

# The twisted $\text{Spin}^c$ Dirac operator on Kähler submanifolds of the complex projective space

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## Abstract

In this paper, we estimate the eigenvalues of the twisted Dirac operator on Kähler submanifolds of the complex projective space  $\mathbb{C}P^m$  and we discuss the sharpness of this estimate for the embedding  $\mathbb{C}P^d \rightarrow \mathbb{C}P^m$ .

**Keywords:**  $\text{Spin}^c$  geometry, Kähler manifolds and submanifolds, twisted Dirac operator, eigenvalue estimates.

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## 1 Introduction

In his Ph.D. thesis [5], N. Ginoux gave an upper bound for the eigenvalues of the twisted Dirac operator for a Kähler spin submanifold  $M^{2d}$  of a Kähler spin manifold  $\widetilde{M}^{2m}$  carrying Kählerian Killing spinors (see Equation (3)). More precisely, he showed that there are at least  $\mu$  eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_\mu$

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of the square of the twisted Dirac operator satisfying

$$\lambda \leq \begin{cases} (d+1)^2 & \text{if } d \text{ is odd,} \\ d(d+2) & \text{if } d \text{ is even.} \end{cases} \quad (1)$$

Here  $\mu$  denotes the dimension of the space of Kählerian Killing spinors on  $\widetilde{M}^{2m}$ . Recall that the normal bundle is endowed with the induced spin structure coming from both manifolds  $M$  and  $\widetilde{M}$ . The idea consists in computing the so-called Rayleigh-quotient applied to the Kählerian Killing spinor restricted to the submanifold  $M$ . The upper bound is then deduced by using the min-max principle. This technique was also used by C. Bär in [1] for submanifolds in  $\mathbb{R}^{n+1}$ ,  $\mathbb{S}^{n+1}$  and  $\mathbb{H}^{n+1}$ .

The complex projective space  $\mathbb{C}P^m$  is a spin manifold if and only  $m$  is odd. In this case, the sharpness of the upper bound (1) was studied in [6] for the canonical embedding  $\mathbb{C}P^d \rightarrow \mathbb{C}P^m$ , where  $d$  is also odd. In fact, it is shown that for  $d = 1$ , the upper estimate is optimal for  $m = 3, 5, 7$  while it is not for  $m \geq 9$ .

Kähler manifolds are not necessary spin but every Kähler manifold has a canonical  $\text{Spin}^c$  structure (see Section 2) and any other  $\text{Spin}^c$  structure can be expressed in terms of the canonical one. Moreover, O. Hijazi, S. Montiel and F. Urbano [8] constructed on Kähler-Einstein manifolds with positive scalar curvature,  $\text{Spin}^c$  structures carrying Kählerian Killing spinors. Thus one can consider the result of N. Ginoux for  $\text{Spin}^c$  manifolds.

Section 2 is devoted to recall some basic facts on  $\text{Spin}^c$  structures on Kähler manifolds. In Section 3, we extend the estimate (1) to the eigenvalues of the twisted Dirac operator for a Kähler submanifold of the complex projective space (see Theorem 3.1). Finally, we discuss the sharpness for the embedding  $\mathbb{C}P^d \rightarrow \mathbb{C}P^m$  with different values of  $m$  and  $d$ .

## 2 Kähler Submanifolds of Kähler manifolds

Let  $(M^{2m}, g, J)$  be a Kähler manifold of complex dimension  $m$ . Recall that the complexified tangent bundle splits into the orthogonal sum  $T^{\mathbb{C}}M = T_{1,0}M \oplus T_{0,1}M$  where  $T_{1,0}M$  (resp.  $T_{0,1}M$ ) denotes the eigenbundle of  $T^{\mathbb{C}}M$  corresponding to the eigenvalue  $i$  (resp.  $-i$ ) of the extension of  $J$ . Using this decomposition, we define  $\Lambda^{0,r}M := \Lambda^r(T_{0,1}^*M)$  (resp.  $\Lambda^{r,0}M$ ) as being the

bundle of complex  $r$ -forms of type  $(0, r)$  (resp. of type  $(r, 0)$ ). Recall also that every Kähler manifold has a *canonical*  $\text{Spin}^c$  structure whose complex spinorial bundle is given by  $\Sigma M = \Lambda^{0,*} M = \bigoplus_{r=0}^m \Lambda^{0,r} M$ , where the auxiliary bundle of this  $\text{Spin}^c$  structure is given by  $K_M^{-1}$ . Here  $K_M$  is the canonical bundle of  $M$  defined by  $K_M = \Lambda^{m,0} M$  [4, 11]. On the other hand, the spinor bundle of any other  $\text{Spin}^c$  structure can be written as [4, 8]:

$$\Sigma M = \Lambda^{0,*} M \otimes \mathfrak{L},$$

where  $\mathfrak{L}^2 = K_M \otimes L$  and  $L$  is the auxiliary bundle associated with this  $\text{Spin}^c$  structure. Moreover, the action of the Kähler form  $\Omega$  of  $M$  splits the spinor bundle into [4, 10, 9]:

$$\Sigma M = \bigoplus_{r=0}^m \Sigma_r M,$$

where  $\Sigma_r M$  denotes the eigensubbundle corresponding with the eigenvalue  $i(2r - m)$  of  $\Omega$  with complex rank  $\binom{m}{k}$ . For any vector field  $X \in \Gamma(TM)$  and  $\psi \in \Gamma(\Sigma_r M)$ , we have the following property  $p_{\pm}(X) \cdot \psi \in \Gamma(\Sigma_{r\pm 1} M)$ , where  $p_{\pm}(X) = \frac{1}{2}(X \mp iJX)$ .

Let  $(M^{2d}, g, J)$  be an immersed Kähler submanifold in a Kähler manifold  $(\widetilde{M}^{2m}, g, J)$  carrying the induced complex structure  $J$  (i.e.  $J(TM) = TM$ ) and denote respectively by  $\Omega_{\widetilde{M}}$ ,  $\Omega$  and  $\Omega_N$  the Kähler form of  $\widetilde{M}$ ,  $M$  and of the normal bundle  $NM \rightarrow M$  of the immersion. Since the manifolds  $M$  and  $\widetilde{M}^{2n}$  are Kähler, they carry  $\text{Spin}^c$  structures with corresponding auxiliary line bundles  $L_M$  and  $L_{\widetilde{M}}$ . This induces a  $\text{Spin}^c$  structure on the bundle  $NM$  such that the restricted complex spinor bundle  $\Sigma \widetilde{M}|_M$  of  $\widetilde{M}$  can be identified with  $\Sigma M \otimes \Sigma N$ , where  $\Sigma M$  and  $\Sigma N$  are the spinor bundles of  $M$  and  $NM$  respectively ([1], [7]). Moreover, the auxiliary line bundle  $L_N$  of this  $\text{Spin}^c$  structure on  $NM$  is given by  $L_N := (L_M)^{-1} \otimes (L_{\widetilde{M}})|_M$ . Given connection 1-forms on  $L_M$  and  $L_{\widetilde{M}}$ , they induce a connection  $\nabla := \nabla^{\Sigma M \otimes \Sigma N}$  on  $\Sigma M \otimes \Sigma N$ . Thus one can state a Gauss-type formula for the spinorial Levi-Civita connections  $\widetilde{\nabla}$  and  $\nabla$  on  $\Sigma \widetilde{M}$  and  $\Sigma M \otimes \Sigma N$  respectively [13]. That is, for all  $X \in TM$  and  $\varphi \in \Gamma(\Sigma \widetilde{M}|_M)$ , we have

$$\widetilde{\nabla}_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1}^{2d} e_j \cdot II(X, e_j) \cdot \varphi, \quad (2)$$

where  $(e_j)_{1 \leq j \leq 2d}$  is any local orthonormal basis of  $TM$  and  $II$  is the second fundamental form of the immersion. As a consequence of the Gauss formula, the square of the auxiliary Dirac-type operator  $\widehat{D} := \sum_{j=1}^{2d} e_j \cdot \widetilde{\nabla}_{e_j}$  is related

to the square of the twisted Dirac operator  $D_M^{\Sigma N} := \sum_{j=1}^{2d} e_j \cdot \nabla_{e_j}$  by [5, Lemme 4.1]:

$$\widehat{D}^2 \varphi = (D_M^{\Sigma N})^2 \varphi - d^2 |H|^2 \varphi - d \sum_{j=1}^{2d} e_j \cdot \nabla_{e_j}^N H \cdot \varphi,$$

where  $H := \frac{1}{2d} \text{tr}(II)$  is the mean curvature vector field of the immersion. In our case, the mean curvature vanishes which means that the operators  $\widehat{D}^2$  and  $(D_M^{\Sigma N})^2$  coincide.

In the sequel, take the manifold  $\widetilde{M}$  as the complex projective space  $\mathbb{C}P^m$  endowed with its Fubini-Study metric of constant holomorphic sectional curvature 4. In [8], the authors proved that for every  $q \in \mathbb{Z}$ , such that  $q + m + 1 \in 2\mathbb{Z}$ , there exists a  $\text{Spin}^c$  structure on  $\mathbb{C}P^m$  whose auxiliary line bundle is given by  $\mathcal{L}_m^q$ . Here  $\mathcal{L}_m$  denotes the tautological bundle of  $\mathbb{C}P^m$ . In particular for  $q = -m - 1$  (resp.  $q = m + 1$ ), the  $\text{Spin}^c$  structure is the canonical one (resp. anti-canonical) [12] and for  $q = 0$  it corresponds to the unique spin structure if  $m$  is odd. Let us denote by  $\Sigma^q \mathbb{C}P^m$  the spinor bundle of the corresponding  $\text{Spin}^c$  structure with  $\mathcal{L}^q$  as auxiliary line bundle. For any integer  $r$  in  $\{0, \dots, m + 1\}$  such that  $q = 2r - (m + 1)$ , the bundle  $\Sigma^q \mathbb{C}P^m$  carries a Kählerian Killing spinor field  $\psi = \psi_{r-1} + \psi_r$  satisfying, for all  $X \in \Gamma(T\mathbb{C}P^m)$  [8]

$$\begin{aligned} \widetilde{\nabla}_X \psi_r &= -p_+(X) \cdot \psi_{r-1}, \\ \widetilde{\nabla}_X \psi_{r-1} &= -p_-(X) \cdot \psi_r, \end{aligned} \tag{3}$$

The space of Kählerian Killing spinors is of rank  $\binom{m+1}{r}$ . We point out that for  $r = 0$  (resp.  $r = m+1$ ) the Kählerian Killing spinor is a parallel spinor which is carried by the canonical structure (resp. anti-canonical). Moreover, for  $r = \frac{m+1}{2}$ , i.e.  $m$  is odd, the Kählerian Killing spinor is the usual one lying in  $\Sigma_{\frac{m-1}{2}}^0 \mathbb{C}P^m \oplus \Sigma_{\frac{m+1}{2}}^0 \mathbb{C}P^m$  defined in [9, 10].

### 3 Main result

In this section, we will establish the estimates for the eigenvalues of the twisted Dirac operator of complex submanifolds of the complex projective space. We will test the sharpness of Inequality (4) for the canonical embedding  $\mathbb{C}P^d \rightarrow \mathbb{C}P^m$ . For more details, we refer to [6].

**Theorem 3.1** *Let  $(M^{2d}, g, J)$  be a closed Kähler submanifold of the complex projective space  $\mathbb{C}P^m$ . For  $r \in \{0, \dots, m+1\}$  and  $q = 2r - (m+1)$ , there are at least  $\binom{m+1}{r}$ -eigenvalues  $\lambda$  of  $(D_M^{\Sigma N})^2$  satisfying*

$$\lambda \leq \begin{cases} -(q^2 - (d+1)^2) + 2|q|(m-d) - 1 & \text{if } m-d \text{ is odd} \\ -(q^2 - (d+1)^2) + 2|q|(m-d) & \text{if } m-d \text{ is even.} \end{cases} \quad (4)$$

**Proof.** The proof relies on computing the Rayleigh-quotient

$$\frac{\int_M \operatorname{Re}\langle (D_M^{\Sigma N})^2 \psi, \psi \rangle v_g}{\int_M |\psi|^2 v_g}$$

applied to any non-zero Kählerian Killing spinor  $\psi = \psi_{r-1} + \psi_r$  on  $\mathbb{C}P^m$ . A straightforward computation of the auxiliary Dirac operator leads to

$$\widehat{D}\psi_{r-1} = (q+d+1)\psi_r + i\Omega_N \cdot \psi_r.$$

$$\widehat{D}\psi_r = -(q-d-1)\psi_{r-1} - i\Omega_N \cdot \psi_{r-1}.$$

Summing up the above two equations, we deduce after using the fact that the auxiliary Dirac operator commutes with the normal Kähler form [6], that

$$\widehat{D}^2\psi = -(q^2 - (d+1)^2)\psi - 2iq\Omega^N \cdot \psi + \Omega^N \cdot \Omega^N \cdot \psi.$$

Taking the Hermitian inner product with  $\psi$  and using the fact that the second term can be bounded from above by  $2|q|(m-d)$ , we get our estimates after using  $|\Omega^N \cdot \psi| \geq |\psi|$  if  $m-d$  is odd and 0 otherwise.  $\square$

In the following, we will treat the sharpness through the embedding  $\mathbb{C}P^d \rightarrow \mathbb{C}P^m$ . Recall first that the complex projective space  $\mathbb{C}P^d$  can be seen as the symmetric space  $SU_{d+1}/S(U_d \times U_1)$  where  $S(U_d \times U_1) := \left\{ \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} \mid B \in U_d \right\}$ . The tangent bundle of  $\mathbb{C}P^d$  can be described as a homogeneous bundle which is associated with the  $S(U_d \times U_1)$ -principal bundle  $SU_{d+1} \rightarrow \mathbb{C}P^d$  via the isotropy representation

$$\begin{aligned} \alpha : \quad S(U_d \times U_1) &\longrightarrow U_d \\ \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} &\longmapsto \det(B)B. \end{aligned}$$

For the canonical embedding  $\mathbb{C}P^d \rightarrow \mathbb{C}P^m$ , the normal bundle  $T^\perp \mathbb{C}P^d$  is isomorphic to  $\mathcal{L}_d^* \otimes \mathbb{C}^{m-d}$  where  $\mathcal{L}_d$  is the tautological bundle of  $\mathbb{C}P^d$ . The bundle  $\mathcal{L}_d$  is isomorphic to the homogeneous bundle which is associated with the  $S(U_d \times U_1)$ -principal bundle  $SU_{d+1}$  via the representation

$$\begin{aligned} \rho : \quad S(U_d \times U_1) &\longrightarrow U_1 \\ \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} &\longmapsto (\det(B))^{-1}. \end{aligned}$$

Thus the normal bundle is associated with the  $S(U_d \times U_1)$ -principal bundle  $SU_{d+1} \rightarrow \mathbb{C}P^d$  via the representation

$$\begin{aligned} \rho : \quad S(U_d \times U_1) &\longrightarrow U_{m-d} \\ \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} &\longmapsto \det(B)I_{m-d}. \end{aligned}$$

Now, we endow  $\mathbb{C}P^d$  with a  $\text{Spin}^c$  structure whose auxiliary line bundle is given by  $\mathcal{L}_d^{q'}$  for  $q' \in \mathbb{Z}$ . In this case, its spinor bundle is given by

$$\Sigma^{q'} \mathbb{C}P^d = \Lambda^{0,*} \mathbb{C}P^d \otimes \mathcal{L}_d^{\frac{q'+d+1}{2}}.$$

The existence of  $\text{Spin}^c$  structures on both  $\mathbb{C}P^d$  and  $\mathbb{C}P^m$  induces also a  $\text{Spin}^c$  structure on the normal bundle of the embedding with auxiliary line bundle is given by  $\mathcal{L}_m^q|_{\mathbb{C}P^d} \otimes \mathcal{L}_d^{q'}$  which is isomorphic to  $\mathcal{L}_d^{q-q'}$ . Therefore the Lie-group homomorphism

$$\begin{aligned} \rho : \quad S(U_d \times U_1) &\longrightarrow U_{m-d} \times U_1 \\ \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} &\longmapsto (\det(B)I_{m-d}, \det(B)^{-(q-q')}) \end{aligned}$$

can be lifted through the non-trivial two-fold covering map  $\text{Spin}_{2(m-d)}^c \rightarrow \text{SO}_{2(m-d)} \times U_1$  to the homomorphism

$$\begin{aligned} \tilde{\rho} : \quad S(U_d \times U_1) &\longrightarrow \text{Spin}_{2(m-d)}^c \\ \begin{pmatrix} B & 0 \\ 0 & \det(B)^{-1} \end{pmatrix} &\longmapsto (\det(B))^{-\frac{q-q'+m-d}{2}} j(\det(B)I_{m-d}), \end{aligned}$$

where for any positive integer  $k$ , we recall that  $j : U_k \rightarrow \text{Spin}_{2k}^c$  is given on elements of diagonal form of  $U_k$  as

$$j(\text{diag}(e^{i\lambda_1}, \dots, e^{i\lambda_k})) = e^{\frac{i}{2}(\sum_{j=1}^k \lambda_j)} \tilde{R}_{e_1, J_{e_1}}\left(\frac{\lambda_1}{2}\right) \cdots \tilde{R}_{e_k, J_{e_k}}\left(\frac{\lambda_k}{2}\right).$$

Here  $J$  is the canonical complex structure on  $\mathbb{C}^k$  and  $\tilde{R}_{v,w}(\lambda) = \cos(\lambda) + \sin(\lambda)v \cdot w \in \text{Spin}_{2k}$  is defined for any orthonormal system  $\{v, w\} \in \mathbb{R}^{2k}$ . We point out that the integer  $q - q' + m - d$  is even. Following the similar proof as in [6, Corollary 4.4], the complex spinor bundle of  $T^\perp \mathbb{C}P^d$  splits into the orthogonal sum

$$\Sigma(T^\perp \mathbb{C}P^d) \cong \bigoplus_{s=0}^{m-d} \binom{m-d}{s} \mathcal{L}_d^{\frac{q-q'+m-d}{2}-s},$$

where for each  $s \in \{0, \dots, m-d\}$ , the factor  $\binom{m-d}{s}$  stands the multiplicity which the line bundle  $\mathcal{L}_d^{\frac{q-q'+m-d}{2}-s}$  appears in the splitting. This gives the following decomposition

$$\begin{aligned} \Sigma \mathbb{C}P^d \otimes \Sigma^\perp \mathbb{C}P^d &\simeq \bigoplus_{s=0}^{m-d} \binom{m-d}{s} \Sigma \mathbb{C}P^d \otimes \mathcal{L}_d^{\frac{q-q'+m-d}{2}-s} \\ &\simeq \bigoplus_{s=0}^{m-d} \binom{m-d}{s} \Lambda^{0,*} \mathbb{C}P^d \otimes \mathcal{L}_d^{\frac{d+1+q'}{2}} \otimes \mathcal{L}_d^{\frac{q-q'+m-d}{2}-s} \\ &\simeq \bigoplus_{s=0}^{m-d} \binom{m-d}{s} \Lambda^{0,*} \mathbb{C}P^d \otimes \mathcal{L}_d^{\frac{m+1+q}{2}-s}. \end{aligned}$$

We point out here that the above decomposition does not depend on the  $\text{Spin}^c$  structure chosen on  $\mathbb{C}P^d$ , since no power in  $q'$  appears. In [3], the authors proved that (see also [6, 2]):

**Proposition 3.2** *Let  $\mathbb{C}P^d$  be the complex projective space of constant holomorphic sectional curvature 4 endowed with a  $\text{Spin}^c$  structure whose spinor bundle is given by  $\Lambda^{0,*} \mathbb{C}P^d \otimes \mathcal{L}^v$ , for some  $v \in \mathbb{Z}$ , i.e., whose auxiliary line bundle is given by  $\mathcal{L}_d^{2v-(d+1)}$ . Then, the spectrum of the square of the Dirac operator is given by the eigenvalue 0 if  $v \leq 0$  or  $v \geq d+1$  and by*

$$\lambda^2 = 4(l+v)(l-k+d),$$

where  $l \in \mathbb{N}$ ,  $l+v \geq k+1$  and  $0 \leq k \leq d-1$ . Moreover, the multiplicity of  $\lambda^2$  is given by

$$\frac{2(l+d)!(l+v-k-1+d)!(2l+v-k+d)}{l!k!d!(l+v-k-1)!(d-k-1)!(l+v)(l+d-k)}$$

and the multiplicity of 0 by  $\frac{(|v|+d)!}{d!|v|!}$  if  $v \leq 0$  and by  $\frac{(v-1)!}{d!(v-d-1)!}$  if  $v \geq d+1$ .

In order to find the spectrum of the square of the twisted Dirac operator corresponding with the embedding  $\mathbb{C}P^d \rightarrow \mathbb{C}P^m$ , one should replace  $v$  in Proposition 3.2 by  $\frac{m+1+q}{2} - s$  and in this case, the eigenvalue of the square of the twisted Dirac operator is given by 0 if  $\frac{m+1+q}{2} - s \leq 0$  or if  $\frac{m+1+q}{2} - s \geq d + 1$  and by  $4(l + \frac{m+1+q}{2} - s)(l - k + d)$  for  $0 \leq s \leq m - d$ ,  $0 \leq k \leq d - 1$  and  $l + \frac{m+q+1}{2} - s \geq k + 1$ .

Let us consider particular values for  $d, m$  and  $q$  in order to check the optimality. For  $d = 1, m = 2$  and  $q = 1$ , by Theorem 3.1, there are at least 3 eigenvalues of the square of the twisted Dirac operator satisfying the estimate  $\lambda \leq 4$ . The multiplicity of zero is 1 and the multiplicity of the eigenvalue 4 is 4 which means that the estimate is optimal. For  $d = 1, m = 3$  and  $q = 2$ , by Theorem 3.1, there are at least 4 eigenvalues of the square of the twisted Dirac operator satisfying the estimate  $\lambda \leq 8$ . The multiplicity of zero is 3. The multiplicity of the eigenvalue 4 is 4 and the multiplicity of the eigenvalue 8 is 6 which means that the estimate is not optimal. For  $d = 2, m = 3$  and  $q = 2$ , there are at least 4 eigenvalues of the square of the twisted Dirac operator satisfying the estimate  $\lambda \leq 8$ . The multiplicity of zero is 1 and the multiplicity of the eigenvalue 8 is 6 which means that the estimate is optimal.

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## References

- [1] C. Bär, *Extrinsic bounds for eigenvalues of the Dirac operator*, Ann. Glob. Anal. Geom. **16** (1998), no. 2, 573-596.
- [2] M. Ben Halima, *Spectrum of twisted Dirac operators on the complex projective space  $\mathbb{P}^{2q+1}(\mathbb{C})$* , Comment. Math. Univ. Carolin. **49** (2008), no. 3, 437-445.
- [3] B. P. Dolan, I. Huet, S. Murray and D. O'Connor *A Universal Dirac operator and noncommutative spin bundles over fuzzy complex projective spaces*, JHEP **03** (2008), 29, arXiv:0711.1347[hep-th].

- [4] Th. Friedrich, *Dirac operator's in Riemannian geometry*, Graduate studies in mathematics, Volume 25, American Mathematical Society.
- [5] N. Ginoux, *Opérateurs de Dirac sur les sous-variétés*, PhD thesis, Université Henri Poincaré, Nancy, 2002.
- [6] N. Ginoux and G. Habib, *The spectrum of the twisted Dirac operator on Kähler submanifolds of the complex projective space*, Manuscripta Math. **137** (2012), 215-231.
- [7] N. Ginoux and B. Morel, *On eigenvalue estimates for the submanifold Dirac operator*, Internat. J. Math. **13** (2002), no. 5, 533-548.
- [8] O. Hijazi, S. Montiel and F. Urbano, *Spin<sup>c</sup> geometry of Kähler manifolds and the Hodge Laplacian on minimal Lagrangian submanifolds*, Math. Z. **253**, Number 4 (2006) 821-853.
- [9] O. Hijazi, *Eigenvalues of the Dirac operator on compact Kähler manifolds*, Commun. Math. Phys. **160**, 563-579 (1994).
- [10] K.-D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature*, Ann. Glob. Anal. Geom. **4**, (1986) no.3, 291-325.
- [11] B. Mellor, *Spin<sup>c</sup>-manifolds*, unpublished paper.
- [12] A. Moroianu, *Lectures on Kähler Geometry*, London Mathematical Society Student Texts 69, Cambridge University Press, Cambridge, 2007.
- [13] R. Nakad, *The Energy-Momentum tensor on Spin<sup>c</sup> manifolds*, IJGMMP Vol. 8, No. 2, 2011.