}{\mathscr{W}\big(v_m(\cdot,k),u_m^{[n]}(\cdot,k)\big)+\kappa\frac{\Gamma(1-m)}{\Gamma(1+m)}\frac{k^{2m}}{2^{2m}}\mathscr{W}_{-m}(k)}\begin{cases}(u_m^{[n]}+\kappa\frac{\Gamma(1-m)}{\Gamma(1+m)}u_{-m})(x,k)v_m(y,k), & x<y,\\ v_m(x,k)(u_m^{[n]}+\kappa\frac{\Gamma(1-m)}{\Gamma(1+m)}u_{-m})(y,k), & y<x.\end{cases}$$

6. Closed realizations of L_{m^2}

In this section we describe realizations of L_{m^2} as closed operators on $L^2]0,\infty[$. We will see that under certain assumptions on the perturbation Q one can impose the boundary condition at 0 in a similar way as in the unperturbed case. If we fix Q, it is also natural to organize these operators in holomorphic families, analogous to the holomorphic families studied in [12].

In the first two subsections we recall the basics of the theory of 1d Schrödinger operators and their boundary conditions.

6.1. **1-dimensional Schrödinger operators on the halfline.**

We will follow [13], other references include [20, 21].

Suppose that $]0,\infty[\ni x\mapsto V(x)$ is a function in $L^1_{\rm loc}]0,\infty[$, possibly complex valued. Consider the expression

$$L:=-\partial_x^2+V(x),$$

viewed as a linear map from $AC^1]0,\infty[$ to $L^1_{\rm loc}]0,\infty[$. Let us describe the four most obvious closed realizations of L on $L^2]0,\infty[$.

First define

$$\begin{aligned}\mathcal{D}(L^{\max})&:=\big\{f\in L^2]0,\infty[\cap AC^1]0,\infty[\ \mid\ Lf\in L^2]0,\infty[\big\},\\ \mathcal{D}(L^{\min})&:=\text{the closure of }\{f\in\mathcal{D}(L^{\max})\ \mid\ f=0\text{ near }0\text{ and }\infty\},\\ \mathcal{D}(L^{0})&:=\text{the closure of }\{f\in\mathcal{D}(L^{\max})\ \mid\ f=0\text{ near }0\},\\ \mathcal{D}(L^{\infty})&:=\text{the closure of }\{f\in\mathcal{D}(L^{\max})\ \mid\ f=0\text{ near }\infty\}.\end{aligned}$$

Above, $\mathcal{D}(L^{\max})$ is treated as a Hilbert space with the norm given by $\|f\|_L^2:=\|Lf\|^2+\|f\|^2$. We define

$$L^{\max}:=L\big|_{\mathcal{D}(L^{\max})},\quad L^{\min}:=L\big|_{\mathcal{D}(L^{\min})},\quad L^{0}:=L\big|_{\mathcal{D}(L^{0})},\quad L^{\infty}:=L\big|_{\mathcal{D}(L^{\infty})}.$$

Then $L^{\max}$, $L^{\min}$, L^0 and L^∞ are closed operators satisfying

$$L^{\max}\supset L^{\infty}\supset L^{\min},\quad L^{\max}\supset L^{0}\supset L^{\min}.$$

Let us quote some general results. The following proposition is well-known, it is e.g. proven as Proposition 5.15 of [13]:

Proposition 6.1. *If*

$$\limsup_{c\to\infty}\int_c^{c+1}|V(x)|\mathrm{d}x<\infty, \tag{6.1}$$

then

$$L^{\max}=L^{\infty},\quad L^0=L^{\min}. \tag{6.2}$$

Thus, there is no need to set boundary conditions at infinity.

By [13, Theorem 6.12], we have

Proposition 6.2. *Suppose that (6.2) holds. Then we have the following alternative:*

(i) either $\mathcal{D}(L^{\max})/\mathcal{D}(L^{\min}) = 0$

and $\dim\big\{f \in \mathcal{N}(L-\lambda) \;\mid\; f$ *is square integrable near* $0\big\} \leq 1$ *for all* $\lambda \in \mathbb{C}$ *;*

(ii) or $\mathcal{D}(L^{\max})/\mathcal{D}(L^{\min}) = 2$

and $\dim\{f \in \mathcal{N}(L^{\max}-\lambda) \;\mid\; f$ *is square integrable near* $0\} = 2$ *for all* $\lambda \in \mathbb{C}$.

Until the end of this subsection we suppose that alternative (ii) of Proposition 6.2 holds. Fix $\lambda \in \mathbb{C}$ and $\xi \in C_{\mathrm{c}}[0,\infty[$ equal 1 near 0. Then by [13] we have a direct sum decomposition

$$\mathcal{D}(L^{\max}) = \mathcal{D}(L^{\min}) \oplus \{\xi f \mid f \in \mathcal{N}(L)\}, \tag{6.3}$$

where $\mathcal{N}(L)$ denotes all functions in $AC^1[0,\infty[$ annihilated by L.

We are interested in operators $L^\bullet$ lying "in the middle" between $L^{\min}$ and $L^{\max}$, that is satisfying

$$L^{\min} \subset L^\bullet \subset L^{\max},$$

where both inclusions are of codimension 1. All such operators correspond to one-dimensional subspaces of $\mathcal{D}(L^{\max})/\mathcal{D}(L^{\min})$. To specify such a subspace it is enough to choose

$$r \in \mathcal{D}(L^{\max}), \quad r \notin \mathcal{D}(L^{\min}), \tag{6.4}$$

and to define

$$\mathcal{D}(L^r) := \mathcal{D}(L^{\min}) + \mathbb{C}r, \tag{6.5}$$

$$L^r := L^{\max}\big|_{\mathcal{D}(L^r)}. \tag{6.6}$$

6.2. **Boundary functionals.** We continue to analyze general 1d Schrödinger operators. Until the end of this subsection we assume (6.2). We will give a method to describe boundary conditions which is often more practical than (6.5).

First recall the concept of Wronskian (1.61). If $f, g \in \mathcal{D}(L^{\max})$, then $f, g \in AC^1]0,\infty[\subset C^1]0,\infty[$, hence their Wronskian at $x \in]0,\infty[$, denoted $\mathscr{W}(f,g;x)$, is well defined. Interestingly, the Wronskian extends to the boundary $x = 0$, as follows e.g. from [13, Theorem 4.4]:

Proposition 6.3. *For* $f, g \in \mathcal{D}(L^{\max})$

$$\lim_{x\searrow 0} \mathscr{W}(f,g;x) =: \mathscr{W}(f,g;0)$$

exists and defines a continuous bilinear form. If in addition (6.2) holds, then

$$\mathcal{D}(L^{\min}) = \Big\{f \in \mathcal{D}(L^{\max}) \;\mid\; \mathscr{W}(f,g;0) = 0 \text{ for all } g \in \mathcal{D}(L^{\max})\Big\}.$$

Let us define the *boundary space*

$$\mathcal{B} := \big(\mathcal{D}(L^{\max})/\mathcal{D}(L^{\min})\big)',$$

where the prime denotes the dual.

Let r be as in (6.4). Let $\phi \neq 0$ be a boundary functional (that is, an element of $\mathcal{B}$) such that $\phi(r) = 0$. Obviously,

$$\mathcal{D}(L^r) := \{f \in \mathcal{D}(L^{\max}) \mid \phi(f) = 0\}$$

is equivalent to (6.5).

The boundary functional ϕ can be simply written as

$$\phi = c\mathscr{W}(r, \cdot; 0), \tag{6.7}$$

where $c \neq 0$. Using (6.3) and (6.7) we obtain

Corollary 6.4. *Suppose that the alternative (ii) of Proposition 6.2 holds. Fix $\lambda \in \mathbb{C}$. Then we have a natural isomorphism of $\mathcal{B}$ and $\mathcal{N}(L - \lambda)$:*

$$\mathcal{B} = \{\mathscr{W}(f, \cdot; 0) \mid f \in \mathcal{N}(L - \lambda)\}. \tag{6.8}$$

We will say that a function $f \in C^1]0, \infty[$ *possesses the Wronskian at zero on* $\mathcal{D}(L^{\max})$ if

$$\mathscr{W}(f, g; 0) := \lim_{x \searrow 0} \mathscr{W}(f, g; x), \quad g \in \mathcal{D}(L^{\max}),$$

exists. Proposition 6.3 says that each function in $\mathcal{D}(L^{\max})$ possesses the Wronskian at zero on $\mathcal{D}(L^{\max})$.

In practice, it may be difficult to make explicit an element r in $\mathcal{D}(L^{\max})$ describing the functional ϕ by (6.7). Instead, we can often find a simpler function r_1, not necessarily in $\mathcal{D}(L^{\max})$, which also possesses the Wronskian at zero on $\mathcal{D}(L^{\max})$ and such that

$$\mathscr{W}(r, \cdot; 0) = \mathscr{W}(r_1, \cdot; 0). \tag{6.9}$$

Then instead of (6.5) the domain of L^r can be equivalently characterized as :

$$\mathcal{D}(L^r) := \{f \in \mathcal{D}(L^{\max}) \mid \mathscr{W}(r_1, f; 0) = 0\}. \tag{6.10}$$

6.3. **The maximal and minimal realizations of L_{m^2}.** As everywhere in this paper, we assume that $]0, \infty[\ni x \mapsto Q(x)$ belongs to $L^1_{\mathrm{loc}}]0, \infty[$. For $m \in \mathbb{C}$, set

$$V_{m^2}(x) := \Big(m^2 - \frac{1}{4}\Big)\frac{1}{x^2} + Q(x).$$

We consider the differential expression

$$L_{m^2} := -\partial_x^2 + V_{m^2}(x).$$

as a linear map on $AC^1]0, \infty[$. By applying the definitions of Subsection 6.1, we can introduce the closed operators $L^{\max}_{m^2}$, $L^{\min}_{m^2}$ such that

$$\big(L^{\min}_{m^2}\big)^{\#} = L^{\max}_{m^2}, \qquad \big(L^{\max}_{m^2}\big)^{\#} = L^{\min}_{m^2}.$$

Until the end of this section we assume that $Q \in \mathscr{L}_0^{(\infty)}$.

Proposition 6.5. *Let $m \in \mathbb{C}$. Suppose that*

$$Q \in \mathscr{L}_0^{(0)}, \text{ if } m \neq 0, \qquad Q \in \mathscr{L}_{0,\ln}^{(0)}, \text{ if } m = 0.$$

Then the following holds:

(i) *If $1 \leq |\mathrm{Re}(m)|$, then $L^{\min}_{m^2} = L^{\max}_{m^2}$.*
(ii) *If $|\mathrm{Re}(m)| < 1$, then $\mathcal{D}(L^{\min}_{m^2})$ is a closed subspace of $\mathcal{D}(L^{\max}_{m^2})$ of codimension 2.*

Proof. Obviously, the condition (6.1) holds. Therefore, only the boundary conditions at zero matter.

We can assume that $\mathrm{Re}(m) \geq 0$ and $\mathrm{Re}(k) \geq 0$. For $m \neq 0$, in the space $\mathcal{N}(L_{m^2} + k^2)$ all elements proportional to $u_m(\cdot, k)$ behave as $x^{\frac{1}{2}+m}$, and all other elements of $\mathcal{N}(L_{m^2} + k^2)$, by

Proposition 3.10, behave as $x^{\frac12-m}$. For $m=0$ they behave respectively as $x^{\frac12}$ and $x^{\frac12}\ln(x)$. Both are square integrable iff $|\mathrm{Re}(m)|<1$. Hence

$$\begin{aligned}&\dim\big\{f\in\mathcal{N}(L_{m^2}^{\max}+k^2)\ \mid\ f \text{ is square integrable near } 0\big\}\leq 1 \text{ for all } \mathrm{Re}(k)\geq 0\\&\text{iff}\ \ |\mathrm{Re}(m)|\geq 1;\end{aligned}\tag{6.11}$$

and

$$\begin{aligned}&\dim\{f\in\mathcal{N}(L_{m^2}^{\max}+k^2)\ \mid\ f \text{ is square integrable near } 0\}=2 \text{ for all } \mathrm{Re}(k)\geq 0\\&\text{iff}\ \ |\mathrm{Re}(m)|<1.\end{aligned}\tag{6.12}$$

Now we apply Proposition 6.2: (i) corresponds to (6.11) and (ii) corresponds to (6.12). □

6.4. **Closed realizations of the unperturbed Bessel operator.** Let us recall the basic theory of closed realizations of $L_{m^2}^0$. We will essentially follow [16], except that we will put the superscript 0 on the symbols of various operators.

Let $\mathcal{B}_{m^2}^0$ denote the boundary space of $L_{m^2}^0$. Below we give natural bases of $\mathcal{B}_{m^2}^0$:

$$\mathscr{W}(x^{\frac12-m},\cdot;0),\quad \mathscr{W}(x^{\frac12+m},\cdot;0),\quad m\neq 0;\tag{6.13}$$

$$\mathscr{W}(x^{\frac12},\cdot;0),\quad \mathscr{W}(x^{\frac12}\ln(x),\cdot;0),\quad m=0.\tag{6.14}$$

Note that for $|\mathrm{Re}(m)|<1$,

$$\mathscr{W}(x^{\frac12-m},x^{\frac12+m})=2m,\tag{6.15}$$

$$\mathscr{W}(x^{\frac12},x^{\frac12}\ln(x))=1,\tag{6.16}$$

which implies the linear independence of (6.13) and (6.14).

Let us describe the basic families of closed realizations of Bessel operators. We will use two kinds of definitions of their domains. In what follows, ξ is a smooth compactly supported function equal to 1 near $x=0$.

We have the family of realizations with pure boundary conditions defined for $\mathrm{Re}(m)>-1$:

$$\mathcal{D}(H_m^0):=\mathcal{D}(L_{m^2}^0)+\mathbb{C}x^{\frac12+m}\xi(x)\tag{6.17}$$

$$=\Big\{f\in\mathcal{D}(L_{m^2}^{0,\max})\mid \mathscr{W}(x^{\frac12+m},f;0)=0\Big\},\tag{6.18}$$

$$H_m^0:=L_{m^2}^0\big|_{\mathcal{D}(H_m^0)}.\tag{6.19}$$

We have also two families with mixed boundary conditions: The first is the generic family defined for $-1<\mathrm{Re}(m)<1$, $m\neq 0$, $\kappa\in\mathbb{C}\cup\{\infty\}$:

$$\mathcal{D}(H_{m,\kappa}^0):=\mathcal{D}(L_{m^2}^{0,\min})+\mathbb{C}\big(x^{\frac12+m}+\kappa x^{\frac12-m}\big)\xi(x)\tag{6.20}$$

$$=\Big\{f\in\mathcal{D}(L_{m^2}^{0,\max})\mid \mathscr{W}\big(x^{\frac12+m}+\kappa x^{\frac12-m},f;0\big)=0\Big\},\tag{6.21}$$

$$\mathcal{D}(H_{m,\infty}^0):=\mathcal{D}(H_{-m}^0),\quad H_{m,\kappa}^0:=L_{m^2}^0\big|_{\mathcal{D}(H_{m,\kappa}^0)}.\tag{6.22}$$

The second family corresponds to $m=0$ and depends on $\nu\in\mathbb{C}\cup\{\infty\}$:

$$\mathcal{D}(H_0^{0,\nu}):=\mathcal{D}(L_{m^2}^{0,\min})+\mathbb{C}\big(x^{\frac12}\ln(x)+\nu x^{\frac12}\big)\xi(x)\tag{6.23}$$

$$=\Big\{f\in\mathcal{D}(L_0^{0,\max})\mid \mathscr{W}\big(\nu x^{\frac12}+x^{\frac12}\ln(x),f;0\big)=0\Big\},\tag{6.24}$$

$$\mathcal{D}(H_0^{0,\infty}):=\mathcal{D}(H_0^0),\quad H_0^{0,\nu}:=L_0^0\big|_{\mathcal{D}(H_0^{0,\nu})}.\tag{6.25}$$

The families of closed operators defined in (6.19), (6.22) and (6.25) are clearly independent of the cutoff ξ. They are holomorphic with respect to the parameters m, κ, ν. They are situated between $L_{m^2}^{0,\min}$ and $L_{m^2}^{0,\max}$.

6.5. **Boundary functionals for perturbed Bessel operators.** In this subsection, as well as in Subsections 6.6, 6.7 and 6.8, we analyze boundary conditions near zero and the corresponding closed realizations of perturbed Bessel operators. For definiteness, throughout these four subsections we assume that $Q \in \mathscr{L}_0^{(\infty)}$.

It does not seem practical to define boundary conditions for perturbed Bessel operators analogously as in (6.19), (6.22) and (6.25). In fact, even after imposing stronger conditions on Q, such as in Proposition 3.7, it is not easy to describe explicitly sufficiently well the behavior of elements in $\mathcal{D}(L_{m^2}^{\max})$ near zero. In particular, conditions of Theorem 3.5 in general do not allow us to conclude that $x^{\frac{1}{2}\pm m}\xi(x)$ belongs to $\mathcal{D}(L_{m^2}^{\max})$ and $x^{\frac{1}{2}}\ln(x)\xi(x)$ to $\mathcal{D}(L_0^{\max})$.

Fortunately, we can use the method of (6.18), (6.21) and (6.24) involving the Wronskian at 0. The precise choice of a boundary functional is in general more complicated than in the unperturbed case, as we explain below.

Let us first describe some properties of the minimal domain. In the next proof, we denote by $G_\to = G_{m^2,\to}(-k^2)$ the forward Green's operator associated to L_{m^2} defined in (5.19).

Lemma 6.6.

(i) Let $0 \leq \mathrm{Re}(m) < 1$, $m \neq 0$, $Q \in \mathscr{L}_0^{(0)}$ *and* $h \in \mathcal{D}(L_{m^2}^{\min})$. *Then*

$$h(x) = o(x^{\frac{3}{2}}), \quad \partial_x h(x) = o(x^{\frac{1}{2}}). \tag{6.26}$$

(ii) Let $m = 0$, $Q \in \mathscr{L}_{0,\ln}^{(0)}$ *and* $h \in \mathcal{D}(L_{m^2}^{\min})$. *Then*

$$h(x) = o(x^{\frac{3}{2}}\ln(x)), \quad \partial_x h(x) = o(x^{\frac{1}{2}}\ln(x)). \tag{6.27}$$

Proof. Let $h \in \mathcal{D}(L_{m^2}^{\min})$ and $c > 0$. Let $\mathrm{Re}(k) \geq 0$ be such that $\mathscr{W}_m(k) \neq 0$ (k exists by Corollary 5.4). Using e.g. [13, Proposition 7.3] we know that there exists $f \in L^2]0, c[$ such that

$$h\big|_{]0,c[} = G_\to(-k^2)f\big|_{]0,c[}.$$

By e.g. [13, Proposition 7.5]

$$|G_\to(-k^2)f(x)| \leq \frac{1}{2}\big(|w_m(x,k)|\|u_m(\cdot,k)\|_x + |u_m(x,k)|\|w_m(\cdot,k)\|_x\big)\|f\|_x,$$

$$|\partial_x G_\to(-k^2)f(x)| \leq \frac{1}{2}\big(|\partial_x w_m(x,k)|\|u_m(\cdot,k)\|_x + |\partial_x u_m(x,k)|\|w_m(\cdot,k)\|_x\big)\|f\|_x,$$

where $\|f\|_x := (\int_0^x |f(y)|^2 \mathrm{d}y)^{\frac{1}{2}}$. By Proposition 4.12, for small x,

$$|w_m(x,k)| \lesssim x^{\frac{1}{2}-\mathrm{Re}(m)}, \quad |\partial_x w_m(x,k)| \lesssim x^{-\frac{1}{2}-\mathrm{Re}(m)}, \quad \text{if } m \neq 0,$$

$$|w_0(x,k)| \lesssim x^{\frac{1}{2}}|\ln(x)|, \quad |\partial_x w_m(x,k)| \lesssim x^{-\frac{1}{2}}|\ln(x)|, \quad \text{if } m = 0,$$

while, by Proposition 3.6,

$$|u_m(x,k)| \lesssim x^{\frac{1}{2}+\mathrm{Re}(m)}, \quad |\partial_x u_m(x,k)| \lesssim x^{-\frac{1}{2}+\mathrm{Re}(m)}.$$

This yields the estimates (6.26)–(6.27). □

Lemma 6.7.

(i) *Let* $0 \le \mathrm{Re}(m) < 1$, $m \ne 0$ *and* $Q \in \mathscr{L}_0^{(0)}$. *Let* $g \in AC^1]0,\infty[$ *be such that* $g(x) = o(x^{\frac{1}{2}+\mathrm{Re}(m)})$ *and* $\partial_x g(x) = o(x^{-\frac{1}{2}+\mathrm{Re}(m)})$. *Then*

$$\mathscr{W}(g,f;0) = 0 \quad \text{for all } f \in \mathcal{D}(L_{m^2}^{\max}).$$

(ii) *The same holds if* $m = 0$, $Q \in \mathscr{L}_{0,\ln}^{(0)}$ *and* $g \in AC^1]0,\infty[$ *satisfies* $g(x) = o(x^{\frac{1}{2}}|\ln(x)|^{-1})$ *and* $\partial_x g(x) = o(x^{-\frac{1}{2}}|\ln(x)|^{-1})$.

Proof. Fix any $k \in \mathbb{C}$. Every $f \in \mathcal{D}(L_{m^2}^{\max})$ can be written as $f = \xi f_0 + f_1$ where $f_0 \in \mathcal{N}(L_{m^2}^{\max} + k^2)$ and $f_1 \in \mathcal{D}(L_{m^2}^{\min})$. Now (i) follows by Lemma 6.6(i) and Proposition 3.10(i), and (ii) follows by Lemma 6.6(ii) and Proposition 3.10(ii). □

In what follows, we will denote by $\mathcal{B}_{m^2}$ the space of boundary functionals of L_{m^2} with a given perturbation Q. We will describe convenient bases of $\mathcal{B}_{m^2}$. In other words, we will find pairs of linearly independent functionals on $\mathcal{D}(L_{m^2}^{\max})$ that vanish on $\mathcal{D}(L_{m^2}^{\min})$.

Note that cases (i)(a) and (iii)(a) of the next theorem have quite weak assumptions on the perturbation, however their non-principal boundary functionals depend on an arbitrary parameter a.

Theorem 6.8.

(i) *Let* $0 < \mathrm{Re}(m) < 1$.

(a) *Assume* $Q \in \mathscr{L}_0^{(0)}$. *Let* $a > 0$ *be small enough as in Proposition 3.15. Then*

$$\mathscr{W}(u_{-m}^{\bowtie(a)}(\cdot,0),\cdot;0), \quad \mathscr{W}(x^{\frac{1}{2}+m},\cdot;0), \tag{6.28}$$

is a basis of $\mathcal{B}_{m^2}$.

(b) *Suppose that the assumption is strengthened to* $Q \in \mathscr{L}_\varepsilon^{(0)}$ *for some* $\varepsilon > 0$ *(but we drop the assumption on* a*). Let* n *be a non-negative integer such that* $\mathrm{Re}(m) \le \frac{(n+1)\varepsilon}{2}$. *Then*

$$\mathscr{W}(u_{-m}^{0[n]},\cdot;0), \quad \mathscr{W}(x^{\frac{1}{2}+m},\cdot;0) \tag{6.29}$$

is a basis of $\mathcal{B}_{m^2}$.

(c) *If we assume* $0 < \varepsilon$, $0 < \mathrm{Re}(m) \le \frac{\varepsilon}{2}$ *and* $Q \in \mathscr{L}_\varepsilon^{(0)}$, *then*

$$\mathscr{W}(x^{\frac{1}{2}-m},\cdot;0), \quad \mathscr{W}(x^{\frac{1}{2}+m},\cdot;0) \tag{6.30}$$

is a basis of $\mathcal{B}_{m^2}$.

(ii) *Let* $\mathrm{Re}(m) = 0$, $m \ne 0$. *Assume* $Q \in \mathscr{L}_0^{(0)}$. *Then*

$$\mathscr{W}(x^{\frac{1}{2}-m},\cdot;0), \quad \mathscr{W}(x^{\frac{1}{2}+m},\cdot;0) \tag{6.31}$$

is a basis of $\mathcal{B}_{m^2}$.

(iii) *Let* $m = 0$.

(a) *Assume* $Q \in \mathscr{L}_{0,\ln}^{(0)}$. *Let* $a > 0$ *be small enough as in Proposition 3.15. Then*

$$\mathscr{W}(p_0^{\triangle(a)},\cdot;0), \quad \mathscr{W}(u_0(\cdot,0),\cdot;0) \tag{6.32}$$

is a basis of $\mathcal{B}_0$.

(b) *Suppose that the assumption on Q is strengthened to $Q \in \mathscr{L}^{(0)}_{0,\ln^2}$ (but we drop the condition on a). Then*

$$\mathscr{W}(p_0, \cdot; 0), \quad \mathscr{W}(x^{\frac12}, \cdot; 0) \tag{6.33}$$

is a basis of $\mathcal{B}_0$.

(c) *If the assumption is further strengthened to $Q \in \mathscr{L}^{(0)}_\varepsilon$ for some $\varepsilon > 0$, then*

$$\mathscr{W}(x^{\frac12}\ln(x), \cdot; 0), \quad \mathscr{W}(x^{\frac12}, \cdot; 0) \tag{6.34}$$

is a basis of $\mathcal{B}_0$.

Proof. (i) Recall that in Propositions 3.6 and 3.15 we introduced the functions $u_m(\cdot, 0), u^{\bowtie(a)}_{-m}(\cdot, 0)$ spanning $\mathcal{N}(L_{m^2})$. Therefore, by (6.8),

$$\mathscr{W}(u^{\bowtie(a)}_{-m}(\cdot, 0), \cdot; 0), \quad \mathscr{W}(u_m(\cdot, 0), \cdot; 0) \tag{6.35}$$

is a basis of $\mathcal{B}_{m^2}$. Using (3.18)–(3.19) and Lemma 6.7(i) we see that

$$\mathscr{W}(u_m(\cdot, 0), \cdot; 0) = \frac{1}{\Gamma(m+1)}\mathscr{W}(x^{\frac12+m}, \cdot; 0). \tag{6.36}$$

Therefore, we can replace $u_m(\cdot, 0)$ by $x^{\frac12+m}$, obtaining the basis (6.28).

Assume now that $Q \in \mathscr{L}^{(0)}_\varepsilon$ for some $\varepsilon > 0$ and suppose that n is a positive integer, $0 < \varepsilon < \frac{2}{n+1}$, $0 < \mathrm{Re}(m) \le \frac{\varepsilon(n+1)}{2}$ and $Q \in \mathscr{L}^{(0)}_\varepsilon$. By Proposition 3.17, we have

$$\mathscr{W}(u^{\bowtie(a)}_{-m}(\cdot, 0), \cdot; 0) = \mathscr{W}(u^{0[n]}_{-m}, \cdot; 0) + c^{(a)[n]}_m \mathscr{W}(u_m(\cdot, 0), \cdot; 0),$$

for some constant $c^{(a)[n]}_m$ depending on the parameters. Therefore, (6.29) is also a basis of $\mathcal{B}_{m^2}$.

If $2\mathrm{Re}(m) \le \varepsilon$, then we can take $n = 0$:

$$u^{0[0]}_{-m}(\cdot, 0) = u^0_{-m}(\cdot, 0) = \frac{x^{\frac12-m}}{\Gamma(1-m)} \tag{6.37}$$

Then we can replace $\mathscr{W}(u^{0[0]}_{-m}(\cdot, 0), \cdot; 0)$ with $\mathscr{W}(x^{\frac12-m}, \cdot; 0)$ obtaing the basis (6.30).

(ii) By Proposition 3.6, both $u_m(\cdot, 0)$ and $u_{-m}(\cdot, 0)$ are well defined and span $\mathcal{N}(L_{m^2})$. By (6.8), we obtain a basis of $\mathcal{B}_{m^2}$

$$\mathscr{W}(u_m(\cdot, 0), \cdot; 0), \quad \mathscr{W}(u_{-m}(\cdot, 0), \cdot; 0).$$

Now (6.36) is still valid so that we can replace $u_{\pm m}(\cdot, 0)$ by $x^{\frac12\pm m}$, obtaining the basis (6.31)

(iii) In Propositions 3.6 and 3.25 we constructed functions $u_0(\cdot, 0)$ and $p^{\triangle(a)}_0(\cdot, 0)$ spanning $\mathcal{N}(L_0)$. It follows from (6.8) that (6.32) is a basis of $\mathcal{B}_0$.

If we strengthen the assumption to $Q \in \mathscr{L}^{(0)}_{0,\ln^2}$, then in Proposition 3.7 we defined $p_0(\cdot, 0) \in \mathcal{N}(L_0)$. The functions $p_0(\cdot, 0)$ and $u_0(\cdot, 0)$ span $\mathcal{N}(L_0)$. Therefore, by (6.8),

$$\mathscr{W}(u_0(\cdot, 0), \cdot; 0), \quad \mathscr{W}(p_0(\cdot, 0), \cdot; 0).$$

is a basis of $\mathcal{B}_0$. Besides, by Proposition 3.9 and Lemma 6.7(ii) we have

$$\mathscr{W}(u_0(\cdot, 0), \cdot; 0) = \mathscr{W}(x^{\frac12}, \cdot; 0).$$

Therefore, (6.33) is a basis of $\mathcal{B}_0$.

If the assumption is further strengthened to $Q \in \mathscr{L}_\varepsilon^{(0)}$ with $\varepsilon > 0$, then by Proposition 3.7 and Lemma 6.7(ii) we have

$$\mathscr{W}(p_0, \cdot; 0) = \mathscr{W}(x^{\frac{1}{2}}\ln(x), \cdot; 0).$$

Therefore, (6.34) is a basis of $\mathcal{B}_0$. □

Remark 6.9. *Let $Q(x) = -\frac{\beta}{x}\mathbb{1}_{]0,1]}(x)$ as in Remark 3.21. Taking $n = 1$ in the previous theorem, it follows from that remark that for $0 < \mathrm{Re}(m) < 1$,*

$$\mathscr{W}\Big(x^{\frac{1}{2}-m}\Big(1 - \frac{\beta x}{1-2m}\Big), \cdot; 0\Big), \quad \mathscr{W}(x^{\frac{1}{2}+m}, \cdot; 0)$$

forms a basis of $\mathcal{B}_{m^2}$.

The next table summarizes the bases of $\mathcal{B}_{m^2}$ that we constructed, depending on the values of $-1 < \mathrm{Re}(m) < 1$ and on the conditions on Q.

	$-1 < \mathrm{Re}(m) < 0$	$\mathrm{Re}(m) = 0$		$0 < \mathrm{Re}(m) < 1$
		$m \neq 0$	$m = 0$	
$Q \in \mathscr{L}_0^{(0)}$	$x^{\frac{1}{2}-m}$, $u_m^{\bowtie(a)}$	$x^{\frac{1}{2}-m}$, $x^{\frac{1}{2}+m}$	?	$u_{-m}^{\bowtie(a)}$, $x^{\frac{1}{2}+m}$
$Q \in \mathscr{L}_{0,\ln}^{(0)}$	$x^{\frac{1}{2}-m}$, $u_m^{\bowtie(a)}$	$x^{\frac{1}{2}-m}$, $x^{\frac{1}{2}+m}$	$p_0^{\triangle(a)}$, $x^{\frac{1}{2}}$	$u_{-m}^{\bowtie(a)}$, $x^{\frac{1}{2}+m}$
$Q \in \mathscr{L}_{0,\ln^2}^{(0)}$	$x^{\frac{1}{2}-m}$, $u_m^{\bowtie(a)}$	$x^{\frac{1}{2}-m}$, $x^{\frac{1}{2}+m}$	p_0, $x^{\frac{1}{2}}$	$u_{-m}^{\bowtie(a)}$, $x^{\frac{1}{2}+m}$
$Q \in \mathscr{L}_\varepsilon^{(0)}$	$-\frac{(j+1)\varepsilon}{2} \leq \mathrm{Re}(m) \leq 0$	$x^{\frac{1}{2}-m}$, $x^{\frac{1}{2}+m}$	$x^{\frac{1}{2}}\ln(x)$, $x^{\frac{1}{2}}$	$0 \leq \mathrm{Re}(m) \leq \frac{(j+1)\varepsilon}{2}$
	$x^{\frac{1}{2}-m}$, $u_m^{0[j]}$			$u_{-m}^{0[j]}$, $x^{\frac{1}{2}+m}$

Table 3. *Bases of the boundary space $\mathcal{B}_{m^2}$ of L_{m^2}.* In each case, we write g_1, g_2 if $\mathscr{W}(g_1, \cdot; 0)$, $\mathscr{W}(g_2, \cdot; 0)$ is a basis of $\mathcal{B}_{m^2}$. To shorten notations, we write $u_m^{\bowtie(a)}$ for $u_m^{\bowtie(a)}(\cdot, 0)$ and likewise for other functions. Note that $u_m^{\bowtie(a)}$ and $p_0^{\triangle(a)}$ depend on an arbitrary parameter a. Each line corresponds to a condition on Q, from the minimal one $Q \in \mathscr{L}_0^{(0)}$ to the strongest one $Q \in \mathscr{L}_\varepsilon^{(0)}$, with $\varepsilon > 0$ (beside the condition $Q \in \mathscr{L}_0^{(\infty)}$ which we everywhere assume). In the last line, $j \in \{0, \dots, n\}$, where n is the smallest nonnegative integer such that $\frac{(n+1)\varepsilon}{2} \geq 1$.

As mentioned above, in the last line, for $j = 0$, $u_m^{0[0]}$ can be replaced by $x^{\frac{1}{2}+m}$ and $u_{-m}^{0[0]}$ can be replaced by $x^{\frac{1}{2}-m}$. For growing values or $\mathrm{Re}(m) > 0$, the picture is then that, to pass from the region $R_j := \{\frac{j\varepsilon}{2} < \mathrm{Re}(m) \leq \frac{(j+1)\varepsilon}{2}\}$ to R_{j+1}, one needs to add a further term to $u_{-m}^{0[j]}$ in order to still have an element of $\mathcal{B}_{m^2}$. Of course, we could also use $u_{-m}^{0[n]}$ in the whole region $0 < \mathrm{Re}(m) < 1$ for any n such that $\frac{(n+1)\varepsilon}{2} \geq 1$, but then all the terms of order $o(x^{\frac{1}{2}+m})$ in $u_{-m}^{0[n]}$ are irrelevant.

6.6. **The perturbed Bessel operator with pure boundary conditions.** In this subsection we introduce the most natural family of perturbed Bessel operators. It is parallel to what in the unperturbed case was called the family of homogeneous Bessel operators. (In the perturbed case the homogeneity is no longer true, therefore the name has to be changed).

Let $m \in \mathbb{C}$, $-1 < \mathrm{Re}(m)$. We assume that

$$\begin{aligned} &Q \in \mathscr{L}_{\varepsilon}^{(0)}, \quad m \neq 0, \quad \varepsilon = \max\big(0, -2\mathrm{Re}(m)\big); \\ &Q \in \mathscr{L}_{0,\ln^2}^{(0)}, \quad m = 0. \end{aligned}$$

We can then define

$$\begin{aligned} \mathcal{D}(H_m) &:= \Big\{ f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathscr{W}(x^{\frac12+m}, f; 0) = 0 \Big\}, \\ H_m &:= L_{m^2}\big|_{\mathcal{D}(H_m)}, \end{aligned}$$

which we will call the *perturbed Bessel operator with pure boundary conditions.*

Using Theorem 6.8 we see that the operator H_m is closed,

$$\begin{aligned} L_{m^2}^{\min} = H_m = L_{m^2}^{\max}, &\quad \mathrm{Re}(m) \geq 1; \\ L_{m^2}^{\min} \subset H_m \subset L_{m^2}^{\max}, &\quad -1 < \mathrm{Re}(m) < 1; \end{aligned}$$

and both inclusions are of codimension 1.

Proposition 6.10. *Suppose that the assumptions on Q stated at the beginning of this subsection hold. Let $\mathrm{Re}(k) > 0$. Then $k^2 \notin \sigma(H_m)$ if and only if $\mathscr{W}_m(k) \neq 0$. Moreover, the operator $G_{m,\bowtie}(-k^2)$ defined in (5.21) is then bounded and*

$$G_{m,\bowtie}(-k^2) = (k^2 + H_m)^{-1}.$$

Proof. We use [13, Proposition 7.7] together with the asymptotic behavior near 0 of $u_m(x,k)$ established in Propositions 3.6. □

In the following theorem we fix a perturbation Q and consider the operator valued family H_m. In the definition of regularity we use the concept of a holomorphic family of closed operators recalled in Appendix B. Moreover, the continuity of a family of closed operators should be understood in the weak resolvent sense.

Theorem 6.11.

(i) Let $2 > \varepsilon > 0$ and suppose that $Q \in \mathscr{L}_{\varepsilon}^{(0)}$. Then

$$\Big\{ -\frac{\varepsilon}{2} \leq \mathrm{Re}(m) \Big\} \ni m \mapsto H_m \tag{6.38}$$

is regular.

(ii) Let $Q \in \mathscr{L}_{0}^{(0)}$. Then

$$\Big\{ \mathrm{Re}(m) \geq 0,\, m \neq 0 \Big\} \ni m \mapsto H_m, \tag{6.39}$$

is regular. If we strengthen the assumption to $Q \in \mathscr{L}_{0,\ln^2}^{(0)}$, then we can include $m = 0$ in (6.39).

Proof. In view of Proposition 6.10, we can proceed as in the proof of Theorem 3.10 in [12]: It suffices to use Propositions 3.8, 4.6 and 5.1. □

Note that for $\mathrm{Re}(m) \geq 1$ the operator H_m is the unique closed realization of L_{m^2}. Theorem 6.11 shows that the holomorphic function

$$\{\mathrm{Re}(m) > 1\} \ni m \mapsto H_m$$

has an analytic continuation to a larger region, (6.38) or (6.39), where the width of the additional strip depends on the assumption on the potential.

6.7. **The perturbed Bessel operator with mixed boundary conditions I.** In this subsection we describe closed realizations of perturbed Bessel operators with mixed boundary conditions under sufficiently strong conditions on the perturbation, which guarantee that these realizations are very similar to the unperturbed case.

Let $m \neq 0$, $|\mathrm{Re}(m)| < 1$, $\varepsilon = 2|\mathrm{Re}(m)|$ and $Q \in \mathscr{L}_\varepsilon^{(0)}$. For $\kappa \in \mathbb{C} \cup \{\infty\}$ we set

$$\begin{aligned}
\mathcal{D}(H_{m,\kappa}) &:= \Big\{ f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathscr{W}\big(x^{\frac12+m} + \kappa x^{\frac12-m}, f; 0\big) = 0 \Big\}, \quad \kappa \in \mathbb{C},\\
\mathcal{D}(H_{m,\infty}) &:= \Big\{ f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathscr{W}\big(x^{\frac12-m}, f; 0\big) = 0 \Big\},\\
H_{m,\kappa} &:= L_{m^2}\big|_{\mathcal{D}(H_{m,\kappa})}.
\end{aligned}$$

If $m = 0$ we assume $Q \in \mathscr{L}_{0,\ln^2}^{(0)}$. For $\nu \in \mathbb{C} \cup \{\infty\}$ we set

$$\begin{aligned}
\mathcal{D}(H_0^\nu) &:= \Big\{ f \in \mathcal{D}(L_0^{\max}) \mid \mathscr{W}\big(\nu x^{\frac12} + p_0, f; 0\big) = 0 \Big\}, \quad \nu \in \mathbb{C},\\
\mathcal{D}(H_0^\infty) &:= \mathcal{D}(H_0),\\
H_0^\nu &:= L_0\big|_{\mathcal{D}(H_0^\nu)}.
\end{aligned}$$

Note that if $Q \in \mathscr{L}_\varepsilon^{(0)}$ with $\varepsilon > 0$, then

$$\mathcal{D}(H_0^\nu) := \Big\{ f \in \mathcal{D}(L_0^{\max}) \mid \mathscr{W}\big(\nu x^{\frac12} + x^{\frac12}\ln(x), f; 0\big) = 0 \Big\}.$$

Clearly, the operators $H_{m,\kappa}$, H_0^ν are closed,

$$L_{m^2}^{\min} \subset H_{m,\kappa} \subset L_{m^2}^{\max}, \tag{6.40}$$

$$L_0^{\min} \subset H_0^\nu \subset L_0^{\max}, \tag{6.41}$$

and both inclusions in (6.40) and (6.41) are of codimension 1.

One can compute the resolvents of $H_{m,\kappa}$ in the same way as for H_m.

Proposition 6.12. *Let* $\mathrm{Re}(k) > 0$.

(i) Let $m \neq 0$, $|\mathrm{Re}(m)| < 1$, $\varepsilon = 2|\mathrm{Re}(m)|$, $Q \in \mathscr{L}_\varepsilon^{(0)}$ *and* $\kappa \in \mathbb{C} \cup \{\infty\}$. *We have* $k^2 \notin \sigma(H_{m,\kappa})$ *if and only if* $\mathscr{W}_m(k) + \kappa \frac{\Gamma(1-m)}{\Gamma(1+m)} \frac{k^{2m}}{2^{2m}} \mathscr{W}_{-m}(k) \neq 0$. *Besides, the operator* $G_{m,\bowtie,\kappa}(-k^2)$ *defined in* (5.22) *is then bounded and*

$$G_{m,\bowtie,\kappa}(-k^2) = (k^2 + H_{m,\kappa})^{-1}.$$

(ii) Let $m = 0$, $Q \in \mathscr{L}_{0,\ln^2}^{(0)}$ *and* $\nu \in \mathbb{C} \cup \{\infty\}$. *We have* $k^2 \notin \sigma(H_m^\nu)$ *if and only if* $\mathscr{W}(w_0, \nu u_0 + p_0) \neq 0$. *Besides, the operator* $G_0^\nu(-k^2)$ *with kernel defined in* (5.24) *is then bounded and*

$$G_{0,\bowtie}^\nu(-k^2) = (k^2 + H_0^\nu)^{-1}.$$

Proof. The argument is the same as in the proof of Proposition 6.10. □

Let us fix a perturbation Q and consider the operator valued families with mixed boundary conditions.

Theorem 6.13.

(i) Let $2 > \varepsilon > 0$ and $Q \in \mathscr{L}_\varepsilon^{(0)}$. Then

$$\Big\{(m,\kappa) \mid |\mathrm{Re}(m)| \le \frac{\varepsilon}{2},\ \kappa \in \mathbb{C} \cup \{\infty\},\ (m,\kappa) \neq (0,-1)\Big\} \ni (m,\kappa) \mapsto H_{m,\kappa} \tag{6.42}$$

is regular. Besides, $H_{m,\kappa} = H_{-m,\kappa^{-1}}$.

(ii) Let $m = 0$. Suppose that $Q \in \mathscr{L}_{0,\ln^2}^{(0)}$. Then

$$\mathbb{C} \cup \{\infty\} \ni \nu \mapsto H_0^\nu, \tag{6.43}$$

is analytic.

Proof. This follows as in Theorem 6.11. □

Remark 6.14. *Proposition 3.11(ii) in [12] (in the case $Q = 0$) shows that $(m,\kappa) \mapsto H_{m,\kappa}$ cannot be extended by continuity at $(0,-1)$.*

6.8. **The perturbed Bessel operator with mixed boundary conditions II.** As discussed in Theorem 6.8 we can define closed realizations of L_{m^2} under much weaker conditions on Q than those in the previous subsection. Let us choose a nonnegative integer n. A natural method to describe them is by using the boundary functionals defined by $u_{-m}^{0[n]}$.

Let $0 \le \mathrm{Re}(m) < 1$, $m \neq 0$, $\varepsilon = \frac{2\mathrm{Re}(m)}{n+1}$ and $Q \in \mathscr{L}_\varepsilon^{(0)}$. For $\kappa \in \mathbb{C} \cup \{\infty\}$ we set

$$\mathcal{D}(H_{-m,\kappa}^{[n]}) := \Big\{ f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathscr{W}\Big(u_{-m}^{0[n]} + \kappa \frac{\Gamma(1-m)}{\Gamma(1+m)} x^{\frac{1}{2}+m}, f; 0\Big) = 0 \Big\}, \quad \kappa \in \mathbb{C},$$

$$\mathcal{D}(H_{-m,\infty}^{[n]}) := \Big\{ f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathscr{W}\big(x^{\frac{1}{2}+m}, f; 0\big) = 0 \Big\},$$

$$H_{-m,\kappa}^{[n]} := L_{m^2}\big|_{\mathcal{D}(H_{-m,\kappa})}.$$

Clearly, the operators $H_{-m,\kappa}^{[n]}$ are closed,

$$L_{m^2}^{\min} \subset H_{-m,\kappa}^{[n]} \subset L_{m^2}^{\max} \tag{6.44}$$

and both inclusions in (6.44) are of codimension 1.

Note that in the particular case $n = 0$ we have $H_{-m,\kappa}^{[0]} = H_{-m,\kappa}$.

Proposition 6.15. *Let $0 \le \mathrm{Re}(m) < 1$, $m \neq 0$, $\varepsilon = \frac{2\mathrm{Re}(m)}{n+1}$ and $Q \in \mathscr{L}_\varepsilon^{(0)}$. We have $k^2 \notin \sigma(H_{-m,\kappa}^{[n]})$ if and only if $\mathscr{W}(v_m(\cdot,k), u_{-m}^{[n]}(\cdot,k)) + \kappa \frac{\Gamma(1-m)}{\Gamma(1+m)} \mathscr{W}_m(k) \neq 0$. Besides, the operator $G_{m,\bowtie,\kappa}^{[n]}(-k^2)$ defined in (5.25) is then bounded and*

$$G_{m,\bowtie,\kappa}^{[n]}(-k^2) = (k^2 + H_{-m,\kappa}^{[n]})^{-1}.$$

Proof. The argument is the same as in the proof of Proposition 6.10. □

The following theorem can be compared with Theorem 6.13:

Theorem 6.16. *Let $1 > \frac{\varepsilon(n+1)}{2} > 0$ and $Q \in \mathscr{L}_\varepsilon^{(0)}$. Then*

$$\Big\{ -\frac{\varepsilon(n+1)}{2} \ge -\mathrm{Re}(m) \ge 0, m \neq 0 \Big\} \times (\mathbb{C} \cup \{\infty\}) \ni (-m,\kappa) \mapsto H_{-m,\kappa}^{[n]} \tag{6.45}$$

is regular.

6.9. **Scattering length.** Suppose that

$$Q \in \mathscr{L}_{\delta}^{(\infty)}, \quad \text{if } 0 \le \mathrm{Re}(m), \quad m \neq 0, \quad \delta = 1 + 2\mathrm{Re}(m);$$
$$Q \in \mathscr{L}_{1,\ln^2}^{(\infty)}, \quad \text{if } m = 0.$$

(Note that we do not impose conditions on Q near 0 apart from the usual local integrability) By Propositions 4.8 and 4.10, under these assumptions the space $\mathcal{N}(L_{m^2})$ possesses a distinguished basis

$$q_{-m}, \quad q_m, \quad m \neq 0;$$
$$q_0, \quad q_{0,\ln}, \quad m = 0,$$

Therefore, the boundary space $\mathcal{B}_{m^2}$ has a basis

$$\mathscr{W}(q_{-m}, \cdot; 0), \quad \mathscr{W}(q_m, \cdot; 0), \quad m \neq 0; \tag{6.46}$$
$$\mathscr{W}(q_0, \cdot; 0), \quad \mathscr{W}(q_{0,\ln}, \cdot; 0), \quad m = 0. \tag{6.47}$$

Suppose that $H_\bullet$ is one of the realizations of the Bessel operator such that

$$L_{m^2}^{\min} \subsetneq H_\bullet \subsetneq L_{m^2}^{\max}.$$

As we discussed above, to define $H_\bullet$ we need to fix a non-zero boundary functional. So far, we tried to express boundary functionals in terms of the asymptotics of functions near zero, as in Subsection 6.5.

In quantum mechanics one often prefers to describe realizations of perturbed Bessel operators using (6.46) and (6.47). We say that the *scattering length* of $H_\bullet$ is $a \in \mathbb{C}$ if

$$\mathcal{D}(H_\bullet) = \{f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathscr{W}(q_m - a q_{-m}, f; 0) = 0\}, \quad m \neq 0; \tag{6.48}$$
$$\mathcal{D}(H_\bullet) = \{f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathscr{W}(q_{0,\ln} - a q_0, f; 0) = 0\}, \quad m = 0. \tag{6.49}$$

If

$$\mathcal{D}(H_\bullet) = \{f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathscr{W}(q_{-m}, f; 0) = 0\}, \quad m \neq 0; \tag{6.50}$$
$$\mathcal{D}(H_\bullet) = \{f \in \mathcal{D}(L_{m^2}^{\max}) \mid \mathscr{W}(q_0, f; 0) = 0\}, \quad m = 0, \tag{6.51}$$

then we say that the scattering length of $H_\bullet$ is $a = \infty$.

Appendix A. Integral operators

In this appendix we recall a well-known property of Volterra operators that was used in Section 3. We start with the following easy lemma:

Lemma A.1. *Suppose that $(x, y) \mapsto K(x, y)$ is a measurable function on $]0, \infty[\times]0, \infty[$ such that*

$$\sup_x \int_0^\infty |K(x, y)| \mathrm{d}y =: C < \infty.$$

Then the operator K defined by

$$Kf(x) := \int_0^\infty K(x, y) f(y) \mathrm{d}y$$

is bounded on $L^\infty]0, \infty[$ and $\|K\| \le C$.

Given an integral operator K as in the previous lemma and $a > 0$, the operator $K^{(a)}$ is defined as an operator on $L^\infty]0,a[$ by the kernel

$$K^{(a)}(x,y) := \theta(a-x)\theta(a-y)K(x,y).$$

The operator $K^{(a)}$ is called *the compression of* K *to* $]0,a[$.

We will say that the operator K with the kernel $K(x,y)$ is a *forward*, respectively *backward Volterra operator* if $K(x,y) = 0$ for $x < y$, respectively $x > y$. The following proposition can be proven by an induction argument.

Proposition A.2. *Suppose that* K *is a forward Volterra operator and*

$$\int_0^\infty |K(x,y)|\mathrm{d}y =: C(x) < \infty.$$

Then for any $a > 0$, $K^{(a)}$ *is bounded and* $\mathbb{1}^{(a)} + K^{(a)}$ *is invertible on* $L^\infty]0,a[$. *Besides, for all* $x > 0$, $n \in \mathbb{N}$,

$$|(K^n f)(x)| \leq \frac{1}{n!}C(x)^n \operatorname*{ess\,sup}_{y<x} |f(y)|.$$

so that for $f \in L^\infty]0,a[$ *the series*

$$(\mathbb{1} + K)^{-1} f(x) = \sum_{n=0}^{\infty} (-K)^n f(x)$$

is convergent.

Appendix B. Holomorphic families of closed operators

In this appendix we recall the concept of a holomorphic family of operators on a complex Banach space $\mathcal{H}$.

Let Θ be an open subset of $\mathbb{C}^d$. We say that a family $\{B(z)\}_{z\in\Theta}$ of bounded operators on $\mathcal{H}$ is a *holomorphic family of bounded operators* if for any $f, g \in \mathcal{H}$

$$\Theta \ni z \mapsto (f|B(z)g) \tag{B.1}$$

is holomorphic. Note that this is equivalent to a weaker condition: $\{B(z)\}_{z\in\Theta}$ is locally bounded on Θ and there exists a dense subspace $\mathcal{D} \subset \mathcal{H}$ such that, for all $f, g \in \mathcal{D}$, the map (B.1) is holomorphic.

One can also introduce another concept: that of *holomorphic families of closed operators*. We will not give here its general definition, which can be found e.g. in [7, 18, 26] and will not be used here. We will restrict ourselves to defining this concept for families that have nonempty resolvent set.

More precisely, suppose that $\{H(z)\}_{z\in\Theta}$ is a function with values in closed operators on $\mathcal{H}$. Suppose that for any $z_0 \in \Theta$, there exist $\lambda \in \mathbb{C}$ and a neighborhood $\Theta_0 \subset \Theta$ of z_0 such that, for all $z \in \Theta_0$, λ is in the resolvent set of $H(z)$. Then we say that $\{H(z)\}_{z\in\Theta}$ is holomorphic if for all such Θ_0 the map $\Theta_0 \ni z \mapsto (H(z) - \lambda)^{-1} \in \mathcal{L}(\mathcal{H})$ is holomorphic as a family of bounded operators.

Appendix C. Technical lemma

The following easy lemma was used several times in the main part of the manuscript.

Lemma C.1. *Let $a > 0$ or $a = \infty$. Let $f \in L^1_{\mathrm{loc}}]0, a[$ and $h :]0, a[\to \mathbb{R}$ be a positive increasing function such that $h(x) \to 0$ as $x \to 0$ and*

$$\int_0^a h(x)|f(x)|\mathrm{d}x < \infty.$$

Then

$$\int_x^a f(y)\mathrm{d}y = o\big(h(x)^{-1}\big), \quad x \to 0.$$

Proof. Let

$$w(x, y) := h(x)|f(y)|\mathbb{1}_{[x,a[}(y).$$

Clearly, for a.e. $y \in]0, a[$, $w(x, y) \to 0$ as $x \to 0$. Moreover, since h is increasing,

$$w(x, y) \le h(y)|f(y)|,$$

for all $x, y \in]0, a[$. Since $y \mapsto h(y)|f(y)|$ is integrable on $]0, a[$ by assumption, the dominated convergence theorem implies that

$$\int_0^a w(x, y)\mathrm{d}y \to 0, \quad x \to 0.$$

This proves the lemma. □

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