BLOWING-UP SOLUTIONS FOR SECOND-ORDER CRITICAL ELLIPTIC EQUATIONS: THE IMPACT OF THE SCALAR CURVATURE

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Abstract. Given a closed manifold (M^n, g), $n \geq 3$, Olivier Druet [5,7] proved that a necessary condition for the existence of energy-bounded blowing-up solutions to perturbations of the equation

$$\Delta_g u + h_0 u = u^{\frac{n+2}{n-2}},\ u > 0 \text{ in } M$$

is that $h_0 \in C^1(M)$ touches the Yamabe potential somewhere when $n \geq 4$ (the condition is different for $n = 6$). In this paper, we prove that Druet's condition is also sufficient provided we add its natural differentiable version. For $n \geq 6$, our arguments are local. For the low dimensions $n \in \{4, 5\}$, our proof requires to introduce a suitable mass that is defined only where Druet's condition holds. This mass carries global information both on h_0 and (M, g).

1. Introduction and main results

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ without boundary and $h_0 \in C^p(M)$, $1 \leq p \leq \infty$. We consider the equation

$$\Delta_g u + h_0 u = u^{2^\star - 1},\ u > 0 \text{ in } M, \tag{1}$$

where $\Delta_g := -\text{div}_g(\nabla)$ is the Laplace–Beltrami operator and $2^\star := \frac{2n}{n-2}$. We investigate the existence of families $(h_\epsilon)_{\epsilon>0} \in C^p(M)$ and $(u_\epsilon)_{\epsilon>0} \in C^2(M)$ satisfying

$$\Delta_g u_\epsilon + h_\epsilon u_\epsilon = u_\epsilon^{2^\star - 1},\ u_\epsilon > 0 \text{ in } M \text{ for all } \epsilon > 0, \tag{2}$$

and such that $h_\epsilon \to h_0$ in $C^p(M)$ and $\max_M u_\epsilon \to \infty$ as $\epsilon \to 0$. We say that $(u_\epsilon)_{\epsilon>0}$ *blows up* at some point $\xi_0 \in M$ as $\epsilon \to 0$ if for all $r > 0$, $\lim_{\epsilon\to 0} \max_{B_r(\xi_0)} u_\epsilon = +\infty$. Druet [5,7] obtained the following necessary condition for blow-up:

Theorem 1.1 (Druet [5, 7])**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. Let $h_0 \in C^1(M)$ be such that $\Delta_g + h_0$ is coercive. Assume that there exist families $(h_\epsilon)_{\epsilon>0} \in C^1(M)$ and $(u_\epsilon)_{\epsilon>0} \in C^2(M)$ satisfying (2) and such that $h_\epsilon \to h_0$ strongly in $C^1(M)$ and $u_\epsilon \rightharpoonup u_0$ weakly in $L^{2^\star}(M)$. Assume that $(u_\epsilon)_\epsilon$ blows-up. Then there exists $\xi_0 \in M$ such that $(u_\epsilon)_\epsilon$ blows up at ξ_0 and*

$$(h_0 - c_n \,\text{Scal}_g)(\xi_0) = 0 \text{ if } n \neq 6 \text{ and } (h_0 - c_n \,\text{Scal}_g - 2u_0)(\xi_0) = 0 \text{ if } n = 6. \tag{3}$$

Furthermore, if $n \in \{4, 5\}$, then $u_0 \equiv 0$.

Here $c_n := \frac{n-2}{4(n-1)}$ and Scal_g is the Scalar curvature of (M, g). This result does not hold in dimension $n = 3$. Indeed, Hebey–Wei [12] constructed examples of blowing-up solutions to (2) on the standard sphere $(\mathbb{S}^3, g_0)$, which are bounded in $L^{2^\star}(\mathbb{S}^3)$ but do not satisfy (3).

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This paper is concerned with the converse of Theorem 1.1 in dimensions $n \geq 4$. For the sake of clarity, we state separately our results in the cases $u_0 \equiv 0$ in dimension $n \geq 4$ (Theorem 1.2) and $u_0 > 0$ in dimension $n \geq 6$ (Theorem 1.3):

Theorem 1.2 ($u_0 \equiv 0$)**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. Let $h_0 \in C^p(M)$, $1 \leq p \leq \infty$, be such that $\Delta_g + h_0$ is coercive. Assume that there exists a point $\xi_0 \in M$ such that*

$$(h_0 - c_n \operatorname{Scal}_g)(\xi_0) = |\nabla (h_0 - c_n \operatorname{Scal}_g)(\xi_0)| = 0. \tag{4}$$

Then there exist families $(h_\epsilon)_{\epsilon>0} \in C^p(M)$ and $(u_\epsilon)_{\epsilon>0} \in C^2(M)$ satisfying (2) and such that $h_\epsilon \to h_0$ strongly in $C^p(M)$, $u_\epsilon \rightharpoonup 0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_{\epsilon>0}$ blows up at ξ_0.

Theorem 1.3 ($u_0 > 0$)**.** *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 6$. Let $h_0 \in C^p(M)$, $1 \leq p \leq \infty$, be such that $\Delta_g + h_0$ is coercive. Assume that there exist a solution $u_0 \in C^2(M)$ of (1) and a point $\xi_0 \in M$ such that*

$$\begin{cases} (h_0 - c_n \operatorname{Scal}_g)(\xi_0) = |\nabla (h_0 - c_n \operatorname{Scal}_g)(\xi_0)| = 0 & \text{if } n \neq 6; \\ (h_0 - 2u_0 - c_6 \operatorname{Scal}_g)(\xi_0) = |\nabla (h_0 - 2u_0 - c_6 \operatorname{Scal}_g)(\xi_0)| = 0 & \text{if } n = 6; \end{cases} \tag{5}$$

Then there exist families $(h_\epsilon)_{\epsilon>0} \in C^p(M)$ and $(h_\epsilon)_{\epsilon>0} \in C^2(M)$ satisfying (2) and such that $h_\epsilon \to h_0$ strongly in $C^p(M)$, $u_\epsilon \rightharpoonup u_0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_{\epsilon>0}$ blows up at ξ_0.

Compared with Theorem 1.1, we have assumed here that condition (3) is also satisfied at order 1. However, this stronger condition is actually expected to be necessary for the existence of blowing-up solutions (see Theorem 12.1 in the last section of this paper and the discussion in Druet [7, Section 2.5]). Note that we do not make any nondegeneracy assumptions, neither on the solution u_0, nor on the critical point ξ_0.

We refer to Section 11 for examples of functions h_0 and u_0 satisfying the assumptions of Theorem 1.3. Recently, Premoselli–Thizy [20] obtained a beautiful example of blowing-up solutions showing that in dimension $n \in \{4, 5\}$, condition (4) may not be satisfied at all blow-up points.

When $h_0 \equiv c_n \operatorname{Scal}_g$, that is when (1) is the Yamabe equation, several examples of blowing-up solutions have been obtained. In the perturbative case, that is when $h_\epsilon \not\equiv c_n \operatorname{Scal}_g$, examples of blowing-up solutions have been obtained by Druet–Hebey [8], Esposito–Pistoia–Vétois [10], Morabito–Pistoia–Vaira [19], Pistoia–Vaira [21] and Robert–Vétois [24]. In the nonpertubative case $h_\epsilon \equiv c_n \operatorname{Scal}_g$, we refer to Brendle [1] and Brendle–Marques [2] regarding the non-compactness of Yamabe metrics. When solutions blow-up not only pointwise but also in energy, the function $h_0 - c_n \operatorname{Scal}_g$ may not vanish (see Chen–Wei–Yan [3] for $n \geq 5$ and Vétois–Wang [29] for $n = 4$).

When there does not exist any blowing-up solutions to the equations (2), then equation (1) is *stable*. We refer to the survey of Druet [7] and the book of Hebey [11] for exhaustive studies of the various concepts of stability. Stability also arises in the Lin–Ni–Takagi problem (see for instance del Pino–Musso–Roman–Wei [4] for a recent reference on this topic). In Geometry, stability is linked to the problem of compactness of the Yamabe equation (see Schoen [27, 28], Li–Zhu [17], Druet [6], Marques [18], Li–Zhang [15, 16], Khuri–Marques–Schoen [13]).

Let us give some general considerations about the proofs. Theorem 1.1 yields *local* information on blow-up points. It is essentially the consequence of the concentration of the L^2–norm of the solutions at one of the blow-up points when $n \geq 4$. However, in our construction, the problem may be *both local and global.* Indeed, we reduce the problem to finding critical points of a functional defined on a finite-dimensional space. The first term in the asymptotic expansion of the reduced functional is local. This is due to the L^2–concentration of the standard bubble in the definition of our ansatz. The second term in the expansion plays a decisive role for obtaining critical points. For the high dimensions $n \geq 6$, this term is also local (see e.g. (30)). However, for $n \in \{4, 5\}$, the second term is global and we are then compelled to introduce a suitable notion of mass, which carries global information on h_0 and (M, g), and to add a corrective term to the standard bubble (see (49)) in order to obtain a reasonable expansion (see e.g. (60)). Unlike the case where $n = 3$ or $h_0 \equiv c_n \operatorname{Scal}_g$, the mass is not defined at all points in the manifold, but only at the points where the condition (5) is satisfied.

More precisely, Theorems 1.2 and 1.3 are consequences of Theorems 1.4 and 1.5 below. The latter are the core results of our paper. In these theorems, we fix a linear perturbation $h_\epsilon = h_0 + \epsilon f$ for some function $f \in C^p(M)$. Furthermore, we specify the behavior of the blowing-up solutions that we obtain. Let $H_1^2(M)$ be the completion of $C^\infty(M)$ for the norm $\|\cdot\|_{H_1^2} := \|\cdot\|_2 + \|\nabla \cdot\|_2$. We say that $(u_\epsilon)_\epsilon$ blows up with one bubble at some point $\xi_0 \in M$ if $u_\epsilon = u_0 + U_{\delta_\epsilon,\xi_\epsilon} + \mathrm{o}(1)$ as $\epsilon \to 0$ in $H_1^2(M)$, where $u_0 \in H_1^2(M)$ is such that $u_\epsilon \rightharpoonup u_0$ weakly in $H_1^2(M)$, $U_{\delta_\epsilon,\xi_\epsilon}$ is as in (17), $(\delta_\epsilon, \xi_\epsilon) \to (0, \xi_0)$ and $o(1) \to 0$ strongly in $H_1^2(M)$ as $\epsilon \to 0$. For convenience, for every $h_0, u_0 \in C^0(M)$, we define

$$\varphi_{h_0} := h_0 - c_n \operatorname{Scal}_g \text{ and } \varphi_{h_0,u_0} := \begin{cases} h_0 - c_n \operatorname{Scal}_g & \text{if } n \neq 6 \\ h_0 - 2u_0 - c_n \operatorname{Scal}_g & \text{if } n = 6. \end{cases} \tag{6}$$

Our first result deals with the case where $u_0 \equiv 0$ in dimension $n \geq 4$:

Theorem 1.4. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. Let $h_0 \in C^p(M)$, $p \geq 2$, be such that $\Delta_g + h_0$ is coercive. Assume that there exists a point $\xi_0 \in M$ satisfying (4). Assume in addition that ξ_0 is a nondegenerate critical point of $h_0 - c_n \operatorname{Scal}_g$ and*

$$K_{h_0}(\xi_0) := \left\{ \begin{array}{ll} m_{h_0}(\xi_0) & \text{if } n = 4, 5 \\ \Delta_g \left(h_0 - c_n \operatorname{Scal}_g\right)(\xi_0) + \dfrac{c_n}{6} |\operatorname{Weyl}_g(\xi_0)|_g^2 & \text{if } n \geq 6 \end{array} \right\} \neq 0, \tag{7}$$

where $m_{h_0}(\xi_0)$ is the mass of $\Delta_g + h_0$ at the point ξ_0 (see Proposition-Definition 6.1), and Weyl_g is the Weyl curvature tensor of the manifold. We fix a function $f \in C^p(M)$ such that $f(\xi_0) \times K_{h_0}(\xi_0) > 0$. Then for small $\epsilon > 0$, there exists $u_\epsilon \in C^2(M)$ satisfying

$$\Delta_g u_\epsilon + (h_0 + \epsilon f) u_\epsilon = u_\epsilon^{2^\star - 1} \text{ in } M,\ u_\epsilon > 0, \tag{8}$$

and such that $u_\epsilon \rightharpoonup 0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_\epsilon$ blows up with one bubble at ξ_0.

The definition of $K_{h_0}(\xi_0)$ outlines the major difference between high- and low-dimensions that was mentioned above: for $n \geq 6$, it is a local quantity, but for $n \in \{4, 5\}$, it carries global information (see Section 6 for more discussions).

Next we deal with the case where $u_0>0$ in dimension $n\geq 6$. We assume that u_0 is *nondegenerate*, that is, for every $\phi\in H_1^2(M)$,

$$\Delta_g\phi+h_0\phi=(2^\star-1)u_0^{2^\star-2}\phi\iff\phi\equiv 0. \tag{9}$$

Theorem 1.5. *Let (M,g) be a compact Riemannian manifold of dimension $n\geq 6$. Let $h_0\in C^p(M)$, $p\geq 2$, be such that Δ_g+h_0 is coercive. Assume that there exist a nondegenerate solution $u_0\in C^2(M)$ to equation (1) and $\xi_0\in M$ satisfying (5). Assume in addition that ξ_0 is a nondegenerate critical point of φ_{h_0,u_0} and*

(10)
$$K_{h_0,u_0}(\xi_0):=\left\{\begin{array}{ll}\Delta_g\varphi_{h_0,u_0}(\xi_0)+\frac{c_6}{6}|\operatorname{Weyl}_g(\xi_0)|_g^2 & \text{if } n=6\\ u_0(\xi_0) & \text{if } 7\leq n\leq 9\\ 672u_0(\xi_0)+\Delta_g\varphi_{h_0,u_0}(\xi_0)+\frac{c_{10}}{6}|\operatorname{Weyl}_g(\xi_0)|_g^2 & \text{if } n=10\\ \Delta_g\varphi_{h_0,u_0}(\xi_0)+\frac{c_n}{6}|\operatorname{Weyl}_g(\xi_0)|_g^2 & \text{if } n\geq 11\end{array}\right\}\neq 0.$$

We fix a function $f\in C^p(M)$ such that

$$K_{h_0,u_0}(\xi_0)\times\left\{\begin{array}{ll}\left[f+2(\Delta_g+h_0-2u_0)^{-1}(fu_0)\right](\xi_0) & \text{if } n=6\\ f(\xi_0) & \text{if } n>6\end{array}\right\}>0. \tag{11}$$

Then for small $\epsilon>0$, there exists $u_\epsilon\in C^2(M)$ satisfying (8) and such that $u_\epsilon\rightharpoonup u_0$ weakly in $L^{2^\star}(M)$ and $(u_\epsilon)_\epsilon$ blows up with one bubble at ξ_0.

The paper is organized as follows. In Section 2, we introduce our notations and discuss the general setting of the problem, including general C^1-estimates. In Sections 3, 4 and 5, we then compute a C^1-asymptotic expansion of the energy functional in the case where $n\geq 6$. In Section 6, we discuss the specific setting of dimensions $n\in\{4,5\}$ and we define the mass of Δ_g+h_0 in this case. In Section 7, we then deal with the C^1-asymptotic expansion of the energy functional when $n\in\{4,5\}$. In Sections 8 and 9, we complete the proofs of Theorems 1.4 and 1.5, respectively. In Section 10, we then prove Theorems 1.2 and 1.3. In Section 11, we discuss the question of existence of functions h_0 and u_0 satisfying the assumptions of Theorem 1.3. Finally, in Section 12, we deal with the necessity of condition (4) on the gradient. The interested reader can find some computational details in [26].

2. Notations and general setting

We follow the notations and definitions of Robert–Vétois [23].

2.1. Euclidean setting.

We define

$$U_{1,0}(x):=\left(\frac{\sqrt{n(n-2)}}{1+|x|^2}\right)^{\frac{n-2}{2}}\quad\text{for all } x\in\mathbb{R}^n,$$

so that $U_{1,0}$ is a positive solution to the equation $\Delta_{\text{Eucl}}U=U^{2^\star-1}$ in $\mathbb{R}^n$, where Eucl stands for the Euclidean metric. For every $\delta>0$ and $\xi\in\mathbb{R}^n$, we define

$$U_{\delta,\xi}(x):=\delta^{-\frac{n-2}{2}}U\left(\delta^{-1}(x-\xi)\right)=\left(\frac{\sqrt{n(n-2)}\delta}{\delta^2+|x-\xi|^2}\right)^{\frac{n-2}{2}}\quad\text{for all } x\in\mathbb{R}^n. \tag{12}$$

As one checks,

$$\partial_\delta U_{\delta,\xi}(x) = \sqrt{n(n-2)}^{\frac{n-2}{2}} \frac{n-2}{2} \delta^{\frac{n-2}{2}-1} \frac{|x-\xi|^2 - \delta^2}{(\delta^2 + |x-\xi|^2)^{n/2}} \tag{13}$$

$$\text{and } \partial_{\xi_i} U_{\delta,\xi}(x) = \sqrt{n(n-2)}^{\frac{n-2}{2}} (n-2)\delta^{\frac{n-2}{2}} \frac{(x-\xi)_i}{(\delta^2 + |x-\xi|^2)^{n/2}} \tag{14}$$

for all $i = 1, \ldots, n$. We denote $p = (p_0, p_1, \ldots, p_n) := (\delta, \xi) \in (0,\infty) \times \mathbb{R}^n$.

2.2. **Riemannian setting.** We fix $N > n-2$ to be chosen large later. It follows from Lee–Parker [14] that there exists a function $\Lambda \in C^\infty(M \times M)$ such that, defining $\Lambda_\xi := \Lambda(\xi, \cdot)$, we have

$$\Lambda_\xi > 0,\ \Lambda_\xi(\xi) = 1 \text{ and } \nabla\Lambda_\xi(\xi) = 0 \text{ for all } \xi \in M \tag{15}$$

and, defining the metric $g_\xi := \Lambda_\xi^{2^\star-2} g$ conformal to g, we have

$$\begin{gathered} \mathrm{Scal}_{g_\xi}(\xi) = 0,\ \nabla \mathrm{Scal}_{g_\xi}(\xi) = 0,\ \Delta_g \mathrm{Scal}_{g_\xi}(\xi) = \frac{1}{6}|\mathrm{Weyl}_g(\xi)|_g^2 \\ \text{and } dv_{g_\xi}(x) = (1 + \mathrm{O}(|x|^N))\, dx \text{ via the chart } \exp_\xi^{g_\xi} \text{ around } 0, \end{gathered} \tag{16}$$

where dx is the Euclidean volume element, dv_{g_ξ} is the Riemannian volume element of (M, g_ξ) and $\exp_\xi^{g_\xi}$ is the exponential chart at ξ for the metric g_ξ. There exists $r_0 > 0$ such that the injectivity radius of the metric g_ξ satisfies $i_{g_\xi}(M) \geq 3r_0$ for all $\xi \in M$. We let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(t) = 1$ for $t \leq r_0$, $\chi(t) = 0$ for $t \geq 2r_0$ and $0 \leq \chi \leq 1$. For every $\delta > 0$ and $\xi \in M$, we define the bubble

$$U_{\delta,\xi}(x) := \chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x) \left(\frac{\delta\sqrt{n(n-2)}}{\delta^2 + d_{g_\xi}(x,\xi)^2} \right)^{\frac{n-2}{2}}. \tag{17}$$

2.3. **General reduction theorem.** For every $1 \leq q \leq \infty$, we let $\|\cdot\|_q$ be the usual norm of $L^q(M)$. For every $h \in C^0(M)$, we define

$$J_h(u) := \frac{1}{2}\int_M \left(|\nabla u|_g^2 + hu^2\right) dv_g - \frac{1}{2^\star}\int_M u_+^{2^\star}\, dv_g \text{ for all } u \in H_1^2(M),$$

where $u_+ := \max(u, 0)$. We let $(\delta, \xi) \mapsto B_{h,\delta,\xi} = B_h(\delta,\xi)$ be a function in $C^1((0,\infty) \times M, H_1^2(M))$ such that for every $\delta > 0$, there exists $\epsilon(\delta) > 0$ independent of h and ξ such that

$$\|B_{h,\delta,\xi}\|_{H_1^2} + \delta\|\partial_p B_{h,\delta,\xi}\|_{H_1^2} < \epsilon(\delta) \text{ for all } p = (\delta,\xi) \in (0,\infty) \times M \tag{18}$$

and $\epsilon(\delta) \to 0$ as $\delta \to 0$. The function $B_{h,\delta,\xi}$ will be fixed later. We also let $\tilde{u}_0 \in C^2(M)$. We define

$$W_{h,\tilde{u}_0,\delta,\xi} := \tilde{u}_0 + U_{\delta,\xi} + B_{h,\delta,\xi}.$$

We fix a point $\xi_0 \in M$ and a function $h_0 \in C^0(M)$ such that $\Delta_g + h_0$ is coercive. We let $u_0 \in C^2(M)$ be a solution of the equation

$$\Delta_g u_0 + h_0 u_0 = u_0^{2^\star-1},\ u_0 \geq 0 \text{ in } M.$$

It follows from the strong maximum principle that either $u_0 \equiv 0$ or $u_0 > 0$. We assume that u_0 is nondegenerate (see (9)). It then follows from [23] that there exist $\epsilon_0 > 0$, $U_0 \subset M$ a small open neighborhood of ξ_0 and $\Phi_{h,\tilde{u}_0} \in C^1((0,\epsilon_0) \times U_0, H_1^2(M))$ such that, when $\|h - h_0\|_\infty < \epsilon_0$ and $\|\tilde{u}_0 - u_0\|_{C^2} < \epsilon_0$, we have

$$\|\Phi_{h,\tilde{u}_0,\delta,\xi}\|_{H_1^2} \leq C\left\|R_{\delta,\xi}\right\|_{\frac{2n}{n+2}} \tag{19}$$

for all $(\delta,\xi)\in(0,\epsilon_0)\times U_0$, where $C>0$ does not depend on $(h,\tilde u_0,\delta,\xi)$, $\Phi_{h,\tilde u_0,\delta,\xi}:=\Phi_{h,\tilde u_0}(\delta,\xi)$ and

$$R_{\delta,\xi}:=(\Delta_g+h)W_{h,\tilde u_0,\delta,\xi}-(W_{h,\tilde u_0,\delta,\xi})_+^{2^\star-1}. \tag{20}$$

Furthermore, for every $(\delta_0,\xi_0)\in(0,\epsilon_0)\times U_0$, we have

$$\begin{aligned}&J_h'(W_{h,\tilde u_0,\delta_0,\xi_0}+\Phi_{h,\tilde u_0,\delta_0,\xi_0})=0\\&\iff(\delta_0,\xi_0)\text{ is a critical point of }(\delta,\xi)\mapsto J_h(W_{h,\tilde u_0,\delta,\xi}+\Phi_{h,\tilde u_0,\delta,\xi})\end{aligned} \tag{21}$$

$$\text{and } J_h(W_{h,\tilde u_0,\delta,\xi}+\Phi_{h,\tilde u_0,\delta,\xi})=J_h\left(W_{h,\tilde u_0,\delta,\xi}\right)+\mathrm{O}(\|\Phi_{h,\tilde u_0,\delta,\xi}\|^2_{H_1^2}) \tag{22}$$

uniformly in $(\delta,\xi)\in(0,\epsilon_0)\times U_0$ and $(h,\tilde u_0)$ such that $\|h-h_0\|_\infty<\epsilon_0$ and $\|\tilde u_0-u_0\|_{C^2}<\epsilon_0$. Moreover, assuming that

$$|B_{h,\delta,\xi}(x)|+\delta|\partial_pB_{h,\delta,\xi}(x)|\le C(U_{\delta,\xi}(x)+\delta\tilde U_{\delta,\xi}(x))\text{ for all }x\in M, \tag{23}$$

$$\text{where }\tilde U_{\delta,\xi}(x):=\left(\frac{\delta\sqrt{n(n-2)}}{\delta^2+d_{g_\xi}(x,\xi)^2}\right)^{\frac{n-2}{2}},$$

we have (see Esposito–Pistoia–Vétois [10])

$$\begin{aligned}\partial_pJ_h(W+\Phi)=\partial_pJ_h(W)&+\mathrm{O}(\delta^{-1}\|\Phi\|_{H_1^2}(\|R\|_{\frac{2n}{n+2}}+\delta\|\partial_pR\|_{\frac{2n}{n+2}}+\|\Phi\|_{H_1^2}))\\&+\mathrm{O}(\mathbf{1}_{n\ge7}\delta^{-1}\|\Phi\|_{H_1^2}^{2^\star-1}),\end{aligned} \tag{24}$$

where, to avoid unnecessarily heavy notations, we drop the indices $(h,\tilde u_0,\delta,\xi)$, so that $W:=W_{h,\tilde u_0,\delta,\xi}$, $\Phi:=\Phi_{h,\tilde u_0,\delta,\xi}$, $R=R_{\delta,\xi}$, etc.

2.4. **Conventions:**

- The differentiation in (δ,ξ) is denoted by ∂_p, and the differentiation in $x\in M$ (or $\mathbb{R}^n$) by ∂_x.
- For every $\xi\in U_0$, we identify the tangent space $T_\xi M$ with $\mathbb{R}^n$.
- C denotes a positive constant that depends on n, (M,g), $\xi_0\in M$, the functions $h_0,u_0\in C^2(M)$ and $A>0$ such that $\|h\|_{C^2}<A$ and $\lambda_1(\Delta_g+h)>1/A$. When $u_0>0$, we assume that $\|u_0\|_{C^2}<A$ and $u_0>1/A$.
- For every $f,g\in\mathbb{R}$, the notations $f=\mathrm{O}(g)$ and $f=\mathrm{o}(g)$ will stand for $|f|\le C\,|g|$ and $|f|\le C\epsilon(h,\delta,\xi)\,|g|$, respectively, where $\epsilon(h,\delta,\xi)\to0$ as $h\to h_0$ in $C^2(M)$, $\delta\to0$ and $\xi\to\xi_0$.

3. Energy and remainder estimates: the case $n\ge6$ and $u_0\equiv\tilde u_0\equiv0$

In this section, we consider the case $n\ge6$ and $u_0\equiv\tilde u_0\equiv0$. We set $B_{h,\delta,\xi}\equiv0$. Then $W_{h,\tilde u_0,\delta,\xi}=W_{\delta,\xi}\equiv U_{\delta,\xi}$. We prove the following estimates for $R=R_{\delta,\xi}$:

Proposition 3.1. *Assume that $n\ge6$ and $u_0\equiv\tilde u_0\equiv0$. Then*

$$\|R\|_{\frac{2n}{n+2}}+\delta\|\partial_pR\|_{\frac{2n}{n+2}}\le C\begin{cases}\delta^2+D_{h,\xi}\delta^2\left(\ln(1/\delta)\right)^{2/3}&\text{if }n=6\\\delta^{\frac{n-2}{2}}+D_{h,\xi}\delta^2&\text{if }7\le n\le9\\\delta^4\left(\ln(1/\delta)\right)^{3/5}+D_{h,\xi}\delta^2&\text{if }n=10\\\delta^4+D_{h,\xi}\delta^2&\text{if }n\ge11,\end{cases} \tag{25}$$

where $D_{h,\xi}:=\|h-h_0\|_\infty+d_g(\xi,\xi_0)^2$.

Proof of Proposition 3.1. Let $L_g := \Delta_g + c_n \operatorname{Scal}_g$ be the conformal Laplacian. For a metric $g' = w^{4/(n-2)}g$ conformal to g ($w \in C^\infty(M)$ is positive), the conformal invariance law gives that $L_{g'}\phi = w^{-(2^\star-1)}L_g(w\phi)$ for all $\phi \in C^\infty(M)$. Therefore, we have

$$R = (\Delta_g + h)U - U^{2^\star-1} = \Lambda_\xi^{2^\star-1}(\Delta_{g_\xi}(\Lambda_\xi^{-1}U) - (\Lambda_\xi^{-1}U)^{2^\star-1}) + \hat{h}_\xi U,$$

where φ_h is as in (6) and

$$\hat{h}_\xi := \varphi_h + c_n\Lambda_\xi^{2^\star-2}\operatorname{Scal}_{g_\xi}. \tag{26}$$

Via the exponential chart, using the radial symmetry of $U_{\delta,0} : \mathbb{R}^n \to \mathbb{R}$, we obtain

$$\Delta_{g_\xi}(\Lambda_\xi^{-1}U) - (\Lambda_\xi^{-1}U)^{2^\star-1} = \Delta_{\mathrm{Eucl}}U_{\delta,0} + \frac{\partial_r\sqrt{|g_\xi|}}{\sqrt{|g_\xi|}}\partial_r U_{\delta,0} - U_{\delta,0}^{2^\star-1} = \frac{\partial_r\sqrt{|g_\xi|}}{\sqrt{|g_\xi|}}\partial_r U_{\delta,0}$$

around 0. It then follows from (16) that

$$R(x) = \hat{h}_\xi(x)U(x) + \delta^{\frac{n-2}{2}}\Theta_{\delta,\xi}(x), \text{ where } |\Theta_{\delta,\xi}(x)| + |\partial_p\Theta_{\delta,\xi}(x)| \le C \tag{27}$$

for all $(\delta,\xi) \in (0,\infty) \times U_0$ and $x \in M$. Computations then yield (25). This ends the proof of Proposition 3.1. □

Plugging together (22), (19), (24) and (25), we obtain

$$J_h(W+\Phi) = J_h(W) + \mathrm{O}\begin{pmatrix} \delta^4 + D_{h,\xi}^2\delta^4(\ln(1/\delta))^{4/3} & \text{if } n=6 \\ \delta^{n-2} + D_{h,\xi}^2\delta^4 & \text{if } 7 \le n \le 9 \\ \delta^8(\ln(1/\delta))^{6/5} + D_{h,\xi}^2\delta^4 & \text{if } n = 10 \\ \delta^8 + D_{h,\xi}^2\delta^4 & \text{if } n \ge 11 \end{pmatrix} \tag{28}$$

$$\text{and } \partial_{p_i}J_h(W+\Phi) = \partial_{p_i}J_h(W) \tag{29}$$
$$+ \mathrm{O}\left(\delta^{-1}\right)\begin{cases} \delta^4 + D_{h,\xi}^2\delta^4(\ln(1/\delta))^{4/3} & \text{if } n=6 \\ (\delta^{\frac{n-2}{2}} + D_{h,\xi}\delta^2)^{2^\star-1} & \text{if } 7 \le n \le 9 \\ (\delta^4(\ln(1/\delta))^{3/5} + D_{h,\xi}\delta^2)^{2^\star-1} & \text{if } n = 10 \\ (\delta^4 + D_{h,\xi}\delta^2)^{2^\star-1} & \text{if } n \ge 11 \end{cases}$$

for all $i = 0,\dots,n$. We now estimate $J_h(W+\Phi)$:

Proposition 3.2. *Assume that $n \ge 6$ and $u_0 \equiv \tilde{u}_0 \equiv 0$. Then*

$$J_h(W+\Phi) = \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx + \frac{1}{2}\varphi_h(\xi)\delta^2\int_{\mathbb{R}^n} U_{1,0}^2\,dx \tag{30}$$
$$-\frac{1}{4n}\begin{cases} 24^2\omega_5 K_{h_0}(\xi_0)\delta^4\ln(1/\delta) + \mathrm{O}(\delta^4(\mathrm{o}(\ln(1/\delta) + D_{h,\xi}^2(\ln(1/\delta))^{4/3})) & \text{if } n=6 \\ K_{h_0}(\xi_0)\delta^4\int_{\mathbb{R}^n}|x|^2U_{1,0}^2\,dx + \mathrm{o}(\delta^4) & \text{if } n \ge 7 \end{cases}$$

as $\delta \to 0$, $\xi \to \xi_0$ and $h \to h_0$ in $C^2(M)$, where $K_{h_0}(\xi_0)$ is as in (7).

Proof of Proposition 3.2. Integrating by parts, we obtain

$$J_h(U) = \frac{1}{2}\int_M[(\Delta_g+h)U - U^{2^\star-1}]U\,dv_g + \frac{1}{n}\int_M U^{2^\star}\,dv_g.$$

It follows from (27) and the volume estimate (16) that

$$J_h(U) = \frac{1}{2}\int_M \hat{h}_\xi U^2 dv_g + \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\, dx + \mathrm{O}(\delta^{n-2}).$$

With the change of metric, (16) and (17) yield

$$\int_M \hat{h}_\xi U^2 dv_g = \int_{B_{r_0}(\xi)} \hat{h}_\xi U^2 dv_g + \mathrm{O}(\delta^{n-2}) = \int_{B_{r_0}(0)} A_{h,\xi} U_{\delta,0}^2\, dx + \mathrm{O}(\delta^{n-2}),$$

where $A_{h,\xi}(x) := (\hat{h}_\xi \Lambda_\xi^{2-2^\star})(\exp_\xi^{g_\xi}(x))$. Using the radial symmetry of $U_{\delta,0}$ together with Taylor expansions and the explicit form of $U_{\delta,0}$, we obtain

$$J_h(U) = \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\, dx + \frac{1}{2}\varphi_h(\xi)\delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\, dx - \frac{1}{4n}\begin{cases} 24^2\omega_5 K_{h_0}(\xi_0)\delta^4\ln(1/\delta) + \mathrm{o}\left(\delta^4\ln(1/\delta)\right) & \text{if } n=6 \\ K_{h_0}(\xi_0)\delta^4 \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2\, dx + \mathrm{o}(\delta^4) & \text{if } n \geq 7. \end{cases} \tag{31}$$

Plugging together (28) and (31), we obtain (30). This proves Proposition 3.2. □

With some extra care, arguing similarly as Esposito–Pistoia–Vétois [10], see also additional details are in [26], we also obtain the differentiable version:

Proposition 3.3. *Assume that $n \geq 6$ and $u_0 \equiv \tilde{u}_0 \equiv 0$. Then*

$$\partial_\delta J_h(W+\Phi) = \varphi_h(\xi)\delta \int_{\mathbb{R}^n} U_{1,0}^2 dx - \frac{1}{n}\begin{cases} 24^2\omega_5 K_{h_0}(\xi_0)\delta^3\ln(1/\delta) + \mathrm{o}\left(\delta^3\ln(1/\delta)\right) + \mathrm{O}(D_{h,\xi}^2\delta^3(\ln(1/\delta))^{4/3}) & \text{if } n=6 \\ K_{h_0}(\xi_0)\delta^3 \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2\, dx + \mathrm{o}(\delta^3) + \mathrm{O}(D_{h,\xi}^{2^\star-1}\delta^{\frac{n+6}{n-2}}) & \text{if } n \geq 7 \end{cases} \tag{32}$$

and

$$\partial_{\xi_i} J_h(W+\Phi) = \frac{1}{2}\partial_{\xi_i}\varphi_h(\xi)\delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\, dx + \mathrm{O}\begin{pmatrix} \mathrm{o}(\delta^3\ln(1/\delta)) + \mathrm{O}(D_{h,\xi}^2\delta^3\,(\ln(1/\delta))^{4/3}) & \text{if } n=6 \\ \mathrm{o}(\delta^3) + \mathrm{O}(D_{h,\xi}^{2^\star-1}\delta^{\frac{n+6}{n-2}}) & \text{if } n\geq 7 \end{pmatrix} \tag{33}$$

for all $i = 1,\dots,n$, as $\delta \to 0$, $\xi \to \xi_0$ and $h \to h_0$ in $C^2(M)$.

Theorem 1.4 for $n \geq 6$ will be proved in Section 8.

4. Energy and remainder estimates: the case $n \geq 7$ and $u_0, \tilde{u}_0 > 0$

In this section, we assume that $u_0, \tilde{u}_0 > 0$ and $n \geq 7$, that is $2^\star - 1 < 2$. As in the previous case, we set $B_{h,\delta,\xi} \equiv 0$, so that $W_{h,\tilde{u}_0,\delta,\xi} = W_{\tilde{u}_0,\delta,\xi} \equiv \tilde{u}_0 + U_{\delta,\xi}$ and we are in the framework of Section 2. We prove the following estimates for $R = R_{\delta,\xi}$:

Proposition 4.1. *Assume that $n \geq 7$ and $u_0, \tilde{u}_0 > 0$. Then*

$$\|R\|_{\frac{2n}{n+2}} \leq C\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_\infty + C(D_{h,\xi} + \delta^2 + \delta^{\frac{n-6}{2}})\delta^2 \text{ and } \|\partial_p R\|_{\frac{2n}{n+2}} \leq C\delta, \tag{34}$$

where $D_{h,\xi}$ is as in (25).

Proof of Proposition 4.1. We have

$$R = (\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}) + R^0 - ((\tilde{u}_0 + U)^{2^\star-1} - \tilde{u}_0^{2^\star-1} - U^{2^\star-1})$$
$$\text{and } \partial_{p_i} R = \partial_{p_i} R^0 - (2^\star - 1)((\tilde{u}_0 + U)^{2^\star-2} - U^{2^\star-2})\partial_{p_i} U$$

for all $i = 1, \dots, n$, where $R^0 := \Delta_g U + hU - U^{2^\star-1}$. Arguing as in [22], we then obtain (34), which proves Proposition 4.1. □

Plugging (34) together with (22), (19) and (24), we obtain

$$J_h(W+\Phi) = J_h(W) + \mathrm{O}(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_\infty^2 + D_{h,\xi}^2 \delta^4 + \delta^8 + \delta^{n-2}) \tag{35}$$

$$\text{and } \partial_{p_i} J_h(W+\Phi) = \partial_{p_i} J_h(W) + \mathrm{O}(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_{\frac{2n}{n+2}} \delta \tag{36}$$
$$+ \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_\infty^{2^\star-1} \delta^{-1} + (D_{h,\xi} + \delta^2 + \delta^{\frac{n-6}{2}})^{2^\star-1} \delta^{\frac{n+6}{n-2}} + D_{h,\xi}\delta^3 + \delta^5 + \delta^{n/2})$$

for all $i = 0, \dots, n$. We now estimate $J_h(W+\Phi)$:

Proposition 4.2. *Assume that $n \geq 7$ and $u_0, \tilde{u}_0 > 0$. Then*

$$J_h(W+\Phi) = J_h(\tilde{u}_0) + \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\, dx + \frac{1}{2}\varphi_h(\xi)\delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\, dx \tag{37}$$
$$- \frac{1}{4n} K_{h_0}(\xi_0)\delta^4 \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2\, dx + \mathrm{o}(\delta^4) - u_0(\xi_0)\delta^{\frac{n-2}{2}} \int_{\mathbb{R}^n} U_{1,0}^{2^\star-1} dx$$
$$+ \mathrm{O}(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_\infty^2 + \delta^{\frac{n-2}{2}}(\|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_\infty + \|\tilde{u}_0 - u_0\|_\infty + \mathrm{o}(1)))$$

as $\delta \to 0$, $\xi \to \xi_0$ and $h \to h_0$ in $C^2(M)$.

Proof of Proposition 4.2. We first write

$$J_h(\tilde{u}_0 + U) = J_h(\tilde{u}_0) + J_h(U) - \int_M \tilde{u}_0 U^{2^\star-1} dv_g + \int_M (\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}) U\, dv_g$$
$$- \frac{1}{2^\star}\int_M ((\tilde{u}_0 + U)^{2^\star} - \tilde{u}_0^{2^\star} - U^{2^\star} - 2^\star \tilde{u}_0^{2^\star-1} U - 2^\star \tilde{u}_0 U^{2^\star-1})\, dv_g.$$

We then argue as in [22] to estimate the last term in the right-hand side of this identity. The third term is estimated as in [25]. The second term is estimated by using the expansion (31). Then (35) yields (37), thus proving Proposition 4.2. □

In the same spirit, arguing again as in Esposito-Pistoia–Vétois [10], see again details in [26], we obtain the following estimates for the derivatives of $J_h(W+\Phi)$:

Proposition 4.3. *Assume that $n \geq 7$ and $u_0, \tilde{u}_0 > 0$. Then*

$$\partial_\delta J_h(W+\Phi) = \varphi_h(\xi)\delta \int_{\mathbb{R}^n} U_{1,0}^2\, dx - \frac{1}{n} K_{h_0}(\xi_0)\delta^3 \int_{\mathbb{R}^n} |x|^2 U_{1,0}^2\, dx + \mathrm{o}(\delta^3) \tag{38}$$
$$- \frac{n-2}{2} u_0(\xi_0)\delta^{\frac{n-4}{2}} \int_{\mathbb{R}^n} U_{1,0}^{2^\star-1} dx + \mathrm{O}(\delta^{\frac{n-4}{2}}(\|\tilde{u}_0 - u_0\|_\infty + \mathrm{o}(1))$$
$$+ \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_\infty \delta + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_\infty^{2^\star-1}\delta^{-1} + D_{h,\xi}^{2^\star-1}\delta^{\frac{n+6}{n-2}})$$

$$\text{and } \partial_{\xi_i} J_h(W+\Phi) = \frac{1}{2}\partial_{\xi_i}\varphi_h(\xi)\delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\, dx + \mathrm{o}(\delta^3) \tag{39}$$
$$+ \mathrm{O}(\delta^{\frac{n-4}{2}}(\|\tilde{u}_0 - u_0\|_\infty + \mathrm{o}(1)) + \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_\infty \delta$$
$$+ \|\Delta_g \tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^{2^\star-1}\|_\infty^{2^\star-1}\delta^{-1} + D_{h,\xi}^{2^\star-1}\delta^{\frac{n+6}{n-2}})$$

for all $i = 1, \dots, n$, *as* $\delta \to 0$, $\xi \to \xi_0$ *and* $h \to h_0$ *in* $C^2(M)$.

Theorem 1.5 for $n \geq 7$ will be proved in Section 9.

5. Energy and remainder estimates: the case $n = 6$ and $u_0, \tilde{u}_0 > 0$

In this section, we assume that $u_0, \tilde{u}_0 > 0$ and $n = 6$, that is $2^\star - 1 = 2$. Here again, we set $B_{h,\delta,\xi} \equiv 0$, so that $W_{h,\tilde{u}_0,\delta,\xi} = W_{\tilde{u}_0,\delta,\xi} \equiv \tilde{u}_0 + U_{\delta,\xi}$ and we are in the framework of Section 2. The remark underlying this section is that since $2^\star - 1 = 2$,

$$\Delta_g(u_0 + U) + h(u_0 + U) - (u_0 + U)^2 = \Delta_g U + (h - 2u_0)U - U^2.$$

Therefore, to obtain a good approximation of the blowing-up solution, it suffices to subtract a perturbation of $2u_0$ to the potential. With this remark, setting

$$\bar{h} := h - 2\tilde{u}_0 \text{ and } \bar{h}_0 := h_0 - 2u_0$$

and arguing as in the case where $n \geq 7$, we obtain the following:

Proposition 5.1. *Assume that* $n = 6$ *and* $u_0, \tilde{u}_0 > 0$. *Then*

$$\begin{aligned}(40)\quad & J_h(W + \Phi) = J_h(\tilde{u}_0) + \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx + \frac{1}{2}\varphi_{h,\tilde{u}_0}(\xi)\delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\,dx \\ & -24\omega_5 K_{h_0,u_0}(\xi_0)\delta^4\ln(1/\delta) + \mathrm{O}(\|\Delta_g\tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty^2 + \|\Delta_g\tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty\delta^2) \\ & + \mathrm{O}(\delta^4\ln(1/\delta)(\mathrm{o}(1) + \overline{D}_{h,\xi}^2(\ln(1/\delta))^{1/3})),\end{aligned}$$

$$\begin{aligned}(41)\quad & \partial_\delta J_h(W + \Phi) = \varphi_{h,\tilde{u}_0}(\xi)\delta\int_{\mathbb{R}^n} U_{1,0}^2\,dx - 96\omega_5 K_{h_0,u_0}(\xi_0)\delta^3\ln(1/\delta) \\ & + \mathrm{O}(\|\Delta_g\tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty\delta + \|\Delta_g\tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty^2\delta^{-1}) \\ & + \mathrm{O}(\delta^3\ln(1/\delta)(\mathrm{o}(1) + \overline{D}_{h,\xi}^2(\ln(1/\delta))^{1/3}))\end{aligned}$$

$$\begin{aligned}(42)\quad & \text{and } \partial_{\xi_i} J_h(W + \Phi) = \frac{1}{2}\partial_{\xi_i}\varphi_{h,\tilde{u}_0}(\xi)\delta^2\int_{\mathbb{R}^n} U_{1,0}^2\,dx \\ & + \mathrm{O}(\|\Delta_g\tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty\delta + \|\Delta_g\tilde{u}_0 + h\tilde{u}_0 - \tilde{u}_0^2\|_\infty^2\delta^{-1}) \\ (43)\quad & + \mathrm{O}(\delta^3\ln(1/\delta)(\mathrm{o}(1) + \overline{D}_{h,\xi}^2(\ln(1/\delta))^{1/3}))\end{aligned}$$

for all $i = 1, \dots, n$, *as* $\delta \to 0$, $\xi \to \xi_0$ *and* $h \to h_0$ *in* $C^2(M)$, *where* $\varphi_{h,\tilde{u}_0}$ *and* $K_{h_0,u_0}(\xi_0)$ *are as in* (6), (10) *and* $\overline{D}_{h,\xi} := \|\bar{h} - \bar{h}_0\|_\infty + d_g(\xi, \xi_0)^2$.

Here again, additional details are in [26].

6. Setting and definition of the mass in dimensions $n = 4, 5$

Following the computations in (27), we obtain the following:

Lemma 6.1. *There exist two smooth functions* $(\xi, x) \mapsto f_i(\xi, x)$, $i = 1, 2$, *on* $M \times M$ *such that for every* $f : \mathbb{R}^n \to \mathbb{R}$ *that is radially symmetric, we have*

$$\begin{aligned}(\Delta_g + h)(\chi(r)\Lambda_\xi(x)f(r)) = \Lambda_\xi(x)^{2^\star - 1}\chi\Delta_{\mathrm{Eucl}}(f(r)) + f_1(\xi, x)f'(r) + f_2(\xi, x)f(r) \\ + \hat{h}_\xi\chi(x)\Lambda_\xi(x)f(r)\end{aligned}$$

for all $x \in M\backslash\{\xi\}$, *where* $r := d_{g_\xi}(x, \xi)$ *and* $\hat{h}_\xi$ *is as in* (26). *Furthermore,* $f_i(\xi, x) = 0$ *when* $d_g(x, \xi) \geq r_0$ *and there exists* $C_N > 0$ *such that*

$$|f_1(\xi, x)(x)| \leq C_N d_g(x, \xi)^{N-1} \text{ and } |f_2(\xi, x)| \leq C_N d_g(x, \xi)^{N-2} \text{ for all } x, \xi \in M.$$

We define

$$\Gamma_\xi(x) := \frac{\chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)}{(n-2)\omega_{n-1} d_{g_\xi}(x,\xi)^{n-2}}$$

for all $x \in M\setminus\{\xi\}$. It follows from Lemma 6.1 and the definition (12) that

(44)

$$\Delta_g U_{\delta,\xi} + hU_{\delta,\xi} = U_{\delta,\xi}^{2^\star-1} + F_\delta(\xi,x)\delta^{\frac{n-2}{2}} + \hat{h}_\xi U_{\delta,\xi}$$

$$\text{and } (\Delta_g + h)\Gamma_\xi = \delta_\xi + \frac{F_0(\xi,x)}{k_n} + \hat{h}_\xi\Gamma_\xi, \text{ where } k_n := (n-2)\omega_{n-1}\sqrt{n(n-2)}^{\frac{n-2}{2}},$$

δ_ξ is the Dirac mass at ξ and $(t,\xi,x) \to F_t(\xi,x)$ is of class C^p on $[0,\infty)\times M\times M$, with p being as large as we want provided we choose N large enough. This includes $t = 0$ and, therefore,

$$\lim_{t\to 0} F_t = F_0 \text{ in } C^p(M\times M). \tag{45}$$

For every $t \geq 0$, we define $\beta_{h,t,\xi} \in H_1^2(M)$ as the unique solution to

$$(\Delta_g + h)\beta_{h,t,\xi} = -\frac{F_t(\xi,\cdot)}{k_n} - \hat{h}_\xi \begin{cases} \dfrac{U_{t,\xi}}{k_n t^{\frac{n-2}{2}}} & \text{if } t > 0 \\ \Gamma_\xi & \text{if } t = 0. \end{cases} \tag{46}$$

Since $N > n-2$ and $n \leq 5$, the right-hand-side is uniformly bounded in $L^q(M)$ for some $q > \frac{2n}{n+2}$, independently of $t \geq 0$, $\xi \in U_0$ and $h \in C^2(M)$ satisfying $\|h\|_\infty < A$ and $\lambda_1(\Delta_g + h) > 1/A$. Therefore, $\beta_{h,t,\xi}$ is well defined and we have

$$\|\beta_{h,t,\xi} - \beta_{h,0,\xi}\|_{H_1^2} = \mathrm{o}(1) \text{ as } t \to 0 \tag{47}$$

uniformly in ξ and h. Furthermore, we have $\beta_{h,t,\xi} \in C^2(M)$ when $t > 0$. As one checks, with these definitions, we obtain that

$$G_{h,\xi} := \Gamma_\xi + \beta_{h,0,\xi}$$

is the Green's function of the operator $\Delta_g + h$ at the point ξ.

Proposition-Definition 6.1. *Assume that $n \in \{4,5\}$ and $N > n-2$. Let $h \in C^2(M)$ be such that $\Delta_g + h$ is coercive. Assume that there exists $\xi \in M$ such that $\varphi_h(\xi) = |\nabla\varphi_h(\xi)| = 0$, where φ_h is as in (6). Then $\beta_{h,0,\xi} \in C^0(M)$. Furthermore, $\beta_{h,0,\xi}(\xi)$ does not depend on the choice of $N > n-2$ and g_ξ satisfying (15) and (16). We then define the mass of $\Delta_g + h$ at the point ξ as $m_h(\xi) := \beta_{h,0,\xi}(\xi)$.*

Proof of Proposition-Definition 6.1. We have $\hat{h}_\xi(x)\Gamma_\xi(x) = \mathrm{O}(d_g(x,\xi)^{4-n})$ since $\varphi_h(\xi) = |\nabla\varphi_h(\xi)| = 0$. We also have $F_0(\xi,x) = \mathrm{O}(d_g(x,\xi)^{N-n})$. When $N > n$, this implies that $\beta_{h,0,\xi} \in C^0(M)$. One has that $\beta_{h,0,\xi}(\xi)$ is independent of N and g_ξ (see Lee–Parker [14]). This ends the proof of Proposition-Definition 6.1. □

Note that when $h \equiv c_n \,\mathrm{Scal}_g$, the mass $m_{c_n \mathrm{Scal}_g}(\xi)$ is defined for all $\xi \in M$, and one recovers the concept of mass of Schoen–Yau when $n = 3,4,5$. When $h \equiv c_n \,\mathrm{Scal}_g$ and the manifold is locally conformally flat, here again, one recovers the concept of Schoen–Yau. We now prove a differentiation result that will allow us to obtain Theorem 1.2:

Proposition 6.1. *Assume that $n \in \{4,5\}$. Let $h \in C^2(M)$ be such that $\Delta_g + h$ is coercive. Assume that there exists $\xi \in M$ such that $\varphi_h(\xi) = |\nabla \varphi_h(\xi)| = 0$. Let $H \in C^2(M)$ be such that $H(\xi) = |\nabla H(\xi)| = 0$. Then $m_{h+\epsilon H}(\xi)$ is well defined for small $\epsilon \in \mathbb{R}$ and differentiable in ϵ. Furthermore,*

$$\partial_\epsilon (m_{h+\epsilon H}(\xi))_{|0} = -\int_M H G_{h,\xi}^2 \, dv_g.$$

Proof of Proposition 6.1. We set $G_{h,\xi} = G_{c_n \operatorname{Scal}_g, \xi} + \hat{\beta}_{h,\xi}$, where $\hat{\beta}_{h,\xi}$ is such that

$$(\Delta_g + h)\hat{\beta}_{h,\xi} = -\varphi_h G_{c_n \operatorname{Scal}_g, \xi}, \; \hat{\beta}_{h,\xi} \in H_1^2(M). \tag{48}$$

Under the assumptions of the proposition, we have $\hat{\beta}_{h,\xi} \in C^0(M)$ and

$$\hat{\beta}_{h,\xi}(\xi) = -\int_M \varphi_h G_{c_n \operatorname{Scal}_g, \xi} G_{h,\xi} \, dv_g.$$

As one checks, we have $m_h(\xi) = m_{c_n \operatorname{Scal}_g}(\xi) - \hat{\beta}_{h,\xi}(\xi)$. Elliptic theory gives that $h \mapsto \hat{\beta}_{h,\xi}$ is differentiable. Differentiating (48) with respect to h in the direction H yields

$$\partial_\epsilon (m_{h+\epsilon H}(\xi))_{|\epsilon=0} = -\int_M H G_{h,\xi}^2 \, dv_g.$$

This ends the proof of Proposition 6.1. □

7. Energy and remainder estimates in dimensions $n = 4, 5$

Here we assume that $n \leq 5$, $u_0 \equiv \tilde{u}_0 \equiv 0$ and (4) is satisfied. We define

$$W_{h,\tilde{u}_0,\delta,\xi} = W_{h,\delta,\xi} := U_{\delta,\xi} + B_{h,\delta,\xi}, \text{ where } B_{h,\delta,\xi} := k_n \delta^{\frac{n-2}{2}} \beta_{h,\delta,\xi}. \tag{49}$$

Our first step is to obtain estimates for $\beta_{h,\delta,\xi}$ and its derivatives in $H_1^2(M)$:

Proposition 7.1. *For $n \in \{4,5\}$, let $B_{h,\delta,\xi}$ be as in (49). Then (18) holds.*

Proof of Proposition 7.1. It follows from (47) that $\|\beta_{h,\delta,\xi}\|_{H_1^2} \leq C$. Differentiating (46) in ξ_i, $i = 1, \dots, n$, we obtain

$$(\Delta_g + h)(\partial_{\xi_i} \beta_{h,\delta,\xi}) = -\frac{1}{k_n} \left(\partial_{\xi_i} F_\delta(\xi, \cdot) + \partial_{\xi_i} \hat{h}_\xi \frac{U_{\delta,\xi}}{\delta^{\frac{n-2}{2}}} + \hat{h}_\xi \frac{\partial_{\xi_i} U_{\delta,\xi}}{\delta^{\frac{n-2}{2}}} \right).$$

Using the explicit pointwise control that we have for the right-hand side of this identity, elliptic theory yields (18) for the derivative in ξ_i. We then apply the same method to estimate the derivative in δ. This proves Proposition 7.1. □

We now need a pointwise control for $\beta_{h,\delta,\xi}$ and its derivatives.

Proposition 7.2. *We have*

$$|\beta_{h,\delta,\xi}(x)| \leq C \begin{cases} 1 + \left|\ln\left(\delta^2 + d_g(x,\xi)^2\right)\right| & \text{if } n = 4 \\ \left(\delta^2 + d_g(x,\xi)^2\right)^{-1/2} & \text{if } n = 5, \end{cases} \tag{50}$$

$$|\partial_\delta \beta_{h,\delta,\xi}(x)| \leq C + C D_{h,\xi} \delta \ln(1/\delta) \left(\delta^2 + d_g(x,\xi)^2\right)^{-\frac{n-2}{2}} \quad \text{and} \tag{51}$$

$$|\partial_{\xi_i} \beta_{h,\delta,\xi}(x)| \leq C + C \begin{cases} D_{h,\xi} \left(\delta^2 + d_g(x,\xi)^2\right)^{-1/2} & \text{if } n = 4 \\ \left|\ln(\delta^2 + d_g(x,\xi)^2)\right| + D_{h,\xi}(\delta^2 + d_g(x,\xi)^2)^{-1} & \text{if } n = 5 \end{cases} \tag{52}$$

for all $i = 1, \dots, n$, where $D_{h,\xi}$ is as in (25).

Proof of Proposition 7.2. Green's representation formula together with (46) yields
(53)
$$\beta_{h,\delta,\xi}(x) = -\int_M G_{h,x}(y)\left(\frac{F_\delta(\xi,y)}{k_n} + \hat{h}_\xi \frac{\chi(d_{g_\xi}(y,\xi))\Lambda_\xi(y)}{(n-2)\omega_{n-1}(\delta^2 + d_{g_\xi}(y,\xi)^2)^{\frac{n-2}{2}}}\right) dv_g(y)$$

for all $x \in M$. Standard estimates of the Green's function give $0 < G_{h,x}(y) \leq C d_g(x,y)^{2-n}$ for all $x, y \in M$, $x \neq y$. Proposition 7.2 then follows from these estimates together with (45) and Giraud's lemma (see [9]). □

When the mass is defined at ξ, that is $\varphi_h(\xi) = |\nabla \varphi_h(\xi)| = 0$, then $\beta_{h,\delta,\xi}$ is bounded. Otherwise, it is not. For instance, when $n = 5$, it behaves like $(h - c_n \mathrm{Scal}_g)(\xi)\, d_g(x,\xi)^{-1}$ as $x \to \xi$ and $\delta \to 0$.

It follows from Proposition 7.2 that (18) is satisfied and therefore, we are in the framework of Section 2. Since $n \leq 5$, we then obtain

$$J_h(W + \Phi) = J_h(W) + \mathrm{O}(\|R\|^2_{\frac{2n}{n+2}}) \tag{54}$$
$$\text{and } \partial_p J_h(W + \Phi) = \partial_p J_h(W) + \mathrm{O}(\delta^{-1}\|R\|_{\frac{2n}{n+2}}(\|R\|_{\frac{2n}{n+2}} + \delta\|\partial_p R\|_{\frac{2n}{n+2}})), \tag{55}$$

where $R = R_{\delta,\xi}$ is as in (20). We prove the following estimates for R:

Proposition 7.3. *We have*

$$\|R\|_{\frac{2n}{n+2}} + \delta\|\partial_p R\|_{\frac{2n}{n+2}} \leq C \begin{cases} \delta^2 \ln(1/\delta) & \text{if } n = 4 \\ D_{h,\xi}\delta^2 \ln(1/\delta) + \delta^2 & \text{if } n = 5. \end{cases} \tag{56}$$

Proof of Proposition 7.3. Note that since $n < 6$, we have $2^\star > 3$. The definitions (46), (49) combined with (44) yield

$$R = U^{2^\star-1} - (U+B)_+^{2^\star-1} = -(2^\star - 1)U^{2^\star-2}B + \mathrm{O}(U^{2^\star-3}B^2 + |B|^{2^\star-1}), \tag{57}$$

where we have used that $U \geq 0$. Therefore,

$$\|R\|_{\frac{2n}{n+2}} \leq C\|U^{2^\star-2}B\|_{\frac{2n}{n+2}} + \||B|^{2^\star-1}\|_{\frac{2n}{n+2}}.$$

As regards the gradient term, letting $i \in \{0, \dots, n\}$, we have

$$\partial_{p_i} R = -(2^\star - 1)(((U+B)_+^{2^\star-2} - U^{2^\star-2})\partial_{p_i} U + (U+B)_+^{2^\star-2}\partial_{p_i} B).$$

We then obtain

$$\delta|\partial_{p_i} R| \leq C\tilde{U}^{2^\star-2}|B| + C\tilde{U}|B|^{2^\star-2} + C\delta|\partial_{p_i} B|\tilde{U}^{2^\star-2},$$

where $\tilde{U} = \tilde{U}_{\delta,\xi}$ is as in (23). Since $B = k_n \delta^{\frac{n-2}{2}}\beta$, Proposition 7.2 together with (50) and long but easy computations yields (56), thus proving Proposition 7.3. □

With (56), the estimates (54) and (55) become

$$J_h(W + \Phi) = J_h(W) + O\begin{pmatrix} \delta^4 (\ln(1/\delta))^2 & \text{if } n = 4 \\ \delta^4 + D^2_{h,\xi}\delta^4(\ln(1/\delta))^2 & \text{if } n = 5 \end{pmatrix} \tag{58}$$
$$\text{and } \partial_{p_i} J_h(W + \Phi) = \partial_{p_i} J_h(W) + \mathrm{O}\begin{pmatrix} \delta^3 (\ln(1/\delta))^2 & \text{if } n = 4 \\ \delta^3 + D^2_{h,\xi}\delta^3(\ln(1/\delta))^2 & \text{if } n = 5 \end{pmatrix}. \tag{59}$$

Proposition 7.4. *We have*

$$J_h(W+\Phi)=\frac{1}{n}\int_{\mathbb{R}^n}U_{1,0}^{2^\star}\,dx+\frac{1}{2}\varphi_h(\xi)\left\{\begin{array}{ll}8\omega_{n-1}\delta^2\ln(1/\delta) & \text{if } n=4\\ \delta^2\int_{\mathbb{R}^n}U_{1,0}^2\,dx & \text{if } n=5\end{array}\right\}-\frac{k_n^2}{2}m_{h_0}(\xi_0)\delta^{n-2}+\mathrm{o}(\delta^{n-2}) \tag{60}$$

as $\delta\to 0$, $\xi\to\xi_0$ and $h\to h_0$ in $C^2(M)$.

Proof of Proposition 7.4. We have

$$J_h(W)=\frac{1}{2}\int_M RW\,dv_g+\left(\frac{1}{2}-\frac{1}{2^\star}\right)\int_M W_+^{2^\star}\,dv_g. \tag{61}$$

Using that $U\geq 0$, we obtain

$$W_+^{2^\star}=(U+B)_+^{2^\star}=U^{2^\star}+2^\star BU^{2^\star-1}+\mathrm{O}\left(B^2U^{2^\star-2}+|B|^{2^\star}\right). \tag{62}$$

Plugging (57) and (62) into (61), and using (23), we obtain

$$J_h(W)=\frac{1}{n}\int_M U^{2^\star}dv_g-\frac{1}{2}\int_M BU^{2^\star-1}dv_g+\mathrm{O}\left(\int_M(\tilde{U}^{2^\star-2}B^2+\tilde{U}|B|^{2^\star-1}+|B|^{2^\star})\,dv_g\right).$$

Since $B=k_n\delta^{\frac{n-2}{2}}\beta$, the pointwise estimate (50) and the definition (12) yield

$$J_h(W)=\frac{1}{n}\int_{\mathbb{R}^n}U_{1,0}^{2^\star}\,dx-\frac{1}{2}\int_M BU^{2^\star-1}dv_g+O\left(\begin{array}{ll}\delta^4\left(\ln(1/\delta)\right)^3 & \text{if } n=4\\ \delta^4 & \text{if } n=5\end{array}\right).$$

The definitions (46) and (49) of β and B yield

$$\Delta_g B+hB=U^{2^\star-1}-(\Delta_g U+hU)\text{ in } M. \tag{63}$$

Therefore, we obtain

$$\int_M BU^{2^\star-1}dv_g=\int_M\left(|\nabla B|^2+hB^2\right)dv_g-\delta^{\frac{n-2}{2}}\int_M F_\delta(\xi,\cdot)U\,dv_g-\int_M\hat{h}_\xi U^2dv_g.$$

Since $B=k_n\delta^{\frac{n-2}{2}}\beta$, using (47) and (45) together with integration theory yields

$$\int_M BU^{2^\star-1}dv_g=\delta^{n-2}k_n^2\left(\int_M\left(|\nabla\beta_{h,0,\xi}|^2+h\beta_{h,0,\xi}^2\right)dv_g-\frac{1}{k_n}\int_M F_0(\xi,\cdot)\Gamma_\xi\,dv_g\right)-\int_M\hat{h}_\xi U^2dv_g+\mathrm{o}(\delta^{n-2}). \tag{64}$$

Taking the exponential chart at ξ, we write

$$\int_M\hat{h}_\xi U^2dv_g=\hat{h}_\xi(\xi)\int_M U^2dv_g+\partial_{\xi_i}\hat{h}_\xi(\xi)\int_M x^iU^2dv_g+\int_M(\hat{h}_\xi-\hat{h}_\xi(\xi)-\partial_{\xi_i}\hat{h}_\xi(\xi)x^i)U^2dv_g.$$

As one checks, there exists $C>0$ such that

$$|\hat{h}_\xi-\hat{h}_\xi(\xi)-\partial_{\xi_i}\hat{h}_\xi(\xi)x^i|U^2\leq C\delta^{n-2}d_g(\xi,x)^{6-2n}$$

for all $x, \xi \in M$, $x \neq \xi$. Since $n < 6$ and ξ remains in a neighborhood of ξ (so that the exponential chart remains nicely bounded), integration theory then yields

$$\int_M (\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i}\hat{h}_\xi(\xi)x^i)U^2 dv_g = \delta^{n-2}k_n^2 \int_M (\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i}\hat{h}_\xi(\xi)x^i)\Gamma_\xi^2 dv_g + \mathrm{o}(\delta^{n-2}).$$

Furthermore, letting $\xi \to \xi_0$, $h \to h_0$ and using (4), we obtain

$$\int_M (\hat{h}_\xi - \hat{h}_\xi(\xi) - \partial_{\xi_i}\hat{h}_\xi(\xi)x^i)U^2 dv_g = \delta^{n-2}k_n^2 \int_M (\hat{h_0})_{\xi_0}\Gamma_{\xi_0}^2\, dv_g + \mathrm{o}(\delta^{n-2}). \tag{65}$$

Via the exponential chart, using the radial symmetry of U, we obtain

$$\int_M x^i U^2 dv_g = \mathrm{O}\left(\int_{B_{r_0}(0)} |x|^2 \left(\frac{\delta}{\delta^2 + |x|^2}\right)^{n-2} dx\right) = \mathrm{O}(\delta^{n-2})$$

since $n < 6$. Using estimates on $U_{\delta,\xi}$ together with the above estimates, we obtain

$$\int_M \hat{h}_\xi U^2 dv_g = \hat{h}_\xi(\xi)\left\{\begin{matrix} 8\omega_{n-1}\delta^2\ln(1/\delta) & \text{if } n=4 \\ \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\, dx & \text{if } n=5 \end{matrix}\right\} + \delta^{n-2}k_n^2 \int_M (\hat{h_0})_{\xi_0}\Gamma_{\xi_0}^2\, dv_g + \mathrm{o}(\delta^{n-2}).$$

Combining this estimate with (64), we obtain

$$\int_M BU^{2^\star-1}dv_g = -\hat{h}_\xi(\xi)\left\{\begin{matrix} 8\omega_{n-1}\delta^2\ln(1/\delta) & \text{if } n=4 \\ \delta^2 \int_{\mathbb{R}^n} U_{1,0}^2\, dx & \text{if } n=5 \end{matrix}\right\} + \delta^{n-2}k_n^2 I_{h_0,\xi_0} + \mathrm{o}(\delta^{n-2}),$$

$$\text{where } I_{h_0,\xi_0} := \int_M \left(|\nabla\beta_{h_0,0,\xi_0}|^2 + h_0\beta_{h_0,0,\xi_0}^2\right) dv_g - \frac{1}{k_n}\int_M F_0(\xi,\cdot)\Gamma_{\xi_0}\, dv_g - \int_M (\hat{h_0})_{\xi_0}\Gamma_{\xi_0}^2\, dv_g. \tag{66}$$

Integrating by parts and using the definition (46), we obtain

$$I_{h_0,\xi_0} = \int_M G_{h_0,\xi_0}(\Delta_g\beta_{h_0,0,\xi_0} + h_0\beta_{h_0,0,\xi_0})\, dv_g = \beta_{h_0,0,\xi_0}(\xi_0) = m_{h_0}(\xi_0).$$

Putting these results together yields (7.4), which proves Proposition 7.4. □

With similar arguments as in the proof of Proposition 7.4, we obtain the estimates of the derivatives:

Proposition 7.5. *We have*

$$\partial_\delta J_h(W+\Phi) = \varphi_h(\xi)\left\{\begin{matrix} 8\omega_{n-1}\delta\ln(1/\delta) & \text{if } n=4 \\ \delta \int_{\mathbb{R}^n} U_{1,0}^2\, dx & \text{if } n=5 \end{matrix}\right\} - \frac{n-2}{2}k_n^2 m_{h_0}(\xi_0)\delta^{n-3} + \mathrm{o}(\delta^{n-3}) \tag{67}$$

$$\text{(68)} \qquad \textit{and } \partial_{\xi_i} J_h(W+\Phi) = \frac{1}{2}\partial_{\xi_i}\varphi_h(\xi)\begin{Bmatrix} 8\omega_{n-1}\delta^2\ln(1/\delta) & \textit{if } n=4 \\ \delta^2\int_{\mathbb{R}^n} U_{1,0}^2\,dx & \textit{if } n=5 \end{Bmatrix} + \mathrm{O}\begin{pmatrix} \delta^2 + D_{h,\xi}\delta^2\ln(1/\delta) & \textit{if } n=4 \\ \delta^3 + D_{h,\xi}\delta^2 & \textit{if } n=5 \end{pmatrix}$$

for all $i=1,\dots,n$, *as* $\delta\to 0$, $\xi\to\xi_0$ *and* $h\to h_0$ *in* $C^2(M)$.

Here again, we refer the reader to [26] for additional details.

8. Proof of Theorem 1.4

We let $h_0, f \in C^p(M)$, $p\geq 2$, and $\xi_0\in M$ satisfy the assumptions of Theorem 1.4. For small $\epsilon>0$ and $\tau\in\mathbb{R}^n$, we define

$$\text{(69)} \qquad h_\epsilon := h_0 + \epsilon f \text{ and } \xi_\epsilon(\tau) := \exp_{\xi_0}^{g_{\xi_0}}(\sqrt{\epsilon}\tau).$$

We fix $R>0$ and $0<a<b$ to be chosen later.

8.1. Proof of Theorem 1.4 for $n\geq 6$. In this case, we let $(\delta_\epsilon)_{\epsilon>0}>0$ be such that $\delta_\epsilon\to 0$ as $\epsilon\to 0$. We define

$$\text{(70)} \qquad \delta_\epsilon(t) := \delta_\epsilon t \text{ and } F_\epsilon(t,\tau) := J_{h_\epsilon}(U_{\delta_\epsilon(t),\xi_\epsilon(\tau)} + \Phi_{h_\epsilon,0,\delta_\epsilon(t),\xi_\epsilon(\tau)})$$

for all $(t,\tau)\in(a,b)\times\mathbb{R}^n$ such that $|\tau|<R$. With (4), we obtain

$$\varphi_{h_\epsilon}(\xi_\epsilon(\tau)) = \frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)[\tau,\tau]\epsilon + f(\xi_0)\epsilon + \mathrm{o}(\epsilon)$$
$$\text{and } \nabla\varphi_{h_\epsilon}(\xi_\epsilon(\tau)) = \nabla^2\varphi_{h_0}(\xi_0)[\tau,\cdot]\sqrt{\epsilon} + \mathrm{o}(\sqrt{\epsilon})$$

as $\epsilon\to 0$ uniformly in $|\tau|<R$. We first assume that $n\geq 7$. In this case, we set $\delta_\epsilon := \sqrt{\epsilon}$. It follows from (30) that

$$\text{(71)} \qquad \lim_{\epsilon\to 0}\frac{F_\epsilon(t,\tau) - \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx}{\epsilon^2} = E_0(t,\xi) \text{ in } C^0_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^n),$$
$$\text{where } E_0(t,\tau) := C_n\left(\frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)[\tau,\tau] + f(\xi_0)\right)t^2 - D_n K_{h_0}(\xi_0)t^4,$$

for some constants $C_n, D_n>0$. Furthermore, we have

$$\partial_t F_\epsilon(t,\tau) = \sqrt{\epsilon}\left(\partial_\delta J_{h_\epsilon}(U_{\delta_\epsilon(t),\xi_\epsilon(\tau)} + \Phi_{\delta_\epsilon(t),\xi_\epsilon(\tau)})\right)$$
$$\text{and } \partial_{\tau_i} F_\epsilon(t,\tau) = \sqrt{\epsilon}\left(\partial_{\xi_i} J_{h_\epsilon}(U_{\delta_\epsilon(t),\xi_\epsilon(\tau)} + \Phi_{s\delta_\epsilon(t),\xi_\epsilon(\tau)})\right).$$

Therefore, it follows from (32) and (33) that the limit in (71) actually holds in $C^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^n)$. Assuming that $f(\xi_0)\times K_{h_0}(\xi_0)>0$, we then obtain $t_0>0$ such that $(t_0,0)$ is a nondegenerate critical point of E_0. Then, there exists a critical point $(t_\epsilon,\tau_\epsilon)$ of F_ϵ such that $(t_\epsilon,\tau_\epsilon)\to(t_0,0)$ as $\epsilon\to 0$. Then (21) yields

$$u_\epsilon := U_{\delta_\epsilon(t_\epsilon),\xi_\epsilon(\tau_\epsilon)} + \Phi_{h_\epsilon,0,\delta_\epsilon(t_\epsilon),\xi_\epsilon(\tau_\epsilon)}$$

is a solution to (8) satisfying the conclusion of Theorem 1.4 when $n\geq 7$. In the case where $n=6$, the proof is similar by choosing δ_ϵ such that $\delta_\epsilon^2\ln(1/\delta_\epsilon)=\epsilon$. □

8.2. **Proof of Theorem 1.4 for $n \in \{4,5\}$.** When $n \in \{4,5\}$, we define

$$F_\epsilon(t,\tau) := J_{h_\epsilon}(U_{\delta_\epsilon(t),\xi_\epsilon(\tau)} + B_{h_\epsilon,\delta_\epsilon(t),\xi_\epsilon(\tau)} + \Phi_{h_\epsilon,0,\delta_\epsilon(t),\xi_\epsilon(\tau)}),$$

where $\delta_\epsilon(t)$ will be chosen differently depending on the dimension.

Case $n = 5$. In this case, we set $\delta_\epsilon(t) := t\epsilon$. It follows from (60) that

$$\lim_{\epsilon\to 0} \frac{F_\epsilon(t,\tau) - \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx}{\epsilon^3} = E_0(t,\xi) \text{ in } C^0_{\text{loc}}((0,\infty)\times\mathbb{R}^n),$$

$$\text{where } E_0(t,\tau) := C_5\left(\frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)(\tau,\tau) + f(\xi_0)\right)t^2 - \frac{k_5^2}{2}m_{h_0}(\xi_0)t^3.$$

It follows from the C^1-estimates of Proposition 7.5 that the convergence holds in $C^1_{\text{loc}}((0,\infty)\times\mathbb{R}^n)$. Assuming that $f(\xi_0)\times m_{h_0}(\xi_0) > 0$, we conclude as when $n \geq 7$.

Case $n = 4$. We set $\delta_\epsilon(t) := e^{-t/\epsilon}$. The C^1-estimates of Proposition 7.5 yield

$$\lim_{\epsilon\to 0}\left(-\epsilon\delta_\epsilon(t)^{-2}\partial_t F_\epsilon(t,\tau), \delta_\epsilon(t)^{-2}\partial_\tau F_\epsilon(t,\tau)\right) = (\psi_0(t,\tau),\psi_1(t,\tau))$$

in $C^0_{\text{loc}}((0,\infty)\times\mathbb{R}^n)$, where

$$\psi_0(t,\tau) := C_4\left(\frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)(\tau,\tau) + f(\xi_0)\right)t - \frac{n-2}{2}k_n^2 m_{h_0}(\xi_0)$$

$$\text{and } \psi_1(t,\tau) := \frac{1}{2}C_4\nabla^2\varphi_{h_0}(\xi_0)[\tau,\cdot]t.$$

Arguing as in Esposito–Pistoia–Vétois [10], we then obtain the existence of a critical point of J_{h_ϵ} which satisfies the conclusion of Theorem 1.4. □

9. Proof of Theorem 1.5

We let $h_0, f \in C^p(M)$, $p \geq 2$, $u_0 \in C^2(M)$ and $\xi_0 \in M$ satisfy the assumptions of Theorem 1.5. We let h_ϵ be as in (8). We let $\xi_\epsilon(\tau)$ and $\delta_\epsilon(t)$ be as in (69) and (70). Since u_0 is nondegenerate, there exists $(u_{0,\epsilon})_{\epsilon>0} \in C^2(M)$ such that

$$\Delta_g u_{0,\epsilon} + h_\epsilon u_{0,\epsilon} = u_{0,\epsilon}^{2^\star-1},\ u_{0,\epsilon} > 0 \text{ in } M \text{ for small } \epsilon > 0 \tag{72}$$

and then $\|u_{0,\epsilon} - u_0\|_{C^2} \leq C\epsilon$. We let $0 < a < b$, $R > 0$ to be fixed later. We define

$$F_\epsilon(t,\tau) := J_{h_\epsilon}(u_{0,\epsilon} + U_{\delta_\epsilon(t),\xi_\epsilon(\tau)} + \Phi_{h_\epsilon,u_{0,\epsilon},\delta_\epsilon(t),\xi_\epsilon(\tau)})$$

for all $(t,\tau) \in (a,b)\times\mathbb{R}^n$ such that $|\tau| < R$. With (37), we obtain for $n \geq 7$,

$$F_\epsilon(t,\tau) = J_{h_\epsilon}(u_{0,\epsilon}) + \frac{1}{n}\int_{\mathbb{R}^n} U_{1,0}^{2^\star}\,dx + C_n\left(\frac{1}{2}\nabla^2\varphi_{h_0}(\xi_0)(\tau,\tau) + f(\xi_0)\right)t^2\epsilon\delta_\epsilon^2$$
$$+ \text{o}(\epsilon\delta_\epsilon^2) - D_n K_{h_0}(\xi_0)t^4\delta_\epsilon^4 + \text{o}(\delta_\epsilon^4) - B_n u_0(\xi_0)t^{\frac{n-2}{2}}\delta_\epsilon^{\frac{n-2}{2}} + \text{o}(\delta_\epsilon^{\frac{n-2}{2}})$$

as $\epsilon \to 0$ uniformly in $a < t < b$ and $|\tau| < R$, where $B_n, C_n, D_n > 0$. We set $\delta_\epsilon := \epsilon^{\frac{2}{n-6}}$ if $7 \leq n \leq 10$ and $\delta_\epsilon := \sqrt{\epsilon}$ if $n \geq 11$ and then argue as in the proof of Theorem 1.4. In the case where $n = 6$, remarking that $2^\star - 1 = 2$, differentiating (72) in ϵ and using the nondegeneracy of u_0, we obtain

$$(\partial_\epsilon u_{0,\epsilon})_{|0} = -(\Delta_g + h_0 - 2u_0)^{-1}(f u_0).$$

It follows that $\varphi_{h_\epsilon,u_\epsilon} = h_\epsilon - 2u_{0,\epsilon} - c_n\,\text{Scal}_g = \varphi_{h_0,u_0} + \tilde{f}\epsilon + \text{o}(\epsilon)$ as $\epsilon \to 0$, where $\tilde{f} := f + 2(\Delta_g + h_0 - 2u_0)^{-1}(f u_0)$. We then conclude as in the previous cases. □

10. Proofs of Theorems 1.2 and 1.3

10.1. **Proof of Theorem 1.2.** We let $h_0 \in C^p(M)$, $1 \leq p \leq \infty$, and $\xi_0 \in M$ be such that $\Delta_g + h_0$ is coercive and (4) is satisfied. We first easily construct a suitable approximation $(\tilde{h}_\epsilon)_{\epsilon>0} \in C^{\max\{2,p\}}(M)$ such that $\tilde{h}_\epsilon \to h_0$ in $C^p(M)$ as $\epsilon \to 0$, $\varphi_{\tilde{h}_\epsilon}(\xi_0) = \varphi_{h_0}(\xi_0) = 0$, $|\nabla\varphi_{\tilde{h}_\epsilon}(\xi_0)| = |\nabla\varphi_{h_0}(\xi_0)| = 0$ and for small $\epsilon > 0$, ξ_0 is a nondegenerate critical point of $\varphi_{\tilde{h}_\epsilon}$. Using Proposition 6.1, we can assume moreover that for small $\epsilon > 0$, $K_{\tilde{h}_\epsilon}(\xi_0) \neq 0$ and the sign of $K_{\tilde{h}_\epsilon}(\xi_0)$ is independent of ϵ. We fix $f_0 \in C^\infty(M)$ such that $f_0(\xi_0) \times K_{\tilde{h}_\epsilon}(\xi_0) > 0$ for small $\epsilon > 0$. It then follows from Theorem 1.4 that there exist $\alpha_\epsilon > 0$ and $(\tilde{u}_{\epsilon,\alpha})_{0<\alpha<\alpha_\epsilon}$ such that

$$\Delta_g \tilde{u}_{\epsilon,\alpha} + (\tilde{h}_\epsilon + \alpha f_0)\tilde{u}_{\epsilon,\alpha} = \tilde{u}_{\epsilon,\alpha}^{2^\star-1},\ \tilde{u}_{\epsilon,\alpha} > 0 \text{ in } M,$$

$\tilde{u}_{\epsilon,\alpha} \rightharpoonup 0$ weakly in $L^{2^\star}(M)$ and $(\tilde{u}_{\epsilon,\alpha})_\alpha$ blows up with one bubble at ξ_0 as $\alpha \to 0$. A diagonal argument then yields $(h_\epsilon)_\epsilon$ and $(u_\epsilon)_\epsilon$ such that Theorem 1.2 holds. □

10.2. **Proof of Theorem 1.3.** We let $h_0 \in C^p(M)$, $1 \leq p \leq \infty$, such that $\Delta_g + h_0$ is coercive, $u_0 \in C^2(M)$, $u_0 > 0$ and $\xi_0 \in M$ such that (1) and (5) are satisfied.

Lemma 10.1. *There exists a neighborhood Ω_0 of ξ_0 and families $(\tilde{h}_\epsilon)_{\epsilon>0} \in C^p(M)$ and $(\tilde{u}_\epsilon)_{\epsilon>0} \in C^2(M)$ such that $\tilde{h}_\epsilon \to h_0$ in $C^p(M)$, $\tilde{u}_\epsilon \to u_0$ in $C^2(M)$, $\tilde{h}_\epsilon \equiv h_0$ and $\tilde{u}_\epsilon \equiv u_0$ in Ω_0 and $\tilde{u}_\epsilon$ is a nondegenerate solution of*

$$\Delta_g \tilde{u}_\epsilon + \tilde{h}_\epsilon \tilde{u}_\epsilon = \tilde{u}_\epsilon^{2^\star-1},\ \tilde{u}_\epsilon > 0 \text{ in } M \text{ for all } k \in \mathbb{N}.$$

Proof of Lemma 10.1. For all $v \in C^{p+2}(M)$ such that $v > -u_0$, we define $u(v) := u_0 + v$ and $h(v)$ such that

$$\Delta_g u(v) + h(v)u(v) = u(v)^{2^\star-1} \text{ in } M.$$

By elliptic regularity, we have $h(v) \to h_0$ in $C^p(M)$ and $u(v) \to u_0$ in $C^2(M)$ as $v \to 0$ in $C^{p+2}(M)$. We assume by contradiction that for every neighborhood Ω of ξ_0, there exists a neighborhood V_Ω of 0 in $C^{p+2}(M)$ such that for every $v \in V_\Omega$, if $v \equiv 0$ in Ω, then $u(v)$ is degenerate i.e. there exists $\phi(v) \in K_v \setminus \{0\}$, where

$$K_v := \{\phi \in H_1^2(M) :\ \Delta_g\phi + h(v)\phi = (2^\star - 1)u(v)^{2^\star-2}\phi \text{ in } M\}.$$

We can assume that $\phi(v) \in \mathbb{S}_{K_v} := \{\phi \in K_v : \|\phi\|_{H_1^2} = 1\}$. Then there exists $\phi_v \in K_0$ and $(t_k)_{k\in\mathbb{N}} > 0$ such that $t_k \to 0$ and $\phi(t_k v) \to \phi_v$ strongly in $C^1(M)$, so $\phi_v \in \mathbb{S}_{K_0}$. We then define $\psi_k(v) := t_k^{-1}(\phi(t_k v) - \phi_v)$. As one checks,

$$\Delta_g\psi_k(v) + h_0\psi_k(v) = (2^\star - 1)u_0^{2^\star-2}\psi_k(v) + f_k(v)\phi(t_k v) \text{ in } M \tag{73}$$

for a suitable sequence $(f_k)_k$ satisfying $f_k(v) = u_0^{-1}L_0(v) + \mathrm{o}(1)$ as $k \to +\infty$, where

$$L_0(v) := \Delta_g v + h_0 v - (1 - (2^\star - 2)^2)u_0^{2^\star-2}v.$$

We then obtain that there exists $\psi_v \in K_0^\perp$ such that $\Pi_{K_0^\perp}(\psi_k(v)) \rightharpoonup \psi_v$ weakly in $H_1^2(M)$, where $\Pi_{K_0^\perp}$ is the projection onto $K_0^\perp$. Passing to the limit in (73) yields

$$\Delta_g\psi_v + h_0\psi_v = (2^\star - 1)u_0^{2^\star-2}\psi_v + u_0^{-1}L_0(v)\phi_v \text{ in } M.$$

Since $\phi_v \in K_0$, multiplying this equation by ϕ_v and integrating by parts yields

$$\int_M u_0^{-1}L_0(v)\phi_v^2\,dv_g = 0. \tag{74}$$

We now construct v contradicting (74). For every $\epsilon > 0$, we choose $\Omega := B_\epsilon(\xi_0)$ and we consider the neighborhood $V_{B_\epsilon(\xi_0)}$ of 0 in $C^{p+2}(M)$ and $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(t) = 0$ for $t \leq 1$ and $\chi(t) = 1$ for $t \geq 2$. We define

$$v_\epsilon(x) := C_\epsilon \chi(d_g(x, \xi_0)/\epsilon) u_0(x) \text{ for all } x \in M \text{ and } \epsilon > 0.$$

For $C_\epsilon \to 0$ suitably chosen, $v_\epsilon \equiv 0$ in $B_\epsilon(\xi_0)$ and $v_\epsilon \in V_{B_\epsilon(\xi_0)}$, so the above analysis applies. Up to a subsequence, we then obtain the existence of $\phi_0 \in K_0$ such that

$$\lim_{\epsilon \to 0} \phi_{v_\epsilon} = \phi_0 \neq 0 \text{ in } C^2(M).$$

Applying (74) to ϕ_{v_ϵ}, integrating by parts and passing to the limit then yields a contradiction since $C_\epsilon^{-1} v_\epsilon \to u_0$ in $L^2(M)$. This ends the proof of Lemma 10.1. □

We can now end the proof of Theorem 1.3. Letting Ω_0, $(\tilde{h}_\epsilon)_{\epsilon>0}$ and $(\tilde{u}_\epsilon)_{\epsilon>0}$ be given by Lemma 10.1, we have $\varphi_{\tilde{h}_\epsilon, \tilde{u}_\epsilon} \equiv \varphi_{h_0, u_0}$ in Ω_0 and so $\varphi_{\tilde{h}_\epsilon, \tilde{u}_\epsilon}(\xi_0) = |\nabla \varphi_{\tilde{h}_\epsilon, \tilde{u}_\epsilon}(\xi_0)| = 0$. Theorem 1.3 then follows by mimicking the proof of Theorem 1.2.

11. Examples of h_0 and u_0 satisfying the assumptions of Theorem 1.3.

Proposition 11.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. Then there exists $\epsilon_0 \geq 0$ depending only on n and (M, g) such that $\epsilon_0 > 0$ if (M, g) is not conformally diffeomorphic to the standard sphere, $\epsilon_0 = 0$ otherwise, and for every $\varphi_0 \in C^p(M)$, $1 \leq p \leq \infty$, satisfying $\varphi_0(\xi_0) = |\nabla \varphi_0(\xi_0)| = 0$ and*

$$\begin{cases} \varphi_0 \leq \epsilon_0 \text{ and } \lambda_1(\Delta_g + \varphi_0 + c_n \operatorname{Scal}_g) > 0 & \text{if } n \neq 6 \\ \lambda_1(\Delta_g + \varphi_0 + c_n \operatorname{Scal}_g) < 0 & \text{if } n = 6, \end{cases}$$

there exists a solution $u_0 \in C^2(M)$ of the equation (1) which satisfies (5) with $h_0 := \varphi_0 + c_n \operatorname{Scal}_g$ if $n \neq 6$ and $h_0 := \varphi_0 + c_n \operatorname{Scal}_g + 2u_0$ if $n = 6$.

Proof of Proposition 11.1: Since (M, g) is aspherical, its Yamabe quotient is below the quotient of the round sphere. This property persists when adding a small perturbation φ_0. It is standard that this property yields the existence of a solution to the problem, which proves the proposition when $n \neq 6$. When $n = 6$, since $2^\star - 1 = 2$, we can rewrite the equation (1) as $\Delta_g u + (h_0 - 2u)u = -u^2$. A classical variational method then yields the existence of a solution to (1). This ends the proof of Proposition 11.1. □

12. Necessity of the condition on the gradient

Theorem 12.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 4$. Let $h_0 \in C^1(M)$ be such that $\Delta_g + h_0$ is coercive. Assume that there exist $(h_\epsilon)_{\epsilon>0} \in C^p(M)$, $(u_\epsilon)_{\epsilon>0} \in C^2(M)$ satisfying (2) and such that $h_\epsilon \to h_0$ strongly in $C^1(M)$. Assume that (M, g) is locally conformally flat. If $(u_\epsilon)_\epsilon$ blows up with one bubble at some point $\xi_0 \in M$ and $u_\epsilon \rightharpoonup 0$ weakly as $\epsilon \to 0$, then (4) holds true.*

Proof of Theorem 12.1. Theorem 1.1 yields $\varphi_{h_0}(\xi_0) = 0$. With the conformal flatness, we can assume that $\xi_0 \in \mathbb{R}^n$ and there exists $(\hat{u}_\epsilon)_\epsilon \in C^2(B_2(\xi_0))$ such that

$$\Delta_{\text{Eucl}} \hat{u}_\epsilon + \hat{h}_\epsilon \hat{u}_\epsilon = \hat{u}_\epsilon^{2^\star - 1}, \; \hat{u}_\epsilon > 0 \text{ in } B_2(\xi_0) \subset \mathbb{R}^n, \; \hat{h}_\epsilon := (h_\epsilon - c_n \operatorname{Scal}_g) \Lambda^{2-2^\star}$$

for some function $\Lambda > 0$. It follows from [9] that for some constant $C > 0$ we have

$$\frac{1}{C} \left(\frac{\delta_\epsilon}{\delta_\epsilon^2 + |x - \xi_\epsilon|^2} \right)^{\frac{n-2}{2}} \leq \hat{u}_\epsilon(x) \leq C \left(\frac{\delta_\epsilon}{\delta_\epsilon^2 + |x - \xi_\epsilon|^2} \right)^{\frac{n-2}{2}} \text{ for all } x \in M \tag{75}$$

for small $\epsilon > 0$ and $\xi_\epsilon \to \xi_0$. Differentiating the Pohozaev identity on $B_1(\xi_\epsilon)$ yields

$$\frac{1}{2}\int_{B_1(\xi_\epsilon)} \partial_{x_i}\hat{h}_\epsilon \hat{u}_\epsilon^2\, dx$$
$$= \int_{\partial B_1(\xi_\epsilon)} \left(\frac{(x-\xi_\epsilon)_i}{|x-\xi_\epsilon|} \left(\frac{|\nabla \hat{u}_\epsilon|^2 + \hat{h}_\epsilon \hat{u}_\epsilon^2}{2} - \frac{\hat{u}_\epsilon^{2^\star}}{2^\star} \right) - \left\langle \frac{x-\xi_\epsilon}{|x-\xi_\epsilon|}, \nabla \hat{u}_\epsilon \right\rangle \partial_{x_i} \hat{u}_\epsilon \right) d\sigma(x).$$

By standard elliptic theory and (75), we then obtain

$$\int_{B_1(\xi_\epsilon)} \partial_{x_i}\hat{h}_\epsilon \hat{u}_\epsilon^2\, dx = \mathrm{O}(\delta_\epsilon^{n-2}) \text{ as } \epsilon \to 0. \tag{76}$$

Estimating the left-hand side of (76) with (75) then gives $\nabla\varphi_{h_0}(\xi_0) = 0$. This ends the proof of Theorem 12.1. □

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