SHARP SOLVABILITY CONDITIONS FOR A FOURTH ORDER EQUATION WITH PERTUBATION

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Abstract. Let B be the unit ball of $\mathbb{R}^n$, $n \geq 5$, and $\rho : \mathbb{R} \to \mathbb{R}$ a smooth function. We consider the following critical problem

$$\begin{cases} \Delta^2 u = |u|^{\frac{8}{n-4}} u + \rho(u) & \text{in } B \\ u \not\equiv 0 \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial B. \end{cases}$$

We give sufficient conditions for the existence of solutions to this problem. These conditions are close to be sharp, as we prove by considering the problem on arbitrary small balls.

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1. Introduction and statement of the results

Let $n \geq 5$. We denote by $B(0,r) \subset \mathbb{R}^n$ the $n-$dimensional ball of radius $r > 0$ and centered at 0. Let $\rho \in C^\infty(\mathbb{R})$ be a smooth function. For $r > 0$, we are interested in finding solutions $u \in C^4(\overline{B}(0,r))$ to the following problem:

$$\begin{cases} \Delta^2 u = |u|^{2^\sharp - 2} u + \rho(u) & \text{in } \overline{B}(0,r) \\ u \not\equiv 0 \\ u = \frac{\partial u}{\partial n} = 0 & \text{on } \partial \overline{B}(0,r). \end{cases} \qquad (E_r)$$

where $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ is the Laplacian with the minus sign convention, $\frac{\partial}{\partial n}$ denotes the normal derivative with respect to the unit outward vector $\vec{n}$, and $2^\sharp = \frac{2n}{n-4}$ is critical from the viewpoint of Sobolev embeddings. More precisely, for $\Omega \subset \mathbb{R}^n$ an open subset, we denote by $H_{2,0}^2(\Omega)$ the standard Sobolev space of second order, that is the completion of $C_c^\infty(\Omega)$, the set of smooth compactly supported functions in Ω, with respect to the norm

$$\|u\|_{H_{2,0}^2(\Omega)} = \sqrt{\int_\Omega (\Delta u)^2 \, dx}.$$

It follows from the Sobolev embedding theorem that $H_{2,0}^2(\Omega)$ is continuously embedded in $L^q(\Omega)$ for $1 \leq q \leq 2^\sharp$, and that this embedding is compact if and only if $1 \leq q < 2^\sharp$. This lack of compactness is one of the main difficulties attached to problem (E_r). Moreover, see [Osw], it can be shown that (E_r) has no positive solution if $\rho \equiv 0$.

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This type of problem was first studied by Brézis and Nirenberg. In [BrNi], Brézis and Nirenberg studied the existence of solutions to the elliptic problem

$$\begin{cases} \Delta u = u^{\frac{n+2}{n-2}} + \rho(u) & \text{in } B_1 \\ u > 0 & \text{in } B_1 \\ u = 0 & \text{on } \partial B_1. \end{cases} \qquad (E')$$

Using test-functions arguments and the moutain-pass lemma of Ambrosetti and Rabinowitz [AmRa], they prove that (E') possesses a solution if $\rho(0) = \rho'(0) = 0$ and $\int_0^{+\infty} \rho(s)s^{-\frac{n}{n-2}}\, ds > 0$. Later on, in view of a nonexistence result of Adimurthi and Yadava in the absence of the above condition, Brézis raised the following question: is the preceding condition a necessary and sufficient condition for the existence of positive solutions for (E') when ρ is compactly supported? A first step in answering this question was carried out by Adimurthi and Yadava [AdYa]. When $n \geq 7$, they prove that there is a specific class of functions ρ for which (E') has a solution if and only if $\int_0^{+\infty} \rho(s)s^{-\frac{n}{n-2}}\, ds \geq 0$. Adimurthi, Mancini and Sandeep [AMS] came back to this problem for a fairly general class of functions ρ. They introduced a new set of conditions for the solvability of (E') in higher dimensions. In particular, using blow-up analysis they showed that there exist functions ρ such that $\int_0^{+\infty} \rho(s)s^{-\frac{n}{n-2}}\, ds = 0$, and problem (E') does not have solutions on arbitrary small balls. We refer to [AMS] for more details.

Let us now return to the study of (E_r). There has been considerable interest in higher order operators since the pioneering work of Chang, Gursky and Yang concerning the Paneitz operator on Riemannian manifolds. We refer for instance to [Cha] for a general survey on such operators. We refer also to [EFJ], [PuSe], [VdV] in the Euclidean context, and [DHL], [HeRo] in the Riemannian context.

In this paper we address questions similar to the ones addressed in [AMS], but concerning the bi-harmonic operator. To be more precise we define

$$\begin{aligned} I_1(\rho) &= \int_0^{+\infty} \rho(s)s^{-\frac{n}{n-4}}\, ds\ ,\ \ I_2(\rho) = \int_0^{+\infty} \rho(s)s^{-\frac{n-2}{n-4}}\, ds\ , \\ I_3(\rho) &= \int_0^{+\infty} r^{-\frac{2n-4}{n-4}} \left[\int_0^r t^{\frac{2}{n-4}} \left(\int_t^{+\infty} \rho(s)s^{-\frac{2n-4}{n-4}}\, ds\right) dt\right]^2 dr \\ &+ \frac{(n-4)^4}{4n(n+2)} \int_0^{+\infty} \rho(t)\frac{1}{t}\, dt, \end{aligned} \qquad (1)$$

when these quantities make sense. We say that $u \in C^4(\overline{B}(0,r))$ is a solution of small energy for (E_r) if it is a solution of (E_r) satisfying

$$\frac{1}{2}\int_{B(0,r)} (\Delta u)^2\, dx - \frac{1}{2^\sharp}\int_{B(0,r)} |u|^{2^\sharp}\, dx - \int_{B(0,r)} \tilde{\rho}(u)\, dx < \frac{2}{nK_0^{\frac{n}{4}}},$$

where $\tilde{\rho}(r) = \int_0^r \rho(t)\, dt$ for $r \in \mathbb{R}$, and $K_0 > 0$ is the best constant in the second order Sobolev inequality. Namely

$$\frac{1}{K_0} = inf\, \frac{\int_{\mathbb{R}^n} (\Delta u)^2\, dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\sharp}\, dx\right)^{\frac{2}{2^\sharp}}},$$

where the infimum is taken over the nonzero compactly supported functions in $\mathbb{R}^n$. We assume in what follows that

$$\begin{array}{l}\rho(0)=\rho'(0)=0\ ,\ \rho' \text{ is bounded}\\ \exists b>\frac{2}{n-4} \text{ such that } |\rho(s)|\le C|s|^{-b} \text{ for all } s\neq 0\end{array} \qquad (H_\rho)$$

Our main result is the following:

Theorem 1.1. *Assume that* $n \geq 13$ *and that* (H_ρ) *holds. If* $I_1(\rho) > 0$*, or if* $I_1(\rho) = 0$ *and* $I_2(\rho) < 0$*, or if* $I_1(\rho) = I_2(\rho) = 0$ *and* $I_3(\rho) > 0$*, then* (E_r) *has a radially symmetrical solution of small energy for all* $r > 0$*. Conversely, if* (E_r) *has a radially symmetrical solution of small energy for all* $r > 0$*, then* $I_1(\rho) \geq 0$ *with the additional properties that if* $I_1(\rho) = 0$*, then* $I_2(\rho) \leq 0$*, and if* $I_1(\rho) = I_2(\rho) = 0$*, then* $I_3(\rho) \geq 0$*.*

When we deal with an arbitrary subset of $\mathbb{R}^n$, the existence part still holds, but the solutions are not necessarily radially symmetrical. The paper is divided as follows. Section 2 is devoted to test-functions estimates. We prove the existence part of theorem 1.1 in section 3. Sections 4, 5 are devoted to the blow-up analysis attached to our problem, and to the proof the second part of theorem 1.1. In section 6, we prove a spectral result we need in section 5. Extensions of theorem 1.1 to the case of a smooth open subset of $\mathbb{R}^n$ and to smaller dimensions are discussed at the end of sections 3 and 5.

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2. Test-functions estimates

We consider a function $\rho \in C^\infty(\mathbb{R})$ satisfying the following conditions:

$$\begin{array}{l}\rho(0)=\rho'(0)=0\ ,\ \rho' \text{ is bounded}\\ \exists b>\frac{2}{n-4} \text{ such that } |\rho(s)|\le C|s|^{-b} \text{ for all } s\neq 0\end{array} \qquad (2)$$

We also define $\tilde\rho(r) = \int_0^r \rho(t)\,dt$, $r > 0$. We denote by B the unit ball of $\mathbb{R}^n$, and for $\alpha > 0$, we consider the following functional

$$J_\alpha(u) = \frac{1}{2}\int_B (\Delta u)^2\,dx - \frac{1}{2^\sharp}\int_B |u|^{2^\sharp}\,dx - \alpha^{2^\sharp}\int_B \tilde\rho\left(\frac{u}{\alpha}\right)\,dx,$$

where $u \in H^2_{2,0}(B)$. We define the function $U \in H^2_{2,0}(\mathbb{R}^n)$ by

$$U(x) = \left(\frac{a_n^2}{a_n^2 + |x|^2}\right)^{\frac{n-4}{2}}, \qquad (3)$$

where $x \in \mathbb{R}^n$, and $a_n = \sqrt[4]{n(n-4)(n^2-4)}$. It is easily checked that U verifies $\Delta^2 U = U^{2^\sharp - 1}$. Moreover, U is an extremal for the second order Sobolev inequality

$$\frac{1}{K_0} = \inf_{u\in H^2_{2,0}(\mathbb{R}^n)-\{0\}} \frac{\int_{\mathbb{R}^n} (\Delta u)^2\,dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\sharp}\,dx\right)^{\frac{2}{2^\sharp}}}, \qquad (4)$$

The value of $K_0 > 0$ and the extremals for (4) are explicitely known. They have been computed by Lieb [Lie], Lions [Lio], and Edmunds-Fortunato-Janelli [EFJ]. For any $\varepsilon > 0$, we define

$$U_\varepsilon(x) = \varepsilon^{-\frac{n-4}{2}} U\left(\frac{x}{\varepsilon}\right) = \left(\frac{a_n^2 \varepsilon}{a_n^2 \varepsilon^2 + |x|^2}\right)^{\frac{n-4}{2}}, \tag{5}$$

where $x \in \mathbb{R}^n$, and a_n is as above. Then

$$\Delta^2 U_\varepsilon = U_\varepsilon^{2^\sharp - 1}. \tag{6}$$

From now on, if $R > 0$ and if $h : B(0,R) \to \mathbb{R}$ is a radially symmetrical function, we write $h(r) = h(|x|)$, where $x \in B(0,R)$ and $|x| = r$. For $\alpha, \varepsilon > 0$, we consider the unique radially symmetrical function $v_{\varepsilon,\alpha} \in C^4(\overline{B})$ solution of the problem:

$$\begin{cases} \Delta^2 v_{\varepsilon,\alpha} = \alpha^{2^\sharp - 1} \rho\left(\frac{U_\varepsilon}{\alpha}\right) & \text{in } B \\ v_{\varepsilon,\alpha} + U_\varepsilon = \frac{\partial(v_{\varepsilon,\alpha} + U_\varepsilon)}{\partial n} = 0 & \text{on } \partial B. \end{cases} \tag{7}$$

This function is explicitely known. We have that

$$\begin{aligned} v_{\varepsilon,\alpha}(r) &= -U_\varepsilon(1) - \frac{C_{\varepsilon,\alpha}}{2n}(1 - r^2) \\ &-\alpha^{2^\sharp - 1} \int_r^1 t^{1-n} \left[\int_0^t s^{n-1} \left[\int_0^s u^{1-n} \left\{\int_0^u \rho\left(\frac{U_\varepsilon}{\alpha}\right) v^{n-1}\, dv\right\} du\right] ds\right] dt \end{aligned} \tag{8}$$

where

$$C_{\varepsilon,\alpha} = -n \frac{\partial U_\varepsilon}{\partial n}(1) - n\alpha^{2^\sharp - 1} \int_0^1 s^{n-1} \left[\int_0^s u^{1-n} \left\{\int_0^u \rho\left(\frac{U_\varepsilon}{\alpha}\right) v^{n-1}\, dv\right\} du\right] ds$$

In the sequel, $a_{\varepsilon,\alpha} = O(b_{\varepsilon,\alpha})$ means that there exists $C > 0$ independant of $\varepsilon > 0$ and $\alpha \in (0,1]$ such that $|a_{\varepsilon,\alpha}| \le C|b_{\varepsilon,\alpha}|$. We write $a_{\varepsilon,\alpha} = o(b_{\varepsilon,\alpha})$ if for any $\eta > 0$, there exists $\varepsilon_0 > 0$ such that $|a_{\varepsilon,\alpha}| \le \eta |b_{\varepsilon,\alpha}|$ for all $\varepsilon \in (0, \varepsilon_0)$ and all $\alpha \in (0,1]$. With (2), it follows from (7) that

$$\|v_{\varepsilon,\alpha}\|_{H_2^2(B)} = o(1). \tag{9}$$

Here and in what follows, $H_k^p(\Omega)$ denotes the Sobolev space of functions $u \in L^p(\Omega)$ such that $\nabla^i u \in L^p(\Omega)$ for $i = 1...k$, where Ω is an open subset of $\mathbb{R}^n$.

This section is devoted to finding estimate on

$$J_\alpha(u_{\varepsilon,\alpha}) = \frac{1}{2}\int_B (\Delta u_{\varepsilon,\alpha})^2\, dx - \frac{1}{2^\sharp}\int_B |u_{\varepsilon,\alpha}|^{2^\sharp}\, dx - \alpha^{2^\sharp}\int_B \tilde{\rho}\left(\frac{u_{\varepsilon,\alpha}}{\alpha}\right) dx,$$

where $u_{\varepsilon,\alpha} = U_\varepsilon + v_{\varepsilon,\alpha} + W_{\varepsilon,\alpha}$, and $W_{\varepsilon,\alpha} \in H_{2,0}^2(B)$ is assumed to be such that

$$\|W_{\varepsilon,\alpha}\|_{H_{2,0}^2(B)} = o\left(\varepsilon^{\frac{n-4}{2}}\right) + o\left(\|\Delta v_{\varepsilon,\alpha}\|_2\right) \tag{10}$$

Here and in the sequel, $\|\cdot\|_p$ denotes the L^p−norm for all $p \ge 1$.

Step 1: We first claim that

$$\int_B U_\varepsilon^{2^\sharp - 1}|v_{\varepsilon,\alpha}|\, dx = o\left(\varepsilon^{\frac{n-4}{2}}\right),\ \int_B U_\varepsilon^{2^\sharp - 2} v_{\varepsilon,\alpha}^2\, dx = o\left(\varepsilon^{n-4} + \|\Delta v_{\varepsilon,\alpha}\|_2^2\right) \tag{11}$$

We prove the claim. We let $v_{\varepsilon,\alpha}^+, v_{\varepsilon,\alpha}^- \in H_{2,0}^2(B)\cap C^4(\overline{B})$ be radially symmetrical functions such that

$$\left\{\begin{array}{ll}\Delta^2 v_{\varepsilon,\alpha}^+ = \alpha^{2^\sharp-1}\rho^+\left(\frac{U_\varepsilon}{\alpha}\right) & \text{in } B\\ v_{\varepsilon,\alpha}^+ = \frac{\partial v_{\varepsilon,\alpha}^+}{\partial n} = 0 & \text{on } \partial B\end{array}\right., \left\{\begin{array}{ll}\Delta^2 v_{\varepsilon,\alpha}^- = \alpha^{2^\sharp-1}\rho^-\left(\frac{U_\varepsilon}{\alpha}\right) & \text{in } B\\ v_{\varepsilon,\alpha}^+ = \frac{\partial v_{\varepsilon,\alpha}^+}{\partial n} = 0 & \text{on } \partial B\end{array}\right.,$$

where $\rho^+(s)=\max\{\rho(s),0\}$ and $\rho^-(s)=\max\{-\rho(s),0\}$ for all $s\in\mathbb{R}$. As stated in Boggio [Bog] (see also Grunau-Sweers [GrSw]), the Green's function on the ball for the bi-harmonic operator with Dirichlet boundary condition is positive. It then follows that $v_{\varepsilon,\alpha}^+$ and $v_{\varepsilon,\alpha}^-$ are nonnegative. We define

$$T_\varepsilon(x) = (U_\varepsilon(1) - \frac{1}{2}\frac{\partial U_\varepsilon}{\partial n}(1)) + \frac{1}{2}\frac{\partial U_\varepsilon}{\partial n}(1)|x|^2 \tag{12}$$

for all $x\in B$. Clearly,

$$\Delta^2 T_\varepsilon = 0 \text{ in } B, \quad \text{ and } T_\varepsilon = U_\varepsilon,\ \frac{\partial T_\varepsilon}{\partial n} = \frac{\partial U_\varepsilon}{\partial n} \text{ on } \partial B.$$

Similarly, $T_\varepsilon>0$ and $U_\varepsilon - T_\varepsilon\in H_{2,0}^2(B)$. Now, integrating by parts, we get that

$$\begin{aligned}\int_B U_\varepsilon^{2^\sharp-1} v_{\varepsilon,\alpha}^+\,dx &= \int_B \Delta^2 U_\varepsilon v_{\varepsilon,\alpha}^+\,dx = \int_B \Delta U_\varepsilon \Delta v_{\varepsilon,\alpha}^+\,dx\\ &= \int_B \Delta(U_\varepsilon - T_\varepsilon)\Delta v_{\varepsilon,\alpha}^+\,dx + \int_B \Delta T_\varepsilon \Delta v_{\varepsilon,\alpha}^+\,dx\\ &= \int_B \Delta(U_\varepsilon - T_\varepsilon)\Delta v_{\varepsilon,\alpha}^+\,dx = \int_B (U_\varepsilon - T_\varepsilon)\Delta^2 v_{\varepsilon,\alpha}^+\,dx\\ &= \alpha^{2^\sharp-1}\int_B U_\varepsilon\rho^+\left(\frac{U_\varepsilon}{\alpha}\right)\,dx + O\left(\varepsilon^{\frac{n-4}{2}}\alpha^{2^\sharp-1}\int_B \rho^+\left(\frac{U_\varepsilon}{\alpha}\right)\,dx\right)\end{aligned}$$

With (2), it comes that for all $\nu\in(0,1)$, there exists $C_\nu>0$ such that $\rho^+(s)\le C_\nu s^\nu$ for all $s>0$, and we get that

$$\int_B U_\varepsilon^{2^\sharp-1} v_{\varepsilon,\alpha}^+\,dx = o(\varepsilon^{\frac{n-4}{2}}).$$

Similarly,

$$\int_B U_\varepsilon^{2^\sharp-1} v_{\varepsilon,\alpha}^-\,dx = o(\varepsilon^{\frac{n-4}{2}}).$$

Now, using that $v_{\varepsilon,\alpha} = v_{\varepsilon,\alpha}^+ - v_{\varepsilon,\alpha}^- - T_\varepsilon$, and that $v_{\varepsilon,\alpha}^+, v_{\varepsilon,\alpha}^-, T_\varepsilon\ge 0$, we get that

$$\int_B U_\varepsilon^{2^\sharp-1}|v_{\varepsilon,\alpha}|\,dx = o(\varepsilon^{\frac{n-4}{2}}).$$

This proves the first equation in (11). Now, with Hölder's and Young's inequalities, we get

$$\begin{aligned}\int_B U_\varepsilon^{2^\sharp-2} v_{\varepsilon,\alpha}^2\,dx &\le \left(\int_B U_\varepsilon^{2^\sharp-1}|v_{\varepsilon,\alpha}|\,dx\right)^{\frac{2^\sharp-2}{2^\sharp-1}}\left(\int_B |v_{\varepsilon,\alpha}|^{2^\sharp}\,dx\right)^{\frac{1}{2^\sharp-1}}\\ &= o\left(\varepsilon^{n-4} + \|v_{\varepsilon,\alpha}\|_{2^\sharp}^2\right)\end{aligned}$$

Now, writing $v_{\varepsilon,\alpha} = -T_\varepsilon + (v_{\varepsilon,\alpha}+T_\varepsilon)$ and noting that $v_{\varepsilon,\alpha}+T_\varepsilon\in H_{2,0}^2(B)$, it comes with Sobolev's inequality that

$$\|v_{\varepsilon,\alpha}\|_{2^\sharp} = O\left(\|\Delta v_{\varepsilon,\alpha}\|_2 + \varepsilon^{\frac{n-4}{2}}\right), \tag{13}$$

and then,

$$\int_B U_\varepsilon^{2^\sharp-2}v_{\varepsilon,\alpha}^2\,dx = o\left(\|\Delta v_{\varepsilon,\alpha}\|_2^2+\varepsilon^{n-4}\right)$$

This proves (11) and our claim.

Step 2: We now estimate

$$A_{\varepsilon,\alpha}=\frac{1}{2}\int_B(\Delta u_{\varepsilon,\alpha})^2\,dx-\frac{1}{2^\sharp}\int_B|u_{\varepsilon,\alpha}|^{2^\sharp}\,dx.$$

Since $u_{\varepsilon,\alpha}=U_\varepsilon+v_{\varepsilon,\alpha}+W_{\varepsilon,\alpha}$, we get

$$\int_B(\Delta u_{\varepsilon,\alpha})^2\,dx=\int_B(\Delta U_\varepsilon)^2\,dx+\int_B(\Delta v_{\varepsilon,\alpha})^2\,dx+2\int_B\Delta U_\varepsilon\Delta v_{\varepsilon,\alpha}\,dx$$
$$+\int_B(\Delta W_{\varepsilon,\alpha})^2\,dx+2\int_B\Delta(U_\varepsilon+v_{\varepsilon,\alpha})\Delta W_{\varepsilon,\alpha}\,dx$$

Thanks to Green's formula,

$$\int_B U_\varepsilon^{2^\sharp-1}v_{\varepsilon,\alpha}\,dx=\int_B\Delta^2U_\varepsilon v_{\varepsilon,\alpha}\,dx$$
$$=\int_B\Delta U_\varepsilon\Delta v_{\varepsilon,\alpha}\,dx+\int_{\partial B}\left(\Delta U_\varepsilon\frac{\partial v_{\varepsilon,\alpha}}{\partial n}-v_{\varepsilon,\alpha}\frac{\partial\Delta U_\varepsilon}{\partial n}\right)d\sigma$$
$$=\int_B\Delta U_\varepsilon\Delta v_{\varepsilon,\alpha}\,dx-\int_{\partial B}\left(\Delta U_\varepsilon\frac{\partial U_\varepsilon}{\partial n}-U_\varepsilon\frac{\partial\Delta U_\varepsilon}{\partial n}\right)d\sigma$$

with (7). Now with (5), we get that

$$\int_B\Delta U_\varepsilon\Delta v_{\varepsilon,\alpha}\,dx=\int_B U_\varepsilon^{2^\sharp-1}v_{\varepsilon,\alpha}\,dx+c_n\varepsilon^{n-4}+o(\varepsilon^{n-4})\tag{14}$$

with $c_n=4(n-4)\omega_{n-1}a_n^{2(n-4)}$.

The following inequality will be useful throughout the paper. For all $p>1$, for all $\theta\in(0,\min(1,p-1)]$, there exists $C_{p,\theta}>0$ such that

$$\left||x+y|^p-|x|^p-p|x|^{p-2}xy\right|\le C_{p,\theta}\left(|y|^p+|x|^{p-1-\theta}|y|^{1+\theta}\right),\tag{15}$$

for all $x,y\in\mathbb{R}$.

It now follows from inequality (15) that

$$\left||u_{\varepsilon,\alpha}|^{2^\sharp}-U_\varepsilon^{2^\sharp}-2^\sharp U_\varepsilon^{2^\sharp-1}(v_{\varepsilon,\alpha}+W_{\varepsilon,\alpha})\right|$$
$$\le C\left(|v_{\varepsilon,\alpha}+W_{\varepsilon,\alpha}|^{2^\sharp}+U_\varepsilon^{2^\sharp-2}|v_{\varepsilon,\alpha}+W_{\varepsilon,\alpha}|^2\right)$$

Integrating over B and using (9), (13),(11) and (10), it follows that

$$\int_B|u_{\varepsilon,\alpha}|^{2^\sharp}\,dx=\int_B U_\varepsilon^{2^\sharp}\,dx+2^\sharp\int_B U_\varepsilon^{2^\sharp-1}W_{\varepsilon,\alpha}\,dx+2^\sharp\int_B U_\varepsilon^{2^\sharp-1}v_{\varepsilon,\alpha}\,dx$$
$$+o\left(\|v_{\varepsilon,\alpha}\|_{2^\sharp}^2\right)+O\left(\|W_{\varepsilon,\alpha}\|_{2^\sharp}^2\right)+O\left(\int_B U_\varepsilon^{2^\sharp-2}v_{\varepsilon,\alpha}^2\,dx\right)$$
$$=\int_B U_\varepsilon^{2^\sharp}\,dx+2^\sharp\int_B U_\varepsilon^{2^\sharp-1}W_{\varepsilon,\alpha}\,dx+2^\sharp\int_B U_\varepsilon^{2^\sharp-1}v_{\varepsilon,\alpha}\,dx$$
$$+o\left(\varepsilon^{n-4}\right)+o\left(\|\Delta v_{\varepsilon,\alpha}\|_2^2\right)$$

This equality combined with equalities (14) and (10) leads to

$$A_{\varepsilon,\alpha} = \frac{1}{2}\int_B (\Delta U_\varepsilon)^2\,dx - \frac{1}{2^\sharp}\int_B U_\varepsilon^{2^\sharp}\,dx + \frac{1}{2}\int_B (\Delta v_{\varepsilon,\alpha})^2\,dx - \int_B U_\varepsilon^{2^\sharp-1} W_{\varepsilon,\alpha}\,dx$$
$$+\int_B \Delta(U_\varepsilon + v_{\varepsilon,\alpha})\Delta W_{\varepsilon,\alpha}\,dx + c_n\varepsilon^{n-4} + o(\varepsilon^{n-4}) + o\left(\|\Delta v_{\varepsilon,\alpha}\|_2^2\right)$$

Some straightforward computations give that

$$\int_B U_\varepsilon^{2^\sharp}\,dx = \int_{\mathbb{R}^n} U^{2^\sharp}\,dx + O(\varepsilon^n).$$

Now, noting that

$$\int_B (\Delta U_\varepsilon)^2\,dx = \int_{\mathbb{R}^n} (\Delta U_\varepsilon)^2\,dx - \varepsilon^{n-4}\int_{\mathbb{R}^n - B}\left(\Delta \varepsilon^{\frac{4-n}{2}} U_\varepsilon\right)^2\,dx,$$

it comes with (5) that when $n \geq 5$,

$$\int_B (\Delta U_\varepsilon)^2\,dx = \int_{\mathbb{R}^n} (\Delta U)^2\,dx - c_n\varepsilon^{n-4} + o(\varepsilon^{n-4}).$$

But since

$$\int_{\mathbb{R}^n} (\Delta U)^2\,dx = \int_{\mathbb{R}^n} U^{2^\sharp}\,dx = \frac{1}{K_0^{\frac{n}{4}}}$$

we get that

$$A_{\varepsilon,\alpha} = \frac{2}{nK_0^{\frac{n}{4}}} + \frac{c_n}{2}\varepsilon^{n-4} + \frac{1}{2}\int_B (\Delta v_{\varepsilon,\alpha})^2\,dx - \int_B U_\varepsilon^{2^\sharp-1} W_{\varepsilon,\alpha}\,dx$$
$$+\int_B \Delta(U_\varepsilon + v_{\varepsilon,\alpha})\Delta W_{\varepsilon,\alpha}\,dx + o\left(\|\Delta v_{\varepsilon,\alpha}\|_2^2\right) + o(\varepsilon^{n-4}) \tag{16}$$

Step 3: We now estimate

$$\alpha^{2^\sharp}\int_B \tilde{\rho}\left(\frac{u_{\varepsilon,\alpha}}{\alpha}\right)\,dx.$$

Since ρ' is bounded with (2), there exists $C > 0$ such that

$$|\tilde{\rho}(x+y) - \tilde{\rho}(x) - y\rho(x)| \leq C|y|^2, \tag{17}$$

for all $x, y \in \mathbb{R}$. Hence,

$$\alpha^{2^\sharp}\int_B \tilde{\rho}\left(\frac{u_{\varepsilon,\alpha}}{\alpha}\right)\,dx \tag{18}$$
$$= \alpha^{2^\sharp}\int_B \tilde{\rho}\left(\frac{U_\varepsilon + v_{\varepsilon,\alpha}}{\alpha} + \frac{W_{\varepsilon,\alpha}}{\alpha}\right)\,dx$$
$$= \alpha^{2^\sharp}\int_B \tilde{\rho}\left(\frac{U_\varepsilon + v_{\varepsilon,\alpha}}{\alpha}\right)\,dx + \alpha^{2^\sharp-1}\int_B \rho\left(\frac{U_\varepsilon + v_{\varepsilon,\alpha}}{\alpha}\right)W_{\varepsilon,\alpha}\,dx$$
$$+O\left(\alpha^{2^\sharp-2}\int_B W_{\varepsilon,\alpha}^2\,dx\right) \tag{19}$$

Now, using the fact that ρ' is bounded, (10) and (13), it comes that

$$\left|\alpha^{2^\sharp-1}\int_B \rho\left(\frac{U_\varepsilon+v_{\varepsilon,\alpha}}{\alpha}\right)W_{\varepsilon,\alpha}\,dx-\alpha^{2^\sharp-1}\int_B\rho\left(\frac{U_\varepsilon}{\alpha}\right)W_{\varepsilon,\alpha}\,dx\right|$$
$$=O\left(\alpha^{2^\sharp-2}\int_B|v_{\varepsilon,\alpha}|W_{\varepsilon,\alpha}\,dx\right)$$
$$=O\left(\alpha^{2^\sharp-2}\|v_{\varepsilon,\alpha}\|_{2^\sharp}\|W_{\varepsilon,\alpha}\|_{2^\sharp}\right)=o\left(\|\Delta v_{\varepsilon,\alpha}\|_2^2\right)+o(\varepsilon^{n-4}). \quad (20)$$

But

$$\int_B\Delta(U_\varepsilon+v_{\varepsilon,\alpha})\Delta W_{\varepsilon,\alpha}\,dx=\int_B\Delta^2(U_\varepsilon+v_{\varepsilon,\alpha})W_{\varepsilon,\alpha}\,dx$$
$$=\int_B\left(U_\varepsilon^{2^\sharp-1}+\alpha^{2^\sharp-1}\rho\left(\frac{U_\varepsilon}{\alpha}\right)\right)W_{\varepsilon,\alpha}\,dx. \quad (21)$$

Now, putting together (16), (19), (20), (21), it comes that

$$J_\alpha(u_{\varepsilon,\alpha})=\frac{2}{nK_0^{\frac{n}{4}}}+\frac{c_n}{2}\varepsilon^{n-4}+\frac{1}{2}\|\Delta v_{\varepsilon,\alpha}\|_2^2-\alpha^{2^\sharp}\int_B\tilde\rho\left(\frac{U_\varepsilon+v_{\varepsilon,\alpha}}{\alpha}\right)dx$$
$$+o\left(\|\Delta v_{\varepsilon,\alpha}\|_2^2\right)+o(\varepsilon^{n-4}). \quad (22)$$

With (2), inequality (17) can be refined as follows: for any $s\in(0,1)$

$$|\tilde\rho(x+y)-\tilde\rho(x)-y\rho(x)|\le C(|x|^s+|y|^s)|y|^2,$$

for all $x,y\in\mathbb{R}$, where C depends only on s and ρ. We then get that

$$\left|\tilde\rho\left(\frac{U_\varepsilon+v_{\varepsilon,\alpha}}{\alpha}\right)-\tilde\rho\left(\frac{U_\varepsilon}{\alpha}\right)-\rho\left(\frac{U_\varepsilon}{\alpha}\right)\left(\frac{v_{\varepsilon,\alpha}}{\alpha}\right)\right|\le C\frac{U_\varepsilon^s}{\alpha^s}\frac{v_{\varepsilon,\alpha}^2}{\alpha^2}+\frac{v_{\varepsilon,\alpha}^{2+s}}{\alpha^{2+s}}.$$

Taking $s>0$ small enough, we get with (13), (9) and Hölder and Sobolev inequalities that

$$\alpha^{2^\sharp}\int_B\tilde\rho\left(\frac{U_\varepsilon+v_{\varepsilon,\alpha}}{\alpha}\right)dx=\alpha^{2^\sharp}\int_B\tilde\rho\left(\frac{U_\varepsilon}{\alpha}\right)dx$$
$$+\alpha^{2^\sharp-1}\int_B\rho\left(\frac{U_\varepsilon}{\alpha}\right)v_{\varepsilon,\alpha}\,dx+o\left(\|\Delta v_{\varepsilon,\alpha}\|_2^2\right)+o(\varepsilon^{n-4}). \quad (23)$$

Through some integrations by parts and using (7) and (12), we get that

$$\alpha^{2^\sharp-1}\int_B\rho\left(\frac{U_\varepsilon}{\alpha}\right)(v_{\varepsilon,\alpha}+T_\varepsilon)\,dx=\int_B(\Delta v_{\varepsilon,\alpha})^2\,dx+n\frac{\partial v_{\varepsilon,\alpha}}{\partial n}(1)\int_B\Delta v_{\varepsilon,\alpha}\,dx$$
$$=\int_B(\Delta v_{\varepsilon,\alpha})^2\,dx-n\omega_{n-1}\left(\frac{\partial v_{\varepsilon,\alpha}}{\partial n}(1)\right)^2 \quad (24)$$

It now follows from (2) that for all $\nu\in(0,1)$, there exists $C_\nu>0$ such that $|\rho(s)|\le C_\nu|s|^{1+\nu}$ for all $s\in\mathbb{R}$. We then get with (5) and (12) that

$$\alpha^{2^\sharp-1}\int_B\rho\left(\frac{U_\varepsilon}{\alpha}\right)T_\varepsilon\,dx=o(\varepsilon^{n-4}). \quad (25)$$

Putting inequalities (22), (23), (24), (25) all together and using (5), it comes that

$$J_\alpha(u_{\varepsilon,\alpha})=\frac{2}{nK_0^{\frac{n}{4}}}-\frac{1}{2}\|\Delta v_{\varepsilon,\alpha}\|_2^2-\alpha^{2^\sharp}\int_B\tilde\rho\left(\frac{U_\varepsilon}{\alpha}\right)dx+\frac{n^2-4n+2}{4}c_n\varepsilon^{n-4}$$
$$+o\left(\|\Delta v_{\varepsilon,\alpha}\|_2^2\right)+o(\varepsilon^{n-4}) \quad (26)$$

Step 4: We now estimate

$$\alpha^{2^\sharp}\int_B \tilde{\rho}\left(\frac{U_\varepsilon}{\alpha}\right)\,dx,$$

that is the third term in the RHS of (26). With (5) and some change of variable, we get that

$$\alpha^{2^\sharp}\int_B \tilde{\rho}\left(\frac{U_\varepsilon}{\alpha}\right)\,dx = \omega_{n-1}a_n^n\varepsilon^{\frac{n}{2}}\alpha^{\frac{n}{n-4}}g(\alpha,\varepsilon),$$

where

$$g(\alpha,\varepsilon) = \int_0^{\frac{\alpha^{\frac{1}{n-4}}}{a_n\sqrt{\varepsilon}}}\tilde{\rho}\left(\left(\alpha^{\frac{2}{n-4}}\varepsilon + r^2\right)^{\frac{4-n}{2}}\right)r^{n-1}\,dr. \tag{27}$$

For $n \geq 5$, some standard computations lead to (see for instance [AMS])

$$\frac{\partial g}{\partial\varepsilon}(\alpha,\varepsilon) = -\frac{\alpha^{\frac{n}{n-4}}}{2a_n^n\varepsilon^{\frac{n+2}{2}}}\tilde{\rho}\left(\frac{1}{\alpha}\left(\frac{a_n^2\varepsilon}{1+a_n^2\varepsilon^2}\right)^{\frac{n-4}{2}}\right) \tag{28}$$

$$-\frac{n-4}{2}\alpha^{\frac{2}{n-4}}\int_0^{\frac{\alpha^{\frac{1}{n-4}}}{a_n\sqrt{\varepsilon}}}\left(\alpha^{\frac{2}{n-4}}\varepsilon + r^2\right)^{\frac{2-n}{2}}\rho\left(\left(\alpha^{\frac{2}{n-4}}\varepsilon + r^2\right)^{\frac{4-n}{2}}\right)r^{n-1}\,dr.$$

Similarly,

$$\begin{aligned}
\frac{\partial^2 g}{\partial\varepsilon^2}(\alpha,\varepsilon) &= \frac{n+2}{4a_n^n}\alpha^{\frac{n}{n-4}}\varepsilon^{-\frac{n+4}{2}}\tilde{\rho}\left(\frac{1}{\alpha}\left(\frac{a_n^2\varepsilon}{1+a_n^2\varepsilon^2}\right)^{\frac{n-4}{2}}\right)\\
&+\frac{(n-4)\alpha^{\frac{4}{n-4}}}{4a_n^4\varepsilon^4}\frac{2a_n^2\varepsilon^2-1}{(1+a_n^2\varepsilon^2)^{\frac{n-2}{2}}}\rho\left(\frac{1}{\alpha}\left(\frac{a_n^2\varepsilon}{1+a_n^2\varepsilon^2}\right)^{\frac{n-4}{2}}\right)\\
&-\frac{(n-4)\alpha^{\frac{4}{n-4}}}{4\,(1+a_n^2\varepsilon^2)^{\frac{n-2}{2}}}\rho\left(\frac{1}{\alpha}\left(\frac{a_n^2\varepsilon}{1+a_n^2\varepsilon^2}\right)^{\frac{n-4}{2}}\right)\\
&+\frac{(n-2)(n-4)\alpha^{\frac{4}{n-4}}}{4}\int_0^{\frac{\alpha^{\frac{1}{n-4}}}{a_n\sqrt{\varepsilon}}}\frac{\rho\left(\left(\alpha^{\frac{2}{n-4}}\varepsilon+r^2\right)^{\frac{4-n}{2}}\right)}{\left(\alpha^{\frac{2}{n-4}}\varepsilon+r^2\right)^{\frac{n-2}{2}}}r^{n-3}\,dr,
\end{aligned} \tag{29}$$

and

$$\begin{aligned}
\frac{\partial^3 g}{\partial\varepsilon^3}(\alpha,\varepsilon) &= -\frac{(n+2)(n+4)}{8a_n^n}\frac{\alpha^{\frac{n}{n-4}}}{\varepsilon^{\frac{n+6}{2}}}\tilde{\rho}\left(\frac{1}{\alpha}\left(\frac{a_n^2\varepsilon}{1+a_n^2\varepsilon^2}\right)^{\frac{n-4}{2}}\right)\\
&+\frac{(n-4)(n+10)}{8a_n^4}\frac{\alpha^{\frac{4}{n-4}}}{\varepsilon^5}\rho\left(\frac{1}{\alpha}\left(\frac{a_n^2\varepsilon}{1+a_n^2\varepsilon^2}\right)^{\frac{n-4}{2}}\right)\left(1+O(\varepsilon^2)\right)\\
&-\frac{(n-4)^2}{8a_n^{8-n}}\varepsilon^{\frac{n-14}{2}}\alpha^{\frac{8-n}{n-4}}\rho'\left(\frac{1}{\alpha}\left(\frac{a_n^2\varepsilon}{1+a_n^2\varepsilon^2}\right)^{\frac{n-4}{2}}\right)\left(1+O(\varepsilon^2)\right)\\
&-\frac{(n-4)^2(n-2)}{8}\alpha^{\frac{6}{n-4}}\int_0^{\frac{\alpha^{\frac{1}{n-4}}}{a_n\sqrt{\varepsilon}}}\frac{\rho\left(\left(\alpha^{\frac{2}{n-4}}\varepsilon+r^2\right)^{\frac{4-n}{2}}\right)}{\left(\alpha^{\frac{2}{n-4}}\varepsilon+r^2\right)^{\frac{n-2}{2}}}r^{n-5}\,dr.
\end{aligned}$$

With (2), we get that

$$\frac{\partial^3 g}{\partial \varepsilon^3}(\alpha,\varepsilon) = O\left(\alpha^{-\frac{n-8}{n-4}}\varepsilon^{\frac{n-14}{2}}\right) + O\left(\alpha^{\frac{6}{n-4}}\right) \tag{30}$$

for all $\alpha \in (0,1]$, $\varepsilon > 0$. Then, using the integral Taylor identity, (27), (28), (29), and (30), we get that for $n \geq 13$,

$$\begin{aligned}
&\alpha^{2^\sharp}\int_B \tilde{\rho}\left(\frac{U_\varepsilon}{\alpha}\right) dx = \\
&a_n^n \varepsilon^{\frac{n}{2}}\alpha^{\frac{n}{n-4}}\left(\int_{\mathbb{R}^n}\tilde{\rho}(|x|^{4-n})\,dx - \frac{n-4}{2}\varepsilon\alpha^{\frac{2}{n-4}}\int_{\mathbb{R}^n}\rho(|x|^{4-n})|x|^{2-n}\,dx\right.\\
&\left.+\frac{(n-2)(n-4)}{8}\alpha^{\frac{4}{n-4}}\varepsilon^2\int_{\mathbb{R}^n}\rho(|x|^{4-n})|x|^{-n}\,dx + o\left(\alpha^{\frac{4}{n-4}}\varepsilon^2\right)\right)\\
&+O\left(\alpha^{\frac{8}{n-4}}\varepsilon^{n-4}\right)
\end{aligned} \tag{31}$$

Step 5: This step is devoted to the estimation of $\int_B (\Delta v_{\varepsilon,\alpha})^2\,dx$. It comes from (8) that

$$\begin{aligned}
&\Delta v_{\varepsilon,\alpha}(r) = n\frac{\partial U_\varepsilon}{\partial n}(1)\\
&-n\alpha^{2^\sharp-1}\int_0^1 s^{n-1}\left[\int_s^r t^{1-n}\left\{\int_0^t \rho\left(\frac{U_\varepsilon}{\alpha}\right)u^{n-1}\,du\right\}dt\right]ds
\end{aligned}$$

When $n \geq 7$, and since $\Delta v_{\varepsilon,\alpha}$ is radially symmetrical, we have that

$$\begin{aligned}
&\int_B (\Delta v_{\varepsilon,\alpha})^2\,dx = n\omega_{n-1}\left(\frac{\partial U_\varepsilon}{\partial n}(1)\right)^2\\
&+\omega_{n-1}\alpha^{\frac{2(n+4)}{(n-4)}}\int_0^1 r^{n-1}\left[\int_0^r t^{1-n}\left(\int_0^t s^{n-1}\rho\left(\frac{U_\varepsilon}{\alpha}\right)ds\right)dt\right]^2 dr\\
&-n\omega_{n-1}\alpha^{\frac{2(n+4)}{(n-4)}}\left[\int_0^1 r^{n-1}\int_0^r t^{1-n}\left(\int_0^t s^{n-1}\rho\left(\frac{U_\varepsilon}{\alpha}\right)ds\right)dt\,dr\right]^2\\
&= n\omega_{n-1}\left(\frac{\partial U_\varepsilon}{\partial n}(1)\right)^2\\
&+\omega_{n-1}\alpha^{\frac{2(n+4)}{(n-4)}}\int_0^1 r^{n-1}\left[\int_0^{+\infty} t^{1-n}\left(\int_0^t s^{n-1}\rho\left(\frac{U_\varepsilon}{\alpha}\right)ds\right)dt\right.\\
&\left.-\int_r^{+\infty} t^{1-n}\left(\int_0^t s^{n-1}\rho\left(\frac{U_\varepsilon}{\alpha}\right)ds\right)dt\right]^2 dr\\
&-n\omega_{n-1}\alpha^{\frac{2(n+4)}{(n-4)}}\left[\int_0^1 r^{n-1}\left(\int_0^{+\infty} t^{1-n}\left(\int_0^t s^{n-1}\rho\left(\frac{U_\varepsilon}{\alpha}\right)ds\right)dt\right.\right.\\
&\left.\left.-\int_r^{+\infty} t^{1-n}\left(\int_0^t s^{n-1}\rho\left(\frac{U_\varepsilon}{\alpha}\right)ds\right)dt\right)dr\right]^2
\end{aligned}$$

Then, still for $n \geq 7$, we get that

$$\int_B (\Delta v_{\varepsilon,\alpha})^2\, dx = n\omega_{n-1}\left(\frac{\partial U_\varepsilon}{\partial n}(1)\right)^2$$
$$+\omega_{n-1}\alpha^{\frac{2(n+4)}{(n-4)}}\int_0^1 r^{n-1}\left[\int_r^{+\infty} t^{1-n}\left(\int_0^t s^{n-1}\rho\left(\frac{U_\varepsilon}{\alpha}\right)ds\right)dt\right]^2 dr$$
$$-n\omega_{n-1}\alpha^{\frac{2(n+4)}{(n-4)}}\left[\int_0^1 r^{n-1}\int_r^{+\infty} t^{1-n}\left(\int_0^t s^{n-1}\rho\left(\frac{U_\varepsilon}{\alpha}\right)ds\right)dt\,dr\right]^2$$
$$= A_1(\alpha,\varepsilon) + A_2(\alpha,\varepsilon) - A_3(\alpha,\varepsilon) \tag{32}$$

We estimate each of the terms seperately. First, with (5), we have by direct calculation that

$$A_1(\alpha,\varepsilon) = \frac{n(n-4)}{4}c_n\varepsilon^{n-4} + o(\varepsilon^{n-4}) \tag{33}$$

We now deal with $A_3(\alpha,\varepsilon)$, that is the third term in the RHS. A change of variable gives

$$A_3(\alpha,\varepsilon) = n\omega_{n-1}a_n^{2n+4}\alpha^{\frac{2(n+4)}{n-4}}\varepsilon^{n+2} \tag{34}$$
$$\times\left[\int_0^{\frac{1}{a_n\sqrt{\varepsilon}}} r^{n-1}\left(\int_r^{+\infty} t^{1-n}\left(\int_0^t s^{n-1}\rho\left(\frac{(\varepsilon+s^2)^{\frac{4-n}{2}}}{\alpha}\right)ds\right)dt\right)dr\right]^2$$

Now, when $n \geq 9$, there exists $\nu \in (0,1]$ such that $\nu \in \left(\frac{4}{n-4}, \frac{8}{n-4}\right)$. With (2), it then comes that there exists $C > 0$ such that $|\rho(s)| \leq C|s|^{1+\nu}$ for any $s \neq 0$. Some computations then lead to

$$\left|\int_0^t s^{n-1}\rho\left(\frac{(\varepsilon+s^2)^{\frac{4-n}{2}}}{\alpha}\right)ds\right| \leq \frac{C}{\alpha^{1+\nu}}\left(\mathbf{1}_{t\leq 1}t^n + \mathbf{1}_{t>1}\right),$$

for all $t \geq 0$, $\alpha,\varepsilon \in (0,1)$. Plugging this expression in (34), we get that

$$A_3(\alpha,\varepsilon) = O\left(\varepsilon^n\alpha^{2\left(\frac{8}{n-4}-\nu\right)}\right) = o(\varepsilon^{n-4}) \tag{35}$$

We now deal with $A_2(\alpha,\varepsilon)$. A change of variable gives

$$A_2(\alpha,\varepsilon) = \omega_{n-1}\alpha^{\frac{2(n+4)}{n-4}}a_n^{n+4}\varepsilon^{\frac{n+4}{2}}f(\alpha,\varepsilon) \tag{36}$$

where

$$f(\alpha,\varepsilon) = \int_0^{\frac{1}{a_n\sqrt{\varepsilon}}} r^{n-1}\left[\int_r^{+\infty} t^{1-n}\left(\int_0^t s^{n-1}\rho\left(\frac{1}{\alpha}\left(\varepsilon+s^2\right)^{\frac{4-n}{2}}\right)ds\right)dt\right]^2 dr.$$

With Lebesgue's theorem, and when $n \geq 9$, we get that for any $\alpha > 0$,

$$\lim_{\varepsilon\to 0} f(\alpha,\varepsilon) = \alpha^{-(2^\sharp-1)}\int_0^{+\infty} r^{n-1}\left[\int_r^{+\infty} t^{1-n}\left(\int_0^t s^{n-1}\rho\left(s^{4-n}\right)ds\right)dt\right]^2 dr \tag{37}$$

Similarly to what was done in Step 4, we compute $\frac{\partial f}{\partial\varepsilon}(\alpha,\varepsilon)$ and we find:

$$\frac{\partial f}{\partial\varepsilon}(\alpha,\varepsilon) = O\left(\alpha^{-2}\varepsilon^{\frac{n-14}{2}}\right) + O(\alpha^{-\frac{n+2}{n-4}})$$

when $n \geq 11$. We then get that

$$f(\alpha,\varepsilon) = f(\alpha,0) + O\left(\alpha^{-\frac{n+2}{n-4}}\varepsilon\right) + O\left(\alpha^{-2}\varepsilon^{\frac{n-12}{2}}\right) \tag{38}$$

as soon as $n \geq 13$. Now, (32), (33), (35), (36), (37) and (38) give

$$
\begin{aligned}
&\|\Delta v_{\varepsilon,\alpha}\|_2^2 = \\
&\omega_{n-1}\alpha^{2^\sharp-1}a_n^{n+4}\varepsilon^{\frac{n+4}{2}}\int_0^{+\infty} r^{n-1}\left(\int_r^{+\infty} t^{1-n}\left(\int_0^t s^{n-1}\rho\left(s^{4-n}\right)\,ds\right)dt\right)^2 dr \\
&+\frac{n(n-4)}{4}c_n\varepsilon^{n-4}+o(\varepsilon^{n-4})+O(\alpha^{\frac{16}{n-4}}\varepsilon^{n-4})+O\left(\alpha^{\frac{n+6}{n-4}}\varepsilon^{\frac{n+6}{2}}\right) \qquad (39)
\end{aligned}
$$

for $n \geq 13$. Combining (1), (26), (31), (39), and using the expressions of $I_1(\rho)$, $I_2(\rho)$, $I_3(\rho)$, we get that

$$
\begin{aligned}
J_\alpha(u_{\varepsilon,\alpha}) &= \frac{2}{nK_0^{\frac{n}{4}}}+\frac{(n-4)(n-2)^2\omega_{n-1}a_n^{2(n-4)}}{2}\varepsilon^{n-4} \qquad (40)\\
&-a_n^n\omega_{n-1}\varepsilon^{\frac{n}{2}}\alpha^{\frac{n}{n-4}}\left[\frac{I_1(\rho)}{n}-\frac{I_2(\rho)}{2}\varepsilon\alpha^{\frac{2}{n-4}}+\frac{a_n^4 I_3(\rho)}{2(n-4)^5}\alpha^{\frac{4}{n-4}}\varepsilon^2+o\left(\alpha^{\frac{4}{n-4}}\varepsilon^2\right)\right]\\
&+o(\varepsilon^{n-4})+O(\alpha^{\frac{8}{n-4}}\varepsilon^{n-4})+O\left(\alpha^{\frac{n+6}{n-4}}\varepsilon^{\frac{n+6}{2}}\right),
\end{aligned}
$$

as soon as $n \geq 13$. Some similar arguments lead to the following estimates in smaller dimensions:

$$
\begin{aligned}
J_\alpha(u_{\varepsilon,\alpha}) &= \frac{2}{nK_0^{\frac{n}{4}}}+\frac{(n-4)(n-2)^2\omega_{n-1}a_n^{2(n-4)}}{2}\varepsilon^{n-4}\\
&-a_n^n\omega_{n-1}\varepsilon^{\frac{n}{2}}\alpha^{\frac{n}{n-4}}\left[\frac{I_1(\rho)}{n}+o(1)\right]+o(\varepsilon^{n-4})+O(\alpha^{\frac{8}{n-4}}\varepsilon^{n-4}), \qquad (41)
\end{aligned}
$$

for $n \geq 9$. If we assume that $n \geq 11$, we obtain that

$$
\begin{aligned}
J_\alpha(u_{\varepsilon,\alpha}) &= \frac{2}{nK_0^{\frac{n}{4}}}+\frac{(n-4)(n-2)^2\omega_{n-1}a_n^{2(n-4)}}{2}\varepsilon^{n-4}\\
&-a_n^n\omega_{n-1}\varepsilon^{\frac{n}{2}}\alpha^{\frac{n}{n-4}}\left[\frac{I_1(\rho)}{n}-\frac{I_2(\rho)}{2}\varepsilon\alpha^{\frac{2}{n-4}}+o\left(\alpha^{\frac{2}{n-4}}\varepsilon\right)\right]\\
&+o(\varepsilon^{n-4})+O(\alpha^{\frac{8}{n-4}}\varepsilon^{n-4}). \qquad (42)
\end{aligned}
$$

3. Proof of the theorem - Existence statement

We obtain solutions of problem (E_r) thanks to the Mountain-pass lemma of Ambrosetti and Rabinowitz. We use the following statement of the lemma:

Theorem 3.1 ([AmRa]). *Let $F \in C^1(V,\mathbb{R})$ where $(V,\|.\|)$ is a Banach space. We assume that:*

(i) $F(0)=0$,

(ii) $\exists \lambda, R>0$ such that $F(v)\geq\lambda$ for all $v\in V$ such that $\|v\|=R$,

(iii) $\exists v_0\in V$ such that $\limsup_{t\to+\infty}F(tv_0)<0$.

We let $t_0>0$ large be such that $\|t_0v_0\|>R$ and $F(t_0v_0)<0$, and $\beta=\inf_{\gamma\in\Gamma}\sup F(\gamma(t))$, where $\Gamma=\{\gamma:[0,1]\to V$ s.t. $\gamma(0)=0, \gamma(1)=t_0v_0\}$. Then there exists a sequence (u_n) in V such that

$$F(u_n)\to\beta \quad , \quad F'(u_n)\to 0 \text{ strongly in } V'.$$

Moreover, we have that $\lambda\leq\beta\leq\sup_{t\geq0}F(tv_0)$.

In order to prove the existence of radial solution, we consider the space

$$V = H^2_{2,0}(B) \cap \{v \in H^2_{2,0}(B) \,/\, v \circ \sigma = v, \;\text{ for all } \sigma \in O_n(\mathbb{R})\},$$

where $O_n(\mathbb{R})$ denotes the group of the isometries of the Euclidean $n-$dimensional space $\mathbb{R}^n$. We also consider the functional $F = J_1$ (where J_1 was defined in section 2) defined on V. Clearly (i) of the theorem is satisfied. With (2), we get that point (ii) is satisfied. Point (iii) is clearly satisfied for all $v_0 \in V - \{0\}$. Let $v_0 \in V - \{0\}$. Then, it follows from theorem 3.1 that there exists a sequence $(u_p) \in H^2_{2,0}(B)$ such that

$$J_1(u_p) \to \beta \quad , \quad J_1'(u_p) \to 0 \text{ strongly in } V', \tag{43}$$

when $p \to +\infty$. Here $0 < \beta \le \sup_{t\ge 0} J_1(tv_0)$.

Step 1: We claim that there exists $u \in V$ such that $u_p \rightharpoonup u$ weakly in $H^2_{2,0}(B)$ when $p \to +\infty$. With the additionnal property that

$$u \neq 0 \text{ if } \sup_{t\ge 0} J_1(tv_0) < \frac{2}{nK_0^{\frac{n}{4}}}.$$

We prove the claim. It follows from standard arguments that (u_p) is bounded in $H^2_{2,0}(B)$. Then there exists $u \in H^2_{2,0}(B)$ such that $u_p \rightharpoonup u$ weakly in $H^2_{2,0}(B)$. Clearly $u \in V$. We now assume that

$$\sup_{t\ge 0} J_1(tv_0) < \frac{2}{nK_0^{\frac{n}{4}}}.$$

We prove that $u \not\equiv 0$ by contradiction. We assume that $u_p \rightharpoonup 0$ weakly in $H^2_{2,0}(B)$. We can assume that $u_p \to 0$ in $L^q(B)$ for all $q \in (1, 2^\sharp)$. Then with (2), it comes that

$$J_1(u_p) = \frac{1}{2}\int_B (\Delta u_p)^2 - \frac{1}{2^\sharp}\int_B |u_p|^{2^\sharp}\, dx + o(1) = \beta + o(1)$$

$$\langle J_1'(u_p), u_p\rangle = \int_B (\Delta u_p)^2 - \int_B |u_p|^{2^\sharp}\, dx + o(1) = o(1)$$

These inequalities combined with the optimal Sobolev inequality (4) then lead to

$$\left(\frac{n}{2}\beta\right)^{\frac{2}{2^\sharp}} \le K_0 \frac{n}{2}\beta.$$

Since $\beta > 0$, we get $\beta \ge \frac{2}{nK_0^{\frac{n}{4}}}$. A contradiction. Then $u \not\equiv 0$. The claim is proved.

With (43), we get that for all $\varphi \in C_c^\infty(B)$ radially symmetrical, we have that

$$\int_B \Delta u \Delta\varphi\, dx = \int_B \left(|u|^{2^\sharp-2}u + \rho(u)\right)\varphi\, dx$$

It then follows by standard arguments (see for instance [Heb1]) that this equality occurs for all $\varphi \in C_c^\infty(B)$. And then

$$\Delta^2 u = |u|^{2^\sharp-2}u + \rho(u)$$

in the distribution sense. It then follows from arguments due to Van der Vorst [VdV] and Agmon-Douglis-Nirenberg [ADN] that for any $\nu \in (0,1)$,

$$u \in C^{4,\nu}(\overline{B}),$$

and that

$$\Delta^2 u = |u|^{2^\sharp-2}u + \rho(u), \text{ in } B, \text{ and } u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial B.$$

Now, proving the first part of theorem 1.1 on the unit ball reduces to find some suitable functions $v_0 \in V - \{0\}$ such that

$$\sup_{t\geq 0} J_1(tv_0) < \frac{2}{nK_0^{\frac{n}{4}}}. \tag{44}$$

Situations for which this inequality holds can be found in [EsRo]. We consider the test-functions introduced in section 2. We now let $\varepsilon > 0$ and consider $U_\varepsilon + v_\varepsilon$, where $v_\varepsilon = v_{\varepsilon,1}$ and $U_\varepsilon + v_\varepsilon \in V$ by construction. By standard variational arguments, there exists $t_\varepsilon \in (0, +\infty)$ such that

$$\sup_{t\geq 0} J_1(t(U_\varepsilon + v_\varepsilon)) = J_1(t_\varepsilon(U_\varepsilon + v_\varepsilon)).$$

Step 2: We claim that

$$t_\varepsilon = 1 + o\left(\varepsilon^{\frac{n-4}{2}}\right) + o\left(\|\Delta v_\varepsilon\|_2\right). \tag{45}$$

We prove the claim. It follows from the estimates of section 2 that

$$\int_B \left(\Delta(U_\varepsilon + v_\varepsilon)\right)^2 dx = \int_{\mathbb{R}^n} (\Delta U)^2 dx + o\left(\varepsilon^{\frac{n-4}{2}}\right) + o\left(\|\Delta v_\varepsilon\|_2\right)$$

$$\int_B |U_\varepsilon + v_\varepsilon|^{2^\sharp} dx = \int_{\mathbb{R}^n} U^{2^\sharp} dx + o\left(\varepsilon^{\frac{n-4}{2}}\right) + o\left(\|\Delta v_\varepsilon\|_2\right).$$

Then $J_1(t_\varepsilon(U_\varepsilon+v_\varepsilon)) \geq J_1(U_\varepsilon+v_\varepsilon) = \frac{2}{nK_0^{\frac{n}{4}}} + o(1)$. It then easily follows that $t_\varepsilon \not\to 0$ and $t_\varepsilon \not\to +\infty$. Up to a subsequence, $t_\varepsilon \to t_0 \in (0, +\infty)$. Then,

$$\begin{aligned} 0 &= \frac{d}{dt} J_1(t(U_\varepsilon + v_\varepsilon))_{|t_\varepsilon} \\ &= t_\varepsilon \int_B \left(\Delta(U_\varepsilon + v_\varepsilon)\right)^2 dx - t_\varepsilon^{2^\sharp-1} \int_B |U_\varepsilon + v_\varepsilon|^{2^\sharp} dx + o\left(\varepsilon^{\frac{n-4}{2}}\right) + o\left(\|\Delta v_\varepsilon\|_2\right) \end{aligned}$$

then, $t_\varepsilon - t_\varepsilon^{2^\sharp-1} = o\left(\varepsilon^{\frac{n-4}{2}}\right) + o\left(\|\Delta v_\varepsilon\|_2\right)$. It then follows that $t_0 = 1$, and that (45) holds. This proves the claim.

Step 3: we now prove the first assertion of theorem 1.1. We write

$$u_{\varepsilon,1} = t_\varepsilon(U_\varepsilon + v_\varepsilon) = U_\varepsilon + v_\varepsilon + (t_\varepsilon - 1)(U_\varepsilon + v_\varepsilon).$$

Then (10) is satisfied with $W_{\varepsilon,1} = (t_\varepsilon - 1)(U_\varepsilon + v_\varepsilon)$. Taking $\alpha = 1$, it follows from (40) that inequality (44) is satisfied with $v_0 = U_\varepsilon + v_\varepsilon$, $\varepsilon > 0$ small, provided the hypothesis of the existence statement of the theorem. It follows from Step 1 that there exists a solution to the problem (E_1). The first assertion of the theorem easily follows throughout a rescaling argument.

Remark: with (41) and (42), this result can be extended to the case of an open subset of $\mathbb{R}^n$, and to smaller dimensions. Of course, we cannot recover that the solutions are radially symmetrical in the general case. However, the following result holds:

Proposition 3.1. *Let Ω be a smooth subset of $\mathbb{R}^n$. We assume that (H_ρ) holds, and that one of the following conditions occurs:*
(i) $n \geq 9$ and $I_1(\rho) > 0$,
(ii) $n \geq 11$, $I_1(\rho) = 0$ and $I_2(\rho) < 0$,
(iii) $n \geq 13$, $I_1(\rho) = I_2(\rho) = 0$ and $I_3(\rho) > 0$,
then there exists $u \in C^4(\overline{B})$ a nonzero function such that

$$\Delta^2 u = |u|^{2^\sharp - 2} u + \rho(u) \text{ in } \Omega, \text{ and } u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega.$$

4. Blow-up analysis I

This section and the following are devoted to the proof of the second part of theorem 1.1. We assume that for all $\alpha > 0$, there exists $\hat{u}_\alpha \in C^4(\overline{B}(0,\alpha))$ a smooth positive radially symmetrical function such that

$$\begin{cases} \Delta^2 \hat{u}_\alpha = |\hat{u}_\alpha|^{2^\sharp - 2} \hat{u}_\alpha + \rho(\hat{u}_\alpha) & \text{in } B(0,\alpha) \\ \hat{u}_\alpha \not\equiv 0 & \\ \hat{u}_\alpha = \frac{\partial \hat{u}_\alpha}{\partial n} = 0 & \text{on } \partial B(0,\alpha). \end{cases}$$

and

$$\frac{1}{2} \int_{B(0,\alpha)} (\Delta \hat{u}_\alpha)^2 \, dx - \frac{1}{2^\sharp} \int_{B(0,\alpha)} |\hat{u}_\alpha|^{2^\sharp} \, dx - \int_{B(0,\alpha)} \tilde{\rho}(\hat{u}_\alpha) \, dx < \frac{2}{n K_0^{\frac{n}{4}}}.$$

Here and in the sequel, $\rho \in C^\infty(\mathbb{R})$ and ρ verifies (2). Up to rescaling, there exists $u_\alpha \in C^4(\overline{B})$ radially symmetrical such that

$$\begin{cases} \Delta^2 u_\alpha = |u_\alpha|^{2^\sharp - 2} u_\alpha + \alpha^{2^\sharp - 1} \rho\left(\frac{u_\alpha}{\alpha}\right) & \text{in } B \\ u_\alpha \not\equiv 0 & \\ u_\alpha = \frac{\partial u_\alpha}{\partial n} = 0 & \text{on } \partial B, \\ J_\alpha(u_\alpha) < \frac{2}{n K_0^{\frac{n}{4}}} & \end{cases} \qquad (I_\alpha)$$

where

$$J_\alpha(u) = \frac{1}{2} \int_B (\Delta u_\alpha)^2 \, dx - \frac{1}{2^\sharp} \int_B |u_\alpha|^{2^\sharp} \, dx - \alpha^{2^\sharp} \int_B \tilde{\rho}\left(\frac{u_\alpha}{\alpha}\right) \, dx.$$

Step 1: We claim that $u_\alpha \rightharpoonup 0$ weakly in $H^2_{2,0}(B)$. We prove the claim. It follows from (I_α) that

$$\begin{aligned} J_\alpha(u_\alpha) &= \frac{2}{n} \int_B |u_\alpha|^{2^\sharp} \, dx + \frac{\alpha^{2^\sharp - 1}}{2} \int_B \rho\left(\frac{u_\alpha}{\alpha}\right) u_\alpha \, dx - \alpha^{2^\sharp} \int_B \tilde{\rho}\left(\frac{u_\alpha}{\alpha}\right) \, dx \\ &= \frac{2}{n} \int_B u_\alpha^{2^\sharp} \, dx + o(\|u_\alpha\|_{2^\sharp}) + o(1) < \frac{2}{n K_0^{\frac{n}{4}}}, \end{aligned} \qquad (46)$$

then $\|u_\alpha\|_{2^\sharp} = O(1)$, and with (I_α), $\|u_\alpha\|_{H^2_{2,0}(B)} = O(1)$. Up to a subsequence, we can assume that it goes weakly to $u \in H^2_{2,0}(B)$. Passing through the limit in (I_α), we have that $\Delta^2 u = |u|^{2^\sharp - 2} u$ in the weak sense. Considering that $u_\alpha \rightharpoonup u$ in $L^{2^\sharp}(B)$, we get that

$$\int_B |u|^{2^\sharp} \, dx \leq \liminf_{\alpha \to 0} \int_B |u_\alpha|^{2^\sharp} \, dx \leq \frac{1}{K_0^{\frac{n}{4}}}.$$

We used that $u \in H_{2,0}^2(B) \subset H_{2,0}^2(\mathbb{R}^n)$. We are now left with proving that $u \equiv 0$. We argue by contradiction, and assume that $u \not\equiv 0$. Then, multiplying by $u \in H_{2,0}^2(B)$ and integrating, we get with the Sobolev inequality (4) that

$$\frac{1}{K_0} \leq \frac{\int_{\mathbb{R}^n} (\Delta u)^2 \, dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\sharp} \, dx\right)^{\frac{2}{2^\sharp}}} = \frac{\int_B (\Delta u)^2 \, dx}{\left(\int_B |u|^{2^\sharp} \, dx\right)^{\frac{2}{2^\sharp}}} = \left(\int_B |u|^{2^\sharp} \, dx\right)^{1-\frac{2}{2^\sharp}} \leq \frac{1}{K_0}.$$

In particular equality holds, and $u \in H_{2,0}^2(\mathbb{R}^n)$ is an extremal function for the Euclidean Sobolev inequality. It follows from [EFJ], [Lie], [Lio] that u is smooth and that there exist $\lambda \in \mathbb{R}$, $C \neq 0$ and $\tilde{x} \in \mathbb{R}^n$ such that

$$u(x) = \frac{C}{(\lambda^2 + |x - \tilde{x}|)^{\frac{n-4}{2}}}.$$

A contradiction, since u is zero outside B. Then $u \equiv 0$. The claim is proved.

We go on with the study of the sequence u_α. Using that $|\rho(r)| \leq C|r|$ for some positive constant C and all $r \in \mathbb{R}$ and the system (I_α), we get that

$$\begin{aligned} \int_B (\Delta u_\alpha)^2 \, dx &= \int_B |u_\alpha|^{2^\sharp} \, dx + \alpha^{2^\sharp - 1} \int_B \rho\left(\frac{u_\alpha}{\alpha}\right) u_\alpha \, dx \\ &= \int_B |u_\alpha|^{2^\sharp} \, dx + O\left(\alpha^{2^\sharp - 2} \int_B u_\alpha^2 \, dx\right). \end{aligned}$$

Now, the standard Sobolev inequality asserts that

$$\left(\int_B |u_\alpha|^{2^\sharp} \, dx\right)^{\frac{2}{2^\sharp}} \leq K_0 \int_B (\Delta u_\alpha)^2 \, dx = K_0 \int_B |u_\alpha|^{2^\sharp} \, dx + o\left(\|u_\alpha\|_{2^\sharp}^2\right),$$

and then $\int_B |u_\alpha|^{2^\sharp} \, dx \geq \frac{1}{K_0^{\frac{n}{4}}} + o(1)$. Then with (46), we get that

$$\int_B |u_\alpha|^{2^\sharp} \, dx = \frac{1}{K_0^{\frac{n}{4}}} + o(1). \tag{47}$$

Now, noting that

$$\int_B |u_\alpha|^{2^\sharp} \, dx \leq \left(\sup_B u_\alpha\right)^{2^\sharp - 2} \int_B u_\alpha^2 \, dx$$

and that $u_\alpha \to 0$ in $L^2(B)$, we get that $\sup_B u_\alpha \to +\infty$. Following [Rob] and [FHR], we now let $x_\alpha \in B$ and $\mu_\alpha > 0$ such that

$$u_\alpha(x_\alpha) = \mu_\alpha^{-\frac{n-4}{2}} = \sup_B u_\alpha \to +\infty.$$

For $x \in \mathbb{R}^n$, we now define

$$\bar{u}_\alpha(x) = \mu_\alpha^{\frac{n-4}{2}} u_\alpha(x_\alpha + \mu_\alpha x) \text{ if } x \in B_\alpha = B\left(-\frac{x_\alpha}{\mu_\alpha}, \frac{1}{\mu_\alpha}\right),$$

and $\bar{u}_\alpha(x) = 0$ elsewhere. Clearly, $\bar{u}_\alpha \in H_{2,0}^2(\mathbb{R}^n)$ satisfies the following system:

$$\left\{ \begin{array}{ll} \Delta^2 \bar{u}_\alpha = |\bar{u}_\alpha|^{2^\sharp - 2} \bar{u}_\alpha + \left(\mu_\alpha^{\frac{n-4}{2}} \alpha\right)^{2^\sharp - 1} \rho\left(\frac{\bar{u}_\alpha}{\alpha \mu_\alpha^{\frac{n-4}{2}}}\right) & \text{in } B_\alpha \\ \bar{u}_\alpha \not\equiv 0 & \\ \bar{u}_\alpha = \frac{\partial \bar{u}_\alpha}{\partial n} = 0 & \text{on } \partial B_\alpha, \end{array} \right. \tag{48}$$

Step 2: We now claim that

$$\lim_{\alpha \to 0} \frac{d(x_\alpha, \partial B)}{\mu_\alpha} = +\infty. \tag{49}$$

We prove this claim by contradiction. Assume that

$$\lim_{\alpha \to 0} \frac{d(x_\alpha, \partial B)}{\mu_\alpha} = R \in [0, +\infty).$$

Since

$$\int_{\mathbb{R}^n} (\Delta \bar{u}_\alpha)^2 \, dx = \int_{B_\alpha} (\Delta \bar{u}_\alpha)^2 \, dx = \int_B (\Delta u_\alpha)^2 \, dx = O(1),$$

it comes that $\|\bar{u}_\alpha\|_{H^2_{2,0}(\mathbb{R}^n)}$ is bounded. Then, up to a subsequence, $\bar{u}_\alpha \rightharpoonup \bar{u} \in H^2_{2,0}(\mathbb{R}^n)$. It then follows that

$$\int_{\mathbb{R}^n} |\bar{u}|^{2^\sharp} \, dx \leq \liminf_{\alpha \to 0} \int_{B_\alpha} |\bar{u}_\alpha|^{2^\sharp} \, dx = \liminf_{\alpha \to 0} \int_B |u_\alpha|^{2^\sharp} \, dx = \frac{1}{K_0^{\frac{n}{4}}}.$$

Since u_α is radially symmetrical, we can assume that $x_\alpha = x_0 - R_\alpha \mu_\alpha \vec{n}_{x_0}$, where $x_0 \in \partial B$, $\vec{n}_{x_0}$ is the unit outward vector at x_0 and $R_\alpha \to R$, $R_\alpha > 0$. Clearly, for all $K > 0$ and all $\tilde{R} < R$, there exists $\alpha_0 > 0$ such that

$$\Omega_{K, \tilde{R}} = B(0, K) \cap \{x \in \mathbb{R}^n / (x, \vec{n}_{x_0}) < \tilde{R}\} \subset\subset B_\alpha$$

for all $\alpha \in (0, \alpha_0)$. We denote by $\mathcal{P}_R$ the open half-plane

$$\mathcal{P}_R = \{x \in \mathbb{R}^n \,/\, (x, \vec{n}_{x_0}) < R\}.$$

For all $\varphi \in C_c^\infty(\mathcal{P}_R)$, we define $\hat{\varphi}_\alpha \in C_c^\infty(B)$ such that

$$\varphi(x) = \mu_\alpha^{\frac{n-4}{2}} \hat{\varphi}_\alpha(x_\alpha + \mu_\alpha x),$$

for all $x \in \mathbb{R}^n$. With (I_α) and a change of variable, we have that

$$\int_{\mathbb{R}^n} \Delta \bar{u}_\alpha \Delta \varphi \, dx = \int_{\mathbb{R}^n} \left[|\bar{u}_\alpha|^{2^\sharp - 2} \bar{u}_\alpha + \left(\alpha \mu_\alpha^{\frac{n-4}{2}} \right)^{2^\sharp - 1} \rho \left(\frac{\bar{u}_\alpha}{\alpha \mu_\alpha^{\frac{n-4}{2}}} \right) \right] \varphi \, dx. \tag{50}$$

Letting α go to 0, it comes that

$$\int_{\mathbb{R}^n} \Delta \bar{u} \Delta \varphi \, dx = \int_{\mathbb{R}^n} |\bar{u}|^{2^\sharp - 2} \bar{u} \varphi \, dx,$$

for all $\varphi \in \mathcal{D}(\mathcal{P}_R)$. We now claim that $\bar{u}(x) = 0$ almost everywhere on $\mathbb{R}^n - \bar{\mathcal{P}}_R$. Let $x \in \mathbb{R}^n$ such that $(x, \vec{n}_{x_0}) > R$. Then, for α small, $x_\alpha + \mu_\alpha x \notin B$, and $\bar{u}_\alpha(x) = 0$. Since $\bar{u}_\alpha(x) \to \bar{u}(x)$ almost everywhere, we get that $\bar{u}(x) = 0$ almost everywhere on $\{x \in \mathbb{R}^n \,/\, (x, \vec{n}_{x_0}) > R\}$. This claim is proved. It then follows that $\bar{u} \in H^2_{2,0}(\mathcal{P}_R)$, and that $\int_{\mathbb{R}^n} (\Delta \bar{u})^2 \, dx = \int_{\mathbb{R}^n} |\bar{u}|^{2^\sharp} \, dx$. With some arguments similar to the ones proceeded in the proof of Step 1, we get that $\bar{u} \equiv 0$. We define

$v_\alpha(x) = \bar{u}_\alpha(x + R_\alpha \vec{n}_{x_0})$, $x_0 + \mu_\alpha x \in B$. Clearly, there exists a diffeomorphism $\varphi_\alpha : B(0, R+2) \to \mathcal{U}_\alpha$, where $\mathcal{U}_\alpha$ is an open subset of $\mathbb{R}^n$, such that for any $x = (x_1, ..., x_n) \in B(0, R+2)$,

$$x_0 + \mu_\alpha \varphi_\alpha(x) \in B \Leftrightarrow x_n < 0.$$

We now set $\tilde{v}_\alpha = v_\alpha \circ \varphi_\alpha$. Clearly, there exists a second order operator L_α on $B(0, R+2)$ such that

$$\begin{cases} L_\alpha^2 \tilde{v}_\alpha = |\tilde{v}_\alpha|^{2^\sharp-2}\tilde{v}_\alpha + \left(\alpha\mu_\alpha^{\frac{n-4}{2}}\right)^{2^\sharp-1} \rho\left(\frac{\tilde{v}_\alpha}{\alpha\mu_\alpha^{\frac{n-4}{2}}}\right) & \text{in } B(0,R+2)\cap\{x_n<0\} \\ \tilde{v}_\alpha = \frac{\partial \tilde{v}_\alpha}{\partial n} = 0 & \text{on } B(0,R+2)\cap\{x_n=0\} \end{cases}$$

We can write L_α^2 as follows:

$$L_\alpha^2 = a_\alpha^{ijkl}\partial_{ijkl} + P_\alpha(\nabla, \nabla^2, \nabla^3),$$

where P_α is a polynomial with continuous and uniformly bounded coefficients, and a_{ijkl}^α is also continuous and uniformly bounded with respect to α. Moreover, we have that

$$\frac{1}{2}|X|^4 \leq a_{ijkl}^\alpha X_i X_j X_k X_l \leq 2|X|^4,$$

for all $X \in \mathbb{R}^n$. It then follows from Theorem 15.3 of Agmon-Douglis-Nirenberg [ADN] that for all $p > 1$, there exists $C_p > 0$ such that

$$\|\tilde{v}_\alpha\|_{H_4^p(B(0,R+1)\cap\{x_n<0\})} \leq C_p.$$

Here, we have used that $|\tilde{v}_\alpha| \leq 1$. It then follows that, up to a subsequence, $\tilde{v}_\alpha$ converges to a continuous function in $C^0(\overline{B}(0,R+1))$. But since $\bar{u}_\alpha \rightharpoonup 0$ weakly, it easily comes that $\tilde{v}_\alpha \to 0$ in $C^0(\overline{B}(0,R+1))$. A contradiction, since $1 = \tilde{v}_\alpha(-R_\alpha \vec{n}_{x_0})$. This proves (49) and our claim.

Thanks to (48) and (49), it then follows by standard regularity theory that $\bar{u}_\alpha$ is bounded in $C^{4,\beta}_{loc}(\mathbb{R}^n)$, with $\beta \in (0,1)$. Then, there exists $U_0 \in C^4(\mathbb{R}^n)$ such that

$$\bar{u}_\alpha \to U_0 \text{ in } C^4_{loc}(\mathbb{R}^n). \tag{51}$$

U_0 verifies that $\Delta^2 U_0 = |U_0|^{2^\sharp-2}U_0$, $|U_0(x)| \leq U_0(0) = 1$ for all $x \in \mathbb{R}^n$. With some arguments similar to the ones proceeded in Step 1, it comes that U_0 is an extremal for the Sobolev inequality (4). It follows from [Lin], [HeRo] that

$$U_0(x) = U(x) = \left(\frac{a_n^2}{a_n^2 + |x|^2}\right)^{\frac{n-4}{2}},$$

for all $x \in \mathbb{R}^n$, where U was defined in (3).

Step 3: We now claim that

$$x_\alpha = o(\mu_\alpha). \tag{52}$$

We prove this claim by contradiction. We borrow ideas from Faget [Fag]. We assume that there exists $\eta > 0$ such that $\frac{|x_\alpha|}{\mu_\alpha} \geq \eta$ up to a subsequence. Let $\vec{n}_0 \in \mathbb{R}^n$ such that $|\vec{n}_0| = 1$. Up to a rotation, we can assume that $x_\alpha = |x_\alpha|\vec{n}_0$. We let $N \in \mathbb{N}^\star$ and σ an isometry of $\mathbb{R}^n$ such that $\sigma^i(\vec{n}_0) \neq \vec{n}_0$ for $1 \leq i < N$ and $\sigma^N(\vec{n}_0) = \vec{n}_0$. We let $\delta > 0$ such that

$$\delta < \frac{1}{3}\eta \inf_{\substack{i\neq j \\ 0\leq i,j<N}} |\sigma^i(\vec{n}_0) - \sigma^j(\vec{n}_0)|. \tag{53}$$

We now define $B_\alpha^i = B(\sigma^i(x_\alpha), \delta\mu_\alpha)$ for all $i = 0, ..., N-1$. We claim that $B_\alpha^i \cap B_\alpha^j = \emptyset$ for all $i \neq j \in [0, N-1)$. We prove this claim by contradiction. We assume that there exist $k \neq l \in [0, N-1)$ such that $B_\alpha^k \cap B_\alpha^l \neq \emptyset$. It then follows that

$$|\sigma^k(x_\alpha) - \sigma^l(x_\alpha)| < 2\delta\mu_\alpha.$$

Using that $x_\alpha = |x_\alpha| \vec{n}_0$, it comes that

$$\eta\mu_\alpha \inf_{\substack{i\neq j\\ 0\leq i,j<N}} |\sigma^i(\vec{n}_0) - \sigma^j(\vec{n}_0)| \leq |x_\alpha| \cdot |\sigma^k(\vec{n}_0) - \sigma^l(\vec{n}_0)| < 2\delta\mu_\alpha,$$

a contradiction with (53). This claim is proved. Now, using that u_α is radially symmetrical, we get that

$$\int_B |u_\alpha|^{2^\sharp}\, dx \geq \int_{\cup_{i=0}^{N-1} B_\alpha^i} |u_\alpha|^{2^\sharp}\, dx = \sum_{i=0}^{N-1} \int_{B_\alpha^i} |u_\alpha|^{2^\sharp}\, dx$$
$$\geq N \int_{B(x_\alpha,\delta\mu_\alpha)} |u_\alpha|^{2^\sharp}\, dx = N \int_{B(0,\delta)} |\bar{u}_\alpha|^{2^\sharp}\, dx.$$

Now, using (51) and (47), it comes that

$$N \int_{B(0,\delta)} U^{2^\sharp}\, dx \leq \frac{1}{K_0^{\frac{n}{4}}}$$

for all $N \in \mathbb{N}^\star$. A contradiction with (3) and $\delta > 0$. It then follows that $\frac{|x_\alpha|}{\mu_\alpha} \to 0$, and the claim is proved.

Step 4: We claim that

$$\|u_\alpha - U_{\mu_\alpha}\|_{H_2^2(B)} \to 0. \tag{54}$$

where U_{μ_α} is defined in (5). We prove the claim. We introduce a new rescaled function

$$\tilde{u}_\alpha(x) = \mu_\alpha^{\frac{n-4}{2}} u_\alpha(\mu_\alpha x) \text{ if } x \in B\left(0, \frac{1}{\mu_\alpha}\right),$$

and $\tilde{u}_\alpha(x) = 0$ elsewhere. Clearly, $\tilde{u}_\alpha$ satisfies the following system:

$$\begin{cases} \Delta^2 \tilde{u}_\alpha = |\tilde{u}_\alpha|^{2^\sharp-2}\tilde{u}_\alpha + \left(\mu_\alpha^{\frac{n-4}{2}}\alpha\right)^{2^\sharp-1} \rho\left(\frac{\tilde{u}_\alpha}{\alpha\mu_\alpha^{\frac{n-4}{2}}}\right) & \text{in } B\left(0, \frac{1}{\mu_\alpha}\right)\\ \tilde{u}_\alpha \not\equiv 0 & \\ \tilde{u}_\alpha = \frac{\partial \tilde{u}_\alpha}{\partial n} = 0 & \text{on } \partial B\left(0, \frac{1}{\mu_\alpha}\right), \end{cases}$$

and $\tilde{u}_\alpha$ is radially symmetrical. It follows from (51), (52) that

$$\tilde{u}_\alpha(0) \to 1 \text{ and } \tilde{u}_\alpha \to U \text{ in } C^4_{loc}(\mathbb{R}^n).$$

Let $R > 0$. It then follows that

$$\int_{B(0,R\mu_\alpha)} |u_\alpha|^{2^\sharp}\, dx = \int_{B(0,R)} |\tilde{u}_\alpha|^{2^\sharp}\, dx = \int_{B(0,R)} U^{2^\sharp}\, dx + o(1).$$

Now, by dominated convergence,

$$\lim_{R\to+\infty} \int_{B(0,R)} U^{2^\sharp}\, dx = \int_{\mathbb{R}^n} U^{2^\sharp}\, dx = \frac{1}{K_0^{\frac{n}{4}}}.$$

With (47), it comes that

$$\int_{B-B(0,R\mu_\alpha)} |u_\alpha|^{2^\sharp}\, dx = \varepsilon(R) + o(1),$$

where $\lim_{R\to+\infty}\varepsilon(R)=0$. Similarly,

$$\int_{B-B(0,R\mu_\alpha)}(\Delta u_\alpha)^2\,dx=\varepsilon(R)+o(1),$$

where $\lim_{R\to+\infty}\varepsilon(R)=0$. Now we get that

$$\begin{aligned}&\int_B\left(\Delta(u_\alpha-U_{\mu_\alpha})\right)^2\,dx\\&=\int_{B(0,R\mu_\alpha)}\left(\Delta(u_\alpha-U_{\mu_\alpha})\right)^2\,dx+\int_{B-B(0,R\mu_\alpha)}\left(\Delta(u_\alpha-U_{\mu_\alpha})\right)^2\,dx\\&=\int_{B(0,R)}\left(\Delta(\tilde u_\alpha-U)\right)^2\,dx+O\left(\int_{B-B(0,R\mu_\alpha)}\left((\Delta u_\alpha)^2+(\Delta U_{\mu_\alpha})^2\right)\,dx\right)\\&=o(1)+\varepsilon(R)\end{aligned}$$

with the strong convergence of $\tilde u_\alpha$ on compact subsets. Consequently,

$$\int_B\left(\Delta(u_\alpha-U_{\mu_\alpha})\right)^2\,dx\to 0.$$

Now, clearly, $u_\alpha-U_{\mu_\alpha}\to 0$ in $H_1^2(B)$, the Sobolev space of first order. And then,

$$\|u_\alpha-U_{\mu_\alpha}\|_{H_2^2(B)}\to 0.$$

The claim is proved.

Now, for $\alpha,\varepsilon>0$, we consider the function $v_{\varepsilon,\alpha}\in C^4(\overline{B})$ defined in (7). Following [AMS], we now consider the minimization problem:

$$\inf_{\substack{0<\varepsilon\le 1\\0<a\le 2}}F_{\varepsilon,a}(u_\alpha),$$

where

$$F_{\varepsilon,a}(u_\alpha)=\int_B\left[\Delta\left(u_\alpha-a\left(U_\varepsilon+v_{\varepsilon,\alpha}\right)\right)\right]^2\,dx.\tag{55}$$

With (9) and (54), it comes that

$$\lim_{\alpha\to 0}\inf_{\substack{0<\varepsilon\le 1\\0<a\le 2}}F_{\varepsilon,a}(u_\alpha)=0.\tag{56}$$

We choose $\alpha_0>0$ such that

$$\inf_{\substack{0<\varepsilon\le 1\\0<a\le 2}}F_{\varepsilon,a}(u_\alpha)<\frac13\int_{\mathbb{R}^n}(\Delta U)^2\,dx\text{ and }\int_B(\Delta u_\alpha)^2\,dx>\frac12\int_{\mathbb{R}^n}(\Delta U)^2\,dx$$

for all $\alpha\in(0,\alpha_0)$.

Step 5: We claim that this infimum is attained at $\varepsilon_\alpha,a_\alpha$, and then that $\varepsilon_\alpha\to 0$ and $a_\alpha\to 1$ when $\alpha\to 0$. We prove the claim. We fix $\alpha\in(0,\alpha_0)$. We let $a_p\in(0,2],\varepsilon_p\in(0,1]$ such that

$$\inf_{\substack{0<\varepsilon\le 1\\0<a\le 2}}F_{\varepsilon,a}(u_\alpha)=F_{\varepsilon_p,a_p}(u_\alpha)+o(1),$$

where $o(1)\to 0$ when $p\to+\infty$. If $a_p\to 0$ when $p\to+\infty$, then

$$\inf_{\substack{0<\varepsilon\le 1\\0<a\le 2}}F_{\varepsilon,a}(u_\alpha)=\int_B(\Delta u_\alpha)^2\,dx.$$

A contradiction with the choice of α. Then $a_p \to a_\alpha \in (0,2]$ when $p \to +\infty$. If $\varepsilon_p \to 0$, then for all $\delta \in (0,1)$,

$$\begin{aligned}\int_{B-B(0,\delta)} (\Delta u_\alpha)^2\,dx &= \int_{B-B(0,\delta)} \left[\Delta\left(u_\alpha - a_p\left(U_{\varepsilon_p} + v_{\varepsilon_p,\alpha}\right)\right)\right]^2\,dx + o(1)\\ &\le \inf_{\substack{0<\varepsilon\le 1\\0<a\le 2}} F_{\varepsilon,a}(u_\alpha) + o(1)\\ &< \frac{1}{3}\int_{\mathbb{R}^n} (\Delta U)^2\,dx + o(1)\end{aligned}$$

Letting $p \to +\infty$, we get that $\int_{B-B(0,\delta)} (\Delta u_\alpha)^2\,dx \le \frac{1}{3}\int_{\mathbb{R}^n} (\Delta U)^2\,dx$. Passing to the limit $\delta \to 0$, we get a contradiction with the choice of α. Then $\varepsilon_p \to \varepsilon_\alpha \in (0,1]$ when $p \to +\infty$. So the infimum is attained at $\varepsilon_\alpha, a_\alpha$. Assume that $a_\alpha \to 0$. Then

$$\begin{aligned}\int_B (\Delta u_\alpha)^2\,dx &= F_{\varepsilon_\alpha,a_\alpha}(u_\alpha) + o(1)\\ &= \inf_{\substack{0<\varepsilon\le 1\\0<a\le 2}} F_{\varepsilon,a}(u_\alpha) = o(1)\end{aligned}$$

when $\alpha \to 0$. A contradiction. Then $a_\alpha \not\to 0$. Assume that $\varepsilon_\alpha \to \varepsilon_0 > 0$. We have that

$$\int_{B(0,R\mu_\alpha)} \left[\Delta\left(u_\alpha - a_\alpha\left(U_{\varepsilon_\alpha} + v_{\varepsilon_\alpha,\alpha}\right)\right)\right]^2\,dx = o(1)$$

for all $R > 0$. Passing through the limit with (51), we get that $\int_{B(0,R)} (\Delta U)^2\,dx = 0$ for all $R > 0$. A contradiction. Then $\varepsilon_\alpha \to 0$, and $v_{\varepsilon_\alpha,\alpha} \to 0$ in $H^2_{2,0}(B)$ when $\alpha \to 0$ (see (9)). Now, with (55) and (56),

$$\begin{aligned}\int_B (\Delta u_\alpha)^2\,dx &= \left(\lim_{\alpha\to 0} a_\alpha\right)^2 \int_{\mathbb{R}^n} (\Delta U)^2\,dx + o(1)\\ &= \frac{(\lim_{\alpha\to 0} a_\alpha)^2}{K_0^{\frac{n}{4}}} + o(1)\end{aligned}$$

But with (I_α) and (47), we get that $\int_B (\Delta u_\alpha)^2\,dx = \int_B |u_\alpha|^{2^\sharp}\,dx = \frac{1}{K_0^{\frac{n}{4}}} + o(1)$. Consequently, $a_\alpha \to 1$ when $\alpha \to 0$. This proves our claim.

We now write

$$u_\alpha = a_\alpha\left(U_{\varepsilon_\alpha} + v_{\varepsilon_\alpha,\alpha}\right) + w_\alpha. \tag{57}$$

Clearly $w_\alpha \to 0$ in $H^2_{2,0}(B)$. For the sake of simplicity, we now write $\varepsilon = \varepsilon_\alpha \to 0$ and $v_\alpha = v_{\varepsilon_\alpha,\alpha}$. Differentiating $F_{\varepsilon,a}(u_\alpha)$ with respect to ε and a, we get that

$$\int_B \Delta w_\alpha \Delta(U_\varepsilon + v_\alpha)\,dx = 0, \tag{58}$$

$$\int_B \Delta w_\alpha \Delta \frac{\partial}{\partial\varepsilon}(U_\varepsilon + v_\alpha)\,dx = 0. \tag{59}$$

Next section is devoted to obtaining asymptotic estimates on $\|w_\alpha\|_{H^2_{2,0}(B)} \to 0$ and $1 - a_\alpha \to 0$.

5. Blow-up analysis II

We follow the techniques developed in [AMS].

Step 1: Some integrations by parts and (58) lead to

$$\begin{aligned}\int_B (U_\varepsilon+v_\alpha)\,\Delta^2 u_\alpha\,dx &= \int_B \Delta(U_\varepsilon+v_\alpha)\,\Delta u_\alpha\,dx\\ &= a_\alpha\int_B \Delta(U_\varepsilon+v_\alpha)\,\Delta(U_\varepsilon+v_\alpha)\,dx\\ &= a_\alpha\int_B (U_\varepsilon+v_\alpha)\left(\Delta^2U_\varepsilon+\Delta^2 v_\alpha\right)\,dx\end{aligned}$$

Using (I_α), (7) and (6) we get that

$$\begin{aligned}&\int_B \left(aU_\varepsilon^{2^\sharp-1}-|u_\alpha|^{2^\sharp-2}u_\alpha\right)(U_\varepsilon+v_\alpha)\,dx\\ &= \alpha^{2^\sharp-1}\int_B\left[\rho\left(\frac{u_\alpha}{\alpha}\right)-a_\alpha\rho\left(\frac{U_\varepsilon}{\alpha}\right)\right](U_\varepsilon+v_\alpha)\,dx. \qquad (60)\end{aligned}$$

Clearly, for all $p>1$ there exist $C_p>0$ such that

$$\left||x+y|^{p-1}(x+y)-|x|^{p-1}x\right|\le C_p\left(|y|^p+|x|^{p-1}|y|\right), \qquad (61)$$

for all $x,y\in\mathbb{R}$. Inequalities (11), (61), (13), the definition (57), and some Hölder inequalities lead to

$$\begin{aligned}&\int_B\left(|u_\alpha|^{2^\sharp-2}u_\alpha-a_\alpha U_\varepsilon^{2^\sharp-1}\right)(U_\varepsilon+v_\alpha)\,dx\\ &=\left(a_\alpha^{2^\sharp-1}-a_\alpha\right)\int_B U_\varepsilon^{2^\sharp-1}(U_\varepsilon+v_\alpha)\,dx\\ &+\int_B\left(|u_\alpha|^{2^\sharp-2}u_\alpha-(a_\alpha U_\varepsilon)^{2^\sharp-1}\right)(U_\varepsilon+v_\alpha)\,dx\\ &=(2^\sharp-2)(a_\alpha-1)\int_{\mathbb{R}^n}U_\varepsilon^{2^\sharp}\,dx+O\left(\|w_\alpha\|_{2^\sharp}\right)+o\left(\|\Delta v_\alpha\|_2\right)\\ &+o\left(\varepsilon^{\frac{n-4}{2}}\right)+o(a_\alpha-1) \qquad (62)\end{aligned}$$

Now, since ρ' is bounded, there exists $C>0$ such that $|\rho(x)-\rho(y)|\le C|x-y|$ for all $x,y\in\mathbb{R}$. It then comes that

$$\begin{aligned}&\left|\alpha^{2^\sharp-1}\int_B\left[\rho\left(\frac{u_\alpha}{\alpha}\right)-a_\alpha\rho\left(\frac{U_\varepsilon}{\alpha}\right)\right](U_\varepsilon+v_\alpha)\,dx\right|\\ &\le\alpha^{2^\sharp-1}\int_B\left|\rho\left(\frac{u_\alpha}{\alpha}\right)-\rho\left(\frac{U_\varepsilon}{\alpha}\right)\right|(U_\varepsilon+v_\alpha)\,dx\\ &\quad+|a_\alpha-1|\int_B\left|\rho\left(\frac{U_\varepsilon}{\alpha}\right)\right|(U_\varepsilon+v_\alpha)\,dx\\ &\le C\alpha^{2^\sharp-2}\int_B|u_\alpha-U_\varepsilon|\times|U_\varepsilon+v_\alpha|\,dx+o(a_\alpha-1)\\ &\le C\alpha^{2^\sharp-2}\int_B|(a_\alpha-1)U_\varepsilon+av_\alpha+w_\alpha|\times|U_\varepsilon+v_\alpha|\,dx+o(a_\alpha-1)\\ &=o(a_\alpha-1)+o\left(\|v_\alpha\|_{2^\sharp}\right)+o\left(\|w_\alpha\|_{2^\sharp}\right)\end{aligned}$$

This inequality combined with (13), (60) and (62) then gives that

$$a_\alpha - 1 = O\left(\|w_\alpha\|_{2^\sharp}\right) + o\left(\|\Delta v_\alpha\|_2\right) + o\left(\varepsilon^{\frac{n-4}{2}}\right). \tag{63}$$

Step 2: We go on with the estimates of $\|w_\alpha\|_{H^2_{2,0}(B)}$ and $1 - a_\alpha$. First note that for all $p > 1$, for all $\theta \in (0, \min\{1, p-1\}]$, there exists $C_{p,\theta} > 0$ such that

$$\left||x+y|^{p-1}(x+y) - |x|^{p-1}x - p|x|^{p-1}y\right| \le C_{p,\theta}\left(|y|^p + |x|^{p-1-\theta}|y|^{1+\theta}\right),$$

for any $x, y \in \mathbb{R}$. This inequality and Hölder's inequality give that

$$\int_B w_\alpha |u_\alpha|^{2^\sharp-2}u_\alpha\, dx - \int_B w_\alpha \left(a_\alpha U_\varepsilon\right)^{2^\sharp-1}\, dx = (2^\sharp - 1)\int_B U_\varepsilon^{2^\sharp-2} w_\alpha^2\, dx$$
$$+ o\left(\|v_\alpha\|_{2^\sharp}^2\right) + o\left(\|w_\alpha\|_{2^\sharp}^2\right) + (2^\sharp-1)a_\alpha^{2^\sharp-1}\int_B U_\varepsilon^{2^\sharp-2} v_\alpha w_\alpha\, dx.$$

With some more Hölder inequality, we get that

$$\int_B w_\alpha |u_\alpha|^{2^\sharp-2}u_\alpha\, dx - \int_B w_\alpha \left(a_\alpha U_\varepsilon\right)^{2^\sharp-1}\, dx = (2^\sharp - 1)\int_B U_\varepsilon^{2^\sharp-2} w_\alpha^2\, dx$$
$$+ o\left(\|v_\alpha\|_{2^\sharp}^2\right) + o\left(\|w_\alpha\|_{2^\sharp}^2\right) + O\left(\|w_\alpha\|_{2^\sharp}\|U_\varepsilon\|_{2^\sharp}^{\frac{2^\sharp-2}{2}}\sqrt{\int_B U_\varepsilon^{2^\sharp-2}v_\alpha^2\, dx}\right). \tag{64}$$

Some integrations by parts and (58) give that

$$\begin{aligned}\int_B \left(\Delta w_\alpha\right)^2\, dx &= \int_B \Delta w_\alpha \Delta\left(u_\alpha - a_\alpha\left(U_\varepsilon + v_\alpha\right)\right)\, dx\\ &= \int_B \Delta w_\alpha \Delta u_\alpha\, dx = \int_B w_\alpha \Delta^2 u_\alpha\, dx\end{aligned}$$

and that

$$\int_B w_\alpha \Delta^2\left(U_\varepsilon + v_\alpha\right)\, dx = \int_B \Delta w_\alpha \Delta\left(U_\varepsilon + v_\alpha\right)\, dx = 0.$$

And then

$$\begin{aligned}\int_B \left(\Delta w_\alpha\right)^2\, dx &= \int_B w_\alpha \Delta^2 u_\alpha\, dx - a_\alpha^{2^\sharp-1}\int_B w_\alpha \Delta^2\left(U_\varepsilon + v_\alpha\right)\, dx\\ &= \int_B w_\alpha\left[u_\alpha^{2^\sharp-1}\, dx - \left(a_\alpha U_\varepsilon\right)^{2^\sharp-1}\right]\, dx\\ &\quad + \alpha^{2^\sharp-1}\int_B w_\alpha\left[\rho\left(\frac{u_\alpha}{\alpha}\right) - a_\alpha^{2^\sharp-1}\rho\left(\frac{U_\varepsilon}{\alpha}\right)\right]\, dx\end{aligned} \tag{65}$$

Similarly to what was done in Step 1, we get that

$$\alpha^{2^\sharp-1}\int_B w_\alpha\left[\rho\left(\frac{u_\alpha}{\alpha}\right) - a_\alpha^{2^\sharp-1}\rho\left(\frac{U_\varepsilon}{\alpha}\right)\right]\, dx$$
$$= o(|a_\alpha - 1|\|w_\alpha\|_{H^2_{2,0}(B)}) + o(\|w_\alpha\|_{2^\sharp}^2) + o(\|v_\alpha\|_{2^\sharp}^2).$$

Plugging (64) and this latest equality in (65), and using (6) and (11), it comes that

$$\begin{aligned}\int_B \left(\Delta w_\alpha\right)^2\, dx &= (2^\sharp-1)\int_B U_\varepsilon^{2^\sharp-2}w_\alpha^2\, dx + o(|a_\alpha - 1|\|w_\alpha\|_{H^2_{2,0}(B)}) + o(\|w_\alpha\|_{2^\sharp}^2)\\ &\quad + o(\|\Delta v_\alpha\|_2^2) + o\left(\varepsilon^{n-4}\right).\end{aligned}$$

Now, with (63), it comes that

$$\begin{aligned}\int_B (\Delta w_\alpha)^2\, dx &= (2^\sharp - 1)\int_B U_\varepsilon^{2^\sharp-2} w_\alpha^2\, dx + o(\|w_\alpha\|_{2^\sharp}^2) \\ &\quad + o(\|\Delta v_\alpha\|_2^2) + o\left(\varepsilon^{n-4}\right). \qquad (66)\end{aligned}$$

We now define $\tilde{w}_\alpha \in D_2^2(\mathbb{R}^n)$ such that $\tilde{w}_\alpha(x) = w_\alpha(x)$ on B and $\tilde{w}_\alpha(x) = 0$ elsewhere. We define

$$C_\varepsilon = \frac{\int_{\mathbb{R}^n} \Delta\tilde{w}_\alpha \Delta U_\varepsilon\, dx}{\|\Delta U_\varepsilon\|_2^2},\ C'_\varepsilon = \frac{\int_{\mathbb{R}^n} \Delta\tilde{w}_\alpha \Delta \frac{\partial U_\varepsilon}{\partial\varepsilon}\, dx}{\|\Delta \frac{\partial U_\varepsilon}{\partial \varepsilon}\|_2^2}.$$

Noting that $\tilde{w}_\alpha$ is radially symmetrical, we have that $\int_{\mathbb{R}^n} \Delta\tilde{w}_\alpha \Delta\partial_i U_\varepsilon\, dx = 0$ for all $i = 1...n$. It then follows that $\tilde{w}_\alpha - C_\varepsilon U_\varepsilon - C'_\varepsilon \frac{\partial U_\varepsilon}{\partial\varepsilon}$ is orthogonal to to the space spanned by $U_\varepsilon, \partial_\varepsilon U_\varepsilon, \partial_i U_\varepsilon$, $i = 1...n$. It then follows from proposition 6.1 of section 6 that

$$\int_{\mathbb{R}^n} \left[\Delta\left(\tilde{w}_\alpha - C_\varepsilon U_\varepsilon - C'_\varepsilon \partial_\varepsilon U_\varepsilon\right)\right]^2 dx \geq \lambda_3 \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\left(\tilde{w}_\alpha - C_\varepsilon U_\varepsilon - C'_\varepsilon\partial_\varepsilon U_\varepsilon\right)^2 dx, \qquad (67)$$

where $\lambda_3 > 2^\sharp - 1$ is independant of α. We develop the RHS term, and get

$$\begin{aligned}\int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\left(\tilde{w}_\alpha - C_\varepsilon U_\varepsilon - C'_\varepsilon\partial_\varepsilon U_\varepsilon\right)^2 dx &= \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\tilde{w}_\alpha^2\, dx + C_\varepsilon^2 \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp}\, dx \\ + C_\varepsilon'^2 \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\left(\partial_\varepsilon U_\varepsilon\right)^2 dx - 2C_\varepsilon \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-1}\tilde{w}_\alpha\, dx &- 2C'_\varepsilon \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\partial_\varepsilon U_\varepsilon \tilde{w}_\alpha\, dx \\ + 2C_\varepsilon C'_\varepsilon \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-1}\partial_\varepsilon U_\varepsilon\, dx\end{aligned}$$

Clearly, with (6) and (58),

$$C_\varepsilon = \frac{\int_{\mathbb{R}^n} \Delta\tilde{w}_\alpha \Delta U_\varepsilon\, dx}{\|\Delta U_\varepsilon\|_2^2} = -\frac{\int_B \Delta w_\alpha \Delta v_\alpha\, dx}{\|\Delta U_\varepsilon\|_2^2} = O\left(\|v_\alpha\|_{H_2^2(B)}\|w_\alpha\|_{H_{2,0}^2(B)}\right).$$

But as already noticed, $v_\alpha \to 0$ in $H_2^2(B)$, so $C_\varepsilon = o\left(\|w_\alpha\|_{H_{2,0}^2(B)}\right)$. Then,

$$\begin{aligned}\int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\left(\tilde{w}_\alpha - C_\varepsilon U_\varepsilon - C'_\varepsilon\partial_\varepsilon U_\varepsilon\right)^2 dx &= \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\tilde{w}_\alpha^2\, dx + o\left(\|w_\alpha\|_{H_{2,0}^2(B)}^2\right) \\ + C_\varepsilon'^2 \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\left(\partial_\varepsilon U_\varepsilon\right)^2 dx &- 2C'_\varepsilon \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\partial_\varepsilon U_\varepsilon\tilde{w}_\alpha\, dx\end{aligned}$$

With the equation verified by $\partial_\varepsilon U_\varepsilon$ (see (71)), its expression in (6), the expression of v_α in (7) and (59), we get that

$$C_\varepsilon'^2 \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\left(\partial_\varepsilon U_\varepsilon\right)^2 dx = \frac{\left(\int_B \Delta w_\alpha \Delta\partial_\varepsilon v_\alpha\, dx\right)^2}{(2^\sharp - 1)\int_B (\Delta\frac{\partial U_\varepsilon}{\partial\varepsilon})^2\, dx} = o\left(\|w_\alpha\|_{H_{2,0}^2(B)}^2\right).$$

Since $\|U_\varepsilon\|_{2^\sharp}$ is bounded, we get with Hölder inequality that

$$\begin{aligned}C'_\varepsilon \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\partial_\varepsilon U_\varepsilon\tilde{w}_\alpha\, dx &= O\left(\|w_\alpha\|_{H_{2,0}^2(B)} C'_\varepsilon\sqrt{\int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\left(\partial_\varepsilon U_\varepsilon\right)^2 dx}\right) \\ &= o\left(\|w_\alpha\|_{H_{2,0}^2(B)}^2\right).\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\left(\tilde{w}_\alpha - C_\varepsilon U_\varepsilon - C'_\varepsilon \partial_\varepsilon U_\varepsilon\right)^2\, dx = \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\tilde{w}_\alpha^2\, dx + o\left(\|w_\alpha\|^2_{H^2_{2,0}(B)}\right). \quad (68)$$

Similarly, we get that

$$\int_{\mathbb{R}^n}\left[\Delta\left(\tilde{w}_\alpha - C_\varepsilon U_\varepsilon - C'_\varepsilon \partial_\varepsilon U_\varepsilon\right)\right]^2\, dx = \int_{\mathbb{R}^n}\left(\Delta\tilde{w}_\alpha\right)^2\, dx + o\left(\|w_\alpha\|^2_{H^2_{2,0}(B)}\right). \quad (69)$$

Plugging (68) and (69) into (67), we obtain that

$$\int_{\mathbb{R}^n}\left(\Delta\tilde{w}_\alpha\right)^2\, dx \geq \lambda_3 \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2}\tilde{w}_\alpha^2\, dx + o\left(\|w_\alpha\|^2_{H^2_{2,0}(B)}\right)$$

with $\lambda_3 > 2^\sharp - 1$. Now, plugging this inequality into (66), and using (63), we get that

$$\|w_\alpha\|_{H^2_{2,0}(B)} = o\left(\|\Delta v_\alpha\|_2\right) + o\left(\varepsilon^{\frac{n-4}{2}}\right)$$

$$1 - a_\alpha = o\left(\|\Delta v_\alpha\|_2\right) + o\left(\varepsilon^{\frac{n-4}{2}}\right)$$

Step 3: We now prove the last part of theorem 1.1. It then follows from the estimate (40) of section 2 that

$$\begin{aligned}
J_\alpha(u_\alpha) = &\frac{2}{nK_0^{\frac{n}{4}}}\\
&- a_n^n \omega_{n-1}\varepsilon^{\frac{n}{2}}\alpha^{\frac{n}{n-4}}\left[\frac{I_1(\rho)}{n} - \frac{I_2(\rho)}{2}\varepsilon\alpha^{\frac{2}{n-4}} + \frac{a_n^4 I_3(\rho)}{2(n-4)^5}\alpha^{\frac{4}{n-4}}\varepsilon^2 + o\left(\alpha^{\frac{4}{n-4}}\varepsilon^2\right)\right]\\
&+ \frac{(n-4)(n-2)^2\omega_{n-1}a_n^{2(n-4)}}{2}\varepsilon^{n-4}\\
&+ o(\varepsilon^{n-4}) + O(\alpha^{\frac{8}{n-4}}\varepsilon^{n-4}) + O\left(\alpha^{\frac{n+6}{n-4}}\varepsilon^{\frac{n+6}{2}}\right).
\end{aligned}$$

when $n \geq 13$. With (I_α), it comes that $J_\alpha(u_\alpha) \leq \frac{2}{nK_0^{\frac{n}{4}}}$. The last part of the theorem then follows from the study of the three different cases. This completes the proof of theorem 1.1.

Remark: it follows from (41) and (42) that when $n \geq 9$, we get that $I_1(\rho) \geq 0$. If $n \geq 11$ and $I_1(\rho) = 0$, then $I_2(\rho) \leq 0$. If $n \geq 13$ and $I_1(\rho) = I_2(\rho) = 0$, then $I_3(\rho) \geq 0$.

6. A fourth order eigenvalue problem on $\mathbb{R}^n$

This section is devoted to the proof of the following proposition:

Proposition 6.1. *We consider the following eigenvalue problem:*

$$\Delta^2 u = \lambda U_\varepsilon^{2^\sharp-2} u \text{ on } H^2_{2,0}(\mathbb{R}^n).$$

The first eigenvalue is $\lambda = 1$, and its eigenspace is the one-dimensional space spanned by U_ε. The second eigenvalue is $2^\sharp - 1$. Its eigenspace is $(n+1)-$dimensional space spanned by $\partial_\varepsilon U_\varepsilon, (\partial_i U_\varepsilon)_{i=1,\dots,n}$. The third eigenvalue is $\lambda_3 > 2^\sharp - 1$ and is independant of $\varepsilon > 0$. More, for all $u \in H^2_{2,0}(\mathbb{R}^n)$, the following inequality holds

$$\int_{\mathbb{R}^n}(\Delta u)^2\, dx \geq \lambda_3 \int_{\mathbb{R}^n} U_\varepsilon^{2^\sharp-2} u^2\, dx,$$

as soon as

$$\int_{\mathbb{R}^n} \Delta u \Delta U_\varepsilon \, dx = \int_{\mathbb{R}^n} \Delta u \Delta \partial_i U_\varepsilon \, dx = \int_{\mathbb{R}^n} \Delta u \Delta \partial_\varepsilon U_\varepsilon \, dx = 0,$$

for all $i = 1, ..., n$.

We first consider the function $U_0(x) = \left(\frac{1}{1+|x|^2}\right)^{\frac{n-4}{2}}$. We let $\lambda \in \mathbb{R}$ and $\varphi \in H_{2,0}^2(\mathbb{R}^n)$ such that

$$\Delta^2 \varphi = \lambda U_0^{2^\sharp - 2} \varphi. \tag{70}$$

By standard elliptic theory, it comes that $\varphi \in C^\infty(\mathbb{R}^n)$. We denote by $\mathbb{S}^n$ the unit sphere of $\mathbb{R}^{n+1}$, and we consider the stereographic projection on $\mathbb{S}^n$, that is

$$\begin{array}{cccc} \pi : & \mathbb{S}^n - \{N\} & \to & \mathbb{R}^n \\ & x & \mapsto & \left(\frac{x_1}{1-x_{n+1}}, ..., \frac{x_n}{1-x_{n+1}}\right) \end{array}$$

where $N = (0, ..., 0, 1)$ is the north pole. We denote by h the round metric on $\mathbb{S}^n$. The pull-back of h via π gives that

$$\left(\pi^{-1}\right)^\star h = \psi^{\frac{4}{n-4}} \xi,$$

where ξ is the Euclidean metric on $\mathbb{R}^n$ and $\psi(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-4}{4}}$. On $(\mathbb{S}^n, h)$, the Paneitz-Branson operator is

$$P_h^n = \Delta_h^2 + c_n \Delta_h + d_n,$$

where $\Delta_h = -div_h(\nabla)$ is the Laplace-Beltrami operator on $\mathbb{S}^n$ and

$$c_n = \frac{n^2 - 2n - 4}{2} \quad , \quad d_n = \frac{n(n-4)(n^2-4)}{16}.$$

Branson [Bra] showed that this operator enjoys the following nice property: for all $u \in C^\infty(\mathbb{S}^n)$, we get that

$$(P_h^n u) \circ \pi^{-1} = \frac{1}{\psi^{2^\sharp - 1}} \Delta^2 (\psi u \circ \pi^{-1}).$$

Now, for $\tilde{\varphi} \in C_c^\infty(\mathbb{R}^n)$, we define $\tilde{u} = \frac{\tilde{\varphi} \circ \pi}{\psi \circ \pi} \in C^\infty(\mathbb{S}^n)$. It follows from the preceding conformal law that

$$\int_{\mathbb{S}^n} \tilde{u} P_h^n \tilde{u} \, dv_h = \int_{\mathbb{R}^n} (\Delta \tilde{\varphi})^2 \, dx,$$

and

$$\begin{aligned} \frac{1}{16} \int_{\mathbb{S}^n} \tilde{u}^2 \, dv_h &= \int_{\mathbb{S}^n} \left(\frac{U_0}{\psi}\right)^{2^\sharp - 2} \left(\frac{\tilde{\varphi}}{\psi}\right)^2 dv_h \\ &= \int_{\mathbb{R}^n} U_0^{2^\sharp - 2} \tilde{\varphi}^2 \, dx, \end{aligned}$$

where dv_h denotes the volume element on the standard sphere $(\mathbb{S}^n, h)$. Since $\varphi \in H_{2,0}^2(\mathbb{R}^n)$, we let $\varphi_p \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi_p \to \varphi$ in $H_{2,0}^2(\mathbb{R}^n)$. We consider

$$u_p(x) = \frac{\varphi_p \circ \pi}{\psi \circ \pi} \in C^\infty(\mathbb{S}^n).$$

It follows from the preceding equalities that u_p converges to a function $u \in H_2^2(\mathbb{S}^n)$, and that $u(x) = \frac{\varphi\circ\pi(x)}{\psi\circ\pi(x)}$ for all $x \in \mathbb{S}^n - \{N\}$. Here, $H_2^2(\mathbb{S}^n)$ is the second order Sobolev space obtained by completion of $C^\infty(\mathbb{S}^n)$ for the norm

$$\|v\|^2_{H_2^2(\mathbb{S}^n)} = \int_{\mathbb{S}^n} (\Delta_h v)^2\, dv_h + \int_{\mathbb{S}^n} |\nabla v|_h^2\, dv_h + \int_{\mathbb{S}^n} v^2\, dv_h.$$

We have that

$$P_h^n u = \frac{\lambda}{16} u,$$

on $\mathbb{S}^n - \{N\}$ in the distribution sense. Now, following what was done in [HeRo], we take a cut-off function η_s, $s > 0$ such that $\eta_s \equiv 0$ on $B_h(N,s)$, $\eta_s \equiv 1$ in $\mathbb{S}^n - B_h(N, 2s)$, $\|\nabla^k \eta_s\|_\infty \le Cs^{-k}$ for $k = 0,1,2$ and where C is independant of s. We choose $t \in C^\infty(\mathbb{S}^n)$, and we get that $\eta_s t \to t$ in $H_2^2(\mathbb{S}^n)$. We omit the details that can be found in [HeRo]. As a consequence, we get that

$$P_h^n u = \frac{\lambda}{16} u$$

in $\mathcal{D}'(\mathbb{S}^n)$. It follows from standard elliptic theory that $u \in C^\infty(\mathbb{S}^n)$. It follows from [DHL] and [HeRo] that there exists $\mu \in \mathbb{R}$ an element of the spectrum of Δ_h such that $\frac{\lambda}{16} = \mu^2 + c_n\mu + d_n$. More, the eigenspace associated to $\frac{\lambda}{16}$ is the eigenspace of μ, considered as an eigenvalue of Δ_h. Now for L an operator and $i \in \mathbb{N}^\star$, we denote by $\lambda_i(L)$ the i^{th} eigenvalue of L and $E_i(L)$ the corresponding eigenspace. As stated in Berger-Gauduchon-Mazet [BGM], we have that

$$\begin{array}{ll} \lambda_1(\Delta_h) = 0, & dim(E_1(\Delta_h)) = 1 \\ \lambda_2(\Delta_h) = n, & dim(E_2(\Delta_h)) = n+1 \end{array}$$

Now, coming back to our initial question, we obtain that

$$\begin{array}{ll} \lambda_1(P_h^n) = d_n, & dim(E_1(P_h^n)) = 1 \\ \lambda_2(P_h^n) = n^2 + nc_n + d_n = d_n(2^\sharp - 1), & dim(E_2(P_h^n)) = n+1 \\ \lambda_3(P_h^n) > d_n(2^\sharp - 1) & \end{array}$$

We now come back to the initial problem. We let $\lambda \in \mathbb{R}$ and $\varphi \in H^2_{2,0}(\mathbb{R}^n)$ such that

$$\Delta^2\varphi = \lambda U_\varepsilon^{2^\sharp - 2}\varphi.$$

We define $\tilde\varphi(x) = \varphi(a_n\varepsilon x)$, and then

$$\Delta^2\tilde\varphi = 16 d_n \lambda U_0^{2^\sharp-2}\tilde\varphi.$$

Consequently, the three first eigenvalues of (70) $\lambda_1, \lambda_2, \lambda_3$ and their corresponding eigenspaces E_1, E_2, E_3 verify

$$\begin{array}{ll} \lambda_1 = 1, & dim(E_1) = 1 \\ \lambda_2 = 2^\sharp - 1, & dim(E_2) = n+1 \\ \lambda_3 = \frac{\lambda_3(P_h^n)}{d_n} > 2^\sharp - 1 & \text{is independant of } \varepsilon \end{array}$$

Now, as easily checked,

$$\Delta^2 \partial_i U_\varepsilon = (2^\sharp - 1) U_\varepsilon^{2^\sharp-2}\partial_i U_\varepsilon \text{ and } \Delta^2\partial_\varepsilon U_\varepsilon = (2^\sharp - 1)U_\varepsilon^{2^\sharp - 2}\partial_\varepsilon U_\varepsilon, \qquad (71)$$

for all $i = 1, ..., n$, and $\partial_\varepsilon U_\varepsilon, \partial_i U_\varepsilon$ $(i = 1, ..., n)$ are linearly independant. Then the eigenspace of $2^\sharp - 1$ is spanned by these vectors. The one dimensional eigenspace of λ_1 is clearly spanned by U_ε.

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