EXAMPLES OF NON-ISOLATED BLOW-UP FOR PERTURBATIONS OF THE SCALAR CURVATURE EQUATION ON NON LOCALLY CONFORMALLY FLAT MANIFOLDS

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Abstract

Solutions to scalar curvature equations have the property that all possible blow-up points are isolated, at least in low dimensions. This property is commonly used as the first step in the proofs of compactness. We show that this result becomes false for some arbitrarily small, smooth perturbations of the potential.

1. Introduction and statement of the results

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 3$. Given a sequence $(h_\varepsilon)_{\varepsilon>0} \in C^\infty(M)$, we are interested in the existence of multi peaks positive solutions $(u_\varepsilon)_{\varepsilon>0} \in C^\infty(M)$ to the family of critical equations

$$\Delta_g u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^\star-1} \text{ in } M \text{ for all } \varepsilon > 0, \tag{1}$$

where $\Delta_g := -\mathrm{div}_g(\nabla)$ is the Laplace-Beltrami operator, and $2^\star := \frac{2n}{n-2}$ is the critical Sobolev exponent. We say that the family $(u_\varepsilon)_\varepsilon$ blows up as $\varepsilon \to 0$ if $\lim_{\varepsilon\to 0} \|u_\varepsilon\|_\infty = +\infty$. Blowing-up families to equations like (1) are described precisely by Struwe [19] in the energy space $H_1^2(\mathrm{M})$: namely, if the Dirichlet energy of u_ε is uniformly bounded with respect to ε, then there exists $u_0 \in C^\infty(M)$, there exists $k \in \mathbb{N}$, there exists k families $(\xi_{i,\varepsilon})_\varepsilon \in M$ and $(\mu_{i,\varepsilon})_\varepsilon \in (0, +\infty)$ such that

$$u_\varepsilon = u_0 + \sum_{i=1}^k \left(\frac{\sqrt{n(n-2)}\mu_{i,\varepsilon}}{\mu_{i,\varepsilon}^2 + d_g(\cdot, \xi_{i,\varepsilon})^2} \right)^{\frac{n-2}{2}} + o(1), \tag{2}$$

where $\lim_{\varepsilon\to 0} o(1) = 0$ in $H_1^2(M)$ and $\lim_{\varepsilon\to 0} \mu_{i,\varepsilon} = 0$ for all $i = 1, ..., k$. In this situation, we say that u_ε develops k peaks when $\varepsilon \to 0$.

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We say that $\xi_0 \in M$ is a blow-up point for $(u_\varepsilon)_\varepsilon$ if $\lim_{\varepsilon\to 0} \max_{B_r(\xi_0)} u_\varepsilon = +\infty$ for all $r > 0$. It follows from elliptic theory that the blow-up points of a family of solutions $(u_\varepsilon)_\varepsilon$ to (1) satisfying (2) is exactly $\{\lim_{\varepsilon\to 0} \xi_{i,\varepsilon} / \, i = 1, .., k\}$.

Following the terminology introduced by Schoen [17], $\xi_0 \in M$ is an isolated point of blow-up for $(u_\varepsilon)_\varepsilon$ if there exists $(\xi_\varepsilon)_\varepsilon \in M$ such that

- ξ_ε is a local maximum point of u_ε for all $\varepsilon > 0$,
- $\lim_{\varepsilon\to 0} \xi_\varepsilon = \xi_0$,
- there exist $C, \bar r > 0$ s.t. $d_g(x, \xi_\varepsilon)^{\frac{n-2}{2}} u_\varepsilon(x) \leq C$ for all $x \in B_{\bar r}(\xi_0)$,
- $\lim_{\varepsilon\to 0} \max_{B_r(\xi_0)} u_\varepsilon = +\infty$ for all $r > 0$.

The notion has proved to be very useful in the analysis of critical equations. Let $c_n := \frac{n-2}{4(n-1)}$ and R_g be the scalar curvature of (M, g). Compactness for the Yamabe equation

$$\Delta_g u + c_n R_g u = u^{2^\star - 1} \tag{3}$$

when $n \leq 24$ (the full result is due to Kuhri–Marques–Schoen [11]) is established by proving first that the sole possible blow-up points for (3) are isolated, see Schoen [17,18], Li–Zhu [14], Druet [6], Marques [15], Li–Zhang (Theorem 1.1 in [13]), and Kuhri–Marques–Schoen [11]). When $n \geq 25$, there are examples of non-compactness of equation (3) (Brendle [2] and Brendle–Marques [3]).

In this note, we address the questions to know whether or not blow-up solutions for (1) do exist, and whether or not they necessarily have isolated blow-up points. When $h_\varepsilon \leq c_n R_g$, blow-up does not occur for $n \leq 5$ as shown by Druet [6] (except for the conformal class of the round sphere). When the potential is allowed to be above the scalar curvature, blow-up is possible: we refer to Druet–Hebey [7] for examples of non-isolated blow-up on the sphere with C^1-perturbations of the scalar curvature term in (3), and to Esposito–Pistoia–Vétois [10] for examples of isolated blow-up on general compact manifolds with arbitrary smooth perturbations of the scalar curvature. We present in this note examples of non-isolated blow-up points for smooth perturbations of the scalar curvature term in (3). This is the subject of the following theorem.

Theorem 1.1. *Let (M, g) be a non-locally-conformally flat compact Riemannian manifold of dimension $n \geq 6$ with positive Yamabe invariant. We fix $\xi_0 \in M$ such that the Weyl tensor at ξ_0 is such that $Weyl_g(\xi_0) \neq 0$. We let $k \geq 1$ and $r \geq 0$ be two integers. Then there exists $(h_\varepsilon)_{\varepsilon>0} \in C^\infty(M)$ such that $\lim_{\varepsilon\to 0} h_\varepsilon = c_n R_g$ in $C^r(M)$, and there exists $(u_\varepsilon)_{\varepsilon>0} \in C^\infty(M)$ a family of solutions to*

$$\Delta_g u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^\star - 1} \text{ in } M \text{ for all } \varepsilon > 0,$$

such that $(u_\varepsilon)_\varepsilon$ develops k peaks at the blow-up point ξ_0. Moreover, ξ_0 is an isolated blow-up point if and only if $k = 1$.

In particular, Theorem 1.1 applies for $M := \mathbb{S}^p \times \mathbb{S}^q$ $(p, q \geq 3)$ endowed with the product metric. In this case, any point can be a blow-up point since the Weyl tensor never vanishes on $\mathbb{S}^p \times \mathbb{S}^q$.
As a consequence, when dealing with general perturbed equations like (1), one has to deal with the delicate situation of the accumulation of peaks at a single point. The C^0-theory by Druet–Hebey–Robert [9] addresses this question in the a priori setting and L^∞-norm. We refer also to Druet [5] and Druet–Hebey [8] where the analysis of the radii of interaction of multi peaks solutions is performed.

The choice of this note is to perturb the potential $c_n R_g$ of the equation. Alternatively, one can fix the potential $c_n R_g$ and multiply the nonlinearity $u^{2^\star - 1}$ by smooth functions then leading to consider Kazdan-Warner type equations: in this slightly different context, Chen–Lin [4] and Brendle (private communication) have constructed non-isolated local blow-up respectively in the flat case and in the Riemannian case.

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2. Proofs

The proof of Theorem 1.1 relies on a Lyapunov-Schmidt reduction. We fix $\xi_0 \in M$ such that $\mathrm{Weyl}_g(\xi_0) \neq 0$. It follows from the classical conformal normal coordinates theorem of Lee–Parker [12] that there exists $\Lambda \in C^\infty(M \times M)$ such that for any $\xi \in M$,

$$R_{g_\xi}(\xi) = 0,\ \nabla R_{g_\xi}(\xi) = 0,\ \text{and}\ \Delta_{g_\xi} R_{g_\xi}(\xi) = \frac{1}{6}\left|\mathrm{Weyl}_g(\xi)\right|_g^2,$$

where $\Lambda_\xi := \Lambda(\xi, \cdot)$ and $g_\xi := \Lambda_\xi^{4/(n-2)} g$. Without loss of generality, up to a conformal change of metric, we assume that $g_{\xi_0} = g$. We let $r_0 > 0$ be such that $r_0 < i_{g_\xi}(M)$ for all $\xi \in M$ compact, where $i_{g_\xi}(M)$ is the injectivity radius of M with respect to the metric g_ξ. We let $\chi \in C^\infty(\mathbb{R})$ be such that $\chi(t) = 1$ for $t \leq r_0/2$ and $\chi(t) = 0$ for $t \geq r_0$. We define a bubble centered at ξ with parameter δ as:

$$W_{\delta,\xi} := \chi(d_g(\cdot,\xi))\Lambda_\xi \left(\frac{\sqrt{n(n-2)}\delta}{\delta^2 + d_{g_\xi}(\cdot,\xi)^2}\right)^{\frac{n-2}{2}}.$$

We fix an integer $k \geq 1$. Given $\alpha > 1$ and $K > 0$, we define the set

$$\mathcal{D}^{(k)}_{\alpha,K}(\delta) := \Big\{ ((\delta_i)_i, (\xi_i)_i) \in (0,\delta)^k \times M^k \Big/ \frac{1}{\alpha} < \frac{\delta_i}{\delta_j} < \alpha\ ;\ \frac{d_g(\xi_i,\xi_j)^2}{\delta_i \delta_j} > K \text{ for } i \neq j \Big\}.$$

For any $h \in C^0(M)$, we define the functional:

$$J_h(u) := \frac{1}{2}\int_M (|\nabla u|_g^2 + hu^2)\,dv_g - \frac{1}{2^\star}\int_M u_+^{2^\star}\,dv_g$$

for all $u \in H_1^2(M)$. For $((\delta_i)_i, (\xi_i)_i) \in \mathcal{D}^{(k)}_{\alpha,K}$, we define the error

$$R_{(\delta_i)_i,(\xi_i)} := \big\|(\Delta_g + h)\big(\textstyle\sum_{i=1}^k W_{\delta_i,\xi_i}\big) - \big(\sum_{i=1}^k W_{\delta_i,\xi_i}\big)^{2^\star-1}\big\|_{\frac{2n}{n+2}}$$

The classical Lyapunov-Schmidt finite-dimensional reduction yields the following:

Proposition 2.1. *We fix $\alpha > 1$, $\eta > 0$, $C_0 > 0$ such that $\|h\|_\infty \le C_0$ and $\lambda_1(\Delta_g + h) \ge C_0^{-1}$. Then there exists $K_0 = K_0((M,g),\alpha,C_0,\eta) > 0$, $\delta_0 = \delta_0((M,g),\alpha,C_0,\eta) > 0$ and $\phi \in C^1(\mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0), H_1^2(M))$ such that*

- *$R_{(\delta_i)_i,(\xi_i)} < \eta$ for all $(\delta_i)_i, (\xi_i) \in \mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$,*
- *$u((\delta_i)_i,(\xi_i)) := \sum_{i=1}^k W_{\delta_i,\xi_i} + \phi((\delta_i)_i,(\xi_i))$ is a critical point of J_h iff $((\delta_i)_i,(\xi_i))$ is a critical point of $((\delta_i)_i,(\xi_i)) \mapsto J_h(u((\delta_i)_i,(\xi_i)))$ in $\mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$,*
- *$\|\phi((\delta_i)_i,(\xi_i))\|_{H_1^2} = O(R_{(\delta_i)_i,(\xi_i)})$,*
- *$J_h(u((\delta_i)_i,(\xi_i)_i)) = J_h(\sum_{i=1}^k W_{\delta_i,\xi_i}) + O(R^2_{(\delta_i)_i,(\xi_i)})$.*

Here, $|O(1)| \le C((M,g),\alpha,C_0)$ uniformly in $\mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$.

This result is essentially contained in the existing litterature. It is a particular case of the general reduction theorem in Robert–Vétois [16]. We also refer to Esposito–Pistoia–Vétois [10] and to the general framework by Ambrosetti–Badiale [1] for nondegenerate critical manifolds.

From now on, we fix $((\delta_i)_i,(\xi_i)_i) \in \mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$. Standard computations yield

$$J_h\Big(\sum_{i=1}^k W_{\delta_i,\xi_i}\Big) = \sum_{i=1}^k J_h(W_{\delta_i,\xi_i}) + \Big(\sum_{i\neq j}\int_M (\nabla W_{\delta_i,\xi_i}, \nabla W_{\delta_j,\xi_j})_g + hW_{\delta_i,\xi_i}W_{\delta_j,\xi_j}\,dv_g\Big) - \frac{1}{2^\star}\int_M \Big(\Big(\sum_{i=1}^k W_{\delta_i,\xi_i}\Big)^{2^\star} - \sum_{i=1}^k W_{\delta_i,\xi_i}^{2^\star}\Big)\,dv_g$$

and

$$\int_M \Big(\Big(\sum_{i=1}^k W_{\delta_i,\xi_i}\Big)^{2^\star} - \sum_{i=1}^k W_{\delta_i,\xi_i}^{2^\star}\Big)\,dv_g = O\Big(\sum_{i\neq j}\int_{W_{\delta_i,\xi_i}\le W_{\delta_j,\xi_j}} W_{\delta_i,\xi_i}W_{\delta_j,\xi_j}^{2^\star-1}\,dv_g\Big).$$

Choosing K_0 larger if necessary, there exists $c_1 = c_1(\alpha,K_0) > 0$ such that for any $i \neq j$ and $x \in M$ such that $W_{\delta_i,\xi_i}(x) \le W_{\delta_j,\xi_j}(x)$, we

have that $d_{g_{\xi_i}}(x,\xi_i) \geq c_1(d_g(\xi_i,\xi_j)+d_g(x,\xi_j))$. Therefore, we get that $W_{\delta_i,\xi_i}(x) \leq c_2\delta_j^{(n-2)/2} d_g(\xi_i,\xi_j)^{2-n}$ for all such x, for some constant $c_2 = c_2(\alpha,K_0) > 0$. Consequently, a rough upper bound yields

$$J_h\Big(\sum_{i=1}^k W_{\delta_i,\xi_i}\Big) = \sum_{i=1}^k J_h(W_{\delta_i,\xi_i}) + O\Big(\sum_{i\neq j}\Big(\frac{\delta_i\delta_j}{d_g(\xi_i,\xi_j)^2}\Big)^{\frac{n-2}{2}}\Big)$$

and

$$R_{(\delta_i)_i,(\xi_i)} \leq \sum_{i=1}^k \|(\Delta_g+h)W_{\delta_i,\xi_i} - W_{\delta_i,\xi_i}^{2^\star-1}\|_{\frac{2n}{n+2}} + O\Big(\sum_{i\neq j}\Big(\frac{\delta_i\delta_j}{d_g(\xi_i,\xi_j)^2}\Big)^{\frac{n-2}{4}}\Big)$$

uniformly in $\mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$. Moreover, see Proposition 2.3 in Esposito–Pistoia–Vétois [10], we have that

$$\begin{aligned} J_h(W_{\delta,\xi}) = \frac{K_n^{-n}}{n}\Bigg(1 &+ \frac{2(n-1)}{(n-2)(n-4)}(h-c_nR_g)(\xi)\delta^2 \\ &+ O(\|h-c_nR_g\|_{C^1})\delta^3 \\ &- |Weyl_g(\xi)|_g^2\begin{cases} \frac{1}{64}\delta^4\ln\frac{1}{\delta} + O(\delta^4) & \text{when } n=6 \\ \frac{1}{24(n-4)(n-6)}\delta^4 + O(\delta^5) & \text{when } n\geq 7 \end{cases}\Bigg) \end{aligned}$$

and

$$\begin{aligned} &\|(\Delta_g+h)W_{\delta,\xi} - W_{\delta,\xi}^{2^\star-1}\|_{\frac{2n}{n+2}} \\ &\qquad \leq C\delta^2\begin{cases} 1 + \|h-c_nR_g\|_{C^0}\left(\ln\frac{1}{\delta}\right)^{2/3} & \text{when } n=6 \\ \sqrt{\delta} + \|h-c_nR_g\|_{C^0} & \text{when } n\geq 7. \end{cases} \end{aligned}$$

Here again, $|O(1)| \leq C((M,g),\alpha,C_0)$ uniformly in $\mathcal{D}^{(k)}_{\alpha,K_0}(\delta_0)$.

We now choose the $(\delta_i), (\xi_i)'s$ and the function h. For any $\varepsilon > 0$, we let $\delta_\varepsilon > 0$ be such that

$$\delta_\varepsilon^2 \ln\frac{1}{\delta_\varepsilon} = \varepsilon \text{ when } n=6 \text{ and } \delta_\varepsilon^2 = \varepsilon \text{ when } n\geq 7.$$

We let $H \in C^\infty(\mathbb{R}^n)$ be such that

- $H(x) = -1$ for all $|x| > 2$,
- H admits k distinct strict local maxima at $p_{i,0} \in B_1(0)$ for $i = 1,...,k$,
- $H(p_{i,0}) > 0$ for all $i = 1,...,k$.

We let $\tilde{r} > 0$ be such that for any $i \in \{1,...,k\}$, the maximum of H on $B_{2\tilde{r}}(p_{i,0})$ is achieved exactly at $p_{i,0}$ and such that $|p_{i,0} - p_{j,0}| \geq 3\tilde{r}$ for all $i \neq j$. We let $(\mu_\varepsilon)_\varepsilon \in (0,+\infty)$ be such that $\lim_{\varepsilon\to 0}\mu_\varepsilon = 0$ and

$$(|\ln\varepsilon|)^{-1/4} = o(\mu_\varepsilon) \text{ when } n=6 \text{ and } \varepsilon^{\frac{n-6}{2(n-2)}} = o(\mu_\varepsilon) \text{ when } n\geq 7,$$

where both limits are taken when $\varepsilon \to 0$. As one can check, $\delta_\varepsilon = o(\mu_\varepsilon)$ when $\varepsilon \to 0$. We define

$$h_\varepsilon(x) := c_n R_g(x) + \varepsilon H\left(\mu_\varepsilon^{-1} \exp_{\xi_0}^{-1}(x)\right) \text{ for all } x \in M.$$

Here, the exponential map is taken with respect to the metric g and after assimilation to $\mathbb{R}^n$ of the tangent space at ξ_0: this definition makes sense for $\varepsilon > 0$ small enough. For $(t_i)_i \in (0,+\infty)^k$ and $(p_i)_i \in (\mathbb{R}^n)^k$, we define

$$\tilde{u}_\varepsilon((t_i)_i, (p_i)_i) := u\left((t_i\delta_\varepsilon)_i, (\exp_{\xi_0}(\mu_\varepsilon p_i))_i\right) \text{ with } h \equiv h_\varepsilon.$$

The above estimates and the choice of the parameters yield

$$\lim_{\varepsilon\to 0} \frac{J_{h_\varepsilon}(\tilde{u}_\varepsilon((t_i)_i,(p_i)_i)) - k\frac{K_n^{-n}}{n}}{\varepsilon\delta_\varepsilon^2} = \sum_{i=1}^k F_n(t_i,p_i) \tag{4}$$

in $C^0_{loc}((0,+\infty)^k \times \prod_{i=1}^k B_r(p_{i,0}))$, where

$$F_n(t,p) := \frac{2(n-1)}{(n-2)(n-4)} H(p) t^2 - d_n |Weyl_g(\xi_0)|_g^2 t^4$$

for $(t,p) \in (0,+\infty)\times\mathbb{R}^n$, with $d_6 = \frac{1}{64}$ and $d_n := \frac{1}{24(n-4)(n-6)}$ for $n \geq 7$. As easily checked, up to choosing the $t_i's$ in suitable compact intervals $I_1, ..., I_k$, the right-hand-side of (4) has a unique maximum point in the interior of $\prod_{i=1}^k I_i \times \prod_{i=1}^k B_{\tilde{r}}(p_{i,0})$. As a consequence, for $\varepsilon > 0$ small enough, $J_{h_\varepsilon}(\tilde{u}_\varepsilon((t_i)_i,(p_i)_i))$ admits a critical point, $((t_{i,\varepsilon})_i, (p_{i,\varepsilon})_i) \in (\alpha,\beta)^k \times \prod_{i=1}^k B_{\tilde{r}}(p_{i,0})$ for some $0 < \alpha < \beta$ independent of ε. Defining $\xi_{i,\varepsilon} := \exp_{\xi_0}(\mu_\varepsilon p_{i,\varepsilon})$ for all $i = 1, ..., k$, there exists $c_0 > 0$ such that $d(\xi_{i,\varepsilon}, \xi_{i,\varepsilon}) \geq c_0\mu_\varepsilon$ for all $i \neq j \in \{1, ..., k\}$ and all $\varepsilon > 0$ small enough. Defining $u_\varepsilon := \tilde{u}_\varepsilon((t_{i,\varepsilon})_i, (p_{i,\varepsilon})_i)$, it follows from Proposition 2.1 and the strong maximum principle that

$$\Delta_g u_\varepsilon + h_\varepsilon u_\varepsilon = u_\varepsilon^{2^\star - 1} \text{ in } M$$

for $\varepsilon > 0$ small enough. In addition to the hypotheses above, we require that $\varepsilon = o(\mu_\varepsilon^r)$ when $\varepsilon \to 0$, which yields $\lim_{\varepsilon\to 0} h_\varepsilon = c_n R_g$ in $C^r(M)$.

We prove that $(u_\varepsilon)_\varepsilon$ develops no isolated blow-up point when $k \geq 2$. We argue by contradiction. Moser's iterative scheme yields the convergence of u_ε to 0 in $C^2_{loc}(M \setminus \{\xi_0\})$. We then get that the isolated blow-up point is ξ_0, and thus that there exists $r_1 > 0$ and $(\xi_\varepsilon)_\varepsilon \in M$ such that $\lim_{\varepsilon\to 0} \xi_\varepsilon = \xi_0$ and there exists $C > 0$ such that

$$d_g(x,\xi_\varepsilon)^{\frac{n-2}{2}} u_\varepsilon(x) \leq C \text{ for all } \varepsilon > 0 \text{ and } x \in B_{r_1}(\xi_0). \tag{5}$$

For any $i = 1, .., k$, we recall that $\xi_{i,\varepsilon} := \exp_{\xi_0}(\mu_\varepsilon p_{i,\varepsilon})$ and we define

$$\tilde{u}_{i,\varepsilon}(x) := (\delta_\varepsilon t_{i,\varepsilon})^{\frac{n-2}{2}} u_\varepsilon(\exp_{\xi_{i,\varepsilon}}(\delta_\varepsilon t_{i,\varepsilon} x))$$

for all $|x| < r_0/(2\delta_\varepsilon t_{i,\varepsilon})$. It follows from standard elliptic theory that

$$\lim_{\varepsilon\to 0} \tilde{u}_{i,\varepsilon} = \left(\frac{\sqrt{n(n-2)}}{1+|\cdot|^2}\right)^{\frac{n-2}{2}} \text{ in } C^2_{loc}(\mathbb{R}^n). \tag{6}$$

Moreover, if $\delta_\varepsilon = o(d_g(\xi_{i,\varepsilon}, \xi_\varepsilon))$ when $\varepsilon \to 0$, inequality (5) yields the convergence of $\tilde{u}_{i,\varepsilon}$ to 0 in $C^0_{loc}(\mathbb{R}^n)$: a contradiction to (6). Therefore, $d_g(\xi_\varepsilon, \xi_{i,\varepsilon}) = O(\delta_\varepsilon)$ when $\varepsilon \to 0$ for all $i = 1, ..., k$, and then $d_g(\xi_{i,\varepsilon}, \xi_{j,\varepsilon}) = O(\delta_\varepsilon) = o(\mu_\varepsilon)$ when $\varepsilon \to 0$ for all $i \neq j$. This contradicts the fact that $d_g(\xi_{i,\varepsilon}, \xi_{j,\varepsilon}) \geq c_0\mu_\varepsilon$ when $k \geq 2$. This proves the non-simpleness when $k \geq 2$.

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