SOME REMARKS ON CALABI-YAU AND HYPER-KÄHLER FOLIATIONS

GEORGES HABIB AND LUIGI VEZZONI

Abstract. We study Riemannian foliations whose transverse Levi-Civita connection ∇ has special holonomy. In particular, we focus on the case where $\mathrm{Hol}(\nabla)$ is contained either in $\mathrm{SU}(n)$ or in $\mathrm{Sp}(n)$. We prove a Weitzenböck formula involving complex basic forms on Kähler foliations and we apply this formula for pointing out some properties of transverse Calabi-Yau structures. This allows us to prove that links provide examples of compact simply-connected contact Calabi-Yau manifolds. Moreover, we show that a simply-connected compact manifold with a Kähler foliation admits a transverse hyper-Kähler structure if and only if it admits a compatible transverse hyper-Hermitian structure. This latter result is the "foliated version" of a theorem proved by Verbitsky in [49]. In the last part of the paper we adapt our results to the Sasakian case, showing in addition that a compact Sasakian manifold has trivial transverse holonomy if and only if it is a compact quotient of the Heisenberg Lie group.

Contents

1. Introduction

Riemannian foliations were introduced by B. Reinhart in [40] and are a natural generalization of Riemannian submersions. Roughly speaking, a *Riemannian foliation* on a manifold M is a decomposition of M into submanifolds given by local Riemannian submersions to a base Riemannian manifold T whose metric is invariant by the transition maps. Riemannian foliations are characterized by the existence of a Riemannian metric on the whole manifold whose restriction to the normal bundle depends only on the transverse variables of a local chart. One of the basic tool for studying the geometry of Riemannian foliations is the holonomy group of the so-called *transverse Levi-Civita connection* ∇. This connection is defined as the pull-back of the Levi-Civita connection of the base manifold by the local submersions. Many additional structures on a foliated manifold can be described in terms of the holonomy group of ∇. For instance, *transverse Kähler structures* are defined as Riemannian foliations having the holonomy

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group of ∇ contained in $\mathrm{U}(n)$, where $q = 2n$ is the codimension of the foliation. Kähler foliations play an important role in many different geometrical contexts: for instance, Sasakian structures and Vaisman metrics induce a Kähler foliation.

In this paper, we investigate the geometry of Riemannian foliations having special transverse holonomy. In particular, we focus on the case of a foliated manifold having either $\mathrm{Hol}(\nabla) \subseteq \mathrm{SU}(n)$ or $\mathrm{Hol}(\nabla) \subseteq \mathrm{Sp}(n)$. The case $\mathrm{Hol}(\nabla) \subseteq \mathrm{SU}(n)$ corresponds to the geometry of *Calabi-Yau* foliations, while $\mathrm{Hol}(\nabla) \subseteq \mathrm{Sp}(n)$ to *hyper-Kähler* foliations. Besides other reasons, our study is motivated by the El Kacimi paper [13] containing the foliated version of the Calabi-Yau theorem. Examples of Calabi-Yau foliations are provided by submersions over Calabi-Yau manifolds, desingularizations of Calabi-Yau orbifolds and contact Calabi-Yau structures (see [46]); while examples of hyper-Kähler foliations can be obtained by considering submersions over hyper-Kähler manifolds, desingularizations of hyper-Kähler orbifolds, 3-cosymplectic structures (see e.g. [5, Section 13.1]) and by the connected sum of some copies of $S^2 \times S^3$ (see [6, 10] and the last paragraph of the present paper).

As a first result of the paper, we provide a Weitzenböck formula for Kähler foliations (see theorem 3.1 in section 3). This formula allows us to establish some analogies between foliated Calabi-Yau manifolds and classical Calabi-Yau manifolds (see section 4). As main result about hyper-Kähler foliations, we prove that every simply-connected compact manifold carrying a Kähler foliation and a compatible transverse hyper-complex structure actually admits a hyper-Kähler foliation. That is the foliated version of a theorem of Verbitsky (see [49]). A key ingredient in the proof of this last result is the existence and uniqueness of a special connection having skew-symmetric transverse torsion on every manifold carrying a Hermitian foliation. The existence of this connection in contact metric manifolds was showed by Friedrich and Ivanov in [19]. In the last part of the paper we consider Sasakian manifolds. We prove that a transversally flat compact Sasakian manifold is always a compact quotient of the Heisenberg group (see theorem 6.2); we point out that some links provide examples of compact simply-connected contact Calabi-Yau manifolds and we adapt some results proved in the first part to the Sasakian case.

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2. Preliminaries

In this section, we recall some basic materials about foliations, transverse structures and basic cohomology; we refer to [47, 34, 5] and the references therein for detailed expositions about these topics.

2.1. **Transverse structures on foliations.** Let M be a smooth manifold. A foliation $\mathcal{F}$ on M of codimension q can be defined as an open cover $\{U_k\}$ of M together with a family of submersions $f_k\colon U_k \to T$ over a manifold T of dimension q (called the base of the foliation) such that whenever $U_j \cap U_k \neq \emptyset$ there exists a diffeomorphism $\gamma_{jk}\colon f_j(U_j \cap U_k) \to f_k(U_j \cap U_k)$ with

$$f_j = \gamma_{jk} \circ f_k\,.$$

The basic examples of foliations are provided by global submersions. A *transverse structure* on a foliated manifold $(M, \mathcal{F})$ is by definition a geometric structure on the base manifold T which is invariant by the transition maps γ_{jk}. A foliation is called *Riemannian* if it admits a transverse Riemannian metric g_Q. In contrast to the non-foliated case the existence of a transverse metric is not always guaranteed. Given a foliated manifold $(M, \mathcal{F})$, we denote by L the subbundle of TM induced by $\mathcal{F}$; by Q the normal bundle TM/L; by $\pi\colon TM \to Q$ the natural projection and for a section X of TM we usually set $X_Q := \pi(X)$. A transverse metric on $\mathcal{F}$ induces a metric g_Q along the fibers of Q which is always *holonomy invariant*, meaning

$$\mathcal{L}_X g_Q = 0 \tag{1}$$

for every $X \in \Gamma(L)$, where $\mathcal{L}$ denotes the Lie derivative. By using the projection π we can regard g_Q as a symmetric tensor on M which can be always "completed" to a global metric g on M. This means that there exists a Riemannian metric g on M such that

$$g(X,Y) = g_Q(X_Q, Y_Q)$$

for every sections X, Y of $L^\perp$. Such a global metric g can be explicitly defined by starting from an arbitrary Riemannian metric g' of M and by setting

$$g(X,Y) := g'(X_L, Y_L) + g_Q(X_Q, Y_Q)$$

for every $X, Y \in \Gamma(TM)$, where the subscript L here denotes the projection onto $\Gamma(L)$ with respect to g'. On the other hand, given a foliated Riemannian manifold $(M, \mathcal{F}, g)$, the metric g always induces a metric g_Q along the fibers of $Q := TM/L \simeq L^\perp$. This g_Q makes $\mathcal{F}$ a Riemannian foliation if and only if it satisfies the *holonomy invariance* condition (1). In this case g is called a *bundle-like* metric.

Given a Riemannian foliation $(\mathcal{F}, g_Q)$ on a manifold M, some special transverse structures can be characterized in terms of the holonomy of the so-called *transversal Levi-Civita connection*. This connection is defined as the unique connection ∇ on Q preserving g_Q and having vanishing transverse torsion, i.e.

$$\nabla_X Y_Q - \nabla_X Y_Q = [X,Y]_Q$$

for all $X, Y \in \Gamma(TM)$. The connection ∇ can be defined in an explicit way in terms of the Levi-Civita connection ∇^g of a bundle-like metric g inducing g_Q by setting

$$\nabla_X s = \begin{cases} [X, \sigma(s)]_Q & \text{if} \quad X \in \Gamma(L), \\ (\nabla^g_X \sigma(s))_Q & \text{if} \quad X \in \sigma(Q), \end{cases} \tag{2}$$

for every $s \in \Gamma(Q)$, where $\sigma\colon \Gamma(Q) \to \Gamma(L^\perp)$ is the natural isomorphism (see e.g. [47]). This last description of ∇ does not depend on the choice of the metric g. The curvature R^∇ of ∇ vanishes along the leaves of $\mathcal{F}$ (see [47]): that is, $X \lrcorner R^\nabla = 0$ for all $X \in \Gamma(L)$. That allows us to define the *transverse Ricci tensor* as

$$\mathrm{Ric}^\nabla(s_1, s_2) = \sum_{k=1}^{q} g_Q(R^\nabla(\tilde{e}_k, \tilde{s}_1)s_2, e_k)$$

for every $s_1, s_2 \in \Gamma(Q)$, where $\{e_k\}$ is an arbitrary orthonormal frame of $\Gamma(Q)$ and $\tilde{e}_k$ and $\tilde{s}_1$ are arbitrary vector fields on M projecting onto e_k and s_1, respectively.

Now we recall the definition of the *basic cohomology complex.* A differential form α on a foliated manifold $(M, \mathcal{F})$ is called *basic* if it is constant along the leaves of $\mathcal{F}$, i.e. if it satisfies

$$X \lrcorner \alpha = 0\,, \quad X \lrcorner d\alpha = 0\,,$$

where $X \lrcorner$ denotes the contraction along $X \in \Gamma(L)$. It is straightforward to see that the exterior derivative d preserves the set of basic forms $\Lambda_B(M)$ and its restriction d_B to this set is used to define the so-called basic cohomology groups $H^r_B(M)$ (see e.g. [34, 47]). From now until the end of this section, we assume that M compact and $\mathcal{F}$ is transversally oriented.

The orientation of $\mathcal{F}$ induces a *basic Hodge star operator*

$$*_B \colon \Lambda^r_B(M) \to \Lambda^{q-r}_B(M)$$

in the usual way. Moreover, the so-called *characteristic form* $\chi_\mathcal{F}$ is defined as the volume form of the leaves and is locally given by

$$\chi_\mathcal{F}(X_1, \ldots, X_p) = \det\left(g(X_i, E_j)\right)\,, \text{ for } X_r \in \Gamma(TM)$$

where $\{E_k\}_{k=1,\cdots,p}$ is a local oriented orthonormal frame of $\Gamma(L)$ and p is the rank of L. Thus, one may define a natural scalar product on the set of basic forms by setting

$$(\alpha, \beta)_B := \int_M \alpha \wedge *_B \beta \wedge \chi_\mathcal{F}. \tag{3}$$

Let δ_B be the formal adjoint of d_B with respect to the scalar product (3) and $\Delta_B := d_B\delta_B + \delta_B d_B$ the *basic Laplacian operator*. Then Δ_B is a transversally elliptic operator and by [16] its kernel has finite dimension. Moreover, in view of the basic Hodge theorem, $H^r_B(M)$ is isomorphic to $\mathcal{H}^r_B(M) := \ker \Delta_B \cap \Lambda^r_B(M)$

[15, 27]. Therefore the basic cohomology groups have finite dimensions, but in general they do not satisfy Poincaré duality (see [9] for an example). This latter is guaranteed when the foliation is *taut*, i.e. when the leaves are minimal with respect to a suitable bundle-like metric or, equivalently in view of [32], when the top basic cohomology group $H^q_B(M)$ is non-trivial (and then it is 1-dimensional).

Given a foliated manifold with a bundle-like metric $(M,\mathcal{F},g)$, the *mean curvature vector field* is defined as

$$H=\sum_{l=1}^{p}(\nabla^g_{E_l}E_l)_Q$$

where $\{E_l\}_{l=1,\cdots,p}$ is an arbitrary orthonormal frame of $\Gamma(L)$. The 1-form κ dual to H is usually called *the mean curvature form*. Notice that the leaves of $\mathcal{F}$ are minimal with respect to g if and only if κ vanishes. According to [12], it is always possible to find a compatible bundle-like metric g on M whose induced κ is basic. Moreover, when κ is basic it is automatically closed (see [26, 47]) and consequently on a compact simply-connected manifold every Riemannian foliation is taut (see [47]). Moreover, the forms κ and $\chi_{\mathcal{F}}$ are related by the following formula arising from [41]

$$\alpha\wedge d\chi_{\mathcal{F}}=-\alpha\wedge\kappa\wedge\chi_{\mathcal{F}} \tag{4}$$

holding for every basic form α of degree $q-1$. Hence if $(\mathcal{F},g_Q)$ is an orientable taut Riemannian foliation there exists a p-form $\chi_{\mathcal{F}}$ which restricts to a volume along the leaves and satisfies

$$\alpha\wedge d\chi_{\mathcal{F}}=0$$

for every basic $(q-1)$-form α.

2.2. **Transverse Kähler structures.** In this section, we focus on transverse Hermitian and transverse Kähler structures.

Let $(M,\mathcal{F},g_Q)$ be a manifold equipped with a Riemannian foliation. A *transverse complex* structure on $(M,\mathcal{F})$ is, accordingly to the previous section, a complex structure on the base which is invariant by the transition functions. Every transverse complex structure induces an endomorphism J of Q such that

$$J^2=-\mathrm{Id}_Q\,,$$

(i.e. a transverse almost complex structure). The pair (g_Q,J) is said to be a *transverse Hermitian* structure if and only if

$$g_Q(J\cdot,J\cdot)=g_Q(\cdot,\cdot)\,.$$

In this case, the triple $(\mathcal{F},g_Q,J)$ is called a *Hermitian foliation*. The basic 2-form ω obtained as the pull-back to M of the skew-symmetric tensor $g_Q(J\cdot,\cdot)$ is usually called the *fundamental form* of the foliation and it is closed if and only if (g_Q,J) is induced by a transverse Kähler structure.

Given a transverse complex structure J on a foliated manifold $(M,\mathcal{F})$, the complexified normal bundle $Q^{\mathbb{C}}:=Q\otimes\mathbb{C}$ splits into the two eigenbundles $Q^{1,0}$ and $Q^{0,1}$ corresponding to the eigenvalues i and $-i$ of J and we have

$$\Lambda^r(Q^*)\otimes\mathbb{C}=\bigoplus_{i+j=r}\Lambda^{i,j}(Q)\,.$$

Since the space of smooth sections of $\Lambda^r(Q^*)$ is isomorphic to the space of r-forms α on M satisfying $X\lrcorner\alpha=0$ for every $X\in\Gamma(L)$, J induces an operator (which we still denote by J) on complex basic forms. Therefore, the natural decomposition holds

$$\Lambda^r_B(M,\mathbb{C})=\bigoplus_{i+j=r}\Lambda^{i,j}_B(M),$$

and the complex extension of d_B splits as

$$d_B=\partial_B+\bar\partial_B$$

where

$$\partial_B\colon\Lambda^{i,j}_B(M)\to\Lambda^{i+1,j}_B(M)\,,\quad \bar\partial_B:\Lambda^{i,j}_B(M)\to\Lambda^{i,j+1}_B(M)\,.$$

In analogy to the non-foliated case, we have $\partial_B^2=\bar\partial_B^2=0$, $\partial_B\bar\partial_B+\bar\partial_B\partial_B=0$.

Moreover, if (g_Q, J) is a transverse Kähler structure, the transverse Ricci tensor of g_Q is J-invariant and induces the *transverse Ricci form* ρ_B. This latter is the closed basic form given by $\frac{1}{2\pi}\mathrm{Ric}^{\nabla}(J\cdot,\cdot)$ and is equal to

$$\rho_B(\cdot,\cdot) = -\frac{i}{2\pi}\partial_B\bar{\partial}_B\log(G)$$

where $G = \det(g_{r\bar{s}})$ and the functions $g_{r\bar{s}}$ are computed with respect to suitable transverse complex coordinates. The class of ρ_B in $H^2_B(M,\mathbb{R})$ is by definition the *first basic Chern class* of $(\mathcal{F}, J)$ and it is usually denoted by c^1_B. The following important theorem is due to El Kacimi and provides a foliated version of the celebrated Calabi-Yau theorem:

Theorem 2.1 ([13]). *Let $(M,\mathcal{F},g_Q,J)$ be a compact manifold endowed with a taut Kähler foliation and let ω be its fundamental form. If c^1_B is represented by a real basic $(1,1)$-form ρ'_B, then ρ'_B is the basic Ricci form of a unique transverse Kähler form ω' in the same basic cohomology class of ω. In particular, if $c^1_B = 0$, then there exists a transverse Kähler metric having vanishing transverse Ricci tensor.*

Remark 2.2. Let $(M,\mathcal{F},g_Q,J)$ be a Hermitian foliation of real codimension $2n$. Then the first Chern class of $K := \Lambda^{n,0}(Q)$ vanishes if and only if there exists a nowhere vanishing $\eta \in \Lambda^{n,0}(Q)$. Such an η induces a nowhere vanishing section ψ of $\Lambda^{2n}(M,\mathbb{C})$ satisfying $X \lrcorner \psi = 0$ for every section X of L. Since ψ is not necessarily basic, condition $c^1(K) = 0$ does not imply $c^1_B(\mathcal{F}) = 0$ and it is quite natural to ask whether the existence of a nowhere vanishing basic $(n,0)$-form ψ implies $c^1_B(\mathcal{F}) = 0$. This fact is certainly true if M is simply-connected since in this case the same argument as in the non foliated case (see e.g. [2]) allows us to prove that $\rho_B = -i\partial_B\bar{\partial}_B f$, where $f = g_Q(\eta,\bar{\eta})$ is the pointwise norm of the form $\eta \in \Lambda^{n,0}_B(M)$ corresponding to ψ.

An important class of transverse Kähler structures is provided by Sasakian manifolds [4]. These latters are characterized by the existence of a unit Killing vector field ξ on a $(2n+1)$-dimensional Riemannian manifold (M,g) such that the tensor field Φ defined for $X \in \Gamma(TM)$ as $\Phi(X) = \nabla^g_X\xi$ satisfies

1. $\Phi^2 = -\mathrm{Id}_{TM} + \xi^\flat \otimes \xi$,
2. $(\nabla^g_X\Phi)(Y) = g(\xi,Y)X - g(X,Y)\xi$,

where X, Y are vector fields in $\Gamma(TM)$. The vector field ξ is called the *Reeb vector field* and generates a 1-dimensional Riemannian foliation $\mathcal{F}$ such that the restriction of Φ to $\xi^\perp$ gives a transverse Kähler structure with vanishing mean curvature. Usually a Sasakian structure is denoted by a quadruple (ξ,η,Φ,g), where $\eta = \xi^\flat$ is the 1-form dual to ξ. For the geometry of Sasakian manifolds we refer to [42, 43, 5] and the references therein, whilst for the Sasakian-version of theorem 2.1 we refer to [7] and [8].

Many interesting examples of non-Kählerian complex spaces carry a Kähler foliation. For instance, Vaisman manifolds, Calabi-Eckmann manifolds and some Oeljeklaus-Toma manifolds carry Kähler foliations with the transverse Kähler form exact (see e.g. [11, 38] and the references therein).

Another interesting example of Kähler foliations comes from the physical study of A-branes (see [28, 24]) obtained by considering a coisotropic submanifold of a Kähler manifold. A submanifold N of a Kähler manifold (M,ω,J,g) is called *coisotropic* if

$$TN^\omega \subset TN \tag{5}$$

where

$$(T_yN)^\omega = \{v \in T_yM \ : \ \omega(v,w) = 0 \ , \ \text{for all } w \in T_yN\}\,.$$

The coisotropic condition (5) implies that $\mathcal{F}_y := (T_yN)^\omega$ is a distribution on N which is intregrable by the closure of ω. Moreover, the complex structure J always preserves the orthogonal complement of the bundle L induced by $\mathcal{F}$ and we have the following

Proposition 2.3. *Let $N \hookrightarrow (M,\omega,J,g)$ be a coistropic submanifold of a Kähler manifold and let $\mathcal{F}$ be the induced foliation on N. Then the metric g is always bundle-like with respect to $\mathcal{F}$ and $\mathcal{F}$ is locally a trivial Kähler foliation.*

Proof. The metric g is bundle-like if and only if

$$\mathcal{L}_X g(X_1, X_2) = 0$$

for every $X \in \Gamma(L)$ and $X_1, X_2 \in \Gamma(L^\perp)$. Such a relation can be read in terms of the Levi-Civita connection ∇^g of g as

$$g(\nabla^g_{X_1} X, X_2) = -g(\nabla^g_{X_2} X, X_1)\,.$$

Now it is enough to observe that the Kähler condition $\nabla^g \omega = 0$ implies

$$g(\nabla^g_{X_1} X, X_2) = 0$$

since $g(\nabla^g_{X_1} X, X_2) = \omega(\nabla^g_{X_1} X, JX_2)$ and

$$0 = (\nabla^g_{X_1}\omega)(X, JX_2) = X_1\omega(X, JX_2) - \omega(\nabla^g_{X_1} X, JX_2) - \omega(X, \nabla^g_{X_1} JX_2) = -\omega(\nabla^g_{X_1} X, JX_2)\,.$$

□

3. A Weitzenböck formula for transverse Kähler structures

In this section, we establish a transverse Weitzenböck formula for complex-valued basic forms on Kähler foliations and we also derive some vanishing results. Mainly, we follow the classical computations in the non-foliated case as described in [37].

Let $(M, \mathcal{F}, g_Q, J)$ be a compact manifold with a Kähler foliation of codimension q and let g a bundle-like metric on M inducing g_Q. We may assume in view of [12] that the mean curvature form κ of the foliation is a basic (otherwise we can work with the basic component of κ). From now until the end of this section in the indicial expressions the symbol of sum over repeated indices is omitted.

Let $\{e_j\}$ be a local orthonormal frame of $\Gamma(Q)$, denote by $\{e^j\}$ the dual frame in Q^* and let $Z_j = \frac{1}{2}(e_j - iJe_j)$ and $\zeta^j = \frac{1}{2}(e^j + iJe^j)$ Furthermore, given a transverse 1-form α, we denote the correspondent dual section of Q via the transverse metric g_Q by $\alpha^\sharp$. Let us consider the two operators ∂_B and $\bar\partial_B$ given in terms of the transverse Levi-Civita connection ∇ of $(\mathcal{F}, g_Q)$ as

$$\partial_B = \zeta^j \wedge \nabla_{e_j}\,, \qquad \bar\partial_B = \bar\zeta^j \wedge \nabla_{e_j}\,.$$

In [22], the authors established a Bochner-Weitzenböck formula for the Laplacian operator corresponding to the *twisted derivative* $\tilde d_B = d_B - \frac{1}{2}\kappa\wedge$. As mentionned before, we assume that the mean curvature is d_B-closed; in particular we have that $\tilde d_B^2 = 0$. In the same spirit as [22], we modify the operators ∂_B and $\bar\partial_B$ by introducing the following two twisted operators

$$\tilde\partial_B = \partial_B - \frac{1}{4}(\kappa + iJ\kappa)\wedge\,, \quad \tilde{\bar\partial}_B = \bar\partial_B - \frac{1}{4}(\kappa - iJ\kappa)\wedge\,.$$

We readily have that $\tilde d_B = \tilde\partial_B + \tilde{\bar\partial}_B$ and the formal adjoint to $\tilde{\bar\partial}_B$ with respect to the scalar product (3) writes as

$$\left(\tilde{\bar\partial}_B\right)^* = -\bar Z_j \lrcorner \nabla_{e_j} + \frac{1}{4}\left(\kappa^\sharp + iJ\kappa^\sharp\right)\lrcorner\,.$$

Now we state the main result of this section:

Theorem 3.1. *Let $(M, g, \mathcal{F}, J)$ be a compact Riemannian manifold endowed with a Kähler foliation. Assume that the mean curvature κ is basic-harmonic. Then the following Weitzenböck-type formula holds*

$$2\left[(\tilde{\bar\partial}_B)^*\tilde{\bar\partial}_B + \tilde{\bar\partial}_B(\tilde{\bar\partial}_B)^*\right] = \nabla^*\nabla + \frac{1}{4}|\kappa|^2 + \mathfrak{R} + \frac{1}{4}\bar\zeta^j \wedge [(\nabla_{e_j}\kappa)^\sharp + i(J\nabla_{e_j}\kappa)^\sharp]\lrcorner$$
$$+ \frac{i}{2} g(Je^j, \nabla_{e_j}\kappa) - \frac{1}{2}(\nabla_{e_\ell}\kappa - iJ\nabla_{e_\ell}\kappa) \wedge \bar Z_\ell \lrcorner\,,$$

where the third term $\mathfrak{R}$ is

$$\mathfrak{R} = \frac{i}{2} R^\nabla(Je_j, e_j) - \bar\zeta^j \wedge \bar Z_\ell \lrcorner R^\nabla(e_j, e_\ell).$$

Proof. Let x be a fixed point of M and $\{e_i\}_{i=1,\cdots,q}$ be an orthonormal frame of $\Gamma(Q)$ which we may assume to be parallel at x. In order to simplify the notation here we set

$$\tau = \frac{1}{2}(\kappa - iJ\kappa)\,.$$

Then working at x we have

$$2\tilde{\bar{\partial}}_B^*\tilde{\bar{\partial}}_B = \tilde{\bar{\partial}}_B^*\left(2\bar{\zeta}^j \wedge \nabla_{e_j} - \tau\wedge\right) = -\bar{Z}_\ell \lrcorner \nabla_{e_\ell}\left(2\bar{\zeta}^j \wedge \nabla_{e_j} - \tau\wedge\right) + \frac{1}{2}\tau^\sharp\lrcorner\left(2\bar{\zeta}^j \wedge \nabla_{e_j} - \tau\wedge\right)$$

which yields to

$$2\tilde{\bar{\partial}}_B^*\tilde{\bar{\partial}}_B = -\,2\bar{Z}_\ell\lrcorner\left(2\bar{\zeta}^j \wedge \nabla_{e_\ell}\nabla_{e_j} - \nabla_\ell\tau\wedge\, -\tau\wedge\nabla_{e_\ell}\right) + \tau^\sharp\lrcorner\left(2\bar{\zeta}^j\wedge\nabla_{e_j} - \tau\wedge\right).$$

Then we get

$$\begin{aligned}2\tilde{\bar{\partial}}_B^*\tilde{\bar{\partial}}_B =& -(\delta^{\ell j} - ig(Je_j, e_\ell))\nabla_{e_\ell}\nabla_{e_j} + 2\bar{\zeta}^j \wedge \bar{Z}_\ell\lrcorner\nabla_{e_\ell}\nabla_{e_j} + \frac{1}{2}g(\nabla_{e_j}\kappa, e^j) + \frac{i}{2}g(Je^j, \nabla_{e_j}\kappa)\\ &- \nabla_{e_j}\tau\wedge Z_j\lrcorner + \frac{1}{2}g(\kappa, e_j)\nabla_{e_j} - \frac{i}{2}g(J\kappa, e_j)\nabla_{e_j} - \tau\wedge\bar{Z}_j\lrcorner\nabla_{e_j}\\ &+ \frac{1}{2}g(\kappa, e^j)\nabla_{e_j} + \frac{i}{2}g(J\kappa, e^j)\nabla_{e_j} - \bar{\zeta}^j\wedge\tau^\sharp\lrcorner\nabla_{e_j} - \frac{1}{4}|\kappa|^2 + \frac{1}{2}\tau\wedge\tau^\sharp\lrcorner\end{aligned}$$

which gives that

$$\begin{aligned}2\tilde{\bar{\partial}}_B^*\tilde{\bar{\partial}}_B =& \nabla^*\nabla + ig(Je_j, e_\ell)\nabla_{e_\ell}\nabla_{e_j} + 2\bar{\zeta}^j\wedge\bar{Z}_\ell\lrcorner\nabla_{e_\ell}\nabla_{e_j} + \frac{1}{2}\mathrm{div}_Q(\kappa) + \frac{i}{2}g(Je^j, \nabla_{e_j}\kappa)\\ &- \nabla_{e_\ell}\tau\wedge\bar{Z}_\ell\lrcorner - \tau\wedge\bar{Z}_\ell\lrcorner\nabla_{e_\ell} - \bar{\zeta}^j\wedge\tau^\sharp\lrcorner\nabla_{e_j} - \frac{1}{4}|\kappa|^2 + \frac{1}{2}\tau\wedge\tau^\sharp\lrcorner.\end{aligned}$$

In the last equality above, we have made use of the relation

$$\nabla^*\nabla = -\nabla_{e_j}\nabla_{e_j} + \nabla_\kappa.$$

Since we are assuming that the mean curvature form is basic-harmonic, the divergence of κ is thus equal to the square of its norm. Thus, we have

$$\begin{aligned}2\tilde{\bar{\partial}}_B^*\tilde{\bar{\partial}}_B =& \nabla^*\nabla + \frac{1}{4}|\kappa|^2 + \frac{i}{2}g(Je_j, e_\ell)R^\nabla(e_\ell, e_j) + 2\bar{\zeta}^j\wedge\bar{Z}_\ell\lrcorner\nabla_{e_\ell}\nabla_{e_j} + \frac{i}{2}g(Je^j, \nabla_{e_j}\kappa) - \nabla_{e_j}\tau\wedge\bar{Z}_j\lrcorner\\ &\tau\wedge\bar{Z}_\ell\lrcorner\nabla_{e_\ell} - \bar{\zeta}^j\wedge\tau^\sharp\lrcorner\nabla_{e_j} + \frac{1}{2}\tau\wedge\tau^\sharp\lrcorner\\ =& \nabla^*\nabla + \frac{1}{4}|\kappa|^2 + \frac{i}{2}R^\nabla(Je_j, e_j) + 2\bar{\zeta}^j\wedge\bar{Z}_\ell\lrcorner\nabla_{e_j}\nabla_{e_\ell} + \bar{\zeta}^j\wedge\bar{Z}_\ell\lrcorner R^\nabla(e_\ell, e_j) + \frac{i}{2}g(Je^j, \nabla_{e_j}\kappa)\\ &- \nabla_{e_j}\tau\wedge\bar{Z}_j\lrcorner - \tau\wedge\bar{Z}_\ell\lrcorner\nabla_{e_\ell} - \bar{\zeta}^j\wedge\tau^\sharp\lrcorner\nabla_{e_j} + \frac{1}{2}\tau\wedge\tau^\sharp\lrcorner,\end{aligned}$$

which finally gives

$$\begin{aligned}2\tilde{\bar{\partial}}_B^*\tilde{\bar{\partial}}_B =& \nabla^*\nabla + \frac{1}{4}|\kappa|^2 + \mathfrak{R} + 2\bar{\zeta}^j\wedge\bar{Z}_\ell\lrcorner\nabla_{e_j}\nabla_{e_\ell} + \frac{i}{2}g(Je^j, \nabla_{e_j}\kappa) - \nabla_{e_j}\tau\wedge\bar{Z}_j\lrcorner - \tau\wedge\bar{Z}_j\lrcorner\nabla_{e_j}\\ &- \bar{\zeta}^j\wedge\tau^\sharp\lrcorner\nabla_{e_j} + \frac{1}{2}\tau\wedge\tau^\sharp\lrcorner\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}2\tilde{\bar{\partial}}_B(\tilde{\bar{\partial}}_B)^* =& \tilde{\bar{\partial}}_B(-2\bar{Z}_j\lrcorner\nabla_{e_j} + \tau^\sharp\lrcorner) = \zeta^j\wedge\nabla_{e_j}\left(-2\bar{Z}_\ell\lrcorner + \tau^\sharp\lrcorner\right) - \frac{1}{2}\tau\wedge\left(-2\bar{Z}_j\lrcorner\nabla_{e_j} + \tau^\sharp\lrcorner\right)\\ =& -2\bar{\zeta}^j\wedge\bar{Z}_\ell\lrcorner\nabla_{e_j}\nabla_{e_\ell} + \frac{1}{4}\bar{\zeta}^j\wedge\left[\nabla_{e_j}\tau\right]^\sharp\lrcorner + \bar{\zeta}^j\wedge\tau^\sharp\lrcorner\nabla_{e_j} + \tau\wedge\bar{Z}_j\lrcorner\nabla_{e_j} - \frac{1}{2}\tau\wedge\tau^\sharp\lrcorner.\end{aligned}$$

Thus, by taking the sum of the last two equations we get the statement. □

The previous theorem has the following remarkable consequence when it is applied to basic $(p, 0)$-forms:

Corollary 3.2. *Under the hypotheses of theorem* 3.1*, for every form* $\alpha \in \Lambda_B^{p,0}(M)$ *we have*

$$2\bar{\tilde{\partial}}_B^*\bar{\tilde{\partial}}_B\alpha = \nabla^*\nabla\alpha + \frac{1}{4}|\kappa|^2\alpha + \frac{i}{2}\sum_{j=1}^{q} R^\nabla(Je_j, e_j)\alpha - \frac{i}{2}\mathrm{div}_Q(J\kappa)\alpha\,.$$

Another consequence of theorem 3.1 is the following:

Theorem 3.3. *Let* $(M, g, \mathcal{F}, J)$ *be a compact manifold endowed with a Kähler foliation. If the transverse Ricci curvature is negative definite, then* $\mathcal{F}$ *has only trivial transversally holomorphic vector fields.*

Proof. In view of [33], one can modify any bundle-like metric on M to another one with a basic-harmonic mean curvature without changing the transverse metric. We show that every $\xi \in \Lambda_B^{1,0}(M)$ satisfying $\bar\partial_B\xi = 0$ is trivial. Such a ξ satisfies $\bar{\tilde{\partial}}_B\xi = -\frac{1}{4}(\kappa - iJ\kappa)\wedge\xi$ and hence $|\bar{\tilde{\partial}}_B\xi|_H^2 = \frac{1}{8}|\kappa|^2|\xi|^2$. By taking the product with ξ in the Weitzenböck formula and integrating over M we get

$$\frac{1}{4}\int_M |\kappa|^2|\xi|_H^2 v_g = \int_M |\nabla\xi|^2 v_g + \frac{1}{4}\int_M |\kappa|^2|\xi|_H^2 v_g - \int_M H(\mathrm{Ric}^\nabla(\xi), \xi)v_g.$$

In the above identity, we have used $\frac{1}{2}\sum_{j=1}^q R^\nabla(Je_j, e_j)\xi = \mathrm{Ric}^\nabla(J\xi) = i\mathrm{Ric}^\nabla(\xi)$. The assumption on the Ricci curvature to be negative definite implies the statement. □

We point out that theorem 3.3 was already obtained by S.D. Jung and H. Liu in [25] by using another method.

Theorem 3.4. *Let* $(M, g_Q, \mathcal{F}, J)$ *be a compact manifold endowed with a Kähler foliation. If the transverse Ricci curvature vanishes, then every transversally holomorphic* $(p, 0)$*-form is parallel. Moreover, if the transverse Ricci curvature of* M *is positive definite, there every transversally holomorphic* $(p, 0)$*-form on* M *is trivial.*

Proof. We use again the fact that g can be completed to a bundle-like metric on M whose induced κ is basic-harmonic. Let γ be a transversally holomorphic $(p, 0)$-form. Then $\bar\partial_B\gamma = 0$ and thus $\bar{\tilde{\partial}}_B\gamma = -\frac{1}{4}(\kappa - iJ\kappa)\wedge\gamma$ which implies $|\bar{\tilde{\partial}}_B\gamma|_H^2 = \frac{1}{8}|\kappa|^2|\gamma|^2$. Using the Weitzenböck formula and the identities

$$R^\nabla(X, Y)\gamma = R^\nabla(X, Y)e_k \wedge e_k \lrcorner\gamma\,, \quad \frac{1}{2}\sum_{j=1}^{q} R^\nabla(J(e_i), e_i)e_k = \mathrm{Ric}^\nabla(J(e_k)),$$

we get the first part of the statement. The second part of the theorem can be obtained following the same proof as in the non-foliated case (see e.g. [37]). □

4. Transverse Calabi-Yau structures

In this short section, we consider *transverse Calabi-Yau* structures. Such structures can be defined as Riemannian foliations having transverse holonomy contained in $\mathrm{SU}(n)$ and have been considered in [35, 36] where it is proved that in the *taut* case the moduli space is a smooth Hausdorff manifold of finite dimension (i.e. a generalization of the Bogomolov-Tian-Todorov theorem [3, 44, 45] to the foliated case). Moreover, Calabi-Yau foliations can be used for desingularizing Calabi-Yau orbifolds.

We have the following:

Proposition 4.1. *Let* $(M, \mathcal{F}, g_Q, J)$ *be a compact simply-connected manifold carrying a Kähler foliation. Assume* $\rho_Q = 0$*; then* $\mathrm{Hol}(\nabla)$ *is contained in* $\mathrm{SU}(n)$ *and* $\mathcal{F}$ *is a Calabi-Yau foliation.*

Proof. Let $(\mathcal{F}, g_Q, J)$ be a Kähler foliation, $K := \Lambda^{n,0}(Q)$ and let R^K be the curvature of the connection induced by the transverse Levi-Civita connection on K. Fix a global section ψ of K. Then a standard computation yields

$$R^K_{X,Y}\psi = \rho_Q(X, Y)\psi$$

for every pair of smooth vector fields on M. Hence if we assume $\rho_Q = 0$, then we have $R^K = 0$ which is equivalent to require that $\mathrm{Hol}^0(\nabla) \subseteq \mathrm{SU}(n)$. Hence when M is simply-connected, we have $\mathrm{Hol}(\nabla) \subseteq \mathrm{SU}(n)$, as required. □

Combining the El Kacimi theorem 2.1 with the last proposition, we get the following

Corollary 4.2. *Let $(\mathcal{F}, g_Q, J)$ Kähler foliation on a compact simply-connected manifold M and assume $c^1_B = 0$. Then there exists a unique transverse Kähler form ω' in the same basic cohomology class of the fundamental form of g_Q whose transverse Levi-Civita connection ∇' satisfies $\mathrm{Hol}(\nabla') \subseteq \mathrm{SU}(n)$.*

5. Transverse Hyper-Kähler structures

A particular class of Calabi-Yau foliations is provided by hyper-Kähler foliations. These are characterized by a triple (J_1, J_2, J_3) of transverse complex structures on a foliated manifold $(M, \mathcal{F}, g_Q)$ such that $(\mathcal{F}, g_Q, J_r)$ is a Kähler foliation for every $r = 1, 2, 3$ and it is satisfied the quaternionic relation

$$J_1 J_2 = -J_2 J_1 = J_3. \tag{6}$$

Requiring the existence of this structure on a foliated manifold $(M, \mathcal{F})$ is equivalent to requiring the existence of a transverse metric having transverse holonomy group contained in $\mathrm{Sp}(n)$. Moreover, if $(\mathcal{F}, g_Q, J_1)$ is a Kähler foliation, then it is *transversally hyper-Kähler* if and only if there exists a basic $(2,0)$-form Ω such that

$$\Omega^n \neq 0\,, \qquad \nabla\Omega = 0\,.$$

If (J_1, J_2, J_3) is simply a triple of transverse complex structures compatible with a fixed transverse metric and satisfying (6), we refer to (g_Q, J_1, J_2, J_3) as a *transverse hyper-Hermitian* structure. The main result of this section is the following theorem which is the foliated counterpart of the main theorem in [49]:

Theorem 5.1. *Let $(M, \mathcal{F}, g_Q, J_1)$ be a compact simply-connected manifold carrying a Kähler foliation. Assume that there exists a pair (J_2, J_3) of transverse complex structures such that (J_1, J_2, J_3) is a transverse hyper-Hermitian structure. Then there exists a transverse metric g'_Q on $(M, \mathcal{F})$ having holonomy contained in $\mathrm{Sp}(n)$.*

We divide the proof in a sequence of lemmas. The first one is a generalization of theorem 8.2 of [19] to the foliated non-contact case:

Lemma 5.2. *For every Hermitian foliation $(\mathcal{F}, g_Q, J)$ on a manifold M, there exists a unique connection $\widetilde{\nabla}$ on Q preserving (g_Q, J) and such that*

1. *$g_Q(T(X_Q, Y_Q), Z_Q) = -g(T(X_Q, Z_Q), Y_Q)$ for every $X, Y, Z \in \Gamma(TM)$;*
2. *$\widetilde{\nabla}_\xi s = \nabla_\xi s$ for every $s \in \Gamma(Q)$ and $\xi \in \Gamma(L)$,*

where T is the transverse torsion tensor

$$T(X, Y) = \widetilde{\nabla}_X Y_Q - \widetilde{\nabla}_Y X_Q - [X, Y]_Q\,.$$

Proof. Let $\tilde{\nabla}$ be the connection defined explicitly as:

$$g(\widetilde{\nabla}_Z X, Y) = \begin{cases} g_Q(\nabla_Z X, Y) + \frac{1}{2} d\omega(JZ, JX, JY) & \text{if} \quad Z \in \Gamma(Q), \\ g_Q(\nabla_Z X, Y) & \text{if} \quad Z \in \Gamma(L)\,, \end{cases} \tag{7}$$

where ω is the fundamental form of (g_Q, J). First of all, we observe that the connection $\widetilde{\nabla}$ described by formula (7) satisfies conditions 1. and 2. of the statement (here we use that $g_Q(T(X, Y), Z) = -d\omega(JX, JY, JZ)$). On the other hand, it preserves the metric g_Q and the second formula in (7) forces $\widetilde{\nabla}_\xi J = 0$ for $\xi \in \Gamma(L)$. Moreover if X, Y, Z lie in $\Gamma(Q)$, a direct computation gives

$$2g_Q((\nabla_Z J)X, Y) = d\omega(X, JY, JZ) + d\omega(JX, Y, JZ) \tag{8}$$

which implies that $\widetilde{\nabla}$ preserves J. This shows the first part of the proof.

We shall now prove the uniqueness. Assume to have a connection $\widetilde{\nabla}$ preserving (g_Q, J) and satisfying the conditions 1 and 2 of the statement. Therefore, we can write

$$g_Q(\widetilde{\nabla}_Z X, Y) = g_Q(\nabla_Z X, Y) + \frac{1}{2} S(Z, X, Y) \tag{9}$$

for a tensor S defined on $TM \times Q \times Q$. Since ∇ and $\tilde{\nabla}$ both preserve g_Q, the tensor S is skew-symmetric in the last two entries, i.e.

$$S(Z,X,Y) = -S(Z,Y,X)\,.$$

The item 2 implies $S(\xi, X, Y) = 0$ for all $\xi \in \Gamma(L)$. An easy computation involving the torsion of $\tilde{\nabla}$, formula (9) and the item 1 implies

$$S(X,Y,Z) = S(Y,Z,X)$$

for all $X, Y, Z \in \Gamma(Q)$. Using that $\tilde{\nabla}$ preserves the structure J, it is easy to show that the following relation holds

$$S(Z,JX,Y) + S(Z,X,JY) = -2g_Q((\nabla_Z J)X,Y) \tag{10}$$

for $X, Y, Z \in \Gamma(Q)$. By considering the cyclic sum in (10) we get

$$\mathfrak{S}_{Z,X,Y} S(Z,JX,Y) = -\mathfrak{S}_{Z,X,Y} g_Q((\nabla_Z J)X,Y) = -d\omega(Z,X,Y)$$

and

$$\mathfrak{S}_{Z,X,Y} S(JZ,X,JY) = d\omega(JZ,JX,JY).$$

Now the integrability of J gives

$$S(X,Y,Z) = S(JX,JY,Z) + S(JX,Y,JZ) + S(X,JY,JZ)$$

which implies

$$S(X,Y,Z) = d\omega(JZ,JX,JY)\,,$$

as required. □

Lemma 5.3. *Let $(\mathcal{F}, g_Q, J_1, J_2, J_3)$ be a transverse hyper-Hermitian foliation and we denote by ∂_k the ∂_B operator with respect to J_k. If $\tilde{\partial}_2 := -J_2^{-1}\bar{\partial}_1 J_2$, then*

$$\partial_1\tilde{\partial}_2 = -\tilde{\partial}_2\partial_1\,. \tag{11}$$

Proof. The proof of the statement can be obtained by using standard algebraic computations. It is sufficient to check (11) for basic maps and basic $(1,0)$-forms. We show how things work for functions and we omit the proof for 1-forms.

Let f be a basic map and let Z, W be two smooth sections of $Q^{1,0}$. Then we have

$$\begin{aligned}(\partial_1\tilde{\partial}_2 f)(Z,W) =&\ Z(\tilde{\partial}_2 f(W)) - W(\tilde{\partial}_2 f(Z)) - \tilde{\partial}_2 f([Z,W])\\ =&\ ZJ_2W(f) - WJ_2Z(f) - J_2[Z,W](f)\end{aligned}$$

and

$$(\tilde{\partial}_2\partial_1 f)(Z,W) = J_2ZW(f) - J_2WZ(f) + J_2[J_2Z,J_2W](f)\,.$$

Therefore

$$\begin{aligned}(\partial_1\tilde{\partial}_2 + \tilde{\partial}_2\partial_1)(f)(Z,W) &= (ZJ_2W - WJ_2Z - J_2[Z,W] + J_2ZW - J_2WZ + J_2[J_2Z,J_2W])\,(f)\\ &= J_2N_{J_2}(Z,W)(f) = 0\,,\end{aligned}$$

as required. □

Lemma 5.4. *Let $(M, \mathcal{F}, g_Q, J_1)$ be a compact manifold with a taut Calabi-Yau foliation of codimension $4n$. Assume that there exists a pair (J_2, J_3) of transverse complex structures such that (J_1, J_2, J_3) is a transverse hyper-complex structure. Then g_Q has transverse holonomy contained in $\mathrm{Sp}(n)$.*

Proof. Let g_1 be the transverse Riemannian metric

$$g_1(\cdot,\cdot) = \frac{1}{2}g_Q(\cdot,\cdot) + \frac{1}{2}g_Q(J_2\cdot,J_2\cdot)\,.$$

This metric is compatible with each J_k. For each $k = 1, 2, 3$, we denote by F_k the fundamental form of $(\mathcal{F}, g_1, J_k)$ and we let

$$\Omega_1 := \frac{1}{2}\,(F_2 + iF_3)\,.$$

The basic form Ω_1 is of type $(2,0)$ with respect to J_1 and satisfies

$$\Omega_1^n \neq 0\,.$$

First of all we show that Ω_1 is ∂_1-closed, where ∂_1 denotes the ∂_B-operator with respect to J_1. Let Ω_2 the $(2,0)$-component of ω with respect to J_2. The form Ω_2 is basic and from the closure of ω we get

$$\partial_2\Omega_2 = 0,$$

where ∂_2 denotes the ∂_B-operator computed with respect to J_2. On the other hand, an easy computation yields

$$\Omega_2 = \frac{1}{2}(F_1 - iF_3)\,.$$

Taking into account the formula

$$\partial_k\gamma = \frac{1}{2}\big(d + (-1)^r iJ_k dJ_k\big)\gamma$$

which holds for every complex basic form γ of degree r (see e.g. [21] for the non-foliated case), we deduce that condition $\partial_2\Omega_2 = 0$ forces to have

$$J_1\,dF_1 = J_3\,dF_3\,.$$

Let $\tilde\nabla^k$ be the connection with skew-symmetric torsion induced by (g_1, J_k) as in lemma 5.2. Formula (7) now implies that all the $\tilde\nabla^k$'s have the same torsion and thus we get $\tilde\nabla^1 = \tilde\nabla^3$ and $\tilde\nabla^1 J_2 = 0$ using the fact that $J_2 = J_3J_1$. Therefore

$$\tilde\nabla^1 = \tilde\nabla^2 = \tilde\nabla^3$$

which gives $J_2dF_2 = J_3dF_3$ and that means $\partial_1\Omega_1 = 0$. Hence Ω_1 satisfies

$$\Omega_1^n \neq 0\,,\quad \partial_1\Omega_1 = 0\,.$$

Since M is compact and $\mathcal{F}$ is taut we can use the transverse Hodge theory to write

$$\Omega_1 = \Omega + \partial_1\alpha,$$

where Ω is the g_Q-basic harmonic component of Ω_1. Since Ω is transversally holomorphic, theorem 3.4 implies that $\nabla\Omega = 0$. In particular the norm of Ω^n is constant. In order to finish the proof, we have to show that $\Omega_x^n \neq 0$ for every $x \in M$. Assume for a contradiction that $\Omega_x^n = 0$ at a point $x \in M$ and let $\psi = \Omega_1^n$. Since the norm of Ω^n is constant, $\Omega^n \equiv 0$ and

$$\psi = \partial_1\beta$$

for a basic form β. The last step consists in showing that ψ cannot be ∂_1-exact. We will obtain this result by adapting the last step in the proof of the main theorem in [49] to our case. Taking into account the isomorphism $\Lambda_B^{2n,1}(M) \cong \Lambda_B^{2n,0}(M) \wedge \Lambda_B^{0,1}(M)$, there exists a basic $(1,0)$-form θ which is ∂_1-closed such that

$$\partial_1\bar\psi = \theta \wedge \bar\psi\,. \tag{12}$$

Let us consider the complex

$$\Lambda_B^{0,0}(M) \xrightarrow{\partial_1 + \frac{1}{2}\theta} \Lambda_B^{1,0}(M) \xrightarrow{\partial_1 + \frac{1}{2}\theta} \Lambda_B^{2,0}(M) \xrightarrow{\partial_1 + \frac{1}{2}\theta} \cdots \tag{13}$$

where

$$\left(\partial_1 + \frac{1}{2}\theta\right)\alpha = \partial_1\alpha + \frac{1}{2}\theta \wedge \alpha\,.$$

Since θ is ∂_1-closed, we have $\left(\partial_1 + \frac{1}{2}\theta\right)^2 = 0$. In view of [13, thm. 2.8.7] the cohomology of the complex (13) is finite-dimensional and its cohomology groups can be identified with the kernel of the Laplacian operator associated with $D_1 = \partial_1 + \frac{1}{2}\theta$. Now, following the approach of [48], we observe that the operator

$$L\colon \Lambda_B^{*,0}(M) \to \Lambda_B^{*+2,0}(M)$$

defined as $\beta \mapsto \beta \wedge \Omega_1$ preserves the kernel of $D_1D_1^* + D_1^*D_1$, where

$$D_1^*(\beta) := \partial_1^*\beta + \frac{1}{2} *_B (\bar\theta \wedge *_B\beta) = - *_B \bar\partial_1(*_B\beta) + \frac{1}{2}\bar\theta \lrcorner \beta$$

is the formal adjoint to D_1 with respect the scalar product (3) induced by g_1. This basically comes from the following three identities which will be proved afterwards

$$LD_1 - D_1L = 0\,, \tag{14}$$

$$D_1D_2 + D_2D_1 = 0\,, \tag{15}$$

$$LD_1^* - D_1^*L = D_2\,, \tag{16}$$

where

$$D_2(\beta) = -J_2^{-1}\bar\partial_1 J_2(\beta) + \frac{1}{2}J_2(\bar\theta)\wedge\beta = \tilde\partial_2(\beta) + \frac{1}{2}J_2(\bar\theta)\wedge\beta\,.$$

Indeed assuming that equalities (14), (15) and (16) hold, we write

$$\begin{aligned} L(D_1D_1^* + D_1^*D_1) &= LD_1D_1^* + LD_1^*D_1 = D_1LD_1^* + D_1^*LD_1 + D_2D_1\\ &= D_1D_1^*L + D_1D_2 + D_1^*D_1L + D_2D_1 = (D_1D_1^* + D_1^*D_1)L\,, \end{aligned}$$

as required. In particular the map $[f]\mapsto[f\psi]$ induces an isomorphism

$$\ker\, D_1\cap\Lambda_B^{0,0}(M)\to\frac{\Lambda_B^{2n,0}(M)}{D_1(\Lambda_B^{2n-1,0}(M))}.$$

Assume for a contradiction that $\psi = \partial_1\beta$. We can certainly find a nowhere vanishing closed basic $(n,0)$-form η such that

$$\bar\psi = g\bar\eta$$

for a nowhere vanishing basic map g. Then

$$\theta\wedge\bar\psi = \frac{1}{g}\partial_1(g)\wedge\bar\psi\,,$$

i.e. $\partial_1 g = g\theta$. Let $f = g^{-\frac{1}{2}}$; then f gives a non-trivial cohomology class in $\ker D_1\cap\Lambda_B^{0,0}(M)$. Moreover, $[f\psi]=0$, since

$$D_1(f\beta) = f\psi\,.$$

Hence ψ cannot be ∂_1-exact and the claim follows.

In order to finish the proof it remains to show that formulas (14), (15), (16) are true. For the first one, we simply have

$$D_1L\gamma = \partial_1(\gamma\wedge\Omega_1) + \frac{1}{2}\theta\wedge\gamma\wedge\Omega_1 = \partial_1\gamma\wedge\Omega_1 + \left(\frac{1}{2}\theta\wedge\gamma\right)\wedge\Omega_1 = LD_1\gamma$$

for every $\gamma\in\Lambda_B^{k,0}(M)$. The proof of the other two formulas is a bit more involved and we need lemma 5.3 and some linear algebra computations proved as in [49]. First of all, it is easy to show that

$$J_2\bar\psi = \psi\,.$$

Therefore $\bar\partial_1(J_2\bar\psi) = \bar\theta\wedge\psi$ which implies the useful relation

$$\tilde\partial_2\bar\psi = J_2\bar\theta\wedge\bar\psi\,,$$

where $\tilde\partial_2$ is defined in lemma 5.3. Now applying (11) to $\bar\psi$ we easily get formula (15). Formula (16) is equivalent to

$$\big(LD_1^* - D_1^*L\big)\beta - \frac{1}{2}\,J_2(\bar\theta)\wedge\beta = \tilde\partial_2(\beta), \tag{17}$$

which it is enough to be checked for smooth basic maps and $\tilde\partial_2$-closed $(1,0)$-forms. Let f be a smooth basic map, then

$$LD_1^*f = 0$$

and

$$\begin{aligned} -D_1^*Lf - \frac{1}{2}fJ_2\bar\theta &= *_B\left(\bar\partial_1(*_Bf\Omega_1)\right) - \frac{1}{2}f\bar\theta\lrcorner\Omega_1 - \frac{1}{2}fJ_2\bar\theta\\ &= *_B\left(\bar\partial_1(f)\wedge(*_B\Omega_1) + f\bar\theta\wedge *_B\Omega_1\right) - fJ_2\bar\theta = J_2(\bar\partial_1(f)) = \tilde\partial_2 f \end{aligned}$$

where we have used that $*_B\Omega_1 = n(\frac{1}{n!})^2\Omega_1^n \wedge \bar\Omega_1^{n-1}$ and the natural identity

$$*_B(\bar Z \wedge *_B\Omega_1) = \bar Z \lrcorner \Omega_1 = J_2 \bar Z.$$

Now we prove (16) for a basic $\tilde\partial_2$-closed $(1,0)$-form α. We have

$$LD_1^*\alpha = (\partial_1^*\alpha)\,\Omega_1 + \frac{1}{2} *_B (\bar\theta \wedge *_B\alpha)\,\Omega_1 = (\partial_1^*\alpha)\,\Omega_1 + \frac{1}{2} g_1(\bar\theta,\alpha)\Omega_1\,.$$

$$-D_1^*L\alpha = -\,\partial_1^*(\alpha\wedge\Omega_1) - \frac{1}{2} *_B \left(\bar\theta \wedge *_B(\alpha\wedge\Omega_1)\right) = -\partial_1^*(\alpha\wedge\Omega_1) - \frac{1}{2}\bar\theta\lrcorner(\alpha\wedge\Omega_1)\,.$$

$$D_2\alpha = \frac{1}{2} J_2(\bar\theta)\wedge\alpha.$$

Taking into account that α is $\tilde\partial_2$-closed, we also have

$$\begin{cases} (\partial_1^*\alpha)\,\Omega_1 = -\,*_B(\bar\theta\wedge *_B\alpha)\,\Omega_1\,, \\ \partial_1^*(\alpha\wedge\Omega_1) = -\,*_B\left(\bar\theta\wedge *_B(\alpha\wedge\Omega_1)\right) \end{cases}$$

and therefore

$$\begin{aligned}(LD_1^* - D_1^*L)\,\alpha &= -\,\frac{1}{2} *_B(\bar\theta\wedge *_B\alpha) + \frac{1}{2} *_B\left(\bar\theta\wedge *_B(\alpha\wedge\Omega_1)\right) \\ &= \frac{1}{2}\bar\theta\lrcorner(\alpha\wedge\Omega_1) - \frac{1}{2}(\bar\theta\lrcorner\alpha)\Omega_1 = -\frac{1}{2}\alpha\wedge(\bar\theta\lrcorner\Omega_1) = \frac{1}{2}J_2(\bar\theta)\wedge\alpha.\end{aligned}$$

i.e.

$$(LD_1^* - D_1^*L)\,\alpha = D_2\alpha\,,$$

as required. □

Now we are ready to prove theorem 5.1

Proof of theorem 5.1. Assume that there exists a pair (J_2, J_3) of transverse complex structures as in the statement. Let

$$\Omega(\cdot,\cdot) := \frac{1}{2}\left(g_Q(J_2\cdot,\cdot) + i g_Q(J_3\cdot,\cdot)\right).$$

This form can be regarded as a basic form of type $(2,0)$ with respect to J_1; therefore $\psi := \Omega^n$ is a nowhere vanishing *basic* $(2n,0)$-form and remark 2.2 implies that the first basic Chern class of $(M,\mathcal{F},J_1)$ vanishes. Since M is simply-connected, the foliation is taut and theorem 2.1 ensures the existence of a transverse metric g'_Q whose transverse Levi-Civita connection ∇' has holonomy contained in $\mathrm{SU}(2n)$. Hence $(M,\mathcal{F},g'_Q,J_1,J_2,J_3)$ satisfies the hypothesis of lemma 5.4 and g'_Q has transverse holonomy group contained in $\mathrm{Sp}(n)$. □

We remark that in the non-foliated case the statement of theorem 5.1 holds without the assumption on M to be simply-connected (see [49]). Indeed, every compact Kähler manifold with vanishing first Chern class can be covered by a Kähler manifold with holomorphically trivial canonical bundle and this fact allows us to drop the assumption on M to be simply-connected. Unfortunately, it seems that a similar construction cannot be performed in the foliated case.

Examples of hyper-Kähler foliations are provided by submersions on to hyper-Kähler manifolds. Other examples are given in the next section in the set-up of Sasakian manifolds. Moreover, a naive way for constructing hyper-Kähler foliations consists in generalizing proposition 2.3 to the hyper-Kähler case. Indeed, let (M,g,J_1,J_2,J_3) be a hyper-Kähler manifold with induced fundamental forms $(\omega_1,\omega_2,\omega_3)$ and let $i\colon N\hookrightarrow M$ be a submanifold. Assume that N is coisotropic with respect to both ω_1 and ω_2 and

$$(TN)^{\omega_1} = (TN)^{\omega_2}\,.$$

In this case $\mathcal{F}_x := (T_xN)^{\omega_1}$ induces a locally trivial hyper-Kähler foliation on N.

6. Sasakian manifolds with special transverse holonomy

In this section, we adapt the results of the previous sections to the Sasakian case having special transverse holonomy. In addition to the previous part of the paper, we consider also the case of a foliated manifold with trivial transverse holonomy group.

6.1. **Riemannian foliations with** $\mathrm{Hol}(\nabla) = 0$**.** We have the following result:

Theorem 6.1. *Let $(M, \mathcal{F}, g_Q)$ be a compact manifold endowed with a Riemannian foliation of codimension q. The transversal holonomy group is trivial if and only if M is the total space of a fibration over the flat torus $\mathbb{T}^q$.*

Proof. Since the transverse holonomy group is trivial, there exists a global parallel orthonormal frame $\{s_i\}_{i=1,\cdots,q}$ of sections of Q. Thus the 1-forms ω_i dual to s_i regarded as 1-forms on M are d-closed and linearly independent at each point. That means the foliation $\mathcal{F}$ is an $\mathbb{R}^q$-Lie foliation and hence in view of [20, p.154] M is the total space of a fibration over the flat torus. □

Now we consider Sasakian manifolds with trivial transverse holonomy. The key pattern is the following: Let $\mathfrak{h}_{2n+1}$ be the $2n+1$-dimensional Heisenberg Lie algebra whose structure equations are given by the choice of a cobasis $\{e^i\}$ satisfying

$$\begin{cases} de^k = 0\,, \quad k = 1, \dots, 2n\,, \\ de^{2n+1} = e^1 \wedge e^2 + e^3 \wedge e^4 + \cdots + e^{2n-1} \wedge e^{2n}\,. \end{cases}$$

(Concisely $\mathfrak{h}_{2n+1}$ has structure equations $(0, \dots, 0, 12 + \cdots + (n-1)n))$. The simply-connected Lie group H associated to $\mathfrak{h}_{2n+1}$ has the following natural invariant Sasakian structure

$$\xi = e_{2n+1}\,, \quad \eta = e^{2n+1}\,, \quad g = \sum e^k \otimes e^k\,, \quad \Phi = e^1 \otimes e_2 - e^2 \otimes e_1 + \cdots + e^{2n-1} \otimes e_{2n} - e^{2n} \otimes e_{2n-1}$$

$\{e_i\}$ is being the dual basis to $\{e^i\}$. It is standard to check that such a Sasakian structure satisfies $\mathrm{Hol}(\nabla) = 0$. Hence if $\Gamma \subseteq \mathrm{H}$ is a co-compact lattice, the compact manifold $M = \Gamma\backslash\mathrm{H}$ inherits a natural ∇-flat Sasakian structure.

The next result says that every ∇-flat compact Sasakian manifold is of the form $M = \Gamma\backslash\mathrm{H}$, for some lattice Γ. Here we regard Sasakian manifolds as manifolds foliated by the foliation spanned by the Reeb vector field.

Theorem 6.2. *Let (M, ξ, η, Φ, g) be a compact Sasakian manifold. Then the holonomy group of ∇ is trivial if and only if M is a compact quotient of the odd-dimensional Heisenberg Lie group* H *by a lattice and (ξ, η, Φ, g) lifts to an invariant Sasakian structure on* H.

Proof. Let (M, ξ, η, Φ, g) be a Sasakian manifold. The holonomy group of ∇ is trivial if and only if there exists a global transverse unitary frame $\{Z_r\}$ satisfying

$$\nabla_{\overline{Z}_r} Z_k = \nabla_{Z_r} Z_k = \nabla_\xi Z_k = \nabla_\xi \overline{Z}_k = 0\,, \quad r, k = 1, \dots, n\,.$$

Conditions $\nabla_{Z_r} Z_k = \nabla_{Z_r} \overline{Z}_k = 0$ say that

$$\nabla^g_{Z_r} Z_k = 0\,, \quad \nabla^g_{\overline{Z}_r} Z_k = -i\delta_{rk}\,\xi$$

whilst condition $\nabla_\xi Z_k = 0$ can be rewritten in terms of brackets as

$$[Z_k, \xi] = 0\,, \quad k = 1, \dots, n\,.$$

Therefore $\{Z_k\, \xi\}$ induces a frame $\{X_i\}$ on M satisfying $[X_i, X_j] = \sum \lambda_{ij}^k\, X_k$ for some constants λ_{ij}^k. In view of [39], M can be written as a quotient of a simply-connected nilpotent Lie group N by a lattice. The vector fields $\{Z_i, \xi\}$ lift to invariant vector fields on N. Let $\{\zeta^i, \eta\}$, be the dual frame to $\{Z_i, \xi\}$; then

$$d\zeta^i = 0\,, \quad d\eta = i \sum_k \zeta^k \wedge \overline{\zeta}^k$$

and N is the Heisenberg Lie group H, as required. □

Remark 6.3. Notice that from the point of view of transverse geometry, manifolds associated to the Heisenberg group play in Sasakian geometry the role that complex tori play in Kähler geometry.

6.2. **Sasakian manifolds with** $\mathrm{Hol}(\nabla) \subseteq \mathrm{SU}(n)$**.** In [46] the geometry of Sasakian manifolds satisfying $\mathrm{Hol}(\nabla) \subseteq \mathrm{SU}(n)$ was studied. These manifolds were named *contact Calabi-Yau*. The main result of [46] is a generalization of McLean's theorem (see [31]) to the Sasakian context, where the role of Lagrangian submanifolds was replaced by some special Legendrian immersions. Indeed, a *contact Calabi-Yau* manifold can be defined as a $2n+1$-dimensional Sasakian manifold (M, ξ, η, Φ, g) with an additional structure given by a basic closed transverse complex volume form ϵ. In analogy to the classical Calabi-Yau case, the real part of ϵ is a calibration on M (see [23]) whose calibrated submanifolds are given by n-dimensional smooth embeddings $i\colon N \hookrightarrow M$ satisfying

$$\begin{cases} i^*(\eta) = 0 \\ i^*(\Im\mathfrak{m}\,\epsilon) = 0\,. \end{cases}$$

Condition $i^*(\eta) = 0$ says that N is a *Legendrian* submanifold and for this reason such submanifolds were named in [46] *special Legendrian*. The main result of [46] is the following:

Theorem 6.4 (Tomassini-Vezzoni [46])**.** *The moduli space of compact Legendrian submanifolds isotopic to a fixed one is always a* 1*-dimensional smooth manifold.*

Some further results about special Legendrian submanifolds with boundary are pointed out in [30].

In this section, we observe that links provide examples of simply-connected compact contact Calabi-Yau manifolds. In order to show this, we consider the following two results arising from section 4 and the Sasakian version of the El Kacimi theorem (see [8]):

Proposition 6.5. *Let* (M, ξ, η, Φ, g) *be a compact simply-connected null Sasaki* η*-Einstein manifold. Then* $\mathrm{Hol}(\nabla)$ *is contained in* $\mathrm{SU}(n)$*.*

Here we recall that in Sasakian geometry *null* η*-Einstein* means that the transverse Ricci tensor vanishes.

Proposition 6.6. *Let* $(M^{2n+1}, \xi, \eta, \Phi, g)$ *be a compact simply-connected Sasaki manifold with* $c_B^1(\mathcal{F}) = 0$*. Then there exists a basic* 1*-form* ζ *on* (M, ξ) *and a unique Sasakian structure* $(M, \xi', \eta', \Phi', g')$ *such that*

$$\xi' = \xi\,, \quad \eta' = \eta + \zeta\,, \quad \Phi' = \Phi - \zeta \otimes \xi \circ \Phi\,, \quad g' = d\eta' \circ (\mathrm{Id} \otimes \Phi') + \eta' \otimes \eta'$$

and the transverse holonomy group of the metric g' *is contained in* $\mathrm{SU}(n)$*.*

It is known that Links provide examples of simply-connected null Sasakian η-Einstein manifolds. More precisely, given a link $L_f = C_f \cap S^{2n-1}$, where $f = (f_1, \dots, f_p)$ are independent weighted homogeneous polynomials of degrees $(d_1, \dots, d_p)$ and weights $(w_1, \dots, w_p)$, then L_f is $(n-p-1)$-connected (see [29]) and inherits a natural η-Einstein Sasakian structure (ξ, η, Φ, g) induced by the weighted Sasakian structure of the sphere. Since L_f is null whenever $\sum(d_i - w_i) = 0$ (see e.g. [5]), by making use of proposition 6.6, we infer

Proposition 6.7. *Let* L_f *be a link, where* $f = (f_1, \dots, f_p)$ *are independent weighted homogeneous polynomials of degrees* $(d_1, \dots, d_p)$ *and weights* $(w_1, \dots, w_p)$ *and assume* $\sum(d_i - w_i) = 0$*. Then* L_f *carries a contact Calabi-Yau structure.*

6.2.1. *A link with* G_2*-geometry.* Let $(M, \xi, \eta, \Phi, g, \epsilon)$ be a 7-dimensional *contact Calabi-Yau manifold* and consider the 3-form

$$\sigma = \eta \wedge d\eta + \Re\mathfrak{e}\,\epsilon.$$

Then σ induces a G_2-structure on M. Since

$$d * \sigma = 0$$

this induced G_2-structure is always *co-calibrated*. A similar construction can be done in 7-dimensional 3-Sasakian manifolds (see [1]). Therefore, proposition 6.7 readily implies

Corollary 6.8. *Let* L_f *be a link as in proposition* 6.7*. Then* L_f *has a co-calibrated* G_2*-structure.*

6.3. **Sasakian manifolds with** $\mathrm{Hol}(\nabla) \subseteq \mathrm{Sp}(n)$**.** In this subsection we translate theorem 5.1 to the context of Sasakian manifolds.

Condition $\mathrm{Hol}(\nabla) \subseteq \mathrm{Sp}(n)$ for a Sasakian manifold $(M, \xi, \eta, \Phi_1, g)$ is equivalent to requiring the existence of a pair $\Phi_2, \Phi_3 \in \mathrm{End}(TM)$ such that

$$(\xi, \eta, \Phi_k, g) \text{ is a Sasakian structure for } k = 2, 3 \tag{18}$$

and satisfy the transverse quaternionic relations

$$\Phi_1\Phi_2 = -\Phi_2\Phi_1\,, \quad \Phi_1\Phi_2 = \Phi_3\,. \tag{19}$$

Conditions (18) can be alternatively rewritten as

$$\Phi_k^2 = -\mathrm{I} + \eta \otimes \xi \tag{20}$$

$$N_{\Phi_k} = 0 \tag{21}$$

$$g(\Phi_k\cdot, \Phi_k\cdot) = g(\cdot,\cdot) - \eta(\cdot)\eta(\cdot) \tag{22}$$

where N_{Φ_k} is the Nijenhuis tensor

$$N_{\Phi_k}(X, Y) = [\Phi_k X, \Phi_k Y] - \Phi_k[\Phi_k X, Y] - \Phi_k[X, \Phi_k X] + \Phi_k^2[X, Y]\,.$$

The following result is the generalization of theorem 5.1 to the Sasakian context:

Theorem 6.9. *Let* $(M, \xi, \eta, \Phi_1, g)$ *be a compact simply-connected* $4n + 1$*-dimensional Sasakian manifold. Assume there is a pair* $\{J_2, J_3\}$ *of transverse complex structures that together with* (g, Φ_1) *define a transverse hyper-Hermitian structure. Then there exists a Sasakian structure* (ξ, η', Φ', g') *on* M *having transverse holonomy contained in* $\mathrm{Sp}(n)$.

Proof. The existence of $\{J_2, J_3\}$ implies that the first basic Chern class of $(M, \xi, \eta, \Phi_1, g)$ is zero. Then by proposition 6.6, there exists a Sasakian structure $\mathcal{S}' = (\xi, \eta', \Phi', g')$ on M

$$\eta' = \eta + \zeta\,, \quad \Phi' = \Phi - \xi \otimes \zeta \circ \Phi\,, \quad g' = d\eta' \circ (\mathrm{Id} \otimes \Phi') + \eta' \otimes \eta'$$

having transverse holonomy contained in $\mathrm{SU}(2n)$. Lemma 5.4 implies that the transverse Levi-Civita connection of g' has holonomy contained in $\mathrm{Sp}(n)$, as required. □

Note that, since $\mathrm{Sp}(1) = \mathrm{SU}(2)$, corollary 6.6 implies that in dimension 5 every simply-connected Sasakian manifold satisfying $\mathrm{Hol}(\nabla) \subseteq \mathrm{Sp}(1)$ is in fact a compact simply-connected null Sasaki η-Einstein manifolds. These kind of manifolds are classified in [10] where it is shown that a 5-dimensional simply-connected compact manifold admits a null Sasaki η-Einstein structure if and only if it is obtained as a connected sum of k-copies of $S^2 \times S^3$, where $k = 3, \dots, 9$.

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