

Eigenvalues of the transversal Dirac Operator on Kähler Foliations

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Abstract

In this paper, we prove Kirchberg-type inequalities for any Kähler spin foliation. Their limiting-cases are then characterized as being transversal minimal Einstein foliations. The key point is to introduce the transversal Kählerian twistor operators.

1 Introduction

On a compact Riemannian spin manifold (M^n, g_M) , Th. Friedrich [Fri80] showed that any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)} S_0, \quad (1.1)$$

where S_0 denotes the infimum of the scalar curvature of M . The limiting case in (1.1) is characterized by the existence of a *Killing spinor*. As a consequence M is Einstein. K.D. Kirchberg [Kir86] established that, on such manifolds any eigenvalue λ satisfies the inequalities

$$\lambda^2 \geq \begin{cases} \frac{m+1}{4m} S_0 & \text{if } m \text{ is odd,} \\ \frac{m}{4(m-1)} S_0 & \text{if } m \text{ is even.} \end{cases}$$

On a compact Riemannian spin foliation (M, g_M, \mathcal{F}) of codimension q with a bundle-like metric g_M such that the mean curvature κ is a basic coclosed

1-form, S.D. Jung [Jun01] showed that any eigenvalue λ of the transversal Dirac operator satisfies

$$\lambda^2 \geq \frac{q}{4(q-1)} K_0^\nabla, \quad (1.2)$$

where $K_0^\nabla = \inf_M(\sigma^\nabla + |\kappa|^2)$, here σ^∇ denotes the transversal scalar curvature with the transversal Levi-Civita connection ∇ . The limiting case in (1.2) is characterized by the fact that \mathcal{F} is minimal ($\kappa = 0$) and transversally Einstein (see Theorem 3.1). The main result of this paper is the following:

Theorem 1.1 *Let (M, g_M, \mathcal{F}) be a compact Riemannian manifold with a Kähler spin foliation \mathcal{F} of codimension $q = 2m$ and a bundle-like metric g_M . Assume that κ is a basic coclosed 1-form, then any eigenvalue λ of the transversal Dirac operator satisfies:*

$$\lambda^2 \geq \frac{m+1}{4m} K_0^\nabla \quad \text{if } m \text{ is odd,} \quad (1.3)$$

and

$$\lambda^2 \geq \frac{m}{4(m-1)} K_0^\nabla \quad \text{if } m \text{ is even.} \quad (1.4)$$

The limiting case in (1.3) is characterised by the fact that the foliation is minimal and by existence of a transversal Kählerian Killing spinor (see Theorem 4.3). We refer to Theorem 4.4 for the equality case in (1.4).

We point out that Inequality (1.3) was proved by S. D. Jung [JK03] with the additional assumption that κ is *transversally holomorphic*. The author would like to thank Oussama Hijazi for his support.

2 Foliated manifolds

In this section, we summarize some standard facts about foliations. For more details, we refer to [Ton88], [Jun01].

Let (M, g_M) be a $(p+q)$ -dimensional Riemannian manifold and a foliation \mathcal{F} of codimension q and let ∇^M be the Levi-civita connection associated with g_M . We consider the exact sequence

$$0 \longrightarrow L \xrightarrow{\iota} TM \xrightarrow{\pi} Q \longrightarrow 0,$$

where L is the tangent bundle of TM and $Q = TM/L \simeq L^\perp$ the normal bundle. We assume g_M to be a *bundle-like metric* on Q , that means the induced metric g_Q verifies the holonomy invariance condition,

$$\mathcal{L}_X g_Q = 0, \quad \forall X \in \Gamma(L),$$

where \mathcal{L}_X is the Lie derivative with respect to X . Let ∇ be the connection on Q defined by:

$$\nabla_X s = \begin{cases} \pi [X, Y_s], & \forall X \in \Gamma(L), \\ \pi (\nabla_X^M Y_s), & \forall X \in \Gamma(L^\perp), \end{cases}$$

where $s \in \Gamma(Q)$ and Y_s is the unique vector of $\Gamma(L^\perp)$ such that $\pi(Y_s) = s$. The connection ∇ is metric and torsion-free. The curvature of ∇ acts on $\Gamma(Q)$ by :

$$R^\nabla(X, Y)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]}s, \quad \forall X, Y \in \chi(M).$$

The transversal Ricci curvature is defined by:

$$\begin{aligned} \rho^\nabla : \Gamma(Q) &\longrightarrow \Gamma(Q) \\ X &\longmapsto \rho^\nabla(X) = \sum_{j=1}^q R^\nabla(X, e_j) e_j. \end{aligned}$$

Also, we define the transversal scalar curvature :

$$\sigma^\nabla = \sum_{i=1}^q g_Q(\rho^\nabla(e_i), e_i) = \sum_{i,j=1}^q R^\nabla(e_i, e_j, e_j, e_i),$$

where $\{e_i\}_{i=1, \dots, q}$ is a local orthonormal frame of Q and $R^\nabla(X, Y, Z, W) = g_Q(R^\nabla(X, Y)Z, W)$, for all $X, Y, Z, W \in \Gamma(Q)$. The foliation \mathcal{F} is said to be transversally Einstein if and only if

$$\rho^\nabla = \frac{1}{q} \sigma^\nabla \text{Id},$$

with constant transversal scalar curvature. The mean curvature of Q is given by:

$$\kappa(X) = g_Q(\tau, X), \quad \forall X \in \Gamma(Q),$$

where $\tau = \sum_{l=1}^p II(e_l, e_l)$, with $\{e_l\}_{l=1, \dots, p}$ is a local orthonormal frame of $\Gamma(L)$ and II is the second fundamental form of \mathcal{F} defined by:

$$\begin{aligned} II : \Gamma(L) \times \Gamma(L) &\longrightarrow \Gamma(Q) \\ (X, Y) &\longmapsto II(X, Y) = \pi(\nabla_X^M Y). \end{aligned}$$

We define basic r -forms by :

$$\Omega_B^r(\mathcal{F}) = \{\Phi \in \Lambda^r T^*M \mid X \lrcorner \Phi = 0 \text{ and } X \lrcorner d\Phi = 0, \quad \forall X \in \Gamma(L)\},$$

where d is the exterior derivative and X_{\perp} is the interior product. Any $\Phi \in \Omega_B^r(\mathcal{F})$ can be locally written as

$$\sum_{1 \leq j_1 < \dots < j_r \leq q} \beta_{j_1, \dots, j_r} dy_{j_1} \wedge \dots \wedge dy_{j_r},$$

where $\frac{\partial}{\partial x_l} \beta_{j_1, \dots, j_r} = 0$, $\forall l = 1, \dots, p$. With the local expression of basic r -forms, one can verify that κ is closed if \mathcal{F} is isoparametric ($\kappa \in \Omega_B^1(\mathcal{F})$). For all $r \geq 0$,

$$d(\Omega_B^r(\mathcal{F})) \subset \Omega_B^{r+1}(\mathcal{F}).$$

We denote by $d_B = d|_{\Omega_B(\mathcal{F})}$ where $\Omega_B(\mathcal{F})$ is the tensor algebra of $\Omega_B^r(\mathcal{F})$. We have the following formulas:

$$d_B = \sum_{i=1}^q e_i^* \wedge \nabla_{e_i} \quad \text{and} \quad \delta_B = - \sum_{i=1}^q e_i \lrcorner \nabla_{e_i} + \kappa \lrcorner,$$

where δ_B is the adjoint operator of d_B with respect to the induced scalar product and $\{e_i\}_{i=1, \dots, q}$ is a local orthonormal frame of Q .

3 The transversal Dirac operator on Kähler Foliations

In this section, we start by recalling some facts on Riemannian foliations which could be found in [GK91a], [GK91b], [AG97], [Jun01]. For completeness, we also sketch a straightforward proof of Inequality ((1.2)) established in [Jun01] and end by recalling well-known facts (see [Kir86], [Kir96], [Hij94a], [Hij94b], [JK03]) on Kähler spin foliations.

On a foliated Riemannian manifold (M, g_M, \mathcal{F}) , a transversal spin structure is a pair $(\text{Spin}Q, \eta)$ where $\text{Spin}Q$ is a Spin_q -principal fibre bundle over M and η a 2-fold cover such that the following diagram commutes:

$$\begin{array}{ccccc} \text{Spin}Q \times \text{Spin}_q & \longrightarrow & \text{Spin}Q & \longrightarrow & M \\ \downarrow \eta \otimes \text{Ad} & & \downarrow \eta & \nearrow & \\ \text{SO}Q \times \text{SO}_q & \longrightarrow & \text{SO}Q & & \end{array}$$

The maps $\text{Spin}Q \times \text{Spin}_q \longrightarrow \text{Spin}Q$, and $\text{SO}Q \times \text{SO}_q \longrightarrow \text{SO}Q$, are respectively the actions of Spin_q and SO_q on the principal fibre bundles $\text{Spin}Q$ and

SOQ . In this case, \mathcal{F} is called a transversal spin foliation. We define the foliated spinor bundle by: $S(\mathcal{F}) := \text{Spin}Q \times_{\rho} \Sigma_q$, where $\rho : \text{Spin}_q \rightarrow \text{Aut}(\Sigma_q)$, is the complex spin representation and Σ_q is a \mathbb{C} vector space of dimension N with $N = 2^{\lfloor \frac{q}{2} \rfloor}$, where $\lfloor \cdot \rfloor$ stands for the integer part. Recall that the Clifford multiplication \mathcal{M} on $S(\mathcal{F})$ is given by:

$$\begin{aligned} \mathcal{M} : \Gamma(Q) \times \Gamma(S(\mathcal{F})) &\longrightarrow \Gamma(S(\mathcal{F})) \\ (X, \Psi) &\longmapsto X \cdot \Psi. \end{aligned}$$

There is a natural Hermitian product on $S(\mathcal{F})$ such that, for all $X, Y \in \Gamma(Q)$, the following relations are true:

$$\begin{aligned} \langle X \cdot \Psi, \Phi \rangle &= -\langle \Psi, X \cdot \Phi \rangle, \\ X(\langle \Psi, \Phi \rangle) &= \langle \nabla_X \Psi, \Phi \rangle + \langle \Psi, \nabla_X \Phi \rangle, \\ \nabla_Y(X \cdot \Psi) &= (\nabla_Y X) \cdot \Psi + X \cdot (\nabla_Y \Psi), \end{aligned}$$

where ∇ is the Levi-Civita connection on $S(\mathcal{F})$ and $\Psi, \Phi \in \Gamma(S(\mathcal{F}))$.

The transversal Dirac operator [GK91a, GK91b] is locally given by:

$$D_{tr} \Psi = \sum_{i=1}^q e_i \cdot \nabla_{e_i} \Psi - \frac{1}{2} \kappa \cdot \Psi, \quad (3.1)$$

for all $\Psi \in \Gamma(S(\mathcal{F}))$. We can easily prove using Green's theorem [YT90] that this operator is formally self adjoint. Furthermore, in [GK91b] it is proved that if \mathcal{F} is isoparametric and $\delta_B \kappa = 0$, then we have the Schrödinger-Lichnerowicz formula:

$$D_{tr}^2 \Psi = \nabla_{tr}^* \nabla_{tr} \Psi + \frac{1}{4} K_{\sigma}^{\nabla} \Psi,$$

where $K_{\sigma}^{\nabla} = \sigma^{\nabla} + |\kappa|^2$ and

$$\nabla_{tr}^* \nabla_{tr} \Psi = - \sum_{i=1}^q \nabla_{e_i, e_i}^2 \Psi + \nabla_{\kappa} \Psi,$$

with $\nabla_{X, Y}^2 = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$, for all $X, Y \in \Gamma(TM)$. Denote by \mathcal{P} the transversal twistor operator defined by

$$\mathcal{P} : \Gamma(S(\mathcal{F})) \xrightarrow{\nabla_{tr}} \Gamma(Q^* \otimes S(\mathcal{F})) \xrightarrow{\pi} \Gamma(\ker \mathcal{M}),$$

where π is the orthogonal projection on the kernel of the Clifford multiplication \mathcal{M} . With respect to a local orthonormal frame $\{e_1, \dots, e_q\}$, for all $\Psi \in \Gamma(S(\mathcal{F}))$, one has

$$\mathcal{P}\Psi = \sum_{i=1}^q e_i^* \otimes (\nabla_{e_i} \Psi + \frac{1}{q} e_i \cdot D_{tr} \Psi + \frac{1}{2q} e_i \cdot \kappa \cdot \Psi). \quad (3.2)$$

For any spinor field Ψ , one can easily show that

$$\sum_{i=1}^q e_i \cdot \mathcal{P}_{e_i} \Psi = 0. \quad (3.3)$$

Now we give a simple proof of the following theorem:

Theorem 3.1 [Jun01] *Let (M, g_M, \mathcal{F}) be a compact Riemannian manifold with a spin foliation \mathcal{F} of codimension q and a bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$. Assume that $\delta_B \kappa = 0$ and let λ be an eigenvalue of the transversal Dirac operator, then*

$$\lambda^2 \geq \frac{q}{4(q-1)} K_0^\nabla. \quad (3.4)$$

Proof. For all $\Psi \in \Gamma(S(\mathcal{F}))$, we have using Identities (3.2), (3.3) (3.1),

$$|\mathcal{P}\Psi|^2 = |\nabla^{tr} \Psi|^2 - \frac{1}{q} |D_{tr} \Psi|^2 - \frac{1}{q} \Re(D_{tr} \Psi, \kappa \cdot \Psi) - \frac{1}{4q} |\kappa|^2 |\Psi|^2.$$

For any spinor field Φ , we have that $(\Phi, \kappa \cdot \Phi) = -(\kappa \cdot \Phi, \Phi) = -\overline{(\Phi, \kappa \cdot \Phi)}$, so the scalar product $(\Phi, \kappa \cdot \Phi)$ is a pure imaginary function. Hence for any eigenspinor Ψ of the transversal Dirac operator, we obtain

$$\int_M |\mathcal{P}\Psi|^2 + \frac{1}{4q} \int_M |\kappa|^2 |\Psi|^2 = \int_M |\nabla^{tr} \Psi|^2 - \frac{1}{q} \int_M \lambda^2 |\Psi|^2,$$

from which we deduce (3.4) with the help of the Schrödinger-Lichnerowicz formula. Finally, we can easily prove in the limiting case that \mathcal{F} is minimal i.e. $\kappa = 0$, and transversally Einstein. \square

A foliation \mathcal{F} is called Kähler if there exists a complex parallel orthogonal structure $J : \Gamma(Q) \rightarrow \Gamma(Q)$ ($\dim Q = q = 2m$). Let Ω be the associated Kähler, i.e., for all $X, Y \in \Gamma(Q)$, $\Omega(X, Y) = g_Q(J(X), Y) = -g_Q(X, J(Y))$. The Kähler form can be locally expressed as

$$\Omega = \frac{1}{2} \sum_{i=1}^q e_i \cdot J(e_i) = -\frac{1}{2} \sum_{i=1}^q J(e_i) \cdot e_i,$$

and for all $X \in \Gamma(Q)$, we have $[\Omega, X] := \Omega \cdot X - X \cdot \Omega = 2J(X)$. Under the action of the Kähler form, the spinor bundle splits into an orthogonal sum

$$S(\mathcal{F}) = \bigoplus_{r=0}^m S_r(\mathcal{F}),$$

where $S_r(\mathcal{F})$ is an eigenbundle associated with the eigenvalue $i\mu_r = i(2r - m)$ of the Kähler form Ω . Moreover, the spinor bundle of a Kähler spin foliation carries a parallel anti-linear map j satisfying the relations:

$$\begin{aligned} j^2 &= (-1)^{\frac{m(m+1)}{2}} Id, \\ [X, j] &= 0, \\ (j\Psi, j\Phi) &= (\Phi, \Psi), \end{aligned}$$

and we have $j\Psi_r = (j\Psi)_{m-r}$. For all $X \in \Gamma(Q)$, we have

$$p_+(X) \cdot S_r(\mathcal{F}) \subset S_{r+1}(\mathcal{F}) \quad \text{and} \quad p_-(X) \cdot S_r(\mathcal{F}) \subset S_{r-1}(\mathcal{F}),$$

where $p_{\pm}(X) = \frac{X \mp iJ(X)}{2}$. We define the operator \tilde{D}_{tr} by

$$\tilde{D}_{tr}\Psi = \sum_{i=1}^q J(e_i) \cdot \nabla_{e_i} \Psi - \frac{1}{2} J(\kappa) \cdot \Psi.$$

The local expression of \tilde{D}_{tr} is independant of the choice of the local frame and by Green's theorem [YT90], we prove that this operator is self-adjoint. On a Kähler spin foliation, the operators D_{tr} and \tilde{D}_{tr} satisfy:

$$[\Omega, D_{tr}] = 2\tilde{D}_{tr}, \quad (3.5)$$

$$[\Omega, \tilde{D}_{tr}] = -2D_{tr}, \quad (3.6)$$

$$[\Omega, D_{tr}^2] = 0, \quad (3.7)$$

$$D_{tr}\tilde{D}_{tr} + \tilde{D}_{tr}D_{tr} = 0, \quad (3.8)$$

$$\tilde{D}_{tr}^2 = D_{tr}^2. \quad (3.9)$$

We should point out that Equations (3.7), (3.8) and (3.9) are true under the assumptions that \mathcal{F} is isoparametric and $\delta_B \kappa = 0$. Now we define the two operators D_+ and D_- by

$$D_+ = \frac{1}{2}(D_{tr} - i\tilde{D}_{tr}) \quad \text{and} \quad D_- = \frac{1}{2}(D_{tr} + i\tilde{D}_{tr}). \quad (3.10)$$

Furthermore, D_{tr} splits into D_+ and D_- , and we have the two exact sequences:

$$\Gamma(S_m(\mathcal{F})) \xrightarrow{D_-} \dots \Gamma(S_r(\mathcal{F})) \xrightarrow{D_-} \Gamma(S_{r-1}(\mathcal{F})) \xrightarrow{D_-} \dots \Gamma(S_0(\mathcal{F})), \quad (3.11)$$

$$\Gamma(S_0(\mathcal{F})) \xrightarrow{D_+} \dots \Gamma(S_r(\mathcal{F})) \xrightarrow{D_+} \Gamma(S_{r+1}(\mathcal{F})) \xrightarrow{D_+} \dots \Gamma(S_m(\mathcal{F})). \quad (3.12)$$

4 Eigenvalues of the transversal Dirac operator

In this section, we prove Kirchberg-type inequalities by using the transversal Kählerian twistor operators on Kähler spin foliations. We refer to [Kir90], [Kir92].

Definition 4.1 *On a Kähler spin foliation, we define the transversal Kählerian twistor operators by*

$$\mathcal{P}^{(r)} : \Gamma(S_r(\mathcal{F})) \xrightarrow{\nabla^{tr}} \Gamma(Q^* \otimes S_r(\mathcal{F})) \xrightarrow{\pi_r} \Gamma(\ker \mathcal{M}_r),$$

where \mathcal{M}_r is the transversal Clifford multiplication defined by

$$\begin{aligned} \mathcal{M}_r : \Gamma(Q^* \otimes S_r(\mathcal{F})) &\longrightarrow \Gamma(S_{r-1}(\mathcal{F})) \oplus \Gamma(S_{r+1}(\mathcal{F})) \\ X \otimes \Psi_r &\longmapsto p_-(X) \cdot \Psi_r \oplus p_+(X) \cdot \Psi_r. \end{aligned}$$

For all $r \in \{0, \dots, m\}$ and $\Psi_r \in \Gamma(S_r(\mathcal{F}))$, we have

$$\mathcal{P}^{(r)}\Psi_r = \sum_{i=1}^q e_i^* \otimes (\nabla_{e_i}\Psi_r + a_r p_-(e_i) \cdot \mathcal{D}_+\Psi_r + b_r p_+(e_i) \cdot \mathcal{D}_-\Psi_r), \quad (4.1)$$

where $\mathcal{D}_\pm = D_\pm + \frac{1}{2}p_\pm(\kappa)$ with $a_r = \frac{1}{2(r+1)}$ and $b_r = \frac{1}{2(m-r+1)}$. For any spinor field $\Psi_r \in \Gamma(S_r(\mathcal{F}))$, we can easily prove

$$\sum_{i=1}^q e_i \cdot \mathcal{P}_{e_i}^{(r)}\Psi_r = 0. \quad (4.2)$$

Remark 4.2 *For any non zero eigenvalue λ of D_{tr} , there exists a spinor field $\Psi \in \Gamma(S(\mathcal{F}))$ called of type $(r, r+1)$, such that $D_{tr}\Psi = \lambda\Psi$ and $\Psi = \Psi_r + \Psi_{r+1}$, with $r \in \{0, \dots, m-1\}$. By using (3.10), (3.11) and (3.12) it follows that $D_-\Psi_r = D_+\Psi_{r+1} = 0$, $D_-\Psi_{r+1} = \lambda\Psi_r$, $D_+\Psi_r = \lambda\Psi_{r+1}$ and $\|\Psi_r\|_{L^2} = \|\Psi_{r+1}\|_{L^2}$.*

Proof. Let φ be an eigenspinor of D_{tr} . There exists an r such that φ_r does not vanish. Let $\Psi = \frac{1}{\lambda}D_-\varphi_r + \varphi_r$, one can easily get that $D_{tr}\Psi = \lambda\Psi$.

Theorem 4.3 *Let (M, g_M, \mathcal{F}) be a compact Riemannian manifold with a Kähler spin foliation \mathcal{F} of codimension $q = 2m$ and a bundle-like metric g_M*

with $\kappa \in \Omega_B^1(\mathcal{F})$ and $\delta_B \kappa = 0$. Then any eigenvalue λ of the transversal Dirac operator, satisfies

$$\lambda^2 \geq \frac{m+1}{4m} K_0^\nabla. \quad (4.3)$$

If Ψ is an eigenspinor of type $(r, r+1)$ associated with an eigenvalue λ satisfying equality in (4.3), then $r = \frac{m-1}{2}$, the foliation \mathcal{F} is minimal and for all $X \in \Gamma(Q)$, the spinor Ψ satisfies

$$\nabla_X \Psi + \frac{\lambda}{2(m+1)} (X \cdot \Psi - i\varepsilon J(X) \cdot \bar{\Psi}) = 0, \quad (4.4)$$

where $\varepsilon = (-1)^{\frac{m-1}{2}}$, and $\bar{\Psi} := (-1)^r (\Psi_r - \Psi_{r+1})$. As a consequence m is odd and \mathcal{F} is transversally Einstein with non negative constant transversal curvature σ^∇ .

Proof. For all $\Psi_r \in \Gamma(S_r(\mathcal{F}))$, using Identities (4.1) and (4.2), we have

$$\begin{aligned} |\mathcal{P}^{(r)} \Psi_r|^2 &= \sum_{i=1}^q |\mathcal{P}_{e_i}^{(r)} \Psi_r|^2 = \sum_{i=1}^q (\mathcal{P}_{e_i}^{(r)} \Psi_r, \nabla_{e_i} \Psi_r) \\ &= \sum_{i=1}^q (\nabla_{e_i} \Psi_r + a_r p_-(e_i) \cdot \mathcal{D}_+ \Psi_r \\ &\quad + b_r p_+(e_i) \cdot \mathcal{D}_- \Psi_r, \nabla_{e_i} \Psi_r). \end{aligned}$$

Finally we obtain,

$$|\mathcal{P}^{(r)} \Psi_r|^2 = |\nabla^{tr} \Psi_r|^2 - a_r |\mathcal{D}_+ \Psi_r|^2 - b_r |\mathcal{D}_- \Psi_r|^2. \quad (4.5)$$

Let λ be an eigenvalue of D_{tr} and let Ψ an eigenspinor of type $(r, r+1)$. Applying Equality (4.5) to Ψ_r , one gets

$$\begin{aligned} |\mathcal{P}^{(r)} \Psi_r|^2 &= |\nabla^{tr} \Psi_r|^2 - a_r \lambda^2 |\Psi_{r+1}|^2 - a_r \lambda \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r) \\ &\quad - \frac{a_r}{4} |p_+(\kappa) \cdot \Psi_r|^2 - \frac{b_r}{4} |p_-(\kappa) \cdot \Psi_r|^2. \end{aligned}$$

By the Schrödinger-Lichnerowicz formula and by the fact that Ψ_r and Ψ_{r+1} have the same L^2 -norms, we get

$$\begin{aligned} \int_M |\mathcal{P}^{(r)} \Psi_r|^2 + \frac{a_r}{4} \int_M |p_+(\kappa) \cdot \Psi_r|^2 + \frac{b_r}{4} \int_M |p_-(\kappa) \cdot \Psi_r|^2 &= \\ \int_M ((1 - a_r) \lambda^2 - \frac{1}{4} K_\sigma^\nabla) |\Psi_r|^2 - a_r \lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r). \end{aligned} \quad (4.6)$$

Similarly applying (4.5) to Ψ_{r+1} , we obtain

$$\begin{aligned} \int_M |\mathcal{P}^{(r+1)}\Psi_{r+1}|^2 + \frac{a_{r+1}}{4} \int_M |p_+(\kappa) \cdot \Psi_{r+1}|^2 + \frac{b_{r+1}}{4} \int_M |p_-(\kappa) \cdot \Psi_{r+1}|^2 = \\ \int_M \left((1 - b_{r+1})\lambda^2 - \frac{1}{4}K_\sigma^\nabla \right) |\Psi_{r+1}|^2 + b_{r+1}\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r), \end{aligned} \quad (4.7)$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$. In order to get rid the term $\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r)$, since the l.h.s. of (4.6) and (4.7) are non negative, dividing (4.6) by a_r and (4.7) by b_{r+1} then summing up, we find by substituting the values of a_r and b_{r+1} ,

$$\lambda^2 \geq \frac{m+1}{4m} K_0^\nabla.$$

Now, we discuss the limiting case of Inequality (4.3). Dividing (4.6) by a_r and (4.7) by b_{r+1} then summing up as before, and substituting a_r , b_{r+1} and λ^2 by their values, we easily deduce that $\kappa = 0$, $\mathcal{P}^{(r)}\Psi_r = 0$ and $\mathcal{P}^{(r+1)}\Psi_{r+1} = 0$. Hence by (4.6), we find that $\lambda^2 = \frac{1}{4(1-a_r)}\sigma_0 = \frac{m+1}{4m}\sigma_0$ where $\sigma_0 = \inf_M \sigma^\nabla$, then $r = \frac{m-1}{2}$ and m is odd. It remains to prove that Ψ satisfies (4.4). For $r = \frac{m-1}{2}$, by definition of the Kählerian twistor operators, for all $j \in \{1, \dots, q\}$, we obtain

$$\nabla_{e_j}\Psi_r + \frac{\lambda}{m+1}p_-(e_j) \cdot \Psi_{r+1} = 0,$$

and

$$\nabla_{e_j}\Psi_{r+1} + \frac{\lambda}{m+1}p_+(e_j) \cdot \Psi_r = 0.$$

Summing up the two equations, we get (4.4) for $X = e_j$. Using Ricci identity in (4.4), one easily proves that \mathcal{F} is transversally Einstein. \square

Theorem 4.4 *Under the same conditions as in Theorem 4.3 for m even, any eigenvalue λ of the transversal Dirac operator satisfies*

$$\lambda^2 \geq \frac{m}{4(m-1)} K_0^\nabla. \quad (4.8)$$

If Ψ is an eigenspinor of type $(r, r+1)$ associated with an eigenvalue satisfying equality in (4.8), then $r = \frac{m}{2}$, the foliation \mathcal{F} is minimal and Ψ satisfies for all $X \in \Gamma(Q)$,

$$\nabla_X \Psi_{r+1} = -\frac{\lambda}{q}(X - iJX) \cdot \Psi_r. \quad (4.9)$$

Proof. Let Ψ an eigenspinor of type $(r, r+1)$ associated with any eigenvalue λ of the transversal Dirac operator D_{tr} . Recalling Equalities (4.6) and (4.7), we have

$$0 \leq \int_M \left((1 - a_r)\lambda^2 - \frac{1}{4}K_\sigma^\nabla \right) |\Psi_r|^2 - a_r\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r), \quad (4.10)$$

and

$$0 \leq \int_M \left((1 - b_{r+1})\lambda^2 - \frac{1}{4}K_\sigma^\nabla \right) |\Psi_{r+1}|^2 + b_{r+1}\lambda \int_M \Re(\Psi_{r+1, p_+(\kappa)} \cdot \Psi_r). \quad (4.11)$$

Hence if $\lambda \int_M \Re(\Psi_{r+1, p_+(\kappa)} \cdot \Psi_r) \leq 0$, then by (4.11)

$$\lambda^2 \geq \frac{1}{4(1 - b_{r+1})} K_0^\nabla,$$

The antilinear isomorphism j sends $S_r(\mathcal{F})$ to $S_{m-r}(\mathcal{F})$. This allows the choice of μ_r to be non negative (i.e. $r \geq \frac{m}{2}$) where μ_r is the eigenvalue associated with Ψ_r . Then a careful study of the graph of the function $\frac{1}{1-b_{r+1}}$, yields (4.8).

On the other hand if $\lambda \int_M \Re(\Psi_{r+1, p_+(\kappa)} \cdot \Psi_r) > 0$. Applying Equation (4.5) to the spinor $j\Psi$, which is a spinor of type $(m - (r + 1), m - r)$, we find the same inequalities as (4.10) and (4.11), then

$$\lambda^2 > \frac{1}{1 - a_r} \frac{K_0^\nabla}{4}.$$

As before we can choose $\mu_{m-(r+1)} \geq 0$ (i.e. $r \leq \frac{m}{2} - 1$). A careful study of the graph of the function $\frac{1}{1-a_r}$ gives Inequality (4.8).

Now we discuss the limiting case of (4.8). As we have seen, it could not be achieved if $\lambda \int_M \Re(\Psi_{r+1, p_+(\kappa)} \cdot \Psi_r) > 0$, so only the other case should be considered. By (4.7), one has

$$\begin{aligned} & \int_M |\mathcal{P}^{(r+1)}\Psi_{r+1}|^2 + \frac{a_{r+1}}{4} \int_M |p_+(\kappa) \cdot \Psi_{r+1}|^2 \\ & + \frac{b_{r+1}}{4} \int_M |p_-(\kappa) \cdot \Psi_{r+1}|^2 - b_{r+1}\lambda \int_M \Re(\Psi_{r+1, p_+(\kappa)} \cdot \Psi_r) = \\ & (1 - b_{r+1}) \int_M \left(\frac{m}{4(m-1)} K_0^\nabla - \frac{1}{4(1-b_{r+1})} K_\sigma^\nabla \right) |\Psi_{r+1}|^2. \end{aligned}$$

Since $\frac{m}{m-1} = \inf_{r \geq \frac{m}{2}} \frac{1}{1-b_{r+1}}$, and the l.h.s. of (4) is non negative, we deduce that $\kappa = 0$, $\mathcal{P}^{r+1}\Psi_{r+1} = 0$ and $\frac{m}{m-1} = \frac{1}{1-b_{r+1}}$ so $r = \frac{m}{2}$. It remains to show that Equation (4.9) holds. For this, take $X = e_j$ where $\{e_j\}_{j=1, \dots, q}$ is a local orthonormal frame. For $r = \frac{m}{2}$, and by definition of the Kählerian twistor operators, for all $j \in \{1, \dots, q\}$, we obtain

$$\nabla_{e_j} \Psi_{r+1} + \frac{\lambda}{q} (e_j - iJ e_j) \cdot \Psi_r = 0.$$

□

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