Essential norms of Volterra and Cesàro operators on Müntz spaces

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Abstract

We study the properties of the Volterra and Cesàro operators viewed on the L^1-Müntz space M_Λ^1 with range in the space of continuous functions. These operators are neither compact nor weakly compact. We estimate how far from being (weakly) compact they are by computing their (generalized) essential norm. It turns out that this latter does not depend on Λ and is equal to $1/2$.

Key words: Volterra operator, Cesàro operator, Müntz spaces, compact operator, essential norm.
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1 Introduction

Throughout this paper, we denote by $C = C([0,1])$ the space of continuous functions on $[0,1]$ equipped with the supremum norm, and $L^p = L^p([0,1])$ for $p \geq 1$, the usual spaces of integrable functions over $[0,1]$ with respect to the Lebesgue measure.

Let Λ be a strictly increasing sequence of positive numbers. We say that Λ satisfies the *Müntz condition* if and only if $\sum_{\lambda \in \Lambda \setminus \{0\}} 1/\lambda < \infty$. Let M_Λ be the linear space spanned by the monomials x^λ with $\lambda \in \Lambda$, and M_Λ^p the

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closure of M_Λ in L^p $(1 \leq p < \infty)$. The classical theorem of Müntz states that M_Λ^p is a proper subspace of L^p if and only if the Müntz condition holds (see [M], [BE]). This result remains true for the closure of M_Λ in C, denoted by M_Λ^∞. The spaces M_Λ^p $(1 \leq p \leq \infty)$ are called *Müntz spaces* whenever the Müntz condition is fulfilled.

An interesting question in this framework is the study of operators acting on Müntz spaces. In this context, the composition and multiplication operators have been investigated in [A], [AL] and [N], where several results have been obtained for the compactness and the weak compactness. However, these operators do not map the Müntz spaces into themselves in general.

In this paper, we are interested in two classical operators of Hardy-Volterra type, namely the Volterra and Cesàro operators. Recall that the Volterra operator V is defined over the space L^1 by $V(f)(x) = \displaystyle\int_0^x f(t)\, dt$, and is mapped to the space C by the usual Lebesgue theorem. The restriction V_Λ of V to the space M_Λ^1 does not stabilize it, but almost, since $V(M_\Lambda) = M_{1+\Lambda}$. On the other hand, the Cesàro operator Γ is better-behaved relatively to this problem. Indeed, it is defined for $x \in (0,1]$ by $\Gamma(f)(x) = \dfrac{1}{x} V(f)(x)$ for any $f \in L^1$, and therefore $\Gamma(M_\Lambda) = M_\Lambda$. Thanks to the Hardy inequality, it is well-known that the Cesàro operator maps the space L^p into L^p when $p > 1$ but it does not map L^1 into itself. By the Clarkson-Erdös theorem, any $f \in M_\Lambda^1$ is equal to an analytic function $\tilde{f}$ on $[0,1)$, and the a priori ambiguous value $f(0)$ will be defined by taking $f(0) := \tilde{f}(0)$. Therefore, the Cesàro operator on M_Λ^1 is defined by

$$\Gamma_\Lambda(f)(x) = \begin{cases} f(0) & \text{if } x = 0 \\[2mm] \dfrac{1}{x}\displaystyle\int_0^x f(t)dt & \text{if } x \in]0,1]. \end{cases}$$

This can also be written as $\Gamma_\Lambda(f)(x) = \displaystyle\int_0^1 f(xu)du$. We easily see that Γ_Λ maps M_Λ^1 to C.

This paper is divided into four parts. In section 2, we recall some preliminaries on the Müntz spaces and several notions of operator theory. In section 3, we prove a general criterion for getting a lower bound for the essential norm of a bounded operator T (see Proposition 3.2). Recall that the essential norm of an operator T is the distance from T to the space of compact (or weakly compact) operators. Section 4 will be devoted to the study of some properties of the Volterra and Cesàro operators. We show that their essential norm is equal to $\dfrac{1}{2}$, and is thus independent from the sequence Λ

(see Theorems 4.1 and 4.3). In the last section, we study some further types of weak compactness, namely the strict singularity and the finite strict singularity; It turns out that V_Λ and Γ_Λ are finitely strictly singular, and that the growth of their Bernstein numbers is of order $\dfrac{1}{n}$ (Theorem 5.3). Actually, in the case of real valued functions, the value of the nth-Bernstein number is precisely $1/(2n-1)$ (hence does not depend on Λ).

2 Preliminaries

We first recall some basic ingredients of the geometry of Müntz spaces. These results can be found in detail in [BE] and [GL].

Given a strictly increasing sequence of non-negative real numbers $\Lambda = (\lambda_k)_{k=0}^\infty$ such that $\sum_{k=1}^\infty 1/\lambda_k < \infty$, A. Clarkson and P. Erdös (see [GL, p.81] and [S]) proved that any element in the Müntz spaces can be represented as a series of power functions x^λ with $\lambda \in \Lambda$. The full statement of the result is the following:

Proposition 2.1 *Let $\Lambda = (\lambda_k)_{k=0}^\infty$ be a strictly increasing sequence of non-negative real numbers satisfying the Müntz condition. Assume that the gap condition $\inf\{\lambda_{k+1} - \lambda_k : k \in \mathbb{N}\} > 0$ holds. For every function $f \in L^p$, we have $f \in M_\Lambda^p$ if and only if f admits an Erdös decomposition, i.e. for every $x \in [0,1)$, we have*

$$f(x) = \sum_{k=0}^\infty a_k x^{\lambda_k} \, .$$

If the gap condition does not hold, any function $f \in M_\Lambda^p$ can still be represented by an analytic function on $\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < 1\}$ restricted to $(0,1)$. Note that there is no information on the value of f at the point 1, but one can estimate the supremum norm of f over a compact set far from 1, by the L^1-norm of f over a compact set approaching 1 (see [BE, p.185]). Using this last result and the Arzela-Ascoli theorem, the following property of the Müntz spaces was established in [AL, Corollary 2.5] :

Lemma 2.2 *Assume that $(f_n)_n$ is a bounded sequence in M_Λ^1. There is a subsequence of $(f_n)_n$ that converges uniformly on every compact subset of $[0,1)$.*

We recall now several notions of operator theory. We begin with the definition of the essential norm of any bounded operator relatively to a closed subspace of operators.

Definition 2.3 *Let X, Y be two Banach spaces, and $\mathcal{I}$ a closed subspace of operators of the space $B(X,Y)$ of bounded operators from X to Y. The*

essential norm of $T \in B(X,Y)$, relatively to $\mathcal{I}$, is the distance from T to $\mathcal{I}$ defined by

$$\|T\|_{e,\mathcal{I}} = \inf\{\|T - S\|; \ S \in \mathcal{I}\}.$$

This is the canonical norm on the quotient space $B(X,Y)/\mathcal{I}$.

With the same notations as in the above definition, if $\mathcal{I}$ is the space $\mathcal{K}(X,Y)$ of compact operators from X to Y, then $\|T\|_{e,\mathcal{I}}$ will be denoted by $\|T\|_e$, and will be referred to as *the essential norm of T*. If $\mathcal{I}$ is the space $\mathcal{W}(X,Y)$ of weakly compact operators from X to Y, then $\|T\|_{e,\mathcal{I}}$ will be denoted by $\|T\|_{e,w}$. The following fact is a direct consequence of the definitions.

Fact: Let $T : X \to Y$ be an operator, and let X_0 and Y_1 be subspaces of X and Y respectively, such that $T(X) \subset Y_1$. Let T_0 be the operator obtained from T by restricting the domain to X_0, and T_1 be the one obtained from T by restricting the codomain to Y_1. Then

$$\|T_0\|_e \leq \|T\|_e \leq \|T_1\|_e.$$

We recall some other classical notions used in the sequel (see for instance [DJT])

Definition 2.4 *A Banach space X has the Schur property if every weakly convergent sequence is actually norm convergent.*

It is well known that ℓ^1 (and consequently all its subspaces) has the Schur property.

Definition 2.5 *An operator from a Banach space X to a Banach space Y is weakly compact if $T(B_X)$ is relatively weakly compact.*

An operator from a Banach space X to a Banach space Y is Dunford-Pettis (or completely continuous) if it maps any weakly null sequence in X into a norm null sequence in Y.

Let us mention that every compact operator is Dunford-Pettis. On the other hand any operator from a space X with the Schur property is Dunford-Pettis.

Finally, we recall

Definition 2.6 *An operator from a Banach space X to a Banach space Y is nuclear if there exists two sequences (y_n) in Y and (x_n^*) in X^* such that $\sum \|y_n\|.\|x_n^*\| < \infty$ and*

$$T(x) = \sum_{n \geq 0} x_n^*(x) y_n$$

for every $x \in X$.

In other words, a nuclear operator is a summable sum of rank one operators. It is *a fortiori* a compact operator.

3 Lower bound of the essential norm

In this section, we will prove that the essential norm of any operator can be bounded from below by some term involving the "height of a discontinuity" of a function at some point. The idea is based on an argument used in [Z] for the classical essential norm, and in [L2] for the essential norm relatively to the weakly compact operators. For this, we say that a sequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ is a block-subsequence of $(x_n)_{n\in\mathbb{N}}$ if there is a sequence of non empty finite subsets of integers $(I_m)_{m\in\mathbb{N}}$ with $\max I_m < \min I_{m+1}$ and $c_i \in [0,1]$ such that

$$\tilde{x}_m = \sum_{j\in I_m} c_j x_j \quad \text{and} \quad \sum_{j\in I_m} c_j = 1.$$

Lemma 3.1 *Let $T : X \to Y$ be a bounded operator from a Banach space X to another Banach space Y. Let $(x_n)_{n\in\mathbb{N}}$ be a normalized sequence in X and $\alpha > 0$.*

1. *Assume that for any subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ and any $g \in Y$, we have $\overline{\lim}\|T(x_{\varphi(n)}) - g\| \geq \alpha$. Then $\|T\|_e \geq \alpha$.*

2. *Assume that for any block-subsequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ and any $g \in Y$, we have $\overline{\lim}\|T(\tilde{x}_n) - g\| \geq \alpha$. Then $\|T\|_{e,w} \geq \alpha$.*

Proof. We prove only (2) since (1) is similar (and easier). Fix any weakly compact operator $S : X \to Y$ and consider a normalized sequence $(x_n)_{n\in\mathbb{N}}$. From the weak compactness of S, we know that there exists a subsequence of $(S(x_n))_{n\in\mathbb{N}}$ that converges weakly to some $g \in Y$. By the Banach-Mazur theorem, there exists a block-subsequence $(\tilde{x}_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ with $(S(\tilde{x}_n)) \to g$ in norm. Note that $(\tilde{x}_n)_{n\in\mathbb{N}}$ lies in the unit ball of X. Hence, $\|T - S\| \geq \overline{\lim}\|T(\tilde{x}_n) - g\| - \lim\|S(\tilde{x}_n) - g\| \geq \alpha$. The lemma follows. $\square$

Let K be a metric space. We will say that a function $H : K \to \mathbb{C}$ has a *discontinuity at the point t_0 with height $h > 0$* if $\overline{\lim_{\delta\to 0}} \operatorname{diam}\big(H(B(t_0,\delta))\big) \geq h$, where $B(t_0,\delta)$ denotes the open ball centered at t_0 with radius δ. Equivalently, one can find two sequences $(s_n)_{n\in\mathbb{N}}$ and $(s'_n)_{n\in\mathbb{N}}$ both converging to t_0 and satisfying $\overline{\lim}|H(s_n) - H(s'_n)| \geq h$.

Proposition 3.2 *Let $T : X \to C(K)$ be a bounded operator from a Banach space X to the space $C(K)$ of continuous functions over a compact K. We assume that there exists a normalized sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $T(x_n)$ converges pointwise to some function H with discontinuity of height $h > 0$ at some point $t_0 \in X$. Then*

$$\|T\|_{e,w} \geq \frac{h}{2}.$$

Proof. Fix any $g \in C(K)$ and any block-subsequence $(\tilde{x}_n)_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$. With the previous notations, we write $\tilde{x}_m = \sum_{j \in I_m} c_j x_j$, with $\sum_{j \in I_m} c_j = 1$. For every $t \in K$, we have

$$T(\tilde{x}_m)(t) = \sum_{j \in I_m} c_j T(x_j)(t) \longrightarrow H(t)$$

since $\min I_m \to +\infty$. Thus, we obtain that

$$\overline{\lim} \|T(\tilde{x}_n) - g\| \geq \sup_{t \in K} \overline{\lim} |T(\tilde{x}_n)(t) - g(t)| = \sup_{t \in K} |H(t) - g(t)|.$$

Since the function H has a discontinuity at the point t_0 with height $h > 0$, we can find two sequences $s_n \to t_0$ and $s'_n \to t_0$ such that

$$\overline{\lim} |H(s_n) - H(s'_n)| \geq h.$$

Therefore, we get that

$$\overline{\lim} \|T(\tilde{x}_n) - g\| \geq \max \left\{ \overline{\lim} |H(s_n) - g(t_0)|; \overline{\lim} |H(s'_n) - g(t_0)| \right\} \geq \frac{h}{2}$$

otherwise we would have

$$h \leq \overline{\lim} |H(s_n) - H(s'_n)| \leq \overline{\lim} |H(s_n) - g(t_0)| + \overline{\lim} |g(t_0) - H(s'_n)| < h$$

by continuity of g at point t_0. This finishes the proof of the proposition. $\square$

Application : Let K be a (metric) connected compact set, $\theta \colon K \to K$ be a non-constant continuous map, and T be the composition operator from $C(K)$ into itself defined by $f \in C(K) \mapsto f \circ \theta$. If we consider the sequence

$$x_n(t) = \frac{nd(t, \alpha) - 1}{nd(t, \alpha) + 1}$$

where $\alpha \in \mathrm{Im}\theta$, then

$$H(t) = \left(\lim_{n \to \infty} T x_n \right)(t) = \begin{cases} -1, & \text{if } t \in \theta^{-1}(\{\alpha\}); \\ 1, & \text{if not.} \end{cases}$$

The function H has a discontinuity at any point of the boundary of $\theta^{-1}(\{\alpha\})$ (which is non empty by connectedness) with height 2. We get on one hand that $\|T\|_e \geq \|T\|_{e,w} \geq 1$. On the other hand, since $\|T\|_e \leq \|T\| \leq 1$, we deduce $\|T\|_e = \|T\|_{e,w} = 1$.

4 The Volterra and Cesàro operators

In this section, we will study some properties of the Volterra and Cesàro operators. These operators are neither compact nor weakly compact when restricted to the Müntz spaces. We will show that their (generalized) essential norm is equal to $\frac{1}{2}$, independently of the choice of the sequence Λ.

Recall first that the Volterra operator V is defined by

$$V(f) = \int_0^x f(t)dt$$

for any $f \in L^1$. The operator V is clearly well-defined from L^1 to C, and is bounded with norm 1. It also acts as an isometry on the set of positive functions. If considered as an operator from L^p to C with $p > 1$, or from L^1 to L^p with p finite, it is easy to see (via Ascoli's theorem) that V is compact. This implies that it is also compact, when viewed as a map from L^p to itself, for any $p \geq 1$. Considered as an operator from L^1 to C, it is easy to see that V is not compact.

Actually the restriction of V to the Müntz space M_Λ^1 is not weakly compact. Indeed, consider any increasing sequence (λ_n) in Λ and the associated normalized sequence $x \mapsto (\lambda_n + 1) x^{\lambda_n}$ (for $n \in \mathbb{N}$) in $M_\Lambda^1 \subset L^1$. The image of this sequence by V is the sequence $x \mapsto x^{\lambda_n+1}$, which admits no weakly convergent subsequence in C. We first compute the essential norm of the Volterra operator viewed as a map from L^1 to C.

Theorem 4.1 *Let V be the Volterra operator viewed as a map from L^1 to C. Then*

$$\|V\|_e = \|V\|_{e,w} = \frac{1}{2}.$$

Proof. For the upper bound, we find a compact operator $S : L^1 \to C$ such that $\|V - S\| \leq \frac{1}{2}$. For this, we set $S(f) = \frac{1}{2}\int_0^1 f(t)dt$, for $f \in L^1$. The operator S is clearly compact since it has rank 1, and

$$
\begin{aligned}
\|V - S\| &= \sup\left\{\|V(f) - S(f)\|_\infty : \|f\|_1 = 1\right\}\\
&= \sup\left\{\left|\int_0^x f(t)dt - \frac{1}{2}\int_0^1 f(t)dt\right| : \|f\|_1 = 1; x \in [0,1]\right\}\\
&= \sup\left\{\left|\frac{1}{2}\int_0^x f(t)dt - \frac{1}{2}\int_x^1 f(t)dt\right| : \|f\|_1 = 1; x \in [0,1]\right\}\\
&\leq \frac{1}{2}\sup\left\{\int_0^1 |f(t)|dt : \|f\|_1 = 1\right\} = \frac{1}{2}.
\end{aligned}
$$

For the lower bound, consider any increasing sequence $(\gamma_n)_n$ which tends to infinity and define a normalized sequence $g_n(x) = (\gamma_n + 1) x^{\gamma_n}$ in L^1. The

sequence $V(g_n) = x^{\gamma_n + 1}$ converges pointwise to the function H which is equal to 0 on $[0, 1)$ and to 1 at the point 1. The function H has a jump of height 1 at 1, and the result follows by Proposition 3.2. $\square$

Let us mention that the preceding proof actually shows that all the *approximation numbers $a_n(V)$ (i.e.* the distance from V to operators with rank less than $n - 1$) are equal to $\frac{1}{2}$ for every $n \geq 2$ since we proved indeed

$$\frac{1}{2} \geq a_2(V) \geq \cdots \geq a_n(V) \geq \cdots \geq \|V\|_e \geq \frac{1}{2}.$$

We recall that the $a_1(V)$ is equal to $\|V\| = 1$.

In the sequel, we will study the properties of the Cesàro operator Γ. Recall that for any function $f \in L^1$, $\Gamma(f)$ is defined at any point $x \in (0, 1]$ by

$$\Gamma(f)(x) = \frac{1}{x} V(f)(x).$$

As mentioned earlier, the Cesàro operator does not map L^1 into itself. We will then study the restriction Γ_Λ of Γ to the Müntz spaces. Recall that for every $f \in M_\Lambda^1$

$$\Gamma_\Lambda(f)(x) = \begin{cases} f(0) & \text{if } x = 0 \\ \dfrac{1}{x} \displaystyle\int_0^x f(t)dt & \text{if } x \in (0, 1] \end{cases}.$$

It is a clear that $\Gamma_\Lambda(f) \in M_\Lambda^\infty$. Moreover, it has the Erdös decomposition $\displaystyle\sum_{k=0}^\infty a_k \frac{x^{\lambda_k}}{\lambda_k + 1}$ when f has the decomposition $\displaystyle\sum_{k=0}^\infty a_k x^{\lambda_k}$.

Proposition 4.2 *The operator Γ_Λ satisfies the following:*

1. *It is a bounded, one-to-one operator, and its image $\operatorname{Im}\Gamma_\Lambda$ is dense but not closed in M_Λ^∞.*

2. *It is a Dunford-Pettis and non weakly compact operator.*

Proof. The operator Γ_Λ is the composition of the bounded operator V_Λ, and the division operator Q defined from $M_{1+\Lambda}^\infty$ to M_Λ^∞ by

$$Q(f) = \frac{1}{x} f(x).$$

This latter operator is bounded by the closed graph theorem and Proposition 2.1. The operator Γ_Λ is clearly injective. The density of the image is given by the fact that $M_\Lambda \subset \operatorname{Im}\Gamma_\Lambda$. Assume for a contradiction that $\operatorname{Im}\Gamma_\Lambda$ is closed.

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Then $\Gamma_\Lambda : M_\Lambda^1 \mapsto M_\Lambda^\infty$ is onto, thus an isomorphism by the Banach theorem. Let S be a lacunary subsequence of Λ. The restriction of Γ_Λ to M_S^1 is then an isomorphism onto M_S^∞. On the other hand, M_S^1 is isomorphic to the space ℓ^1, and M_S^∞ is isomorphic to the space c of convergent sequences (see Theorem 9.2.2 of [GL]). So ℓ^1 is isomorphic to c, and this is a contradiction since, for instance, the Schur property holds in ℓ^1 and not in c.

Now we show that Γ_Λ is not weakly compact. Assume for a contradiction that it is not the case, and consider the normalized sequence $(f_n)_n$ in M_Λ^1 defined by $f_n(x) = (\lambda_n + 1)x^{\lambda_n}$. There exists a subsequence $(f_{n_k})_k$ such that $\Gamma_\Lambda(f_{n_k}) = x^{\lambda_{n_k}}$ converges weakly (*i.e.* pointwise) to $f \in C$. Hence, $f = 0$ on $[0, 1)$ and $f(1) = 1$ which is a contradiction. Finally, Γ is a Dunford-Pettis operator, because M_Λ^1 has the Schur property (see the remark after Def.2.5). Indeed M_Λ^1 can be realized as an isomorphic copy of a subspace of ℓ^1 (see [W]). $\qquad\square$

Remark 1 *We mention the following useful facts:*

1. *For any $p \geq 1$, the Cesàro operator Γ_Λ viewed from M_Λ^1 to M_Λ^p is a compact operator. Indeed, it is the composition of the identity map i_p from M_Λ^∞ to M_Λ^p, which is compact, with the operator Γ_Λ. The compactness of i_p is guaranteed by Lemma 2.2. Similarly, it follows from the Ascoli theorem that the Cesàro operator viewed from M_Λ^p to M_Λ^∞ is compact for $p > 1$.*

2. *The operator $D : M_\Lambda^\infty \to M_\Lambda^1; f \to (xf)'_{|[0,1)}$ is not a bounded operator, since this would give that Γ_Λ is an isomorphism and $\Gamma_\Lambda^{-1} = D$.*

Theorem 4.3 *Let Λ be an increasing sequence of positive numbers satisfying the Müntz condition. Let Γ_Λ and V_Λ be the Cesàro and Volterra operators, viewed as maps from M_Λ^1 to M_Λ^∞ or C. Then*

1. $\left\|\Gamma_\Lambda\right\|_e = \left\|\Gamma_\Lambda\right\|_{e,w} = \dfrac{1}{2}\,.$

2. $\left\|V_\Lambda\right\|_e = \left\|V_\Lambda\right\|_{e,w} = \dfrac{1}{2}\,.$

To get the upper bound in the both preceding statements, we will use the following proposition, which gives a more general result for some weighted versions of the Volterra-Cesàro type operators. For this, fix a continuous function q on $[0,1]$. We define an operator H_q on M_Λ^1 as follows. For $f \in M_\Lambda^1$,

$$H_q(f)(x) = \begin{cases} q(0)f(0) & \text{if } x = 0 \\[2mm] \dfrac{q(x)}{x}\displaystyle\int_0^x f(t)\,dt & \text{if } x \in (0,1]. \end{cases}$$

If $q(x) = x$, we recover the Volterra operator V_Λ, and if $q = \mathbb{1}$, we recover the Cesàro operator Γ_Λ. We also point out that the range of H_q is included in the space $Q_\Lambda = q.M_\Lambda^\infty$.

Proposition 4.4 *With the preceding notations, we have*

$$d\Big(H_q, \mathcal{K}\big(M_\Lambda^1, Q_\Lambda\big)\Big) \leq \frac{\|q\|_\infty}{2} \quad and \quad d\Big(H_q, \mathcal{K}\big(M_\Lambda^1, C\big)\Big) \leq \frac{|q(1)|}{2}.$$

Proof. For the first inequality, let $\lambda \in \Lambda$ and $\rho \in (0,1)$. Let R and T_ρ be the operators from M_Λ^1 to Q_Λ defined for $f \in M_\Lambda^1$ by

$$R(f)(x) = \frac{q(x)x^\lambda}{2} \int_0^1 f(t)\, dt$$

and

$$T_\rho(f)(x) = \begin{cases} \rho\, q(0)f(0) & \text{if } x = 0 \\[2mm] \dfrac{q(x)}{x} \displaystyle\int_0^{\rho x} f(t)\, dt & \text{if } x \in (0,1] \end{cases}.$$

The operator R is compact since it has rank 1. The operator T_ρ is also compact since it is nuclear. Indeed, it can be written as the sum of an absolutely convergent series of rank one operators in the following way: consider the functionals e_n defined on M_Λ^1 by

$$\mathrm{e}_n \; : \; \sum_{n \geq 0} a_n x^{\lambda_n} \longmapsto a_n.$$

By a result from [BE, p.178], the norm of the map e_n is bounded from above by $C_\rho \rho^{-\lambda_n}$, where C_ρ is a constant depending only on ρ. Hence for every polynomial f in M_Λ^1, we have

$$T_\rho(f)(x) = \sum_{n \geq 0} \mathrm{e}_n(f)\frac{\rho^{\lambda_n+1}}{\lambda_n + 1} x^{\lambda_n} q(x).$$

The operator $f \longmapsto \mathrm{e}_n(f)\dfrac{\rho^{\lambda_n+1}}{\lambda_n + 1} x^{\lambda_n} q(x)$ has rank one, and

$$\sum_{n \geq 0} \frac{\|\mathrm{e}_n\|\rho^{\lambda_n+1}}{\lambda_n + 1} \leq \sum_{n \geq 0} \frac{C_\rho}{\lambda_n + 1} < \infty.$$

We show now that $\underline{\lim}_{\rho \to 1^-} \|H_q - (R + T_\rho)\| \leq \dfrac{\|q\|_\infty}{2}$. For this, fix an arbitrary $\varepsilon \in (0,1)$, and let $c \in (0,1)$ be such that $2 - c^{\lambda+1} \leq (1+\varepsilon)c$. Let f be in the

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unit ball of M_Λ^1, and $x \in (0,1)$.

$$\Big(H_q(f) - R(f) - T_\rho(f)\Big)(x) = \frac{q(x)}{x} \int_{\rho x}^{x} f(t)\, dt - \frac{x^\lambda q(x)}{2} \int_0^1 f(t)\, dt$$

$$= \frac{q(x)}{2} \int_0^1 \varphi_x(t) f(t)\, dt$$

$$(4.1)$$

with $\varphi_x(t) = \dfrac{2}{x} - x^\lambda$ if $t \in [\rho x, x]$ and $\varphi_x(t) = -x^\lambda$ if $t \notin [\rho x, x]$. Now we give an upper bound to the value of $|H_q(f) - R(f) - T_\rho(f)|(x)$ according to the position of x. If $x \leq c$, we apply the first equality in (4.1) and obtain the estimate

$$|H_q(f)(x) - R(f)(x) - T_\rho(f)(x)| \leq (1-\rho)\|q\|_\infty \sup_{t \in [0,c]} |f(t)| + \frac{\|q\|_\infty}{2}\|f\|_1$$

$$\leq \left((1-\rho)N_\varepsilon + \frac{1}{2}\right)\|q\|_\infty \|f\|_1$$

$$\leq \left(\varepsilon + \frac{1}{2}\right)\|q\|_\infty$$

where we used [BE, p.185] to get $N_\varepsilon \geq 1$ such that

$$\sup_{t \in [0,c]} |f(t)| \leq N_\varepsilon \|f\|_1$$

for any $f \in M_\Lambda^1$, and we chose ρ to be equal to $1 - \dfrac{\varepsilon}{N_\varepsilon} \in (0,1)$. If $x \geq c$, we use the second equality in (4.1) which yields

$$|H_q(f)(x) - R(f)(x) - T_\rho(f)(x)| \leq \frac{1}{2}\|\varphi_x\|_\infty \|q\|_\infty \|f\|_1 \leq \frac{1+\varepsilon}{2}\|q\|_\infty.$$

Therefore, since ε is arbitrary, the proof of the first inequality is finished.

The proof for the second inequality works as well replacing the operator R by

$$R_1(f)(x) = \frac{q(1)}{2} \int_0^1 f(t)\, dt.$$

Now the computation follows the lines of the first part but a new function φ_x appears which is given by $\varphi_x(t) = \dfrac{2q(x)}{x} - q(1)$ if $t \in [\rho x, x]$ and $\varphi_x(t) = -q(1)$ if $t \notin [\rho x, x]$. A suitable choice of c relatively to the continuity of q at the point 1 gives $\|\varphi_x\|_\infty \leq |q(1)| + \varepsilon$ for $x \geq c$. On the other hand, we manage the case $x \leq c$ like before to get an upper bound $\varepsilon + \dfrac{|q(1)|}{2}$. $\quad\square$

We note that if $0 \in \Lambda$, the previous proposition is not needed for estimating the distance of the Volterra operator V_Λ to $\mathcal{K}(M_\Lambda^1, M_\Lambda^\infty)$. The short argument of Theorem 4.1 works for this case.

Proof of Theorem 4.3. The lower bound follows from Proposition 3.2, exactly as in Theorem 4.1. Just pick the exponents γ_n in Λ. The upper bound follows immediately from Proposition 4.4. □

5 Strict singularity

In this section, we will study other weak forms of compactness, namely the strict singularity and the finite strict singularity. We will show that V_Λ and Γ_Λ share these types of compactness. Moreover, we will estimate the nth Bernstein numbers of these operators by showing that their growth are of order $\dfrac{1}{n}$. For this, let us recall the two definitions:

Definition 5.1 *An operator T from a Banach space X to a Banach space Y is strictly singular if it never induces an isomorphism on an infinite dimensional (closed) subspace of X. That is for every $\varepsilon > 0$ and every infinite dimensional subspace E of X, there exists v in the unit sphere of E such that $\|T(v)\| \leqslant \varepsilon$.*

This notion is now very classical and widely studied (see [LT, p.75] for instance).

Definition 5.2 *An operator T from a Banach space X to a Banach space Y is finitely strictly singular if for every $\varepsilon > 0$, there exists $N_\varepsilon \geqslant 1$ such that for every subspace E of X with dimension greater than N_ε, there exists v in the unit sphere of E such that $\|T(v)\| \leqslant \varepsilon$.*

This latter definition can be reformulated in terms of the so-called *Bernstein approximation numbers* (see [Pl] for instance). Recall that the nth Bernstein number of an operator T is defined as

$$b_n(T) = \sup_{\substack{E \subset X \\ dim(E)=n}} \ \inf_{\substack{v \in E \\ \|v\|=1}} \ \|T(v)\|.$$

Hence, with this terminology, the operator T is finitely strictly singular if and only if $\big(b_n(T)\big)_{n \geqslant 1}$ belongs to the space c_0 of null sequences. This notion has appeared in the late sixties. For instance, in a paper of V. Milman [Mi], it is proved that the identity from ℓ^p to ℓ^q ($p < q$) is finitely strictly singular (see [CFPTT], [Pl], [L], [LR] for recent results). It is also well-known that

compactness $\implies$ finite strict singularity $\implies$ strict singularity
and that the reverse implications are not true. Moreover, complete continuity is not comparable to finite strict singularity in general. In [L], it is proved that the classical Volterra operator is finitely strictly singular (hence strictly singular) from L^1 to C and moreover the Bernstein numbers satisfy

$$b_n(V) \approx \frac{1}{n}.$$

By restriction, the operator V_Λ is clearly finitely strictly singular as well. It is easy to see that the same property occurs for Γ_Λ. But a natural question then arises: how fast the Bernstein numbers vanish? Surprisingly, it turns out that it does not really depend on Λ and the answer is given in the following theorem. In the statement below, we are interested in V_Λ and Γ_Λ acting from M_Λ^1 to C, but actually it is worth pointing out that it does not change the values of the Bernstein numbers if we consider these operators from M_Λ^1 onto their range.

Theorem 5.3 *The operators V_Λ and Γ_Λ are finitely strictly singular. Moreover the growth of their Bernstein numbers is of order $\dfrac{1}{n}$. For every $n \geq 1$, we have*

$$b_n(V_\Lambda) \geq \frac{1}{2n-1} \quad and \quad b_n(\Gamma_\Lambda) \geq \frac{1}{2n-1}\,.$$

In particular, in the case of real valued functions, we have $b_n(V_\Lambda) = \dfrac{1}{2n-1}\,.$

Proof. Since V is finitely strictly singular ([L]), it is clear by restriction that V_Λ is also finitely strictly singular. Moreover by the fact that the map $f \in M_{1+\Lambda}^\infty \longmapsto \dfrac{1}{x} f \in M_\Lambda^\infty$ is bounded, the operator Γ_Λ is again finitely strictly singular. Moreover, we have that $b_n(V_\Lambda) \leq b_n(V) \leq C_n$ where C_n is equal to $\frac{1}{2n-1}$ or $\frac{\sqrt{2}}{n}$ according to the fact that we consider real or complex valued functions ([L]). Now, let us estimate the lower bound. We mainly follow the ideas of Newman in [Ne, Lemma 2]. Fix $n \geq 1$ and $\varepsilon > 0$ and let $(\lambda'_n)_{n \in \mathbb{N}}$ be a subsequence of Λ of positive numbers going very fast to the infinity such that it satisfies the following condition

$$\prod \left[1 - \frac{2\lambda_j'^2\big(1 + \ln(\lambda'_{j+1})\big)}{\lambda'_{j+1}} \right] \geq 1 - \varepsilon.$$

It is straightforward from [Ne, Lemma 2] to get for every $a_1, \ldots, a_n \in \mathbb{C}$ that

$$\left\| \sum_{k=1}^n a_k x^{\lambda'_k} \right\|_\infty \geq (1-\varepsilon) \max_{1 \leq m \leq n} \left\| \sum_{k=1}^m a_k x^{\lambda'_k} \right\|_\infty \geq (1-\varepsilon) \max_{1 \leq m \leq n} \left| \sum_{k=1}^m a_k \right|.$$

Next, we consider the space E spanned by $x^{\lambda'_1}, \ldots, x^{\lambda'_n}$. Hence, we may write any $f \in E$ as $\sum_{k=1}^n a_k(\lambda'_k + 1) x^{\lambda'_k}$ and we have $\Gamma_\Lambda(f) = \sum_{k=1}^n a_k x^{\lambda'_k}$. Denoting $a = (a_k)_{k=1,\ldots,n}$, and $s_m = \sum_{k=1}^m a_k$ (where $1 \leq m \leq n$), we point out that

$$\|a\|_{\ell^1} = |s_1| + \sum_{m=2}^n |s_m - s_{m-1}| \leq (2n-1) \max_{1 \leq m \leq n} |s_m|.$$

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Therefore, we find

$$\|\Gamma_\Lambda(f)\|_\infty \geq (1 - \varepsilon) \max_{1 \leq m \leq n} |s_m| \geq \frac{(1 - \varepsilon)}{2n - 1} \|a\|_{\ell^1}.$$

Finally, the inequality $\|a\|_{\ell^1} \geq \|f\|_1$ finishes the proof of the theorem. The same argument works as well for the operator V_Λ. $\qquad\square$

Remark 2 *We point out that the lower bound for the Bernstein numbers of the Volterra operator V_Λ turns out to be the same one than the lower bound for V (see [L]). Hence the speed of the Bernstein numbers is independent of Λ (up to uniform constants for complex valued functions).*

References

[A] I. Al Alam, *Essential norms of weighted composition operators on Müntz spaces*, J. Math. Anal. **358** (2009), 273-280.

[AL] I. Al Alam, P. Lefèvre, *Essential norms of weighted composition operators on L^1- Müntz spaces*, Serdica Math. J. **40** (2014), 241-260.

[BE] P. Borwein, T. Erdélyi, *Polynomials and polynomial inequalities*, Springer, Berlin Heidelberg New York, (1995).

[CE] A. Clarkson, P. Erdös, *Approximation by polynomials*, Duke Math. J. **10** (1943), 5-11.

[CFPTT] I. Chalendar, E. Fricain, A. Popov, D. Timotin and V. Troitsky, *Finitely strictly singular operators between James spaces*, J. Funct. Anal. **256** (2009), 1258-1268.

[DJT] J. Diestel, H. Jarchow, A. Tonge, *Absolutely summing operators*, Cambridge University Press (1995).

[E] T. Erdélyi, *The "full Clarkson-Erdös-Schwartz Theorem" on the closure of non-dense Müntz spaces*, Studia Math. **155** (2003), 145-152.

[GL] V. I. Gurariy, W. Lusky, *Geometry of Müntz spaces and related questions*, Lecture Notes in Mathematics, Springer, Berlin Heidelberg New York, (2005).

[L] P. Lefèvre, *The Volterra operator is finitely strictly singular from L^1 to L^∞*, J. Approx. Theory **214** (2017), 1-8.

[L2] P. Lefèvre, *Generalized essential norm of weighted composition operators on some uniform algebras of analytic functions*, Integral Equations and Operator Theory **63** (2009), 557-569.

[LR] P. Lefèvre, L. Rodríguez-Piazza, *Finitely strictly singular operators in harmonic analysis and function theory*, Advances in Math. **255** (2014), 119-152.

[LT] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces. I. Sequence spaces*, Vol. 92, Springer-Verlag, Berlin-New York, (1977).

[Mi] V. Milman, *Operators of class C_0 and C_0^*. (Russian) Teor. Funkcii Funkcional,* Anal. i Prilođen. **10** (1970), 15-26.

[M] Ch. H. Müntz, *Über den Approximationssatz von Weierstrass*, H. A. Schwarz's Festschrift, Berlin, pp. 303-312, (1914).

[N] S. Waleed Noor, *Embeddings of Müntz spaces: Composition operators*, Int. Equa. and Oper. Theo. **73** (2012), 589-602.

[Ne] D. Newman, *A Müntz space having no complement*, J. Approx. Theory **40** (1984), 351-354.

[Pl] A. Plichko, *Superstrictly singular and superstrictly cosingular operators*, Functional analysis and its applications, North-Holland Math. Stud., 197, Elsevier, Amsterdam, (2004).

[S] L. Schwartz, *Etude des sommes d'exponentielles*, Hermann, paris, (1959).

[W] D. Werner, *A remark about Müntz spaces*, http ://page.mi.fu-berlin.de/werner/preprints/muentz.pdf.

[Z] L. Zheng, *The essential norms and spectra of composition operators on H^∞*, Pacific J. Math. **203**, 503-510 (2002).