GREEN'S FUNCTION FOR A SINGULAR HARDY-TYPE OPERATOR WITH BOUNDARY SINGULARITY

INFORMAL NOTE

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ABSTRACT. This note is devoted to the construction of the Green's function for coercive operators like $-\Delta-(\gamma|x|^{-2}+h)$ on a smooth domain Ω with singularity $0\in\partial\Omega$. We prove existence and asymptotics when Ω is a bounded domain. We also prove existence and asymptotics when Ω is the half-space $\mathbb{R}^n_-$ and $h\equiv 0$.

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1. MAIN RESULT

Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n\geq 3$, such that $0\in\partial\Omega$. The model space for such domains is $\mathbb{R}^n_- := \{x=(x_1,...,x_n)\in\mathbb{R}^n / x_1<0\}$. The Hardy inequality for the half-space is

$$\inf_{u\in H_0^1(\mathbb{R}^n_-)\setminus\{0\}}\frac{\int_{\mathbb{R}^n_-}|\nabla u|^2\,dx}{\int_{\mathbb{R}^n_-}\frac{u^2}{|x|^2}\,dx}=\frac{n^2}{4},$$

where, for any domain $\Omega\subset\mathbb{R}^n$, $H_0^1(\Omega)$ is the completion of $C_c^\infty(\Omega)$ for the norm $u\mapsto\|\nabla u\|_2$. We refer to Ghoussoub-Robert [1] for discussions and further references on such Hardy inequalities. We fix $h\in L^\infty(\Omega)$ and $\gamma\in\mathbb{R}$. We assume that the operator $-\Delta-(\gamma|x|^{-2}+h)$ is coercive, that is there exists $c>0$ such that

$$\int_\Omega\left(|\nabla u|^2-\left(\frac{\gamma}{|x|^2}+h\right)u^2\right)dx\geq c\int_\Omega u^2\,dx$$

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for all $u \in H_0^1(\Omega)$. It follows from the proof of Proposition 3.1 in Ghoussoub-Robert [1] that a necessary condition for coercivity is that $\gamma \leq n^2/4$.

Definition 1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \partial\Omega$. We fix $\gamma < n^2/4$ and $h \in C^{0,\theta}(\overline{\Omega})$, $\theta \in (0,1)$ such that $-\Delta - (\gamma|x|^{-2} + h)$ is coercive. We say that $G : \Omega \times \Omega \setminus \{(x,x)/\, x \in \Omega\}$ is a Green's function for $-\Delta - \gamma|x|^{-2} - h$ if*

- *For any $p \in \Omega$, $G_p := G(p, \cdot) \in L^1(\Omega)$.*
- *For all $f \in C_c^\infty(\Omega)$ and all $p \in \Omega$, then*

$$\varphi(p) = \int_\Omega G_p(x) f(x)\, dx.$$

where $\varphi \in H_0^1(\Omega) \cap C^0(\Omega)$ is the unique solution to

$$-\Delta\varphi - \left(\frac{\gamma}{|x|^2} + h(x)\right)\varphi = f \text{ in } \Omega \,;\; \varphi_{|\partial\Omega} = 0.$$

Our main result is the following:

Theorem 1 (Existence)**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. We fix $\gamma < \frac{n^2}{4}$. We let $h \in C^{0,\theta}(\overline{\Omega})$ be such that $-\Delta - \gamma|x|^{-2} - h$ is coercive. Then there exists a Green's function for $-\Delta - \gamma|x|^{-2} - h$. Moreover,*

(a) *The Green's function G is unique, $G_p \in C^{2,\theta}(\overline{\Omega} \setminus \{0,p\})$ and $G_p > 0$ for all $p \in \Omega$.*

(b) *For all $p \in \Omega$ and all $\eta \in C_c^\infty(\mathbb{R}^n \setminus \{p\})$, we have that $\eta G_p \in H_0^1(\Omega)$.*

(c) *For all $f \in L^{\frac{2n}{n+2}}(\Omega) \cap L^q(\Omega \setminus B_\delta(0))$ for all $\delta > 0$ and some $q > n/2$, then for any $p \in \Omega$, we have that*

$$\varphi(p) = \int_\Omega G_p(x) f(x)\, dx. \tag{1}$$

where $\varphi \in H_0^1(\Omega) \cap C^0(\Omega)$ is the unique solution to

$$-\Delta\varphi - \left(\frac{\gamma}{|x|^2} + h(x)\right)\varphi = f \text{ in } \Omega \,;\; \varphi_{|\partial\Omega} = 0, \tag{2}$$

In particular,

$$\left\{\begin{array}{ll} -\Delta G_p - \left(\frac{\gamma}{|x|^2} + h(x)\right) G_p = 0 & \text{in } \Omega \setminus \{p\} \\ G_p > 0 & \text{in } \Omega \setminus \{p\} \\ G_p = 0 & \text{in } \partial\Omega \setminus \{0\} \end{array}\right. \tag{3}$$

Theorem 2 (Asymptotics)**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. We fix $\gamma < \frac{n^2}{4}$. We let $h \in C^{0,\theta}(\overline{\Omega})$ be such that $-\Delta - \gamma|x|^{-2} - h$ is coercive. Let G be the Green's function for $-\Delta - \gamma|x|^{-2} - h$. Then*

- *For all $p \in \Omega \setminus \{0\}$, there exists $c_0(p) > 0$ such that*

$$G_p(x) \sim_{x \to 0} c_0(p) \frac{d(x, \partial\Omega)}{|x|^{\alpha_-(\gamma)}} \text{ and } G_p(x) \sim_{x \to p} \frac{1}{(n-2)\omega_{n-1}|x-p|^{n-2}} \tag{4}$$

where

$$\alpha_-(\gamma) := \frac{n}{2} - \sqrt{\frac{n^2}{4} - \gamma} \text{ and } \alpha_+(\gamma) := \frac{n}{2} + \sqrt{\frac{n^2}{4} - \gamma}.$$

- *There exists $c > 0$ such that*

$$c^{-1}H_p(x) < G_p(x) < cH_p(x) \text{ for } x \in \Omega - \{0, p\}. \tag{5}$$

where

$$H_p(x) := \left(\frac{\max\{|p|, |x|\}}{\min\{|p|, |x|\}}\right)^{\alpha_-(\gamma)} |x-p|^{2-n} \min\left\{1, \frac{d(x, \partial\Omega)d(p, \partial\Omega)}{|x-p|^2}\right\}. \tag{6}$$

and

(7)

$$|\nabla G_p(x)| \leq c \left(\frac{\max\{|p|, |x|\}}{\min\{|p|, |x|\}}\right)^{\alpha_-(\gamma)} |x-p|^{1-n} \min\left\{1, \frac{d(p, \partial\Omega)}{|x-p|}\right\} \text{ for } x \in \Omega - \{0, p\}.$$

This note is devoted to the proof of Theorems 1 and 2. We also prove (Theorem 3) the existence and asymptotic behavior for the Green's function of the unbounded domain $\mathbb{R}^n_-$ when $h \equiv 0$. The pointwise control (5) will be seen as a consequence of infinitesimal convergence results for the Green's function when $x \to p$, namely Theorems 4, 5 and 6.

The note is organized as follows. In Section 2, we prove the existence of the Green's function (Theorem 1). In Section 3, we prove the upper bound in (5). In Section 4, we prove the existence of the Green's function on the half-space $\mathbb{R}^n_-$. Section 5 is devoted to the proof of the asymptotic behavior of the Green's function when $x \to p$, namely in the neighborhood of the diagonal. The pointwise controls in Theorem 2 are proved in Section 6. The last section is an Appendix for the proof of a technical lemma.

Notations: In the sequel, $C(a, b, c), C_1(a, b, ...)$... will denote a constant depending only on a, b, c. The notation $C, c, ...$ will sometimes denote different constants from line to line, and even in the same line. In order to simplify notations, we will often drop the dependence in the domain Ω and the dimension $n \geq 3$. If $F : A \times B \to \mathbb{R}$ is a fonction, then for any $x \in A$, we define $F_x : B \to \mathbb{R}$ by $F_x(y) := F(x, y)$ for all $y \in B$. Finally, we will write $\mathrm{Diag}(A) := \{(x, x)/\, x \in A\}$ for any set A.

2. Proof of Theorem 1

Fix $\delta_0 > 0$ such that $B_{\delta_0}(0) \subset \Omega$. We let $\eta_\epsilon(x) := \tilde{\eta}(\epsilon^{-1}|x|)$ for all $x \in \mathbb{R}^n$ and $\epsilon > 0$, where $\tilde{\eta} \in C^\infty(\mathbb{R})$ is nondecreasing and such that $\tilde{\eta}(t) = 0$ for $t < 1$ and $\tilde{\eta}(t) = 1$ for $t > 1$. It follows from Lemma 1 (see the Appendix) and the coercivity of $-\Delta - \left(\gamma|x|^{-2} + h\right)$ that there exists $\epsilon_0 > 0$ and $c > 0$ such that such that for all $\varphi \in H_0^1(\Omega)$ and $\epsilon \in (0, \epsilon_0)$,

$$\int_\Omega \left(|\nabla\varphi|^2 - \left(\frac{\gamma\eta_\epsilon}{|x|^2} + h(x)\right)\varphi^2\right) dx \geq c\int_\Omega \varphi^2\, dx.$$

As a consequence, there exists $c > 0$ such that for all $\varphi \in H_0^1(\Omega)$ and $\epsilon \in (0, \epsilon_0)$,

$$\int_\Omega \left(|\nabla\varphi|^2 - \left(\frac{\gamma\eta_\epsilon}{|x|^2} + h(x)\right)\varphi^2\right) dx \geq c\|\varphi\|^2_{H_0^1}. \tag{8}$$

Let $G_\epsilon > 0$ be the Green's function of $-\Delta - \left(\gamma\eta_\epsilon|x|^{-2} + h\right)$ on Ω with Dirichlet boundary condition. The existence follows from the coercivity and the $C^{0,\theta}$ regularity of the potential for any $\epsilon > 0$ (see Robert [3]). In particular, we have that

$$\begin{cases} -\Delta G_\epsilon(x, \cdot) - \left(\frac{\gamma\eta_\epsilon}{|\cdot|^2} + h\right)G_\epsilon(x, \cdot) = 0 & \text{in } \Omega \setminus \{x\} \\ G_\epsilon(x, \cdot) = 0 & \text{on } \partial\Omega \end{cases} \tag{9}$$

Step 1: Integral bounds for G_ϵ. We claim that for all $\delta > 0$ and $1 < q < \frac{n}{n-2}$ and $\delta' \in (0,\delta)$, there exists $C(\delta,q) > 0$ and $C(\delta,\delta') > 0$ such that

$$\|G_\epsilon(x,\cdot)\|_{L^q(\Omega)} \leq C(\delta,q) \text{ and } \|G_\epsilon(x,\cdot)\|_{L^{\frac{2n}{n-2}}(\Omega\setminus B_{\delta'}(x))} \leq C(\delta,\delta') \tag{10}$$

for all $x \in \Omega$, $|x| > \delta$. We prove the claim. We fix $f \in C_c^\infty(\Omega)$ and let $\varphi_\epsilon \in C^{2,\theta}(\overline{\Omega})$ be the solution to the boundary value problem

$$\begin{cases} -\Delta\varphi_\epsilon - \left(\frac{\gamma\eta_\epsilon}{|x|^2} + h(x)\right)\varphi_\epsilon = f & \text{in } \Omega \\ \varphi_\epsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{11}$$

Multiplying the equation by φ_ϵ, integrating by parts on Ω, using (8) and Hölder's inequality, we get that

$$\int_\Omega |\nabla\varphi_\epsilon|^2\,dx \leq C\|f\|_{\frac{2n}{n+2}}\|\varphi_\epsilon\|_{\frac{2n}{n-2}}$$

where $C > 0$ is independent of ϵ, f and φ_ϵ. The Sobolev inequality $\|\varphi\|_{\frac{2n}{n-2}} \leq C\|\nabla\varphi\|_2$ for $\varphi \in H_0^1(\Omega)$ then yields

$$\|\varphi_\epsilon\|_{\frac{2n}{n-2}} \leq C\|f\|_{\frac{2n}{n+2}}$$

where $C > 0$ is independent of ϵ, f and φ_ϵ. Fix $p > n/2$ and $\delta \in (0,\delta_0)$ and $\delta_1, \delta_2 > 0$ such that $\delta_1 + \delta_2 < \delta$, and $x \in \Omega$ such that $|x| > \delta$. It follows from standard elliptic theory that

$$\begin{aligned} |\varphi_\epsilon(x)| &\leq \|\varphi_\epsilon\|_{C^0(B_{\delta_1}(x))} \\ &\leq C\left(\|\varphi_\epsilon\|_{L^{\frac{2n}{n-2}}(B_{\delta_1+\delta_2}(x))} + \|f\|_{L^p(B_{\delta_1+\delta_2}(x))}\right) \\ &\leq C\left(\|f\|_{L^{\frac{2n}{n+2}}(\Omega)} + \|f\|_{L^p(B_{\delta_1+\delta_2}(x))}\right) \end{aligned}$$

where $C > 0$ depends on $p, \delta, \delta_1, \delta_2$, γ and $\|h\|_\infty$. Therefore, Green's representation formula yields

$$\left|\int_\Omega G_\epsilon(x,\cdot)f\,dy\right| \leq C\left(\|f\|_{L^{\frac{2n}{n+2}}(\Omega)} + \|f\|_{L^p(B_{\delta_1+\delta_2}(x))}\right) \tag{12}$$

for all $f \in C_c^\infty(\Omega)$. It follows from (12) that

$$\left|\int_\Omega G_\epsilon(x,\cdot)f\,dy\right| \leq C\cdot\|f\|_{L^p(\Omega)}$$

for all $f \in C_c^\infty(\Omega)$ where $p > n/2$. It then follows from duality arguments that for any $q \in (1, n/(n-2))$ and any $\delta > 0$, there exists $C(\delta,q) > 0$ such that $\|G_\epsilon(x,\cdot)\|_{L^q(\Omega)} \leq C(\delta,q)$ for all $\epsilon < \epsilon_0$ and $x \in \Omega\setminus B_\delta(0)$.

Let $\delta' \in (0,\delta)$ and $\delta_1, \delta_2 > 0$ such that $\delta_1 + \delta_2 < \delta'$. We get from (12) that

$$\left|\int_\Omega G_\epsilon(x,\cdot)f\,dy\right| \leq C\|f\|_{L^{\frac{2n}{n+2}}(\Omega\setminus B_{\delta'}(x))} \tag{13}$$

for all $f \in C_c^\infty(\Omega\setminus B_{\delta'}(x))$. Here again, a duality argument yields (10), which proves the claim in Step 1.

Using the same method, we can get an improvement of the control, the cost being the integrability exponent q. When $q \in (1, n/(n-1))$, we get that $p > n$. Then,

$\|\varphi_\epsilon\|_{C^1(B_{\delta_1}(x)\cap\Omega)}$ is controled by the L^p and $L^{\frac{2n}{n+2}}$ norms. Moreover, $|\varphi_\epsilon(x)| \leq \|\varphi_\epsilon\|_{C^0(B_{\delta_1}(x)\cap\Omega)}d(x,\partial\Omega)$. The argument above then yields

$$\|G_\epsilon(x,\cdot)\|_{L^q(\Omega)} \leq C(\delta,q)d(x,\partial\Omega) \text{ for } q\in\left(1,\frac{n}{n-1}\right). \tag{14}$$

Step 2: Convergence of G_ϵ. Fix $x\in\Omega\setminus\{0\}$. For $0<\epsilon<\epsilon'$, since $G_\epsilon(x,\cdot)$, $G_{\epsilon'}(x,\cdot)$ are C^2 outside x, (9) yields

$$-\Delta(G_\epsilon(x,\cdot)-G_{\epsilon'}(x,\cdot))-\left(\frac{\gamma\eta_\epsilon}{|\cdot|^2}+h\right)(G_\epsilon(x,\cdot)-G_{\epsilon'}(x,\cdot))=\frac{\gamma(\eta_\epsilon-\eta_{\epsilon'})}{|\cdot|^2}G_{\epsilon'}(x,\cdot)$$

in the strong sense. The coercivity (8) then yields $G_\epsilon(x,\cdot)\geq G_{\epsilon'}(x,\cdot)$ for $0<\epsilon<\epsilon'$ if $\gamma\geq 0$, and the reverse inequality if $\gamma<0$. It then follows from the integral bound (10) and elliptic regularity that there exists $G(x,\cdot)\in C^{2,\theta}(\overline{\Omega}\setminus\{0,x\})$ such that

$$\lim_{\epsilon\to 0}G_\epsilon(x,\cdot)=G(x,\cdot)\geq 0 \text{ in } C^{2,\theta}_{loc}(\overline{\Omega}-\{0,x\}). \tag{15}$$

In particular, G is symmetric and

$$-\Delta G(x,\cdot)-\left(\frac{\gamma}{|\cdot|^2}+h\right)G(x,\cdot)=0 \text{ in } \Omega\setminus\{x\} \text{ and } G(x,\cdot)=0 \text{ on } \partial\Omega. \tag{16}$$

Moreover, passing to the limit $\epsilon\to 0$ in (10), (14) and using elliptic regularity, we get that for all $\delta>0$, $1<q<\frac{n}{n-2}$ and $\delta'\in(0,\delta)$, there exist $C(\delta,q)>0$ and $C(\delta,\delta')>0$ such that for all $x\in\Omega$, $|x|>\delta$,

$$\|G(x,\cdot)\|_{L^q(\Omega)}\leq C(\delta,q) \text{ and } \|G(x,\cdot)\|_{L^{\frac{2n}{n-2}}(\Omega\setminus B_{\delta'}(x))}\leq C(\delta,\delta') \tag{17}$$

and

$$\|G(x,\cdot)\|_{L^q(\Omega)} \leq C(\delta,q)d(x,\partial\Omega) \text{ for } q\in\left(1,\frac{n}{n-1}\right). \tag{18}$$

In particular, for any $x\in\Omega\setminus\{0\}$, $G(x,\cdot)\in L^k(\Omega)$ for all $1<k<n/(n-2)$ and $G(x,\cdot)\in L^{2n/(n-2)}(\Omega\setminus B_\delta(x))$ for all $\delta>0$. Moreover, for any $f\in L^{\frac{2n}{n+2}}(\Omega)\cap L^q(\Omega\setminus B_\delta(0))$ for all $\delta>0$ with $q>n/2$, let $\varphi_\epsilon\in H_0^1(\Omega)$ be such that (11) holds. It follows from elliptic theory that $\varphi_\epsilon\in C^{0,\tau}(\overline{\Omega}\setminus\{0\})$ for some $\tau\in(0,1)$ and that for all $\delta_1>0$, there exists $C(\delta_1)>0$ such that $\|\varphi_\epsilon\|_{C^{0,\tau}(\overline{\Omega}\setminus B_{\delta_1}(0))}\leq C(\delta_1)$. We fix $x\in\Omega\setminus\{0\}$. Passing to the limit $\epsilon\to 0$ in the Green identity $\varphi_\epsilon(x)=\int_\Omega G_\epsilon(x,\cdot)f\,dy$ yields

$$\varphi(x)=\int_\Omega G(x,\cdot)f\,dy \text{ for all } x\in\Omega\setminus\{0\} \tag{19}$$

where $\varphi\in H_0^1(\Omega)\cap C^0(\overline{\Omega}\setminus\{0\})$ is the only weak solution to

$$\begin{cases} -\Delta\varphi-\left(\frac{\gamma}{|x|^2}+h(x)\right)\varphi=f & \text{in } \Omega\\ \varphi=0 & \text{on } \partial\Omega\end{cases}$$

Since $G(x,\cdot)\geq 0$, (16) and the strong comparison principle yield $G(x,\cdot)>0$. These points prove that G is a Green's function for the operator and that (c) holds.

We prove point (b). We fix $\eta\in C_c^\infty(\mathbb{R}^n-\{x\})$ such that $\eta(y)=1$ when $y\in B_\delta(0)$ for some $\delta>0$. Then $\eta G_\epsilon(x,\cdot)\in C^{2,\theta}(\overline{\Omega})\cap H_0^1(\Omega)$. It follows from (9) and (15) that

$$-\Delta(\eta G_\epsilon(x,\cdot))-\left(\frac{\gamma\eta_\epsilon}{|\cdot|^2}+h\right)(\eta G_\epsilon(x,\cdot))=\mathbf{1}_{B_\delta(0)^c}f_\epsilon \text{ in } \Omega$$

where $\|f_\epsilon\|_{C^0(\overline{\Omega})} \leq C$ for some $C>0$ and all $\epsilon>0$. Therefore, with the coercivity (8) and the convergence (15), we get that

$$c\|\eta G_\epsilon(x,\cdot)\|_{H_0^1}^2 \leq \int_{\Omega\setminus B_\delta(0)} f_\epsilon \eta G_\epsilon(x,\cdot)\, dy \leq C$$

for all $\epsilon>0$. Reflexivity yields convergence of $(\eta G_\epsilon(x,\cdot))$ in $H_0^1(\Omega)\cap L^2(\Omega)$ as $\epsilon\to 0$ up to extraction. The convergence in C^2 and uniqueness then yields $\eta G(x,\cdot)\in H_0^1(\Omega)$ and $\eta G_\epsilon(x,\cdot)\to \eta G(x,\cdot)$ in $H_0^1(\Omega)$ as $\epsilon\to 0$. The case of a general η is a direct consequence. This proves point (b).

We are now left with proving uniqueness. We let G' be another Green's function. We fix $x\in\Omega$ and we define $H_x := G_x - G'_x$. Then $H_x\in L^1(\Omega)$ and for any $f\in C_c^\infty(\Omega)$, we have that $\int_\Omega H_x f\, dy = 0$. Approximating a compactly supported function by smooth fonctions with compact support, we get that this equality holds for all $f\in C_c^0(\Omega)$. Integration theory then yields $H_x\equiv 0$, and then $G'_x\equiv G_x$. This proves uniqueness. This finishes the proof of (a).

This proves existence and uniqueness of the Green's function and Theorem 1.

3. Theorem 2: proof of the upper bound in (5)

The behavior (4) is a consequence of the classification of solutions to harmonic equations and Theorem 4.1 in Ghoussoub-Robert [1].

In the proof, we will often use sub- and super-solutions to the linear problem. The following existence result is contained in Proposition 4.3 of [1]:

Proposition 1. *Let Ω be a smooth domain and $h\in C^0(\overline{\Omega})$ be a continuous fonction. We fix $\gamma<\frac{n^2}{4}$ and $\alpha\in\{\alpha_-(\gamma),\alpha_+(\gamma)\}$. Then, there exist $r>0$, and $\overline{u}_\alpha,\underline{u}_\alpha\in C^\infty(\overline{\Omega}\setminus\{0\})$ such that*

$$\left\{\begin{array}{rll} \overline{u}_\alpha,\underline{u}_\alpha = 0 & \text{on } \partial\Omega\cap B_r(0)\\ -\Delta\overline{u}_\alpha - \left(\frac{\gamma}{|x|^2}+h\right)\overline{u}_\alpha > 0 & \text{in } \Omega\cap B_r(0)\\ -\Delta\underline{u}_\alpha - \left(\frac{\gamma}{|x|^2}+h\right)\underline{u}_\alpha < 0 & \text{in } \Omega\cap B_r(0). \end{array}\right. \tag{20}$$

Moreover, for some $\tau>0$, we have that, as $x\to 0$, $x\in\Omega$,

$$\overline{u}_\alpha(x) = \underline{u}_\alpha(x)(1+O(|x|^\tau)) = \frac{d(x,\partial\Omega)}{|x|^\alpha}(1+O(|x|^\tau)). \tag{21}$$

Step 3: Upper bound for $G(x,y)$ when one variable is far from 0.

Step 3.1: It follows from (16), elliptic theory, (18) and (17) that for any $\delta>0$, there exists $C(\delta)>0$ such that

$$0 < G(x,y)\leq C(\delta)d(y,\partial\Omega)d(x,\partial\Omega) \text{ for } x,y\in\Omega \text{ s.t. } |x|,|y|>\delta,\ |x-y|>\delta. \tag{22}$$

Step 3.2: We claim that for any $\delta>0$, there exists $C(\delta)>0$ such that

$$|x-y|^{n-2}G(x,y)\leq C(\delta)\min\left\{1,\frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x-y|^2}\right\} \text{ for } x,y\in\Omega \text{ s.t. } |x|,|y|>\delta. \tag{23}$$

Indeed, with no loss of generality, we can assume that $\delta\in(0,\delta_0)$. Let Ω_δ be a smooth domain of $\mathbb{R}^n$ be such that $\Omega\setminus B_{3\delta/4}(0)\subset\Omega_\delta\subset\Omega\setminus B_{\delta/2}(0)$. We fix $x\in\Omega$ such that $|x|>\delta$. Let H_x be the Green's function for $-\Delta-\left(\frac{\gamma}{|x|^2}+h(x)\right)$ in Ω_δ

with Dirichlet boundary condition. Classical estimates (see [3]) yield the existence of $C(\delta) > 0$ such that

$$|x-y|^{n-2}H_x(y) \le C(\delta)\min\left\{1, \frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x-y|^2}\right\} \text{ for all } x,y\in\Omega_\delta.$$

It is easy to check that

$$\left\{\begin{array}{ll} -\Delta(G_x - H_x) - \left(\frac{\gamma}{|\cdot|^2} + h\right)(G_x - H_x) = 0 & \text{weakly in } \Omega_\delta \\ G_x - H_x = 0 & \text{on } (\partial\Omega_\delta)\setminus B_{3\delta/4}(0) \\ G_x - H_x = G_x & \text{on } (\partial\Omega_\delta)\cap B_{3\delta/4}(0). \end{array}\right.$$

Regularity theory then yields that $G_x - H_x \in C^{2,\theta}(\overline{\Omega_\delta})$. It follows from (22) that $G_x(y) \le C_1(\delta)d(y,\partial\Omega)d(x,\partial\Omega)$ on $(\partial\Omega_\delta)\cap B_{3\delta/4}(0)$ for $|x| > \delta$. The comparison principle then yields $G_x(y) - H_x(y) \le C_1(\delta)d(y,\partial\Omega)d(x,\partial\Omega)$ for $y\in\Omega_\delta$ and $|x| > \delta$. The above bound for H_x and (22) then yields (23).

Step 3.3: We now claim that for any $0 < \delta' < \delta$, there exists $C(\delta,\delta') > 0$ such that

(24)
$$|y|^{\alpha_-(\gamma)}G(x,y) \le C(\delta,\delta')d(y,\partial\Omega)d(x,\partial\Omega) \text{ for } x,y\in\Omega \text{ s.t. } |x| > \delta > \delta' > |y|.$$

We let $\delta_1 \in (0,\delta')$ that will be fixed later. We use (22) to deduce that $G_x(y) \le C(\delta,\delta_1)d(x,\partial\Omega)d(y,\partial\Omega)$ for all $x\in\Omega\setminus B_\delta(0)$ and $y\in\partial B_{\delta_1}(0)\cap\Omega$. Since $\delta_1 < |x|$, we have that

$$\left\{\begin{array}{ll} -\Delta G_x - \left(\frac{\gamma}{|x|^2} + h\right)G_x = 0 & \text{in } \Omega\cap B_{\delta_1}(0) \\ 0 \le G_x \le C(\delta,\delta_1)d(y,\partial\Omega)d(x,\partial\Omega) & \text{on } \partial(\Omega\cap B_{\delta_1}(0))\setminus\{0\}. \end{array}\right.$$

We choose a supersolution $\overline{u}_{\alpha_-(\gamma)}$ as in (20) of Proposition 1. It follows from (21) and (22) that for $\delta_1 > 0$, there exists $C(\delta,\delta_1) > 0$ such that $G_x(z) \le C(\delta,\delta_1)d(x,\partial\Omega)u_{\alpha_-}(z)$ for all $z\in\partial(\Omega\cap B_{\delta_1}(0))$. It then follows from the comparison principle that $G_x(y) \le C(\delta,\delta_1)d(x,\partial\Omega)u_{\alpha_-}(y)$ for all $y\in(\Omega\cap B_{\delta_1}(0))\setminus\{0\}$. Combining this with (22) and (20), we obtain (24).

Note that by symmetry, we also get that for any $0 < \delta' < \delta$, there exists $C(\delta,\delta') > 0$ such that

(25)
$$|x|^{\alpha_-(\gamma)}G(x,y) \le C(\delta,\delta')d(x,\partial\Omega)d(y,\partial\Omega) \text{ for } x,y\in\Omega \text{ s.t. } |y| > \delta > \delta' > |x|.$$

Step 4: Upper bound for $G(x,y)$ when both variables approach 0.

We claim first that for all $c_1,c_2,c_3 > 0$, there exists $C(c_1,c_2,c_3) > 0$ such that for $x,y\in\Omega$ such that $c_1|x| < |y| < c_2|x|$ and $|x-y| > c_3|x|$, we have

$$|x-y|^{n-2}G(x,y) \le C(c_1,c_2,c_3)\frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x|^2}. \tag{26}$$

When one of the variables stays far from 0, (26) is a consequence of (22). We now consider a chart at 0, that is $\delta_0 > 0$, $0\in V\subset\mathbb{R}^n$ and $\varphi : B_{2\delta_0}(0)\to V$ a smooth diffeomorphism such that $\varphi(0) = 0$ and

$$\varphi(B_{2\delta_0}(0)\cap\mathbb{R}^n_-) = \varphi(U)\cap\Omega \text{ and } \varphi(B_{2\delta_0}(0)\cap\partial\mathbb{R}^n_-) = \varphi(U)\cap\partial\Omega. \tag{27}$$

Without loss of generality, we can assume that $d\varphi_0 : \mathbb{R}^n\to\mathbb{R}^n = Id_{\mathbb{R}^n}$. In particular, we have that

$$|\varphi(X)| = (1 + O(|X|))|X| \text{ for all } X\in B_{3\delta_0/2}(0). \tag{28}$$

We fix $X \in \mathbb{R}^n_-$ such that $0 < |X| < 3\delta_0/2$. We define

$$H(z) := G_{\varphi(X)}(\varphi(|X|z)) \text{ for } z \in B_{\delta_0/|X|}(0) \setminus \left\{0, \frac{X}{|X|}\right\},$$

so that

$$-\Delta_{g_X} H - \left(\frac{\gamma}{\left(\frac{|\varphi(|X|z|)}{|X|}\right)^2} + |X|^2 h(\varphi(|X|z))\right) H = 0 \text{ in } B_{\delta_0/|X|}(0) \setminus \left\{0, \frac{X}{|X|}\right\}.$$

where $g_X := (\varphi^\star \mathrm{Eucl})_X$ is the pulled-back metric of the Euclidean metric Eucl via the chart φ at the point X. Since $H > 0$, it follows from the Harnack inequality on the boundary (see Proposition 6.3 in Ghoussoub-Robert [1]) that for all $R > 0$ large enough and $r > 0$ small enough, there exist $\delta_1 > 0$ and $C > 0$ independent of $|X| < 3\delta_0/2$ such that

$$\frac{H(z)}{|z_1|} \leq C\frac{H(z')}{|z_1'|} \text{ for all } z, z' \in (B_R(0) \cap \mathbb{R}^n_-) \setminus \left(B_r(0) \cup B_r\left(\frac{X}{|X|}\right)\right),$$

which, via the chart φ, yields

$$\frac{G_x(y)}{d(y, \partial\Omega)} \leq C\frac{G_x(y')}{d(y', \partial\Omega)} \text{ for all } y, y' \in \Omega \cap B_{R|x|/2}(0) \setminus \left(B_{2r|x|}(0) \cup B_{2r|x|}(x)\right). \tag{29}$$

for all $x \in \Omega$ such that $|x| < \delta_0$. We let W be a smooth domain of $\mathbb{R}^n$ such that for some $\lambda > 0$ small enough, we have

$$B_\lambda(0) \cap \Omega \subset W \subset B_{2\lambda}(0) \cap \Omega \text{ and } B_\lambda(0) \cap \partial W = B_\lambda(0) \cap \partial\Omega. \tag{30}$$

We choose a subsolution $\underline{u}_{\alpha_+(\gamma)}$ as in (20) of Proposition 1. It follows from (21) and (22) that for $|x| < \delta_2$ small

$$G_x(z) \geq C(R)|x|^{\alpha_+(\gamma)} \left(\inf_{y \in \Omega \cap \partial B_{R|x|}(0)} \frac{G_x(y)}{d(y, \partial\Omega)}\right) \underline{u}_{\alpha_+(\gamma)}(z) \text{ for all } z \in W \cap \partial B_{R|x|/3}(0).$$

Since $-\Delta G_x - (\gamma|\cdot|^{-2} + h)G_x = 0$ outside 0, it follows from coercivity and the comparison principle that

$$G_x(z) \geq c|x|^{\alpha_+(\gamma)} \left(\inf_{y \in \Omega \cap \partial B_{R|x|}(0)} \frac{G_x(y)}{d(y, \partial\Omega)}\right) \underline{u}_{\alpha_+(\gamma)}(z) \text{ for all } z \in W \setminus B_{R|x|/3}(0).$$

We fix $z_0 \in W \setminus \{0\}$. Then for δ_3 small enough, when $|x| < \delta_3$, it follows from (25) and the Harnack inequality (29) that there exists $C > 0$ independent of x such that

$$G_x(y) \leq C|x|^{-\alpha_+(\gamma) - \alpha_-(\gamma)} d(x, \partial\Omega) d(y, \partial\Omega) \text{ for all } y \in B_{R|x|}(0) \setminus \left(B_{r|x|}(0) \cup B_{r|x|}(x)\right)$$

Taking $r > 0$ small enough and $R > 0$ large enough, we then get (26) for $|x| < \delta_3$. The general case for arbitrary $x \in \Omega \setminus \{0\}$ then follows from (23). This prove (26).

Step 4.2: We claim that for all $c_1, c_2 > 0$, there exists $C(c_1, c_2) > 0$ such that

$$|x - y|^{n-2} G(x, y) \leq C(c_1, c_2) \min\left\{1, \frac{d(x, \partial\Omega) d(y, \partial\Omega)}{|x - y|^2}\right\} \tag{31}$$

for all $x, y \in \Omega$ s.t. $c_1|x| < |y| < c_2|x|$. To prove (31), we distinguish three cases:

Case 1: We assume that

$$|x| \leq C_1 d(x, \partial\Omega) \text{ with } C_1 > 1. \tag{32}$$

We define

$$H(z) := |x|^{n-2} G_x(x + |x|z) \text{ for } z \in B_{1/C_1}(0) \setminus \{0\}.$$

Note that this definition makes sense since for such z, $x + |x|z \in \Omega$. We then have that $H \in C^2(\overline{B_{1/(2C_1)}(0)} \setminus \{0\})$ and

$$-\Delta H - \left(\frac{\gamma}{\left|\frac{x}{|x|} + z\right|^2} + |x|^2 h(x + |x|z) \right) H = \delta_0 \text{ weakly in } B_{1/(2C_1)}(0).$$

We now argue as in the proof of (23). From (26), we have that $|H(z)| \leq C$ for all $z \in \partial B_{1/(2C_1)}(0)$ where C is independent of $x \in \Omega \setminus \{0\}$ satisfying (32). Let Γ_0 be the Green's function of $-\Delta - \left(\frac{\gamma}{\left|\frac{x}{|x|}+z\right|^2} + |x|^2 h(x + |x|z) \right)$ at 0 on $B_{1/(2C_1)}(0)$ with Dirichlet boundary condition. Therefore, $H - \Gamma_0 \in C^2(\overline{B_{1/(2C_1)}(0)})$ and, via the comparison principle, it is bounded by its supremum on the boundary. Therefore $|z|^{n-2} H(z) \leq C$ for all $B_{1/(2C_1)}(0) \setminus \{0\}$ where C is independent of $x \in \Omega \setminus \{0\}$ satisfying (32). Scaling back and using (26), we get $|x - y|^{n-2} G_x(y) \leq C$ for all $x, y \in \Omega \setminus \{0\}$ such that $c_1|x| < |y| < c_2|x|$ and (32) holds. This proves (31) if $d(x, \partial\Omega)d(y, \partial\Omega) \geq |x - y|^2$. If $d(x, \partial\Omega)d(y, \partial\Omega) < |x - y|^2$, we get that $d(x, \partial\Omega) < 2|x - y|$, and then (32) yields $|x| \leq 2C_1|x - y|$, and (31) is a consequence of (26).

This ends the proof of (31) in Case 1.

Case 2: By symmetry, (31) also holds when $|y| \leq C_1 d(y, \partial\Omega)$.

Case 3: We assume that $d(x, \partial\Omega) \leq C_1^{-1}|x|$ and $d(y, \partial\Omega) \leq C_1^{-1}|y|$. We consider a chart at 0, that is $\delta_0 > 0$, $0 \in V \subset \mathbb{R}^n$ and $\varphi : B_{2\delta_0}(0) \to V$ a smooth diffeomorphism such that $\varphi(0) = 0$ and that (27) and (28) hold. We fix $x' \in \mathbb{R}^{n-1}$ such that $0 < |x'| < 3\delta_0/2$.

We assume that $r \leq c_0|x'|$. We define

$$H_y(z) := r^{n-2} G_{\varphi((0,x')+ry)}(\varphi((0, x') + rz)) \text{ for } y, z \in B_{\delta_0/(2r)}(0) \cap \mathbb{R}^n_- \setminus \{0\}.$$

We then have that $H_y \in C^2(\overline{B_{R_0}(0)} \cap \mathbb{R}^n_- \setminus \{0, y\})$ and

$$-\Delta_{g_r} H_y - \left(\frac{\gamma}{\left(\frac{|\varphi((0,x')+rz)}{r}\right)^2} + r^2 h(\varphi((0, x') + rz)) \right) H_y = \delta_y \text{ weakly in } B_{R_0}(0) \cap \mathbb{R}^n_-,$$

where $g_r := (\varphi^\star \mathrm{Eucl})_{(0,x')+rz}$ is the pulled-back metric of the Euclidean metric Eucl via the chart φ at the point $(0, x') + rz$. We now argue as in the proof of (23). From (26), we have that $|H_y(z)| \leq C$ for all $z \in \partial B_{R_0}(0) \cap \mathbb{R}^n_-$ where C is independent of $y \in B_{R_0/2}(0)$ and $r \in (0, \delta_0/4)$. Let Γ_y be the Green's function of $-\Delta_{g_r} - \left(\frac{\gamma}{\left(\frac{|\varphi((0,x')+rz)}{r}\right)^2} + r^2 h(\varphi((0, x') + rz)) \right)$ at y on $B_{c_0/2}(0) \cap \mathbb{R}^n_-$ with Dirichlet boundary condition. Therefore, $H_y - \Gamma_y \in C^2(\overline{B_{c_0/2}(0) \cap \mathbb{R}^n_-})$ and, via the comparison principle, it is bounded by its supremum on the boundary. It follows from (26) and elliptic estimates for Γ_y (see for instance [3]) that $|H_y - \Gamma_y|(z) \leq C|y_1| \cdot |z_1|$ for $z \in \partial(B_{c_0/2}(0) \cap \mathbb{R}^n_-)$ and $y \in B_{c_0/4}(0) \cap \mathbb{R}^n_-$. Applying elliptic estimates, we then get that $|H_y - \Gamma_y|(z) \leq C|y_1|\cdot|z_1|$ for $z \in B_{c_0/2}(0)\cap\mathbb{R}^n_-$ and $y \in B_{c_0/4}(0)\cap\mathbb{R}^n_-$, and since

$$\Gamma_y(z) \leq C|z-y|^{2-n}\min\left\{1, \frac{|y_1|\cdot|z_1|}{|y-z|^2}\right\} \text{ for all } y,z \in B_{c_0/2}(0)\cap\mathbb{R}^n_-$$

(see [3]), we get that

$$|z-y|^{n-2}H_y(z) \leq C\min\left\{1, \frac{|y_1|\cdot|z_1|}{|y-z|^2}\right\} \text{ for all } y,z \in B_{c_0/2}(0)\cap\mathbb{R}^n_-$$

where C is independent of $x' \in B_{\delta_0/2}(0)\setminus\{0\}$. This yields

$$|rz-ry|^{n-2}G_{\varphi((0,x')+ry)}(\varphi((0,x')+rz)) \leq C\min\{1, \frac{|y_1|\cdot|z_1|}{|y-z|^2}\} \tag{33}$$

for $|x'|<\delta_0/3$, $r\leq c_0|x'|$ and $|y|,|z|\leq c_0/4$.

We now prove (31) in the last case. We fix $x\in\Omega\setminus\{0\}$ such that $|x|<\delta_0/3$. We assume that $d(x,\partial\Omega)\leq C_1^{-1}|x|$, $d(y,\partial\Omega)\leq C_1^{-1}|y|$ and $|x-y|\leq\epsilon_0|x|$. We let $(x_1,x'),(y_1,y')\in B_{\delta_0}(0)$ be such that $x=\varphi(x_1,x')$ and $y=\varphi(y_1,y')$. Taking the norm $|(x_1,x')|=|x_1|+|x'|$, we define $r:=\max\{d(x,\partial\Omega),|x-y|\}$. Using that $|X|/2\leq|\varphi(X)|\leq 2|X|$ for $X\in B_{\delta_0}(0)$, up to taking $\epsilon_0>0$ small and $C_1,c_0>1$ large enough, we get that

$$\left|\frac{x_1}{r}\right|\leq\frac{c_0}{4}\ ,\ \left|\left(\frac{y_1}{r},\frac{y'-x'}{r}\right)\right|\leq\frac{c_0}{4} \text{ and } r\leq c_0|x'|.$$

Therefore, (33) applies and we get (31) in Case 3.

We are now in position to conclude. Inequality (31) is a consequence of Cases 1, 2, 3, (23) and (26). This ends the proof of (31).

Step 4.3: We now show that there exists $C>0$ such that

$$|y|^{\alpha_-(\gamma)}|x|^{\alpha_+(\gamma)}G(x,y)\leq Cd(x,\partial\Omega)d(y,\partial\Omega) \text{ for } x,y\in\Omega \text{ such that } |y|<\frac{1}{2}|x|. \tag{34}$$

The proof goes essentially as in (24). For $|x|<\delta$ with $\delta>0$ small, we have that

$$-\Delta G_x - \left(\frac{\gamma}{|\cdot|^2}+h\right)G_x = 0 \text{ in } H^1(\Omega\cap B_{|x|/3}(0))\cap C^2(\overline{\Omega}\cap B_{|x|/3}(0)\setminus\{0\}).$$

It follows from (26) that $G_x(y)\leq C|x|^{-n}d(x,\partial\Omega)d(y,\partial\Omega)$ in $\Omega\cap\partial B_{|x|/3}(0)$. We choose a supersolution $\overline{u}_{\alpha_-(\gamma)}$ as in (20) of Proposition 1. It follows from (21) and (26) that there exists $C>0$ such that

$$G_x(y)\leq C|x|^{-\alpha_+(\gamma)}d(x,\partial\Omega)\overline{u}_{\alpha_-(\gamma)}(y) \text{ for all } y\in\Omega\cap\partial B_{|x|/3}(0).$$

The comparison principle yields that this inequality holds on $\Omega\cap B_{|x|/3}(0)$.

Step 4.4: By symmetry, we conclude that there exists $C>0$ such that

$$|x|^{\alpha_-(\gamma)}|y|^{\alpha_+(\gamma)}G(x,y)\leq Cd(x,\partial\Omega)d(y,\partial\Omega) \text{ for } x,y\in\Omega \text{ s.t. } |x|<\frac{1}{2}|y|. \tag{35}$$

Step 5: Finally, it follows from (34), (35) and (31) that there exists $c>0$ such that

$$G(x,y)\leq c\left(\frac{\max\{|y|,|x|\}}{\min\{|y|,|x|\}}\right)^{\alpha_-(\gamma)}|x-y|^{2-n}\min\left\{1,\frac{d(x,\partial\Omega)d(y,\partial\Omega)}{|x-y|^2}\right\} \tag{36}$$

for all $x, y \in \Omega$, $x \neq y$. This proves the upper bound in (5) of Theorem 2. The lower-bound and the control of the gradient will be proved in Section 6.

4. Green's function for $-\Delta - \gamma|x|^{-2}$ on $\mathbb{R}^n_-$

In this section, we prove the following:

Theorem 3. *Fix $\gamma < \frac{n^2}{4}$. For all $p \in \mathbb{R}^n_- \setminus \{0\}$, there exists $G_p \in L^1(\mathbb{R}^n_-)$ such that*

(i) *$\eta G_p \in H^2_{1,0}(\mathbb{R}^n_-)$ for all $\eta \in C^\infty_c(\mathbb{R}^n - \{p\})$,*

(ii) *For all $\varphi \in C^\infty_c(\mathbb{R}^n_-)$, we have that*

$$\varphi(p) = \int_{\mathbb{R}^n_-} G_p(x)\left(-\Delta\varphi - \frac{\gamma}{|x|^2}\varphi\right) dx, \tag{37}$$

Moreover, if G_p, G'_p satisfy (i) and (ii) and are positive, then there exists $C \in \mathbb{R}$ such that $G_p(x) - G'_p(x) = C|x_1| \cdot |x|^{-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_- \setminus \{0, p\}$.

In particular, there exists one and only one function $\mathcal{G}_p = \mathcal{G}(p, \cdot) > 0$ such that (i) and (ii) hold with $G_p = \mathcal{G}_p$ and

$$\textbf{(iii)}\ \mathcal{G}_p(x) = O\left(\frac{|x_1|}{|x|^{\alpha_+(\gamma)}}\right) \text{ as } |x| \to +\infty.$$

We say that $\mathcal{G}$ is the Green's function for $-\Delta - \gamma|x|^{-2}$ on $\mathbb{R}^n_-$ with Dirichlet boundary condition.

In addition, $\mathcal{G}$ satisfies the following properties:

- *For all $p \in \mathbb{R}^n \setminus \{0\}$, there exists $c_0(p), c_\infty(p) > 0$ such that*

$$\mathcal{G}_p(x) \sim_{x \to 0} \frac{c_0(p)|x_1|}{|x|^{\alpha_-(\gamma)}} \text{ and } \mathcal{G}_p(x) \sim_{x \to \infty} \frac{c_\infty(p)|x_1|}{|x|^{\alpha_+(\gamma)}} \tag{38}$$

and

$$\mathcal{G}_p(x) \sim_{x \to p} \frac{1}{(n-2)\omega_{n-1}|x - p|^{n-2}}. \tag{39}$$

- *There exists $c > 0$ independent of p such that*

$$c^{-1}\mathcal{H}_p(x) \leq \mathcal{G}_p(x) \leq c\mathcal{H}_p(x) \tag{40}$$

where

$$\mathcal{H}_p(x) := \left(\frac{\max\{|p|, |x|\}}{\min\{|p|, |x|\}}\right)^{\beta_-(\gamma)} |x - p|^{2-n} \min\left\{1, \frac{|x_1| \cdot |p_1|}{|x - p|^2}\right\} \tag{41}$$

Proof of Theorem 3: We shall again proceed with several steps.

Step 1: Construction of a positive kernel at a given point: For a fixed $p_0 \in \mathbb{R}^n \setminus \{0\}$, we show that there exists $G_{p_0} \in C^2(\overline{\mathbb{R}^n_-} \setminus \{0, p_0\})$ such that

$$\begin{cases} -\Delta G_{p_0} - \frac{\gamma}{|x|^2} G_{p_0} = 0 & \text{in } \mathbb{R}^n_- \setminus \{0, p_0\} \\ G_{p_0} > 0 & \\ G_{p_0} \in L^{\frac{2n}{n-2}}(B_\delta(0) \cap \mathbb{R}^n_-) & \text{with } \delta := |p_0|/4 \\ G_{p_0} \text{ satisfies } (ii) \text{ with } p = p_0. & \end{cases} \tag{42}$$

Indeed, let $\tilde{\eta} \in C^\infty(\mathbb{R})$ be a nondecreasing function such that $0 \leq \tilde{\eta} \leq 1$, $\tilde{\eta}(t) = 0$ for all $t \leq 1$ and $\tilde{\eta}(t) = 1$ for all $t \geq 2$. For $\epsilon > 0$, set $\eta_\epsilon(x) := \tilde{\eta}\left(\frac{|x|}{\epsilon}\right)$ for all $x \in \mathbb{R}^n$.

We let Ω_1 be a smooth bounded domain of $\mathbb{R}^n$ such that $\mathbb{R}^n_- \cap B_1(0) \subset \Omega_1 \subset \mathbb{R}^n_- \cap B_3(0)$. We define $\Omega_R := R \cdot \Omega_1$ so that $\mathbb{R}^n_- \cap B_R(0) \subset \Omega_R \subset \mathbb{R}^n_- \cap B_{3R}(0)$. We argue as in the proof of (8) to deduce that the operator $-\Delta - \frac{\gamma \eta_\epsilon}{|x|^2}$ is coercive on Ω_R and that there exists $c > 0$ independent of $R, \epsilon > 0$ such that

$$\int_{\Omega_R} \left(|\nabla \varphi|^2 - \frac{\gamma \eta_\epsilon}{|x|^2}\varphi^2\right) dx \geq c \int_{\Omega_R} |\nabla \varphi|^2 \, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega_R).$$

Consider $R, \epsilon > 0$ such that $R > 2|p_0|$ and $\epsilon < \frac{|p_0|}{6}$, and let $G_{R,\epsilon}$ be the Green's function of $-\Delta - \frac{\gamma \eta_\epsilon}{|x|^2}$ in Ω_R with Dirichlet boundary condition. We have that $G_{R,\epsilon} > 0$ since the operator is coercive.

Fix $R_0 > 0$ and $q' \in (1, \frac{n}{n-2})$, then by arguing as in the proof of (10), we get that there exists $C = C(\gamma, p_0, q', R_0)$ such that

$$\|G_{R,\epsilon}(p_0, \cdot)\|_{L^{q'}(B_{R_0}(0) \cap \mathbb{R}^n_-)} \leq C \text{ for all } R > R_0 \text{ and } 0 < \epsilon < \frac{|p_0|}{6}, \tag{43}$$

and

$$\|G_{R,\epsilon}(p_0, \cdot)\|_{L^{\frac{2n}{n-2}}(B_{\delta_0}(0) \cap \mathbb{R}^n_-)} \leq C \text{ for all } R > R_0 \text{ and } 0 < \epsilon < \frac{|p_0|}{6}, \tag{44}$$

where $\delta := |p_0|/4$. Arguing again as in Step 2 of the proof of Theorem 1, there exists $G_{p_0} \in C^2(\overline{\mathbb{R}^n_-} \setminus \{0, p_0\})$ such that

$$\left\{ \begin{array}{ll} G_{R,\epsilon}(p_0, \cdot) \to G_{p_0} \geq 0 & \text{in } C^2_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0, p_0\}) \text{ as } R \to +\infty,\ \epsilon \to 0 \\ -\Delta G_{p_0} - \frac{\gamma}{|x|^2} G_{p_0} = 0 & \text{in } \mathbb{R}^n_- \setminus \{0, p_0\} \\ G_{p_0} \equiv 0 \text{ on } \partial \mathbb{R}^n_- \setminus \{0\} & \\ G_{p_0} \in L^{\frac{2n}{n-2}}(B_\delta(0) \cap \mathbb{R}^n_-) & \end{array} \right. \tag{45}$$

and $\eta G_{p_0} \in H_0^1(\mathbb{R}^n_-)$ for all $\eta \in C_c^\infty(\mathbb{R}^n \setminus \{p_0\})$. Fix $\varphi \in C_c^\infty(\mathbb{R}^n_-)$. For $R > 0$ large enough, we have that $\varphi(p_0) = \int_{\mathbb{R}^n_-} G_{R,\epsilon}(p_0, \cdot)(-\Delta \varphi - \gamma \eta_\epsilon |x|^{-2} \varphi) \, dx$. The integral bounds above yield $x \mapsto G_{p_0}(x)|x|^{-2} \in L^1_{loc}(\mathbb{R}^n_-)$. Therefore, we get

$$\varphi(p_0) = \int_{\mathbb{R}^n_-} G_{p_0}(x) \left(-\Delta \varphi - \frac{\gamma}{|x|^2}\varphi\right) dx \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^n_-). \tag{46}$$

As a consequence, $G_{p_0} > 0$.

Step 2: Asymptotic behavior at 0 and p_0 for solutions to (42). It follows from Theorem 6.1 in Ghoussoub-Robert [1] that either G_{p_0} behaves like $|x_1| \cdot |x|^{-\alpha_-(\gamma)}$ or $|x_1| \cdot |x|^{-\alpha_+(\gamma)}$ at 0. Since $G_{p_0} \in L^{\frac{2n}{n-2}}(B_\delta(0) \cap \mathbb{R}^n_-)$ for some small $\delta > 0$ and $\alpha_-(\gamma) < \frac{n}{2} < \alpha_+(\gamma)$, we get that there exists $c_0 > 0$ such that

$$\lim_{x \to 0} \frac{G_{p_0}(x)}{|x_1| \cdot |x|^{-\alpha_-(\gamma)}} = c_0. \tag{47}$$

Since G_{p_0} is positive and smooth in a neighborhood of p_0, it follows from (46) and the classification of solutions to harmonic equations that

$$G_{p_0}(x) \sim_{x \to p_0} \frac{1}{(n-2)\omega_{n-1}|x - p_0|^{n-2}}. \tag{48}$$

Step 3: Asymptotic behavior at ∞ for solutions to (42)**:** We let

$$\tilde{G}_{p_0}(x) := \frac{1}{|x|^{n-2}} G_{p_0}\left(\frac{x}{|x|^2}\right) \text{ for all } x \in \mathbb{R}^n_- \setminus \left\{0, \frac{p_0}{|p_0|^2}\right\},$$

be the Kelvin's transform of G. We have that

$$-\Delta \tilde{G}_{p_0} - \frac{\gamma}{|x|^2}\tilde{G}_{p_0} = 0 \text{ in } \mathbb{R}^n_- \setminus \left\{0, \frac{p_0}{|p_0|^2}\right\} \; ; \; \tilde{G} \equiv 0 \text{ on } \partial\mathbb{R}^n_- \setminus \{p_0\}.$$

Since $\tilde{G}_{p_0} > 0$, it follows from Theorem 6.1 in [1] that there exists $c_1 > 0$ such that

$$\text{either } \tilde{G}_{p_0}(x) \sim_{x\to 0} c_1 \frac{|x_1|}{|x|^{\alpha_-(\gamma)}} \text{ or } \tilde{G}_{p_0}(x) \sim_{x\to 0} c_1 \frac{|x_1|}{|x|^{\alpha_+(\gamma)}}.$$

Coming back to G_{p_0}, we get that

$$\text{either } G_{p_0}(x) \sim_{|x|\to\infty} c_1 \frac{|x_1|}{|x|^{\alpha_+(\gamma)}} \text{ or } G_{p_0}(x) \sim_{|x|\to\infty} c_1 \frac{|x_1|}{|x|^{\alpha_-(\gamma)}}. \tag{49}$$

Assuming we are in the second case, for any $c \leq c_1$, we define

$$\bar{G}_c(x) := G_{p_0}(x) - c\frac{|x_1|}{|x|^{\alpha_-(\gamma)}} \text{ in } \mathbb{R}^n_- \setminus \{0, p_0\},$$

which satisfy $-\Delta \bar{G}_c - \frac{\gamma}{|x|^2}\bar{G}_c = 0$ in $\mathbb{R}^n_- \setminus \{0, p_0\}$. It follows from (49) and (48) that for $c < c_1$, $\bar{G}_c > 0$ around p_0 and ∞. Using that $\eta\bar{G}_c \in H_0^1(\mathbb{R}^n_-)$ for all $\eta \in C_c^\infty(\mathbb{R}^n \setminus \{p_0\})$, it follows from the coercivity of $-\Delta - \gamma|x|^{-2}$ that $\bar{G}_c > 0$ in $\mathbb{R}^n_- \setminus \{0, p_0\}$ for $c < c_1$. Letting $c \to c_1$ yields $\bar{G}_{c_1} \geq 0$, and then $\bar{G}_{c_1} > 0$. Since $\bar{G}_{c_1}(x) = o(|x_1| \cdot |x|^{-\alpha_-(\gamma)})$ as $|x| \to \infty$, another Kelvin transform and Theorem 6.1 in [1] yield $|x_1|^{-1}|x|^{\alpha_+(\gamma)}\bar{G}_{c_1}(x) \to c_2 > 0$ as $|x| \to \infty$ for some $c_2 > 0$. Then there exists $c_3 > 0$ such that

$$\lim_{x\to 0} \frac{\bar{G}_{c_1}(x)}{|x_1| \cdot |x|^{-\beta_-(\gamma)}} = c_3 > 0 \text{ and } \lim_{x\to\infty} \frac{\bar{G}_{c_1}(x)}{|x_1| \cdot |x|^{-\alpha_+(\gamma)}} = c_2. \tag{50}$$

Since $x \mapsto |x_1| \cdot |x|^{-\alpha_-(\gamma)} \in H^2_{1,loc}(\mathbb{R}^n)$, we get that $\varphi(p) = \int_{\mathbb{R}^n_-} \bar{G}_{c_1}(x)\left(-\Delta\varphi - \frac{\gamma}{|x|^2}\varphi\right) dx$ for all $\varphi \in C_c^\infty(\mathbb{R}^n_-)$.

Step 4: Uniqueness: Let $G_1, G_2 > 0$ be 2 functions such that $(i), (ii)$ hold for $p := p_0$, and set $H := G_1 - G_2$. It follows from Steps 2 and 3 that there exists $c \in \mathbb{R}$ such that $H'(x) := H(x) - c|x_1| \cdot |x|^{-\alpha_-(\gamma)}$ satisfies

$$H'(x) =_{x\to 0} O\left(|x_1| \cdot |x|^{-\alpha_-(\gamma)}\right) \text{ and } H'(x) =_{|x|\to\infty} O\left(|x_1| \cdot |x|^{-\alpha_+(\gamma)}\right). \tag{51}$$

We then have that $\eta H' \in H_0^1(\mathbb{R}^n_-)$ for all $\eta \in C_c^\infty(\mathbb{R}^n \setminus \{p_0\})$ and

$$\int_{\mathbb{R}^n_-} H'(x)\left(-\Delta\varphi - \frac{\gamma}{|x|^2}\varphi\right) dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^n_-).$$

The ellipticity of the Laplacian then yields $H' \in C^\infty(\overline{\mathbb{R}^n_-} \setminus \{0\})$. The pointwise bounds (51) yield that $H' \in H_0^1(\mathbb{R}^n_-)$. Multiplying $-\Delta H' - \frac{\gamma}{|x|^2}H' = 0$ by H', integrating by parts and the coercivity yield $H' \equiv 0$, and therefore, $(G_1 - G_2)(x) = c|x_1| \cdot |x|^{-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_-$. This proves uniqueness.

Step 5: Existence. It follows from Steps 2 and 3 that, up to substracting a multiple of $x \mapsto |x_1| \cdot |x|^{-\alpha_-(\gamma)}$, there exists a unique function $\mathcal{G}_{p_0} > 0$ satisfying (i), (ii) and the pointwise control (iii). Moreover, (47), (48) and (50) yield (38) and (39). As a consequence, (40) holds with $p = p_0$.

For $p \in \mathbb{R}^n \setminus \{0\}$, consider $\rho_p : \mathbb{R}^n_- \to \mathbb{R}^n_-$ a linear isometry fixing $\mathbb{R}^n_-$ such that $\rho_p(\frac{p_0}{|p_0|}) = \frac{p}{|p|}$, and define

$$\mathcal{G}_p(x) := \left(\frac{|p_0|}{|p|}\right)^{n-2} \mathcal{G}_{p_0}\left(\left(\rho_p^{-1}\left(\frac{|p_0|}{|p|}x\right)\right)\right) \text{ for all } x \in \mathbb{R}^n \setminus \{0, p\}.$$

As one checks, $\mathcal{G}_p > 0$ satisfies (i), (ii), (iii), (38), (39) and (40).

The definition of $\mathcal{G}_p$ is independent of the choice of ρ_p. Indeed, for any linear isometry $\rho_{p_0} : \mathbb{R}^n_- \to \mathbb{R}^n_-$ fixing p_0 and $\mathbb{R}^n_-$, $\mathcal{G}_{p_0} \circ \rho_{p_0}^{-1}$ satisfies (i), (ii), (iii), and therefore $\mathcal{G}_{p_0} \circ \rho_{p_0}^{-1} = \mathcal{G}_{p_0}$. The argument goes similarly of any isometry fixing p.

5. Behavior at infinitesimal scale

We prove three convergence theorems to get a comprehensive behavior of the Green's function

Theorem 4. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. We fix $\gamma < \frac{n^2}{4}$. We let $h \in C^{0,\theta}(\overline{\Omega})$ be such that $-\Delta - \gamma|x|^{-2} - h$ is coercive. Let G be the Green's function of $-\Delta - \gamma|x|^{-2} - h$ with Dirichlet boundary condition on $\partial\Omega$. Let $(x_i)_i \in \Omega$ and $(r_i)_i \in (0, +\infty)$ be such that*

$$\lim_{i\to+\infty} r_i = 0 \text{ and } \lim_{i\to+\infty} \frac{d(x_i, \partial\Omega)}{r_i} = +\infty.$$

Then, for all $X, Y \in \mathbb{R}^n$ such that $X \neq Y$, we have that

$$\lim_{i\to+\infty} r_i^{n-2} G(x_i + r_i X, x_i + r_i Y) = \frac{1}{(n-2)\omega_{n-1}} |X - Y|^{2-n}$$

Moreover, the convergence holds in $C^2_{loc}((\mathbb{R}^n)^2 \setminus Diag(\mathbb{R}^n))$.

We now deal with the case when the points approach the boundary. For any $x_0 \in \partial\Omega$, there exists $\delta_0 > 0$, $x_0 \in V \subset \mathbb{R}^n$ and $\varphi : B_{\delta_0}(0) \to V$ a smooth diffeomorphism such that $\varphi(0) = x_0$ and

$$\varphi(B_{2\delta_0}(0) \cap \mathbb{R}^n_-) = \varphi(U) \cap \Omega \text{ and } \varphi(B_{2\delta_0}(0) \cap \partial\mathbb{R}^n_-) = \varphi(U) \cap \partial\Omega. \tag{52}$$

Without loss of generality, we can assume that $d\varphi_0 : \mathbb{R}^n \to \mathbb{R}^n = Id_{\mathbb{R}^n}$.

Theorem 5. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. We fix $\gamma < \frac{n^2}{4}$. We let $h \in C^{0,\theta}(\overline{\Omega})$ be such that $-\Delta - \gamma|x|^{-2} - h$ is coercive. Let G be the Green's function of $-\Delta - \gamma|x|^{-2} - h$ with Dirichlet boundary condition on $\partial\Omega$. Let $(x_i)_i \in \partial\Omega$ and $(r_i)_i \in (0, +\infty)$ and $x_0 \in \partial\Omega$ be such that*

$$\lim_{i\to+\infty} r_i = 0, \lim_{i\to+\infty} x_i = x_0 \in \partial\Omega \text{ and } \lim_{i\to+\infty} \frac{|x_i|}{r_i} = +\infty.$$

We let φ be a chart at x_0 as in (52). We define $x_i' \in \mathbb{R}^{n-1}$ such that $x_i = \varphi(0, x_i')$. Then, for all $X, Y \in \mathbb{R}^n_-$ such that $X \neq Y$, we have that

$$\lim_{i\to+\infty} r_i^{n-2} G(\varphi((0, x_i') + r_i X), \varphi((0, x_i') + r_i Y)) = \frac{1}{(n-2)\omega_{n-1}} \left(|X - Y|^{2-n} - |X - Y^*|^{2-n}\right)$$

where $(Y_1, Y')^* = (-Y_1, Y')$ for $(Y_1, Y') \in \mathbb{R} \times \mathbb{R}^{n-1}$. Moreover, the convergence holds in $C^2_{loc}((\overline{\mathbb{R}^n_-})^2 \setminus Diag(\overline{\mathbb{R}^n_-})\})$.

Theorem 6. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. We fix $\gamma < \frac{n^2}{4}$. We let $h \in C^{0,\theta}(\overline{\Omega})$ be such that $-\Delta - \gamma|x|^{-2} - h$ is coercive. Let G be the Green's function of $-\Delta - \gamma|x|^{-2} - h$ with Dirichlet boundary condition on $\partial\Omega$. Let $(r_i)_i \in (0, +\infty)$ be such that $\lim_{i\to+\infty} r_i = 0$. We let φ be a chart at 0 as in (52). Then, for all $X, Y \in \overline{\mathbb{R}^n_-} \setminus \{0\}$ such that $X \neq Y$, we have that*

$$\lim_{i\to+\infty} r_i^{n-2} G(\varphi(r_i X), \varphi(r_i Y)) = \mathcal{G}(X, Y)$$

where $\mathcal{G}(X,Y) = \mathcal{G}_X(Y)$ is the Green's function for $-\Delta - \gamma|x|^{-2}$ on $\mathbb{R}^n_-$ with Dirichlet boundary condition. Moreover, the convergence holds in $C^2_{loc}((\overline{\mathbb{R}^n_-} \setminus \{0\})^2 \setminus$ Diag$(\overline{\mathbb{R}^n_-} \setminus \{0\}))$.

Proof of Theorem 4: We let $(r_i)_i \in (0, +\infty)$ and $(x_i)_i \in \Omega$ as in the statement of the Theorem. For any $X, Y \in \mathbb{R}^n$, $X \neq Y$, we define

$$G_i(X,Y) := r_i^{n-2} G(x_i + r_i X, x_i + r_i Y)$$

for all $i \in \mathbb{N}$. Since $r_i = o(d(x_i, \partial\Omega))$ as $i \to +\infty$, for any $R > 0$, there exists $i_0 \in \mathbb{N}$ such that this definition makes sense for any $X, Y \in B_R(0)$. Equation (3) yields

$$-\Delta G_i(X,\cdot) - \left(\frac{\gamma}{\left|\frac{x_i}{r_i} + \cdot\right|^2} + r_i^2 h(x_i + r_i \cdot) \right) G_i(X,\cdot) = 0 \text{ in } B_R(0) \setminus \{X\}. \tag{53}$$

The pointwise control (36) writes

$$0 < G_i(X,Y) \leq c \left(\frac{\max\{|x_i + r_i X|, |x_i + r_i Y|\}}{\min\{|x_i + r_i X|, |x_i + r_i Y|\}} \right)^{\alpha_-(\gamma)} |X - Y|^{2-n} \tag{54}$$

for all $X, Y \in B_R(0)$ such that $X \neq Y$. Since $0 \in \partial\Omega$, we have that $d(x_i, \partial\Omega) \leq |x_i|$, and therefore $r_i = o(|x_i|)$ as $i \to +\infty$. Equation (53) and inequality (54) yield

$$-\Delta G_i(X,\cdot) + \theta_i(X,\cdot) G_i(X,\cdot) = 0 \text{ in } B_R(0) \setminus \{X\}.$$

where $\theta_i \to 0$ uniformly in $C^0_{loc}((\mathbb{R}^n)^2)$ and $0 < G_i(X,Y) \leq c|X-Y|^{2-n}$ for all $X, Y \in B_R(0)$ such that $X \neq Y$. It then follows from standard elliptic theory that, up to a subsequence, there exists $G_\infty(X,\cdot) \in C^2(\mathbb{R}^n \setminus \{X\})$ such that $G_i(X,\cdot) \to G_\infty(X,\cdot) \geq 0$ in $C^2_{loc}(\mathbb{R}^n \setminus \{X\})$ and

$$-\Delta G_\infty(X,\cdot) = 0 \text{ in } \mathbb{R}^n \setminus \{X\} \text{ and } G_\infty(X,Y) \leq c|X-Y|^{2-n} \text{ for } X, Y \in \mathbb{R}^n,\ X \neq Y.$$

It then follows from the classification of positive harmonic functions that there exists $\lambda > 0$ such that $G_\infty(X,Y) = \lambda|X-Y|^{2-n}$ for all $X, Y \in \mathbb{R}^n$, $X \neq Y$.

We let $\varphi \in C^\infty_c(\mathbb{R}^n)$. We define $\varphi_i(x) := \varphi(r_i^{-1}(x - x_i))$ for $x \in \Omega$ (this makes sense for i large enough). It follows from (2) that

$$\varphi_i(x_i + r_i X) = \int_\Omega G(x_i + r_i X, y) \left(-\Delta\varphi_i(y) - \left(\frac{\gamma}{|y|^2} + h(y) \right) \varphi_i(y) \right) dy.$$

Via a change of variable, and passing to the limit, we get that

$$\varphi(X) = \int_{\mathbb{R}^n} G_\infty(X,Y) \left(-\Delta\varphi(Y)\right) dy.$$

Since $G_\infty(X,Y) = \lambda|X-Y|^{2-n}$, we get that $\lambda = 1/((n-2)\omega_{n-1})$. Since the limit is unique, the convergence holds without extracting a subsequence. The convergence in $C^2_{loc}((\mathbb{R}^n)^2 \setminus \mathrm{Diag}(\mathbb{R}^n))$ follows from the symmetry of G and elliptic theory. $\square$

Proof of Theorem 5: The proof goes as in the proof of Theorem 4, except that we have to take a chart due to the closeness of the boundary. We let $(r_i)_i \in (0,+\infty)$, $(x_i)_i \in \partial\Omega$ and $x_0 \in \partial\Omega$ as in the statement of the Theorem. We let φ be a chart at x_0 as in (52) (in particular $d\varphi_0 = Id_{\mathbb{R}^n}$) and we set $x_i' \in \mathbb{R}^n$ such that $x_i = \varphi(0, x_i')$. In particular, $\lim_{i\to+\infty} x_i' = 0$. For any $X, Y \in \overline{\mathbb{R}^n_-}$, $X \neq Y$, we define

$$G_i(X,Y) := r_i^{n-2} G(\varphi\left((0,x_i') + r_i X\right), \varphi\left((0,x_i') + r_i Y\right))$$

for all $i \in \mathbb{N}$. Here again, provided X, Y remain in a given compact set, the definition of G_i makes sense for large i. Equation (3) then rewrites
(55)

$$-\Delta_{g_i} G_i(X,\cdot) - \hat{\theta}_i G_i(X,\cdot) = 0 \text{ in } B_R(0) \cap \mathbb{R}^n_- \setminus \{X\} \;;\; G_i(X,\cdot) \equiv 0 \text{ on } \partial\mathbb{R}^n_- \cap B_R(0)$$

where

$$\hat{\theta}_i(Y) := \frac{\gamma}{\left|\frac{\varphi((0,x_i')+r_i Y)}{r_i}\right|^2} + r_i^2 h(\varphi((0,x_i') + r_i Y))$$

and $g_i = \varphi^\star \mathrm{Eucl}((0,x_i') + r_i \cdot)$ is the pull-back of the Euclidean metric. In particular, since $d\varphi_0 = Id_{\mathbb{R}^n}$, we get that $g_i \to \mathrm{Eucl}$ in $C^2_{loc}(\mathbb{R}^n)$. Since $r_i = o(|x_i|)$, we get that $r_i = o(|x_i'|)$ as $i \to +\infty$, and, using again that $d\varphi_0 = Id_{\mathbb{R}^n}$, we get that $\hat{\theta}_i \to 0$ uniformly in $B_R(0) \cap \mathbb{R}^n_-$. The pointwise control (36) rewrite $G_i(X,Y) \leq c|X-Y|^{2-n}$ for all $X, Y \in \mathbb{R}^n_-$, $X \neq Y$. With the same arguments as above, we get that for any $X \in \overline{\mathbb{R}^n_-}$, there exists $G_\infty(X,\cdot) \in C^2(\overline{\mathbb{R}^n_-} \setminus \{X\})$ such that

$$\lim_{i\to+\infty} G_i(X,\cdot) = G_\infty(X,\cdot) \text{ in } C^2_{loc}(\overline{\mathbb{R}^n_-} \setminus \{X\})$$

$$\text{with } \begin{cases} -\Delta G_\infty(X,\cdot) = 0 & \text{in } \mathbb{R}^n_- \setminus \{X\} \\ G_\infty(X,\cdot) \geq 0 & \\ G_\infty(X,\cdot) \equiv 0 & \text{on } \partial\mathbb{R}^n_- \setminus \{X\} \end{cases}$$

and

$$\varphi(X) = \int_{\mathbb{R}^n_-} G_\infty(X,\cdot)(-\Delta\varphi)\, dY \text{ for all } \varphi \in C^\infty_c(\mathbb{R}^n_-).$$

with $0 \leq G_\infty(X,Y) \leq c|X-Y|^{2-n}$ for all $X, Y \in \mathbb{R}^n_-$, $X \neq Y$. Define

$$\Gamma_{\mathbb{R}^n_-}(X,Y) = \frac{1}{(n-2)\omega_{n-1}}\left(|X-Y|^{2-n} - |X-Y^*|^{2-n}\right).$$

As one checks (see for instance [3]), $\Gamma_{\mathbb{R}^n_-}$ satisfies the same properties as G_∞. We set $f := G_\infty(X,\cdot) - \Gamma_{\mathbb{R}^n_-}(X,\cdot)$. As one checks, $f \in C^\infty(\overline{\mathbb{R}^n_-} \setminus \{X\})$, $-\Delta f = 0$ in the distribution sense in $\mathbb{R}^n_-$, $|f| \leq C|X-\cdot|^{2-n}$ in $\mathbb{R}^n_- \setminus \{X\}$ and $f_{\partial\mathbb{R}^n_-} = 0$. Hypoellipticity yields $f \in C^\infty(\overline{\mathbb{R}^n_-})$. Multiplying $-\Delta f$ by f and integrating by parts, we get that $f \equiv 0$, and then $G_\infty(X,\cdot) = \Gamma_{\mathbb{R}^n_-}(X,\cdot)$. As above, this proves the convergence without any extraction. The convergence in $C^2_{loc}((\overline{\mathbb{R}^n_-})^2 \setminus \mathrm{Diag}(\overline{\mathbb{R}^n_-}))$ follows from the symmetry of G and elliptic theory. □

Proof of Theorem 6: Here again, the proof is similar to the two preceding proofs. We let $(r_i)_i \in (0,+\infty)$ such that $\lim_{i\to+\infty} r_i = 0$. We let φ be a chart at 0 as in (52) (in particular $d\varphi_0 = Id_{\mathbb{R}^n}$). For any $X, Y \in \overline{\mathbb{R}^n_-} \setminus \{0\}$, we define

$$G_i(X,Y) := r_i^{n-2} G(\varphi\left(r_i X\right), \varphi\left(r_i Y\right))$$

for all $i \in \mathbb{N}$. Equation (3) rewrites

$$-\Delta_{g_i} G_i(X,\cdot) - \left(\frac{\gamma}{\left|\frac{\varphi(r_i\cdot)}{r_i}\right|^2} + r_i^2 h(\varphi(r_i\cdot)) \right) G_i(X,\cdot) = 0 \text{ in } B_R(0) \cap \mathbb{R}^n_- \setminus \{0, X\}.$$

with $G_i(X,\cdot) \equiv 0$ on $B_R(0) \cap \partial\mathbb{R}^n_-$, where $g_i = \varphi^\star \mathrm{Eucl}(r_i\cdot)$ is the pull-back of the Euclidean metric. In particular, since $d\varphi_0 = Id_{\mathbb{R}^n}$, we get that $g_i \to \mathrm{Eucl}$ in $C^2_{loc}(\mathbb{R}^n)$. The pointwise control (36) writes

$$0 \le G_i(X,Y) \le C \left(\frac{\max\{|X|,|Y|\}}{\min\{|X|,|Y|\}} \right)^{\alpha_-(\gamma)} |X-Y|^{2-n} \text{ for } X,Y \in \mathbb{R}^n_-,\ X \neq Y.$$

It then follows from elliptic theory that $G_i(X,\cdot) \to G_\infty(X,\cdot)$ in $C^2_{loc}(\overline{\mathbb{R}^n_-} \setminus \{0,X\})$. In particular, $G_\infty(X,\cdot)$ vanishes on $\partial\mathbb{R}^n_- \setminus \{0\}$ and

$$0 \le G_\infty(X,Y) \le C \left(\frac{\max\{|X|,|Y|\}}{\min\{|X|,|Y|\}} \right)^{\alpha_-(\gamma)} |X-Y|^{2-n} \text{ for } X,Y \in \mathbb{R}^n_-,\ X \neq Y. \tag{56}$$

Moreover, passing to the limit in Green's representation formula, we get that

$$\varphi(X) = \int_{\mathbb{R}^n_-} G_\infty(X,Y) \left(-\Delta\varphi - \frac{\gamma}{|Y|^2}\varphi \right) dY \text{ for all } \varphi \in C^\infty_c(\mathbb{R}^n_-).$$

Since $G(x,\cdot)$ is locally in $H^1_0(\Omega)$ (see (b) in Theorem 1), we get that $(\eta G_i(X,\cdot))_i$ is uniformly bounded in $H^2_{1,0}(\mathbb{R}^n_-)$ for all $\eta \in C^\infty_c(\mathbb{R}^n \setminus \{X\})$. Up to another extraction, we get weak convergence in $H^2_{1,0}(\mathbb{R}^n_-)$, and then $\eta G_\infty(X,\cdot) \in H^2_{1,0}(\mathbb{R}^n_-)$ for all $\eta \in C^\infty_c(\mathbb{R}^n \setminus \{X\})$. It then follows from Theorem 3 and (56) that $G_\infty(X,\cdot) = \mathcal{G}_X$ is the unique Green's function of $-\Delta - \gamma|x|^{-2}$ on $\mathbb{R}^n_-$ with Dirichlet boundary condition. Here again, the convergence in C^2 follows from elliptic theory. $\square$

6. A lower bound for the Green's function

We let Ω, γ, h be as in Theorems 1 and 2. We let G be the Green's function for $-\Delta - (\gamma|x|^{-2} + h)$ on Ω with Dirichlet boundary condition. We let $(x_i), (y_i)_{i\in\mathbb{N}}$ be such that $x_i, y_i \in \Omega$ and $x_i \neq y_i$ for all $i \in \mathbb{N}$. We also assume that there exists $x_\infty, y_\infty \in \overline{\Omega}$ such that

$$\lim_{i\to+\infty} x_i = x_\infty \text{ and } \lim_{i\to+\infty} y_i = y_\infty$$

and that there exists c_1, c_2 such that

$$\lim_{i\to+\infty} \frac{G(x_i,y_i)}{H(x_i,y_i)} = c_1 \in [0,+\infty] \text{ and } \lim_{i\to+\infty} \frac{|\nabla G_{x_i}(y_i)|}{\Gamma(x_i,y_i)} = c_2 \in [0,+\infty]$$

where $H(x,y)$ is defined in (6) and

$$\Gamma(x,y) := \left(\frac{\max\{|x|,|y|\}}{\min\{|x|,|y|\}} \right)^{\alpha_-(\gamma)} |x-y|^{1-n} \min\left\{1, \frac{d(x,\partial\Omega)}{|x-y|}\right\}$$

for $x,y \in \Omega$, $x \neq y$. Note that $c_1 < +\infty$ by (36). We claim that

$$0 < c_1 \text{ and } 0 \le c_2 < +\infty \tag{57}$$

The lower bound in (5) and the upper bound in (7) both follow from (57).

This section is devoted to proving (57). We distinguish several cases:

Case 1: $x_\infty \neq y_\infty$, $x_\infty, y_\infty \in \Omega$. As one checks, we then have that $\lim_{i\to+\infty} G(x_i,y_i) = G(x_\infty,y_\infty) > 0$. Therefore, we get that $c_1 \in (0,+\infty)$. Concerning the gradient, $\lim_{i\to+\infty} |\nabla G_{x_i}(y_i)| = |\nabla G_{x_\infty}(y_\infty)| \geq 0$ and this yields $c_2 < +\infty$. This proves (57) in Case 1.

Case 2: $x_\infty \in \Omega$ and $y_\infty \in \partial\Omega \setminus \{0\}$. Since x_∞, y_∞ are distinct and far from 0, we have that $G(x_i,y_i) = d(y_i,\partial\Omega)\left(-\partial_\nu G_{x_\infty}(y_\infty) + o(1)\right)$ as $i \to +\infty$, where $\partial_\nu G_{x_\infty}(y_\infty)$ is the normal derivative of $G_{x_\infty} > 0$ at the boundary point y_∞. Hopf's Lemma then yields $\partial_\nu G_{x_\infty}(y_\infty) < 0$. As one checks, we have that $H(x_i,y_i) = (c+o(1))d(y_i,\partial\Omega)$ as $i \to +\infty$. This then yields $0 < c_1 < +\infty$. Concerning the gradient, we get that $\lim_{i\to+\infty} |\nabla G_{x_i}(y_i)| = |\nabla G_{x_\infty}(y_\infty)| \geq 0$ and $\lim_{i\to+\infty} \Gamma(x_i,y_i) \in (0,+\infty)$, which yields $c_2 < +\infty$. This proves (57) in Case 2.

Case 3: $x_\infty \in \Omega$ and $y_\infty = 0 \in \partial\Omega$. It follows from Case 2 above that there exists $c > 0$ such that $G_{x_i}(y) \geq c d(y,\partial\Omega)|y|^{-\alpha_-(\gamma)}$ for all $y \in \partial(\Omega \cap B_{r_0}(0))$. We take the subsolution $\underline{u}_{\alpha_-(\gamma)}$ defined in Proposition 1. With (21), there exists $c' > 0$ such that $G_{x_i}(y) \geq c_1 \underline{u}_{\alpha_-(\gamma)}(y)$ for all $y \in \partial(\Omega \cap B_{r_0}(0))$. Since G_{x_i} is locally in H_0^1 around 0, the comparison principle and (21) yields $G_{x_i}(y) \geq c" d(y,\partial\Omega)|y|^{-\alpha_-(\gamma)}$ for all $y \in \Omega \cap B_{r_0}(0)$. This yields $c_1 > 0$.

We deal with the gradient. We let φ be a chart at 0 as in (52) and we define

$$G_i(y) := r_i^{\alpha_-(\gamma)-1} G_{x_i}(\varphi(r_i y)) \text{ for } y \in \mathbb{R}^n_- \cap B_2(0)$$

with $r_i \to 0$. It follows from (36) that $G_i(y) \leq C|y_1|\cdot|y|^{-\alpha_-(\gamma)}$ for all $y \in \mathbb{R}^n_- \cap B_2(0)$. It follows from (3) that $-\Delta_{g_i} G_i - \left(\gamma|\cdot|^2 + o(1)\right) G_i = 0$ in $\mathbb{R}^n_- \cap B_2(0)$ where $g_i := \varphi^\star \mathrm{Eucl}(r_i \cdot)$ and $o(1) \to 0$ in $L^\infty_{loc}(\mathbb{R}^n)$. Elliptic regularity then yields $|\nabla G_i(y)| \leq C$ for $y \in \mathbb{R}^n_- \cap B_{3/2}(0)$. We now let $r_i := |\tilde{y}_i|$ where $y_i := \varphi(\tilde{y}_i)$, so that $r_i \to 0$. We the have that $|\nabla G_i(\tilde{y}_i/r_i)| \leq C$, which rewrites $|\nabla G_{x_i}(y_i)| \leq C|y_i|^{-\alpha_-(\gamma)}$. By estimating $\Gamma(x_i,y_i)$, we then get that $c_2 < +\infty$. This proves (57) in Case 3.

Case 4: $x_\infty \neq y_\infty$, $x_\infty, y_\infty \in \partial\Omega \setminus \{0\}$. Since x_∞, y_∞ are distinct and far from 0, we have that $G(x_i,y_i) = d(y_i,\partial\Omega)d(x_i,\partial\Omega)\left(\partial_{\nu_x}\partial_{\nu_y} G_{x_\infty}(y_\infty) + o(1)\right)$ as $i \to +\infty$, where ∂_{ν_x} is the normal derivative along the first coordinate, and ∂_{ν_y} is the normal derivative along the second coordinate. Since $y \mapsto G_x(y)$ is positive for $x, y \in \Omega$, $x \neq y$, and solves (3), Hopf's maximum principle yields $-\partial_{\nu_y} G(x,y_\infty) > 0$ for $x \in \Omega$. Moreover, it follows from the symmetry of G that $-\partial_{\nu_y} G(x,y_\infty) > 0$ solves also (3). Another application of Hopf's principle yields $\partial_{\nu_x}\partial_{\nu_y} G_{x_\infty}(y_\infty) > 0$. Estimating independently $H(x_i,y_i)$, we get that $0 < c_1 < +\infty$.

We deal with the gradient. We have that $|\nabla_y G_{x_i}(y_i)| = |\nabla_y (G_{x_i} - G_{\tilde{x_i}})(y_i)|$ where $\tilde{x}_i \in \partial\Omega$ is the projection of x_i on $\partial\Omega$. The C^2-control then yields $|\nabla_y G_{x_i}(y_i)| \leq C d(x_i,\partial\Omega)$. Estimating independently $\Gamma(x_i,y_i)$, we get that $c_2 < +\infty$. This proves (57) in Case 4.

Case 5: $x_\infty \neq y_\infty$, $x_\infty \in \partial\Omega \setminus \{0\}$ and $y_\infty = 0$. It follows from Cases 2 and 4 that $G_{x_i}(y) \geq C d(x_i,\partial\Omega)d(y_i,\partial\Omega)$ for all $y \in \partial(B_{|x_\infty|/2}(0) \cap \Omega)$. Using a subsolution as in Case 3, we get that $G_{x_i}(y) \geq c d(x_i,\partial\Omega)d(y,\partial\Omega)|y|^{-\alpha_-(\gamma)}$ for all $y \in \partial(B_{|x_\infty|/2}(0) \cap \Omega)$. This yields $0 < c_1$.

For the gradient estimate, we choose a chart φ around $y_\infty = 0$ as in (52), and we let $r_i := |\tilde{y}_i| \to 0$ where $y_i = \varphi(\tilde{y}_i)$we define $G_i(y) := r_i^{\alpha_-(\gamma)-1} G_{x_i}(\varphi(r_i y))/d(x_i,\partial\Omega)$ for $y \in \mathbb{R}^n_- \cap B_2(0)$ where $r_i \to 0$. The pointwise control (36) and equation (3) yields the convergence of (G_i) in $C^1_{loc}(\overline{\mathbb{R}^n_-}\cap B_2(0)\setminus\{0\})$ as $i\to+\infty$. The boundedness of $|\nabla G_i|$ yields $c_2<+\infty$. This proves (57) in Case 5.

Since G is symmetric, it follows from Cases 1 to 5 that (57) holds when $x_\infty\neq y_\infty$.

We now deal with the case $x_\infty=y_\infty$, which rewrites $\lim_{i\to+\infty}|x_i-y_i|=0$. Via a rescaling, we are essentially back to the case $x_\infty\neq y_\infty$ via the convergence Theorems 4, 5 and 6.

Case 6: $|x_i-y_i|=o(d(x_i,\partial\Omega))$ **as** $i\to+\infty$**.** We set $r_i:=|x_i-y_i|\to 0$ as $i\to+\infty$ and we define

$$G_i(Y):=r_i^{n-2}G(x_i,x_i+r_iY)\text{ for }Y\in\frac{\Omega-x_i}{r_i}\setminus\{0\}.$$

It follows from Theorem 4 that $G_i\to c_n|\cdot|^{2-n}$ in $C^2_{loc}(\mathbb{R}^n\setminus\{0\})$ as $i\to+\infty$, with $c_n:=((n-2)\omega_{n-1})^{-1}$. We define $Y_i:=\frac{y_i-x_i}{|y_i-x_i|}$, and we then get that $|y_i-x_i|^{n-2}G(x_i,y_i)=G_i(Y_i)\to c_n$ as $i\to+\infty$. Estimating $H(x_i,y_i)$ (and noting that $d(x_i,\partial\Omega)\leq|x_i-0|=|x_i|$), we get that $0<c_1<+\infty$.

The convergence of the gradient yields $|\nabla G_i(Y_i)|\leq C$ for all i. With the original function G and points x_i, y_i, this yields $c_2<+\infty$. This proves (57) in Case 6.

Case 7: $d(x_i,\partial\Omega)=O(|x_i-y_i|)$ **and** $|x_i-y_i|=o(|x_i|)$ **as** $i\to+\infty$**.** Then $\lim_{i\to+\infty}x_i=x_\infty\in\partial\Omega$. We let φ be a chart at x_∞ as in (52) such that $d\varphi_0=Id_{\mathbb{R}^n}$. We let $x_i=\varphi(x_{i,1},x_i')$ and $y_i=\varphi(y_{i,1},y_i')$ where $(x_{i,1},x_i'),(y_{i,1},y_i')\in(-\infty,0)\times\mathbb{R}^{n-1}$ are going to 0 as $i\to+\infty$. In particular $d(x_i,\partial\Omega)=(1+o(1))|x_{i,1}|$ and $d(y_i,\partial\Omega)=(1+o(1))|y_{i,1}|$ as $i\to+\infty$. We define $r_i:=|(y_{i,1},y_i')-(x_{i,1},x_i')|$. In particular $r_i=(1+o(1))|x_i-y_i|$ as $i\to+\infty$. The hypothesis of Case 7 rewrite $x_{i,1}=O(r_i)$ and $r_i=o(|(x_{i,1},x_i')|)$. Consequently, we have that $y_{i,1}=O(r_i)$ and $r_i=o(|x_i'|)$ as $i\to+\infty$. We define

$$G_i(X,Y):=r_i^{n-2}G(\varphi\left((0,x_i')+r_iX\right),\varphi\left((0,x_i')+r_iY\right))$$

for $X,Y\in\mathbb{R}^n_-$ such that $X\neq Y$. It follows from Theorem 5 that

$$\lim_{i\to+\infty}G_i(X,Y)=c_n\left(|X-Y|^{2-n}-|X-Y^*|^{2-n}\right):=\Psi(X,Y)$$

for all $X,Y\in\overline{\mathbb{R}^n_-}$, $X\neq Y$, and this convergence holds in C^2_{loc}. We define $X_i:=(r_i^{-1}x_{i,1},0)$ and $Y_i:=(r_i^{-1}y_{i,1},r_i^{-1}(y_i'-x_i'))$: the definition of r_i yields $X_i\to X_\infty\in\overline{\mathbb{R}^n_-}$ and $Y_i\to Y_\infty\in\overline{\mathbb{R}^n_-}$ as $i\to+\infty$. Therefore, we get that

$$|x_i-y_i|^{n-2}G(x_i,y_i)=(1+o(1))G_i(X_i,Y_i)\to\Psi(X_\infty,Y_\infty)$$

as $i\to+\infty$, and

$$|X_{\infty,1}|=\lim_{i\to+\infty}\frac{|x_{i,1}|}{r_i}=\lim_{i\to+\infty}\frac{d(x_i,\partial\Omega)}{r_i}.\tag{58}$$

Case 7.1: $X_{\infty,1}\neq 0$ *and* $Y_{\infty,1}\neq 0$. We then get that $\lim_{i\to+\infty}|x_i-y_i|^{n-2}G(x_i,y_i)=\Psi(X_\infty,Y_\infty)>0$. Moreover, it follows from (58) that $d(x_i,\partial\Omega)d(y_i,\partial\Omega)=(c+o(1))|x_i-y_i|^2$ as $i\to+\infty$ for some $c>0$. Since $|x_i|=(1+o(1))|y_i|$ as $i\to+\infty$ (this follows from the assumption of Case 7), we get that $\lim_{i\to+\infty}|x_i-y_i|^{n-2}H(x_i,y_i)\in(0,+\infty)$. Then $0<c_1<+\infty$.

Case 7.2: $X_{\infty,1} \neq 0$ and $Y_{\infty,1} = 0$. Then $Y_{i,1} \to 0$ as $i \to +\infty$, and then, there exists $(\tau_i)_i \in (0,1)$ such that $G_i(X_i, Y_i) = Y_{i,1}\partial_{Y_1} G_i(X_i, (\tau_i Y_{i,1}, Y_i'))$. Letting $i \to +\infty$ and using the convergence of G_i in C^1, we get that

$$\begin{aligned} |x_i - y_i|^{n-2} G(x_i, y_i) &= (1+o(1))G_i(X_i, Y_i) = Y_{i,1}\partial_{Y_1} G_i(X_i, \tau_i Y_i) \\ &= \frac{d(y_i, \partial\Omega)}{|x_i - y_i|}\left(-\partial_{Y_1}\Psi(X_\infty, Y_\infty) + o(1)\right) \end{aligned}$$

as $i \to +\infty$. As one checks, $\partial_{Y_1}\Psi(X_\infty, Y_\infty) < 0$. Arguing as in Case 7.1, we get that $0 < c_1 < +\infty$.

Case 7.3: $X_{\infty,1} = Y_{\infty,1} = 0$. As in Case 7.2, there exists $(\tau_i)_i, (\sigma_i)_i \in (0,1)$ such that $G_i(X_i, Y_i) = Y_{i,1} X_{i,1} \partial_{Y_1}\partial_{X_1} G_i((\sigma_i X_{i,1}, X_i')X_i, (\tau_i Y_{i,1}, Y_i'))$. We conclude as above, noting that $\partial_{Y_1}\partial_{X_1}\Psi(X_\infty, Y_\infty) > 0$. Then $0 < c_1 < +\infty$.

The gradient estimate is proved as in Cases 1 to 6. This proves (57) in Case 7.

Case 8: $d(x_i, \partial\Omega) = O(|x_i - y_i|)$, $|x_i| = O(|x_i - y_i|)$ and $|y_i| = O(|x_i - y_i|)$ as $i \to +\infty$. In particular, $x_\infty = y_\infty = 0$. We take a chart at 0 as in Case 7, and we define $(x_{i,1}, x_i'), (y_{i,1}, y_i')$ similarly. We define $r_i := |(y_{i,1}, y_i') - (x_{i,1}, x_i')| = (1+o(1))|x_i - y_i|$ as $i \to +\infty$. We define

$$G_i(X, Y) := r_i^{n-2} G(\varphi(r_i X), \varphi(r_i Y))$$

for $X, Y \in \mathbb{R}^n_-$. It follows from Theorem 6 that $G_i \to \mathcal{G}$ in $C^2_{loc}((\overline{\mathbb{R}^n_-} \setminus \{0\})^2 \setminus \mathrm{Diag}(\overline{\mathbb{R}^n_-} \setminus \{0\}))$, where $\mathcal{G}$ is the Green's function for $-\Delta - \gamma|\cdot|^{-2}$ in $\mathbb{R}^n_-$. Then

$$|x_i - y_i|^{n-2} G(x_i, y_i) = (1+o(1))G_i(X_i, Y_i) = \mathcal{G}(X_\infty, Y_\infty) + o(1)$$

as $i \to +\infty$.

Case 8.1: We assume that $X_{\infty,1} \neq 0$ and $Y_{\infty,1} \neq 0$. Then we get $0 < c_1 < +\infty$ as in Case 7.1.

Case 8.2: We assume that $X_\infty \in \mathbb{R}^n_-$ and $Y_\infty \in \partial\mathbb{R}^n_- \setminus \{0\}$ or $X_\infty, Y_\infty \in \partial\mathbb{R}^n_- \setminus \{0\}$. Then we argue as in Cases 7.2 and 7.3 to get $0 < c_1 < +\infty$ provided $\{\partial_{Y_1}\mathcal{G}(X_\infty, Y_\infty) < 0 \text{ if } X_\infty \in \mathbb{R}^n_- \text{ and } Y_\infty \in \partial\mathbb{R}^n_-\}$ and $\{\partial_{Y_1}\partial_{X_1}\mathcal{G}(X_\infty, Y_\infty) > 0 \text{ if } X_\infty, Y_\infty \in \partial\mathbb{R}^n_-\}$. So we are just left with proving these two inequalities.

We assume that $X_\infty \in \mathbb{R}^n_-$. It follows from Theorem 3 that $\mathcal{G}(X_\infty, \cdot) > 0$ is a solution to $(-\Delta - \gamma|\cdot|^{-2})\mathcal{G}(X_\infty, \cdot) = 0$ in $\mathbb{R}^n_- - \{X_\infty\}$, vanishing on $\partial\mathbb{R}^n_- \setminus \{0\}$. Hopf's maximum principle then yields $-\partial_{Y_1}\mathcal{G}(X_\infty, Y_\infty) > 0$ for $Y_\infty \in \partial\mathbb{R}^n_- \setminus \{0\}$.

We fix $Y_\infty \in \partial\mathbb{R}^n_- \setminus \{0\}$. For $X \in \mathbb{R}^n_-$, we then define $H(X) := -\partial_{Y_1}\mathcal{G}(X, Y_\infty) > 0$ by the above argument. Moreover, $(-\Delta - \gamma|\cdot|^{-2})H = 0$ in $\mathbb{R}^n_-$, vanishing on $\partial\mathbb{R}^n_- \setminus \{0, Y_\infty\}$. Hopf's maximum principle then yields $-\partial_{X_1}H(X_\infty) = \partial_{Y_1}\partial_{X_1}\mathcal{G}(X_\infty, Y_\infty) > 0$ for $X_\infty, Y_\infty \in \partial\mathbb{R}^n_- \setminus \{0\}$

Case 8.3: we assume that $X_\infty = 0$ or $Y_\infty = 0$. Since $|X_\infty - Y_\infty| = 1$, without loss of generality, we can assume that $X_\infty \neq 0$. It follows from Cases 8.1 and 8.2 that there exists $C > 0$ such that

$$C^{-1}\frac{d(x_i, \partial\Omega)}{|x_i|^{n-\alpha_-(\gamma)}}\frac{d(y, \partial\Omega)}{|y|^{\alpha_-(\gamma)}} \leq G_{x_i}(y) \leq C\frac{d(x_i, \partial\Omega)}{|x_i|^{n-\alpha_-(\gamma)}}\frac{d(y, \partial\Omega)}{|y|^{\alpha_-(\gamma)}} \tag{59}$$

for all $y \in \partial(B_{|x_i|/2}(0) \cap \Omega)$. We let $\underline{u}_{\alpha_-(\gamma)}$ be the sub-solution given by Proposition 1. Arguing as in Case 3, it then follows from the comparison principle that (59) holds for $y \in B_{|x_i|/2}(0) \cap \Omega$. Since $|y_i| = o(|x_i|)$, we then get that (59) holds with $y := y_i$. Estimating $H(x_i, y_i)$, we then get that $0 < c_1 < +\infty$.

The gradient estimate is proved as in Cases 1 to 6. This proves (57) in Case 8.

Since G is symmetric, it follows from Cases 7 and 8 that (57) holds when $x_\infty = y_\infty$.

Conclusion: We then get that (57) holds, which proves the initial claim. As noted previously, the lower bound in (5) and the upper bound in (7) both follow from these results. This ends the proof of Theorem 1.

7. Appendix: A technical eigenvalue Lemma

Lemma 1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a smooth bounded domain. Let $(V_k)_k : \Omega \to \mathbb{R}$ and $V_\infty : \Omega \to \mathbb{R}$ be measurable functions and let $(x_k)_k \in \Omega$ be a sequence of points. We assume that*

i) $\lim_{k\to+\infty} V_k(x) = V_\infty(x)$ *for a.e.* $x \in \Omega$,
ii) There exists $C > 0$ such that $|V_k(x)| \leq C|x - x_k|^{-2}$ for all $k \in \mathbb{N}$ and $x \in \Omega$.
iii) $\lim_{k\to+\infty} x_k = 0 \in \partial\Omega$.
iv) For some $\gamma_0 < n^2/4$, there exists $\delta > 0$ such that $|V_k(x)| \leq \gamma_0|x - x_k|^{-2}$ for all $k \in \mathbb{N}$ and $x \in B_\delta(0) \subset \Omega$.
v) The first eigenvalue $\lambda_1(-\Delta + V_k)$ is achieved for all $k \in \mathbb{N}$.

Then

$$\lim_{k\to+\infty} \lambda_1(-\Delta + V_k) = \lambda_1(-\Delta + V_\infty).$$

Proof of Lemma 1: We first claim that $(\lambda_1(-\Delta + V_k))_k$ is bounded. Indeed, fix $\varphi \in H_0^1(\Omega) \setminus \{0\}$ and use the Hardy inequality to write for all $k \in \mathbb{N}$,

$$\lambda_1(-\Delta+V_k) \leq \frac{\int_\Omega (|\nabla\varphi|^2 + V_k\varphi^2)\, dx}{\int_\Omega \varphi^2\, dx} \leq \frac{\int_\Omega (|\nabla\varphi|^2 + C|x - x_k|^{-2}\varphi^2)\, dx}{\int_\Omega \varphi^2\, dx} := M < +\infty$$

For the lower bound, we have for any $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned}
\int_\Omega (|\nabla\varphi|^2 + V_k\varphi^2)\, dx &= \int_\Omega |\nabla\varphi|^2\, dx + \int_{B_\delta(0)} V_k\varphi^2\, dx + \int_{\Omega\setminus B_\delta(0)} V_k\varphi^2\, dx \\
&\geq \int_\Omega |\nabla\varphi|^2\, dx - \gamma_0 \int_{B_\delta(0)} |x - x_k|^{-2}\varphi^2\, dx \\
&\quad -4C\delta^{-2} \int_{\Omega\setminus B_\delta(0)} \varphi^2\, dx \\
&\geq \left(1 - 4\gamma_0/n^2\right) \int_\Omega |\nabla\varphi|^2\, dx - 4C\delta^{-2} \int_\Omega \varphi^2\, dx. \qquad (60)
\end{aligned}$$

Since $\gamma_0 < n^2/4$, we then get that $\lambda_1(-\Delta+V_k) \geq -4C\delta^{-2}$ for large k, which proves the lower bound.

Up to a subsequence, we can now assume that $(\lambda_1(-\Delta + V_k))_k$ converges as $k \to +\infty$. We now show that

$$\liminf_{k\to+\infty} \lambda_1(-\Delta + V_k) \geq \lambda_1(-\Delta + V_\infty). \qquad (61)$$

For $k \in \mathbb{N}$, we let $\varphi_k \in H_0^1(\Omega)$ be a minimizer of $\lambda_1(-\Delta+V_k)$ such that $\int_\Omega \varphi_k^2\, dx = 1$. In particular,

$$-\Delta\varphi_k + V_k\varphi_k = \lambda_1(-\Delta + V_k)\varphi_k \text{ weakly in } H_0^1(\Omega). \qquad (62)$$

Inequality (60) above yields the boundedness of $(\varphi_k)_k$ in $H_0^1(\Omega)$. Up to a subsequence, we let $\varphi \in H_0^1(\Omega)$ such that, as $k \to +\infty$, $\varphi_k \rightharpoonup \varphi$ weakly in $H_0^1(\Omega)$,

$\varphi_k \to \varphi$ strongly in $L^2(\Omega)$ (then $\int_\Omega \varphi^2\,dx = 1$) and $\varphi_k(x) \to \varphi(x)$ for a.e. $x \in \Omega$. Letting $k \to +\infty$ in (62), the hypothesis on (V_k) allow us to conclude that

$$-\Delta\varphi + V_\infty\varphi = \lim_{k\to+\infty} \lambda_1(-\Delta + V_k)\varphi \text{ weakly in } H_0^1(\Omega).$$

Since $\int_\Omega \varphi^2\,dx = 1$ and we have extracted subsequences, we then get (61).

Finally, we prove the reverse inequality. For $\epsilon > 0$, let $\varphi \in H_0^1(\Omega)$ be such that

$$\frac{\int_\Omega (|\nabla\varphi|^2 + V_\infty\varphi^2)\,dx}{\int_\Omega \varphi^2\,dx} \leq \lambda_1(-\Delta + V_\infty) + \epsilon.$$

We have

$$\lambda_1(-\Delta + V_k) \leq \lambda_1(-\Delta + V_\infty) + \epsilon + \frac{\int_\Omega |V_k - V_\infty|\varphi^2\,dx}{\int_\Omega \varphi^2\,dx}.$$

The hypothesis of Lemma 1 allow us to conclude that $\int_\Omega |V_k - V_\infty|\varphi^2\,dx \to 0$ as $k \to +\infty$. Therefore $\limsup_{k\to+\infty} \lambda_1(-\Delta + V_k) \leq \lambda_1(-\Delta + V_\infty) + \epsilon$ for all $\epsilon > 0$. Letting $\epsilon \to 0$, we get the reverse inequality and the conclusion of Lemma 1. □

References

[1] Nassif Ghoussoub and Frédéric Robert, *Hardy-singular boundary mass and Sobolev-critical variational problems*, Anal. PDE **10** (2017), no. 5, 1017–1079.

[2] ———, *The Hardy-Schrödinger operator with interior singularity: the remaining cases*, Calc. Var. Partial Differential Equations **56** (2017), no. 5, Art. 149, 54.

[3] Frédéric Robert, *Existence et asymptotiques optimales des fonctions de Green des opérateurs elliptiques d'ordre deux (Existence and optimal asymptotics of the Green's functions of second-order elliptic operators)* (2010). Unpublished notes.

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