NONDEGENERACY OF POSITIVE SOLUTIONS TO NONLINEAR HARDY-SOBOLEV EQUATIONS

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Abstract. In this note, we prove that the kernel of the linearized equation around a positive energy solution in $\mathbb{R}^n$, $n \geq 3$, to $-\Delta W - \gamma|x|^{-2}V = |x|^{-s}W^{2^\star(s)-1}$ is one-dimensional when $s+\gamma > 0$. Here, $s \in [0,2)$, $0 \leq \gamma < (n-2)^2/4$ and $2^\star(s) = 2(n-s)/(n-2)$.

We fix $n \geq 3$, $s \in [0,2)$ and $\gamma < \frac{(n-2)^2}{4}$. We define $2^\star(s) = 2(n-s)/(n-2)$. We consider a nonnegative solution $W \in C^2(\mathbb{R}^n \setminus \{0\}) \setminus \{0\}$ to

$$-\Delta W - \frac{\gamma}{|x|^2}W = \frac{W^{2^\star(s)-1}}{|x|^s} \text{ in } \mathbb{R}^n \setminus \{0\}. \tag{1}$$

Due to the abundance of solutions to (1), we require in addition that W is an energy solution, that is $W \in D_1^2(\mathbb{R}^n)$, where $D_1^2(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $u \mapsto \|\nabla u\|_2$. Linearizing (1) yields to consider

$$K := \left\{\varphi \in D_1^2(\mathbb{R}^n) / -\Delta\varphi - \frac{\gamma}{|x|^2}\varphi = (2^\star(s)-1)\frac{W^{2^\star(s)-2}}{|x|^s}\varphi \text{ in } D_1^2(\mathbb{R}^n)\right\}$$

Equation (1) is conformally invariant in the following sense: for any $r > 0$, define

$$W_r(x) := r^{\frac{n-2}{2}}W(rx) \text{ for all } x \in \mathbb{R}^n \setminus \{0\},$$

then, as one checks, $W_r \in C^2(\mathbb{R}^n \setminus \{0\})$ is also a solution to (1), and, differentiating with respect to r at $r = 1$, we get that

$$-\Delta Z - \frac{\gamma}{|x|^2}Z = (2^\star(s)-1)\frac{W^{2^\star(s)-2}}{|x|^s}Z \text{ in } \mathbb{R}^n \setminus \{0\},$$

where

$$Z := \frac{d}{dr}W_{r|r=1} = \sum_i x^i\partial_i W + \frac{n-2}{2}W \in D_1^2(\mathbb{R}^n).$$

Therefore, $Z \in K$. We prove that this is essentially the only element:

Theorem 0.1. *We assume that $\gamma \geq 0$ and that $\gamma + s > 0$. Then $K = \mathbb{R}Z$. In other words, K is one-dimensional.*

Such a result is useful when performing Liapunov-Schmidt's finite dimensional reduction. When $\gamma = s = 0$, the equation (1) is also invariant under the translations $x \mapsto W(x - x_0)$ for any $x_0 \in \mathbb{R}^n$, and the kernel K is of dimension $n+1$ (see Rey [6] and also Bianchi-Egnell [1]). After this note was completed, we learnt that Dancer-Gladiali-Grossi [4] proved Theorem 0.1 in the case $s = 0$, and that their proof can be extended to our case, see also Gladiali-Grossi-Neves [5].

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This note is devoted to the proof of Theorem 0.1. Since $\gamma+s>0$, it follows from Chou-Chu [3], that there exists $r>0$ such that $W=\lambda^{\frac{1}{2^\star(s)-2}}U_r$, where

$$U(x):=\left(|x|^{\frac{2-s}{n-2}\alpha_-(\gamma)}+|x|^{\frac{2-s}{n-2}\alpha_+(\gamma)}\right)^{-\frac{n-2}{2-s}}.$$

with

$$\epsilon:=\sqrt{\frac{(n-2)^2}{4}-\gamma}\text{ and }\alpha_\pm(\gamma):=\frac{n-2}{2}\pm\sqrt{\frac{(n-2)^2}{4}-\gamma}.$$

As one checks, $U\in D_1^2(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n\setminus\{0\})$ and

$$-\Delta U-\frac{\gamma}{|x|^2}U=\lambda\frac{U^{2^\star(s)-1}}{|x|^s}\text{ in }\mathbb{R}^n\setminus\{0\},\text{ with }\lambda:=4\frac{n-s}{n-2}\epsilon^2. \tag{2}$$

Therefore, proving Theorem 0.1 reduces to prove that $\tilde{K}$ is one-dimensional, where

$$\tilde{K}:=\left\{\varphi\in D_1^2(\mathbb{R}^n)/-\Delta\varphi-\frac{\gamma}{|x|^2}\varphi=(2^\star(s)-1)\lambda\frac{U^{2^\star(s)-2}}{|x|^s}\varphi\text{ in }D_1^2(\mathbb{R}^n)\right\}$$

I. Conformal transformation.

We let $\mathbb{S}^{n-1}:=\{x\in\mathbb{R}^n/\sum x_i^2=1\}$ be the standard $(n-1)-$dimensional sphere of $\mathbb{R}^n$. We endow it with its canonical metric can. We define

$$\left\{\begin{array}{cccc}\Phi: & \mathbb{R}\times\mathbb{S}^{n-1} & \mapsto & \mathbb{R}^n\setminus\{0\}\\ & (t,\sigma) & \mapsto & e^{-t}\sigma\end{array}\right.$$

The map Φ is a smooth conformal diffeomorphism and $\Phi^\star\text{Eucl}=e^{-2t}(dt^2+\text{can})$. On any Riemannian manifold (M,g), we define the conformal Laplacian as $L_g:=-\Delta_g+\frac{n-2}{4(n-1)}R_g$ where $\Delta_g:=\text{div}_g(\nabla)$ and R_g is the scalar curvature. The conformal invariance of the Laplacian reads as follows: for a metric $g'=e^{2\omega}g$ conformal to g ($\omega\in C^\infty(M)$), we have that $L_{g'}u=e^{-\frac{n+2}{2}\omega}L_g(e^{\frac{n-2}{2}\omega}u)$ for all $u\in C^\infty(M)$. It follows from this invariance that for any $u\in C_c^\infty(\mathbb{R}^n\setminus\{0\})$, we have that

$$(-\Delta u)\circ\Phi(t,\sigma)=e^{\frac{n+2}{2}t}\left(-\partial_{tt}\hat{u}-\Delta_{\text{can}}\hat{u}+\frac{(n-2)^2}{4}\hat{u}\right)(t,\sigma)$$

for all $(t,\sigma)\in\mathbb{R}\times\mathbb{S}^{n-1}$, where $\hat{u}(t,\sigma):=e^{-\frac{n-2}{2}t}u(e^{-t}\sigma)$ for all $(t,\sigma)\in\mathbb{R}\times\mathbb{S}^{n-1}$. In addition, as one checks, for any $u,v\in C_c^\infty(\mathbb{R}^n\setminus\{0\})$, we have that

$$\begin{aligned}\int_{\mathbb{R}^n}(\nabla u,\nabla v)\,dx &= \int_{\mathbb{R}\times\mathbb{S}^{n-1}}\left(\partial_t\hat{u}\partial_t\hat{v}+(\nabla'\hat{u},\nabla'\hat{v})_{\text{can}}+\frac{(n-2)^2}{4}\hat{u}\hat{v}\right)dt\,d\sigma\\ &:= B(\hat{u},\hat{v})\end{aligned} \tag{3}$$

where we have denoted $\nabla'\hat{u}$ as the gradient on $\mathbb{S}^{n-1}$ with respect to the σ coordinate. We define the space H as the completion of $C_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$ for the norm $\|\cdot\|_H:=\sqrt{B(\cdot,\cdot)}$. As one checks, $u\mapsto\hat{u}$ extends to a bijective isometry $D_1^2(\mathbb{R}^n)\to H$.

The Hardy-Sobolev inequality asserts the existence of $K(n,s,\gamma)>0$ such that $\left(\int_{\mathbb{R}^n}\frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}\le K(n,s,\gamma)\int_{\mathbb{R}^n}\left(|\nabla u|^2-\frac{\gamma}{|x|^2}u^2\right)dx$ for all $u\in C_c^\infty(\mathbb{R}^n\setminus\{0\})$. Via the isometry $D_1^2(\mathbb{R}^n)\simeq H$, this inequality rewrites

$$\left(\int_{\mathbb{R}\times\mathbb{S}^{n-1}}|v|^{2^\star(s)}\,dtd\sigma\right)^{\frac{2}{2^\star(s)}}\le K(n,s,\gamma)\int_{\mathbb{R}\times\mathbb{S}^{n-1}}\left((\partial_t v)^2+|\nabla' v|_{\text{can}}^2+\epsilon^2v^2\right)dtd\sigma,$$

for all $v\in H$. In particular, $v\in L^{2^\star(s)}(\mathbb{R}\times\mathbb{S}^{n-1})$ for all $v\in H$.

We define $H_1^2(\mathbb{R})$ (resp. $H_1^2(\mathbb{S}^{n-1})$) as the completion of $C_c^\infty(\mathbb{R})$ (resp. $C^\infty(\mathbb{S}^{n-1})$) for the norm

$$u \mapsto \sqrt{\int_{\mathbb{R}} (\dot{u}^2 + u^2)\, dx} \;\left(\text{resp. } u \mapsto \sqrt{\int_{\mathbb{S}^{n-1}} (|\nabla' u|^2_{\text{can}} + u^2)\, d\sigma}\right).$$

Each norm arises from a Hilbert inner product. For any $(\varphi, Y) \in C_c^\infty(\mathbb{R}) \times C^\infty(\mathbb{S}^{n-1})$, define $\varphi \star Y \in C_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$ by $(\varphi \star Y)(t,\sigma) := \varphi(t)Y(\sigma)$ for all $(t,\sigma) \in \mathbb{R} \times \mathbb{S}^{n-1}$. As one checks, there exists $C > 0$ such that

$$\|\varphi \star Y\|_H \leq C \|\varphi\|_{H_1^2(\mathbb{R})} \|Y\|_{H_1^2(\mathbb{S}^{n-1})} \tag{4}$$

for all $(\varphi, Y) \in C_c^\infty(\mathbb{R}) \times C^\infty(\mathbb{S}^{n-1})$. Therefore, the operator extends continuously from $H_1^2(\mathbb{R}) \times H_1^2(\mathbb{S}^{n-1})$ to H, such that (4) holds for all $(\varphi, Y) \in H_1^2(\mathbb{R}) \times H_1^2(\mathbb{S}^{n-1})$.

Lemma 1. *We fix $u \in C_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$ and $Y \in H_1^2(\mathbb{S}^{n-1})$. We define*

$$u_Y(t) := \int_{\mathbb{S}^{n-1}} u(t,\sigma) Y(\sigma)\, d\sigma = \langle u(t,\cdot), Y \rangle_{L^2(\mathbb{S}^{n-1})} \text{ for all } t \in \mathbb{R}.$$

Then $u_Y \in H_1^2(\mathbb{R})$. Moreover, this definition extends continuously to $u \in H$ and there exists $C > 0$ such that

$$\|u_Y\|_{H_1^2(\mathbb{R})} \leq C \|u\|_H \|Y\|_{H_1^2(\mathbb{S}^{n-1})} \text{ for all } (u, Y) \in H \times H_1^2(\mathbb{S}^{n-1}).$$

Proof of Lemma 1: We let $u \in C_c^\infty(\mathbb{R} \times \mathbb{S}^{n-1})$, $Y \in H_1^2(\mathbb{S}^{n-1})$ and $\varphi \in C_c^\infty(\mathbb{R})$. Fubini's theorem yields:

$$\int_{\mathbb{R}} (\partial_t u_Y \partial_t \varphi + u_Y \varphi)\, dt = \int_{\mathbb{R} \times \mathbb{S}^{n-1}} (\partial_t u \partial_t (\varphi \star Y) + u \cdot (\varphi \star Y))\, dt d\sigma$$

Taking $\varphi := u_Y$, the Cauchy-Schwartz inequality yields

$$\begin{aligned}
&\|u_Y\|^2_{H_1^2(\mathbb{R})} \\
&\leq \sqrt{\int_{\mathbb{R} \times \mathbb{S}^{n-1}} ((\partial_t u)^2 + u^2)\, dt d\sigma} \times \sqrt{\int_{\mathbb{R} \times \mathbb{S}^{n-1}} ((\partial_t (u_Y \star Y))^2 + (u_Y \star Y)^2)\, dt d\sigma} \\
&\leq C \|u\|_H \|u_Y \star Y\|_H \leq C \|u\|_H \|u_Y\|_{H_1^2(\mathbb{R})} \|Y\|_{H_1^2(\mathbb{S}^{n-1})},
\end{aligned}$$

and then $\|u_Y\|_{H_1^2(\mathbb{R})} \leq C \|u\|_H \|Y\|_{H_1^2(\mathbb{S}^{n-1})}$. The extension follows from density. □

II. Transformation of the problem. We let $\varphi \in \tilde{K}$, that is

$$-\Delta \varphi - \frac{\gamma}{|x|^2} \varphi = (2^\star(s) - 1) \lambda \frac{U^{2^\star(s)-2}}{|x|^s} \varphi \text{ weakly in } D_1^2(\mathbb{R}^n).$$

Since $U \in C^\infty(\mathbb{R}^n \setminus \{0\})$, elliptic regularity yields $\varphi \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Moreover, the correspondance (3) yields

$$-\partial_{tt} \hat{\varphi} - \Delta_{\text{can}} \hat{\varphi} + \epsilon^2 \hat{\varphi} = (2^\star(s) - 1) \lambda \hat{U}^{2^\star(s)-2} \hat{\varphi} \tag{5}$$

weakly in H. Note that since $\hat{\varphi}, \hat{U} \in H$ and H is continuously embedded in $L^{2^\star(s)}(\mathbb{R} \times \mathbb{S}^{n-1})$, this formulation makes sense. Since $\varphi \in C^\infty(\mathbb{R}^n \setminus \{0\})$, we get that $\hat{\varphi} \in C^\infty(\mathbb{R} \times \mathbb{S}^{n-1}) \cap H$ and equation (5) makes sense strongly in $\mathbb{R} \times \mathbb{S}^{n-1}$. As one checks, we have that

$$\hat{U}(t,\sigma) = \left(e^{\frac{2-s}{n-2}\epsilon t} + e^{-\frac{2-s}{n-2}\epsilon t}\right)^{-\frac{n-2}{2-s}} \text{ for all } (t,\sigma) \in \mathbb{R} \times \mathbb{S}^{n-1}.$$

In the sequel, we will write $\hat{U}(t)$ for $\hat{U}(t,\sigma)$ for $(t,\sigma) \in \mathbb{R} \times \mathbb{S}^{n-1}$.

The eigenvalues of $-\Delta_{\text{can}}$ on $\mathbb{S}^{n-1}$ are

$$0=\mu_0<n-1=\mu_1<\mu_2<....$$

We let $\mu\geq 0$ be an eigenvalue for $-\Delta_{\text{can}}$ and we let $Y=Y_\mu\in C^\infty(\mathbb{S}^{n-1})$ be a corresponding eigenfunction, that is

$$-\Delta_{\text{can}}Y=\mu Y \text{ in } \mathbb{S}^{n-1}.$$

We fix $\psi\in C_c^\infty(\mathbb{R})$ so that $\psi\star Y\in C_c^\infty(\mathbb{R}\times\mathbb{S}^{n-1})$. Multiplying (5) by $\psi\star Y$, integrating by parts and using Fubini's theorem yields

$$\int_{\mathbb{R}}\left(\partial_t\hat{\varphi}_Y\partial_t\psi+(\mu+\epsilon^2)\hat{\varphi}_Y\psi\right)\,dt=\int_{\mathbb{R}}(2^\star(s)-1)\lambda\hat{U}^{2^\star(s)-2}\hat{\varphi}_Y\psi\,dt,$$

where $\hat{\varphi}_Y\in H_1^2(\mathbb{R})\cap C^\infty(\mathbb{R})$. Then

$$A_\mu\hat{\varphi}_Y=0 \text{ with } A_\mu:=-\partial_{tt}+(\mu+\epsilon^2-(2^\star(s)-1)\lambda\hat{U}^{2^\star(s)-2})$$

where this identity holds both in the classical sense and in the weak $H_1^2(\mathbb{R})$ sense. We claim that

$$\hat{\varphi}_Y\equiv 0 \text{ for all eigenfunction } Y \text{ of } \mu\geq n-1. \tag{6}$$

We prove the claim by taking inspiration from Chang-Gustafson-Nakanishi ([2], Lemma 2.1). Differentiating (2) with respect to $i=1,...,n$, we get that

$$-\Delta\partial_iU-\frac{\gamma}{|x|^2}\partial_iU-(2^\star(s)-1)\lambda\frac{U^{2^\star(s)-2}}{|x|^s}\partial_iU=-\left(\frac{2\gamma}{|x|^4}U+\frac{s\lambda}{|x|^{s+2}}U^{2^\star(s)-1}\right)x_i$$

On $\mathbb{R}\times\mathbb{S}^{n-1}$, this equation reads

$$-\partial_{tt}\hat{\partial_iU}-\Delta_{\text{can}}\hat{\partial_iU}+\left(\epsilon^2-(2^\star(s)-1)\lambda\hat{U}^{2^\star(s)-2}\right)\hat{\partial_iU}=-\sigma_ie^t\left(2\gamma\hat{U}+s\lambda\hat{U}^{2^\star(s)-1}\right)$$

Note that $\hat{\partial_iU}=-V\star\sigma_i$, where $\sigma_i:\mathbb{S}^{n-1}\to\mathbb{R}$ is the projection on the x_i's and

$$V(t):=-e^{-\frac{n-2}{2}t}U'(e^{-t})=e^{(1+\epsilon)t}\left(\alpha_+(\gamma)+\alpha_-(\gamma)e^{2\frac{2-s}{n-2}\epsilon t}\right)\left(1+e^{2\frac{2-s}{n-2}\epsilon t}\right)^{-\frac{n-s}{2-s}}>0$$

for all $t\in\mathbb{R}$. Since $-\Delta_{\text{can}}\sigma_i=(n-1)\sigma_i$ (the σ_i's form a basis of the second eigenspace of $-\Delta_{\text{can}}$), we then get that

$$A_\mu V\geq A_{n-1}V=e^t\left(2\gamma\hat{U}+s\lambda\hat{U}^{2^\star(s)-1}\right)>0 \text{ for all } \mu\geq n-1 \text{ and } V>0.$$

Note that for $\gamma>0$, we have that $\alpha_-(\gamma)>0$, and that for $\gamma=0$, we have that $\alpha_-(\gamma)=0$. As one checks, we have that

$$(i)\ \left\{(\gamma>0 \text{ and } \epsilon>1) \text{ or } \left(\gamma=0 \text{ and } s<\frac{n}{2}\right)\right\}\quad\Rightarrow\quad V\in H_1^2(\mathbb{R})$$
$$(ii)\ \left\{(\gamma>0 \text{ and } \epsilon\leq 1) \text{ or } \left(\gamma=0 \text{ and } s\geq\frac{n}{2}\right)\right\}\quad\Rightarrow\quad V\notin L^2((0,+\infty))$$

Assume that case (i) holds: in this case, $V\in H_1^2(\mathbb{R})$ is a distributional solution to $A_\mu V>0$ in $H_1^2(\mathbb{R})$. We define $m:=\inf\{\int_{\mathbb{R}}\varphi A_\mu\varphi\,dt\}$, where the infimum is taken on $\varphi\in H_1^2(\mathbb{R})$ such that $\|\varphi\|_2=1$. We claim that $m>0$. Otherwise, it follows from Lemma 3 below that the infimum is achieved, say by $\varphi_0\in H_1^2(\mathbb{R})\setminus\{0\}$ that is a weak solution to $A_\mu\varphi_0=m\varphi_0$ in $\mathbb{R}$. Since $|\varphi_0|$ is also a minimizer, and due to the comparison principle, we can assume that $\varphi_0>0$. Using the self-adjointness of A_μ, we get that $0\geq m\int_{\mathbb{R}}\varphi_0V\,dt=\int_{\mathbb{R}}(A_\mu\varphi_0)V\,dt=\int_{\mathbb{R}}(A_\mu V)\varphi_0\,dt>0$, which is a contradiction. Then $m>0$. Since $A_\mu \varphi_Y = 0$, we then get that $\varphi_Y \equiv 0$ as soon as $\mu \geq n-1$. This ends case (i).

Assume that case (ii) holds: we assume that $\varphi_Y \not\equiv 0$. It follows from Lemma 4 that $V(t) = o(e^{-\alpha|t|})$ as $t \to -\infty$ for all $0<\alpha<\sqrt{\epsilon^2+n-1}$. As one checks with the explicit expression of V, this is a contradiction when $\epsilon < \frac{n-2}{2}$, that is when $\gamma > 0$. Then we have that $\gamma = 0$ and $\epsilon = \frac{n-2}{2}$. Since $\frac{n}{2} \leq s < 2$, we have that $n=3$. As one checks, $(\mu + \epsilon^2 - (2^\star(s)-1)\lambda \hat{U}^{2^\star(s)-2}) > 0$ for $\mu \geq n-1$ as soon as $n=3$ and $s \geq 3/2$. Lemma 4 yields $\varphi_Y \equiv 0$, a contradiction. So $\varphi_Y \equiv 0$, this ends case (ii).

These steps above prove (6). Then, for all $t \in \mathbb{R}$, $\hat{\varphi}(t,\cdot)$ is orthogonal to the eigenspaces of μ_i, $i \geq 1$, so it is in the eigenspace of $\mu_0 = 0$ spanned by 1, and therefore $\hat{\varphi} = \hat{\varphi}(t)$ is independent of $\sigma \in \mathbb{S}^{n-1}$. Then

$$-\hat{\varphi}'' + (\epsilon^2 - (2^\star(s)-1)\lambda \hat{U}^{2^\star(s)-2})\hat{\varphi} = 0 \text{ in } \mathbb{R} \text{ and } \hat{\varphi} \in H_1^2(\mathbb{R}).$$

It follows from Lemma 2 that the space of such functions is at most one-dimensional. Going back to φ, we get that $\tilde{K}$ is of dimension at most one, and then so is K. Since $Z \in K$, then K is one dimensional and $K = \mathbb{R}Z$. This proves Theorem 0.1.

III. Auxiliary lemmas.

Lemma 2. *Let $q \in C^0(\mathbb{R})$. Then*

$$dim_{\mathbb{R}}\{\varphi \in C^2(\mathbb{R}) \cap H_1^2(\mathbb{R}) \text{ such that } -\ddot{\varphi} + q\varphi = 0\} \leq 1.$$

Proof of Lemma 2: Let F be this space. Fix $\varphi, \psi \in F \setminus \{0\}$: we prove that they are linearly dependent. Define the Wronskian $W := \varphi\dot{\psi} - \dot{\varphi}\psi$. As one checks, $\dot{W} = 0$, so W is constant. Since $\varphi, \dot{\varphi}, \psi, \dot{\psi} \in L^2(\mathbb{R})$, then $W \in L^1(\mathbb{R})$ and then $W \equiv 0$. Therefore, there exists $\lambda \in \mathbb{R}$ such that $(\psi(0), \dot{\psi}(0)) = \lambda(\varphi(0), \dot{\varphi}(0))$, and then, classical ODE theory yields $\psi = \lambda\varphi$. Then F is of dimension at most one. □

Lemma 3. *Let $q \in C^0(\mathbb{R})$ be such that there exists $A > 0$ such that* $\lim_{t \to \pm\infty} q(t) = A$*, and define*

$$m := \inf_{\varphi \in H_1^2(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} \left(\dot{\varphi}^2 + q\varphi^2\right)\, dt}{\int_{\mathbb{R}} \varphi^2\, dt}.$$

Then either $m > 0$, or the infimum is achieved.

Note that in the case $q(t) \equiv A$, $m = A$ and the infimum is not achieved.

Proof of Lemma 3: As one checks, $m \in \mathbb{R}$ is well-defined. We let $(\varphi_i)_i \in H_1^2(\mathbb{R})$ be a minimizing sequence such that $\int_{\mathbb{R}} \varphi_i^2\, dt = 1$ for all i, that is $\int_{\mathbb{R}} \left(\dot{\varphi}_i^2 + q\varphi_i^2\right)\, dt = m + o(1)$ as $i \to +\infty$. Then $(\varphi_i)_i$ is bounded in $H_1^2(\mathbb{R})$, and, up to a subsequence, there exists $\varphi \in H_1^2(\mathbb{R})$ such that $\varphi_i \rightharpoonup \varphi$ weakly in $H_1^2(\mathbb{R})$ and $\varphi_i \to \varphi$ strongly in $L^2_{loc}(\mathbb{R})$ as $i \to +\infty$. We define $\theta_i := \varphi_i - \varphi$. Since $\lim_{t\to\pm\infty}(q(t) - A) = 0$ and $(\theta_i)_i$ goes to 0 strongly in L^2_{loc}, we get that $\lim_{i\to+\infty} \int_{\mathbb{R}} (q(t) - A)\theta_i^2\, dt = 0$. Using the weak convergence to 0 and that $(\varphi_i)_i$ is minimizing, we get that

$$\int_{\mathbb{R}} \left(\dot{\varphi}^2 + q\varphi^2\right)\, dt + \int_{\mathbb{R}} \left(\dot{\theta}_i^2 + A\theta_i^2\right)\, dt = m + o(1) \text{ as } i \to +\infty.$$

Since $1 - \|\varphi\|_2^2 = \|\theta_i\|_2^2 + o(1)$ as $i \to +\infty$ and $\int_{\mathbb{R}} \left(\dot{\varphi}^2 + q\varphi^2\right)\, dt \geq m\|\varphi\|_2^2$, we get

$$m\|\theta_i\|_2^2 \geq \int_{\mathbb{R}} \left(\dot{\theta}_i^2 + A\theta_i^2\right)\, dt + o(1) \text{ as } i \to +\infty.$$

If $m \leq 0$, then $\theta_i \to 0$ strongly in $H_1^2(\mathbb{R})$, and then $(\varphi_i)_i$ goes strongly to $\varphi \not\equiv 0$ in H_1^2, and φ is a minimizer for m. This proves the lemma. □

Lemma 4. *Let $q \in C^0(\mathbb{R})$ be such that there exists $A > 0$ such that $\lim_{t\to\pm\infty} q(t) = A$ and q is even. We let $\varphi \in C^2(\mathbb{R})$ be such that $-\ddot{\varphi} + q\varphi = 0$ in $\mathbb{R}$ and $\varphi \in H_1^2(\mathbb{R})$.*

- *If $q \geq 0$, then $\varphi \equiv 0$.*
- *We assume that there exists $V \in C^2(\mathbb{R})$ such that*

$$-\ddot{V} + qV > 0\ ,\ V > 0 \text{ and } V \notin L^2((0, +\infty)).$$

Then either $\varphi \equiv 0$ or $V(t) = o(e^{-\alpha|t|})$ as $t \to -\infty$ for all $0 < \alpha < \sqrt{A}$.

Proof of Lemma 4: We assume that $\varphi \not\equiv 0$. We first assume that $q \geq 0$. By studying the monotonicity of φ between two consecutive zeros, we get that φ has at most one zero, and then $\ddot{\varphi}$ has constant sign around $\pm\infty$. Therefore, φ is monoton around $\pm\infty$ and then has a limit, which is 0 since $\varphi \in L^2(\mathbb{R})$. The contradiction follows from studying the sign of $\ddot{\varphi}$, φ. Then $\varphi \equiv 0$ and the first part of Lemma 4 is proved.

We now deal with the second part and we let $V \in C^2(\mathbb{R})$ be as in the statement. We define $\psi := V^{-1}\varphi$. Then, $-\ddot{\psi} + h\dot{\psi} + Q\psi = 0$ in $\mathbb{R}$ with $h, Q \in C^0(\mathbb{R})$ and $Q > 0$. Therefore, by studying the zeros, $\dot{\psi}$ vanishes at most once, and then $\psi(t)$ has limits as $t \to \pm\infty$. Since $\varphi = \psi V$, $\varphi \in L^2(\mathbb{R})$ and $V \notin L^2(0, +\infty)$, then $\lim_{t\to+\infty} \psi(t) = 0$. We claim that $\lim_{t\to-\infty} \psi(t) \neq 0$. Otherwise, the limit would be 0. Then ψ would be of constant sign, say $\psi > 0$. At the maximum point t_0 of ψ, the equation would yield $\ddot{\psi}(t_0) > 0$, which contradicts the maximum. So the limit of ψ at $-\infty$ is nonzero, and then $V(t) = O(\varphi(t))$ as $t \to -\infty$.

We claim that φ is even or odd and φ has constant sign around $+\infty$. Since $t \mapsto \varphi(-t)$ is also a solution to the ODE, it follows from Lemma 2 that it is a multiple of φ, and then φ is even or odd. Since $\dot{\psi}$ changes sign at most once, then ψ changes sign at most twice. Therefore $\varphi = \psi V$ has constant sign around $+\infty$.

We fix $0 < A' < A$ and we let $R_0 > 0$ such that $q(t) > A'$ for all $t \geq R_0$. Without loss of generality, we also assume that $\varphi(t) > 0$ for $t \geq R_0$. We define $b(t) := C_0 e^{-\sqrt{A'}t} - \varphi(t)$ for all $t \in \mathbb{R}$ with $C_0 := 2\varphi(R_0)e^{\sqrt{A'}R_0}$. We claim that $b(t) \geq 0$ for all $t \geq R_0$. Otherwise $\inf_{t\geq R_0} b(t) < 0$, and since $\lim_{t\to+\infty} b(t) = 0$ and $b(R_0) > 0$, then there exists $t_1 > R_0$ such that $\ddot{b}(t_1) \geq 0$ and $b(t_1) < 0$. However, as one checks, the equation yields $\ddot{b}(t_1) < 0$, which is a contradiction. Therefore $b(t) \geq 0$ for all $t \geq R_0$, and then $0 < \varphi(t) \leq C_0 e^{-\sqrt{A'}t}$ for $t \to +\infty$. Lemma 4 follows from this inequality, φ even or odd, and $V(t) = O(\varphi(t))$ as $t \to -\infty$. ☐

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