ADMISSIBLE $Q-$CURVATURES UNDER ISOMETRIES FOR THE CONFORMAL GJMS OPERATORS

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Dedicated to Jean-Pierre Gossez on the occasion of his 65th birthday

1. Introduction and statement of the main result

Let M be a compact manifold of dimension $n\geq 3$ and let $k\geq 1$ be an integer such that $k\leq \frac{n}{2}$ if n is even. In their celebrated work, Graham-Jenne-Mason-Sparling [15] provided a systematic construction of conformally invariant operators (GJMS operators for short) based on the ambient metric of Fefferman-Graham [12, 13]. More precisely, letting $\mathcal{M}$ be the set of Riemannian metrics on M, then for all $g\in\mathcal{M}$, there exists an operator $P_g: C^\infty(M)\to C^\infty(M)$ such that

(i) P_g is a differential operator and $P_g=\Delta_g^k+lot$

(ii) P_g is natural, that is $\varphi^\star P_g=P_{\varphi^\star g}$ for all smooth diffeomorphism $\varphi: M\to M$.

(iii) P_g is self-adjoint with respect to the L^2-scalar product

(iv) Given $\omega\in C^\infty(M)$ and defining $\hat{g}=e^{2\omega}g$, we have that

$$P_{\hat{g}}(f)=e^{-\frac{n+2k}{2}\omega}P_g\left(e^{\frac{n-2k}{2}\omega}f\right)\text{ for all } f\in C^\infty(M). \quad (1)$$

Here $\Delta_g:=-\mathrm{div}_g(\nabla)$ is the Laplace-Beltrami operator and *lot* denotes differential terms of lower order. Point (iii) above is due to Graham-Zworski [16]. For instance, on $\mathbb{R}^n$ endowed with its Euclidean metric ξ, one has that $P_\xi=\Delta_\xi^k$. There is a natural scalar invariant, namely the $Q-$curvature, attached to the operator P_g: this scalar invariant, denoted as Q_g, was initially introduced by Branson and Ørsted [7] for $n=2k=4$ and generalized by Branson [4, 5]. When $k=1$, the GJMS operator is the conformal Laplacian and the $Q-$curvature is the scalar curvature (up to a dimensional constant). When $k=2$, the GJMS operator is the Paneitz operator introduced in [26]. When $n\neq 2k$, the $Q-$curvature is $Q_g:=\frac{2}{n-2k}P_g(1)$: when $n=2k$, the definition is much more subtle and involves a continuation in dimension argument (we refer to the survey Branson-Gover [6] and to Juhl [20] for an exposition in book form). In the spirit of classical problems in conformal geometry, our objective here is to prescribe the $Q-$curvature in a conformal class; that is, given a conformal Riemannian class $\mathcal{C}$ on M and a function $f\in C^\infty(M)$, we investigate the existence of a metric $g\in\mathcal{C}$ such that $Q_g=f$. As one checks (see Proposition 3 below), up to multiplication by a constant, this amounts to finding

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critical points of the perturbation of the Hilbert functional

$$\begin{array}{ccc}\mathcal{C} & \to & \mathbb{R}\\ g & \mapsto & \frac{\int_M Q_g\, dv_g}{V_f(M,g)^{\frac{n-2k}{n}}}\end{array}$$

where $V_f(M,g) := \int_M f\, dv_g$ is the weighted $f-$volume of (M,g). This structure suggests to apply variational methods to prescribe the $Q-$curvature and we define

$$\mu_f(\mathcal{C}) := \inf_{g\in\mathcal{C}} \frac{\int_M Q_g\, dv_g}{V_f(M,g)^{\frac{n-2k}{n}}}.$$

Given a metric $g \in \mathcal{C}$, the conformal class can be described as

$$\mathcal{C} = \{e^{2\omega} g /\, \omega \in C^\infty(M)\}.$$

We assume that $n > 2k$: in this context, it is more convenient to write a metric $\hat{g} \in \mathcal{C}$ as $\hat{g} = u^{\frac{4}{n-2k}} g$ with $u \in C^\infty_+(M)$, the set of positive smooth functions. With this parametrization, we have that

$$\mathcal{C} = \{u^{\frac{4}{n-2k}} g /\, u \in C^\infty_+(M)\},$$

and the relation (1) between P_g and $P_{\hat{g}}$ rewrites

$$P_{\hat{g}}\varphi = u^{1-2^\star} P_g(u\varphi) \tag{2}$$

for all $\varphi \in C^\infty(M)$, where $2^\star := \frac{2n}{n-2k}$. Therefore, taking $\varphi \equiv 1$, we have that

$$P_g u = \frac{n-2k}{2} Q_{\hat{g}} u^{2^\star - 1} \text{ in } M$$

where $\hat{g} = u^{\frac{4}{n-2k}} g$, and then finding a metric in $\mathcal{C}$ with f as $Q-$curvature amounts to solving the variational elliptic equation $P_g u = \frac{n-2k}{2} f u^{2^\star-1}$. Despite this elegant variational structure, this question gives rise to a crucial intrinsic difficulty due to the essence of the problem, that is the conformal invariance of the operator. More precisely, in the spirit of Bourguignon-Ezin [3], Delanoë and the author proved in [9] that

$$\int_M X(Q_g)\, dv_g = 0$$

for all conformal Killing field X on $(M, \mathcal{C})$. When $k = 1$, this is the celebrated Kazdan-Warner obstruction [21] to the scalar curvature problem. In particular, if $\varphi \in C^\infty(\mathbb{S}^n) \setminus \{0\}$ is a first eigenfunction of the Laplace-Beltrami operator on the standard sphere $(\mathbb{S}^n, h)$, then for any $\epsilon \neq 0$, $Q_h + \epsilon\varphi$ is not achived as the $Q-$curvature of a metric in the conformal class of the standard sphere. Therefore, a function can be arbitrarily close to a $Q-$curvature but not be a $Q-$curvature itself: the prescription of the $Q-$curvature is then a highly unstable problem, and its underlying analysis is intricate. We refer again to [9] for considerations on the structure of the set of $Q-$curvatures. In the case $k = 1$ and $n \geq 3$, the problem of prescribing a constant $Q-$curvature is known as the Yamabe problem: it is not the purpose of the present article to make an extensive historical review of the famous resolution of this problem, and we refer to Lee-Parker [22] and the references therein. Concerning fourth order problems, that is for $k = 2$, there has been an intensive litterature on the question: here, we refer to the recent surveys of Branson-Gover [6], Chang [8], Malchiodi [24] and the references therein.

In the sequel, we will say that a function is admissible if it can be achieved as the $Q-$curvature of a metric in a given conformal class. As seen above, some functions on the sphere are not admissible for the standard conformal class. Moser [25] remarked that functions enjoying some symmetries automatically satisfy the Kazdan-Warner identities: indeed, on the standard sphere, given an isometry σ such that $\varphi\circ\sigma=-\varphi$ for all first eigenfunction of the Laplace-Beltrami operator (take $\sigma=-Id$ for instance), then the Kazdan-Warner identity yields 0 for all function invariant by σ. Then, Moser had the idea to impose invariance under a group of isometries to find admissible functions on the sphere for the scalar curvature problem in 2D. This strategy was also used by Escobar-Schoen [11] and Hebey [18] in higher dimensions. In the same spirit, Delanoë and the author [9] proved that a function on the sphere which is close to Q_h and invariant under a group of isometries acting without fixed point is admissible. In the present article, we relax the condition of being close to Q_h by imposing cancelation of some derivatives (see Theorem 3 below). In the specific case $n=2k+1$, very few is required; this is the object our main result:

Theorem 1. *Let $k\geq 1$ and let G be a subgroup of isometries of $(\mathbb{S}^{2k+1},h)$. Let $f\in C^\infty(M)$ be a positive $G-$invariant function and assume that G acts without fixed point (that is $|O_G(x)|\geq 2$ for all $x\in\mathbb{S}^{2k+1}$). Then there exists $g\in[h]$ such that $Q_g=f$ and $G\subset Isom_g(\mathbb{S}^n)$.*

When $k=1,2$, this result is due respectively to Hebey [18] and to the author [27]. This theorem is a particular case of more general results proved on arbitrary conformal manifolds (see Proposition 8 and Theorem 3 below). In this article, we make a general analysis of the operator P_g and of the blow-up phenomenon attached to it on arbitrary conformal manifolds. In the last section, we apply this analysis to the conformal sphere.

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2. Miscellaneous on the operator P_g

The operator P_g can be written (partially) as a divergence form (we refer to Branson-Gover [6]): as a preliminary step, we precise this divergence form that will be useful in the sequel:

Proposition 1. *Let P_g be the conformal GJMS operator. Then for any $l\in\{0,...,k-1\}$, there exists $A_{(l)}(g)$ a smooth T_{2l}^0-tensor field on M such that*

$$P_g=\Delta_g^k+\sum_{l=0}^{k-1}(-1)^l\nabla^{j_l\dots j_1}(A_{(l)}(g)_{i_1\dots i_l j_1\dots j_l}\nabla^{i_1\dots i_l}),\tag{3}$$

where the indices are raised via the musical isomorphism. In addition for any $l\in\{0,...,k-1\}$, $A_{(l)}(g)$ is symmetric in the following sense: $A_{(l)}(g)(X,Y)=$

$A_{(l)}(g)(Y,X)$ for all X,Y T_0^l-tensors on M. In particular, we have that

$$\int_M uP_g(v)\,dv_g = \int_M \left(\Delta_g^{\frac{k}{2}} u \Delta_g^{\frac{k}{2}} v + \sum_{l=0}^{k-1} A_{(l)}(g)(\nabla^l u, \nabla^l v)\right) dv_g \tag{4}$$

for all $u,v\in C^\infty(M)$. Here, we have adopted the convention

$$\Delta_g^{\frac{k}{2}} u \Delta_g^{\frac{k}{2}} v := (\nabla \Delta_g^{\frac{k-1}{2}} u, \nabla \Delta_g^{\frac{k-1}{2}} v)_g$$

when k is odd.

Proof. The proof uses only the self-adjointness of the operator P_g. In the sequel, we note $A^\star$ the adjoint of any operator A with respect to the L^2-product. As a preliminary, we compute the adjoint of some elementary operators. We adopt here Hamilton's convention [17]: the notation $A\star B$ denotes a linear combination of contraction of the tensors A, B, g and g^{-1}. Given B a smooth T_q^0-tensor field on M, we consider the operator $Bu := B\cdot\nabla^q u = B_{i_1\ldots i_q}\nabla^{i_1\ldots i_q} u$ for all $u\in C^\infty(M)$. We claim that

$$B^\star = (-1)^q B + \sum_{l=1}^{q-1} \nabla^l u \star \nabla^{q-l} B.$$

We prove the claim. We let $u,v\in C^\infty(M)$ be two smooth functions on M. Integrating by parts, we have that

$$\begin{aligned}\int_M uBv\,dv_g &= \int_M uB_{i_1\ldots i_q}\nabla^{i_1\ldots i_q} v\,dv_g = (-1)^q \int_M \nabla^{i_q\ldots i_1}(uB_{i_1\ldots i_q}) v\,dv_g\\ &= (-1)^q \int_M \left(B_{i_1\ldots i_q}\nabla^{i_q\ldots i_1} u + \sum_{l=0}^{q-1} \nabla^l u\star\nabla^{q-l} B\right) v\,dv_g.\end{aligned}$$

Therefore, $B^\star$ is defined and

$$B^\star u = (-1)^q B_{i_1\ldots i_q}\nabla^{i_q\ldots i_1} u + \sum_{l=0}^{q-1} \nabla^l u\star\nabla^{q-l} B.$$

For any smooth tensor field T, we define $Asym(T)(X,Y,...) := T(X,Y,...) - T(Y,X,...)$. It follows from the definition of the curvature tensor that

$$Asym(\nabla^2 T) = T\star R,$$

where R is the curvature tensor. Therefore, for any permutation σ of $\{1,...,q\}$, we have that

$$\nabla^q u - \sigma\cdot\nabla^q u = \nabla^{q-2} u\star R, \tag{5}$$

where $\sigma\cdot T$ permutes the variables of the covariant tensor T along σ. Therefore, we have that $\nabla^{i_q\ldots i_1} u - \nabla^{i_1\ldots i_q} u$ is a contraction of $\nabla^{q-2} u$, and therefore we get that $B^\star = (-1)^q B + lot$. This proves the claim.

We are now in position to prove Proposition 1. It follows from the definition of P_g that there exists B, a smooth T_{2k-1}^0-tensor field on M, such that $P_g u = \Delta_g^k u + Bu + lot$ for all $u\in C^\infty(M)$. Since P_g and Δ_g are self-adjoint, we then get that

$$P_g = P_g^\star = \Delta_g^k + B^\star + lot = \Delta_g^k - B + lot$$

since $2k-1$ is odd. In particular, $Bu = lot$ and therefore, $Bu = 0$ for all $u\in C^\infty(M)$.

We now take C a smooth $(2k-2,0)-$tensor field such that $P_g=\Delta_g^k+C\cdot\nabla^{2k-2}+lot$. We define A as the symmetrized tensor of C, that is via coordinates $A(X,Y)=(-1)^{k-1}\frac{1}{2}(C(X,Y)+C(Y,X))$ for all X,Y any $T_0^{k-1}-$tensors on M. As easily checked, since changing the order of differentiation involves only lower order terms via with (5), we have that

$$\begin{aligned}
C\cdot\nabla^{2k-2}u &= C_{i_1\dots i_{k-1}j_1\dots j_{k-1}}\nabla^{i_1\dots i_{k-1}j_1\dots j_{k_1}}u\\
&= (-1)^{k-1}A_{i_1\dots i_{k-1}j_1\dots j_{k-1}}\nabla^{i_1\dots i_{k-1}j_1\dots j_{k-1}}u+\nabla^{2k-4}u\star R\\
&= (-1)^{k-1}A_{i_1\dots i_{k-1}j_1\dots j_{k-1}}\nabla^{j_{k-1}\dots j_1 i_1\dots i_{k-1}}u+\nabla^{2k-4}u\star R\\
&= (-1)^{k-1}\nabla^{j_{k-1}\dots j_1}\left(A_{i_1\dots i_{k-1}j_1\dots j_{k-1}}\nabla^{i_1\dots i_{k-1}}u\right)\\
&\quad+\nabla^{2k-4}u\star R+\sum_{l=1}^{k-1}\nabla^{2k-2-l}u\star\nabla^l A
\end{aligned}$$

and then

$$P_g=\Delta_g^k+(-1)^{k-1}\nabla^{j_{k-1}\dots j_1}\left(A_{i_1\dots i_{k-1}j_1\dots j_{k-1}}\nabla^{i_1\dots i_{k-1}}\right)+lot.$$

Iterating these steps yields (3). Integrating by parts then yields (4). □

Define the norm $\|u\|_{H_k^2}:=\sum_{l=0}^k\|\nabla^l u\|_2$ and the space $H_k^2(M)$ as the completion of $C^\infty(M)$ for the norm $\|\cdot\|_{H_k^2}$. As a consequence of (4), we get that the bilinear form $(u,v)\mapsto\int_M uP_g v\,dv_g$ extends to a continuous symmetrical bilinear form on $H_k^2(M)\times H_k^2(M)$. We say that P_g is coercive if there exists $c>0$ such that

$$\int_M uP_g u\,dv_g\geq c\|u\|_2^2 \text{ for all } u\in H_k^2(M).$$

We then define the norm $\|u\|_{P_g}:=\sqrt{\int_M uP_g u\,dv_g}$ for all $u\in H_k^2(M)$.

Proposition 2. *Assume that P_g is coercive. Then $\|\cdot\|_{P_g}$ is a norm on H_k^2 equivalent to $\|\cdot\|_{H_k^2}$.*

Proof. Clearly $\|\cdot\|_{P_g}$ is a norm and there exists $C>0$ such that $\|\cdot\|_{P_g}\leq C\|\cdot\|_{H_k^2}$. We now argue by contradiction and we assume that the two norms are not equivalent: then there exists $(u_i)_{i\in\mathbb{N}}\in H_k^2(M)$ such that

$$\|u_i\|_{H_k^2}=1 \text{ and } \|u_i\|_{P_g}=o(1) \tag{6}$$

when $i\to+\infty$. Up to a subsequence, still denoted as (u_i), there exists $u\in H_k^2(M)$ such that $u_i\rightharpoonup u$ weakly in $H_k^2(M)$ and $u_i\to u$ strongly in $H_{k-1}^2(M)$ when $i\to+\infty$. The coercivity of P_g yields $\|u_i\|_2=o(1)$ when $i\to+\infty$, and then $u\equiv 0$. Therefore, we have that

$$u_i\rightharpoonup 0 \text{ weakly in } H_k^2(M) \text{ and } u_i\to 0 \text{ strongly in } H_{k-1}^2(M) \tag{7}$$

when $i\to+\infty$. Consequently, (6) rewrites

$$\lim_{i\to+\infty}\int_M|\nabla^k u_i|_g^2\,dv_g=1 \text{ and } \lim_{i\to+\infty}\int_M(\Delta_g^{\frac{k}{2}}u_i)^2\,dv_g=0. \tag{8}$$

The contradiction comes from a Bochner-Lichnerowicz-Weitzenbock type formula. Here again, we use (5). We fix $u,v\in C^\infty(M)$: we have that (the notation $a\equiv b$

means that the terms are equal up to a divergence)

$$\begin{aligned}(\nabla^k u, \nabla^k v)_g &\equiv g^{\alpha_1\beta_1}...g^{\alpha_k\beta_k}\nabla_{\alpha_1...\alpha_k}u\nabla_{\beta_1...\beta_k}v\\ &\equiv -g^{\alpha_1\beta_1}...g^{\alpha_k\beta_k}\nabla_{\beta_1\alpha_1...\alpha_k}u\nabla_{\beta_2...\beta_k}v\\ &\equiv -g^{\alpha_1\beta_1}...g^{\alpha_k\beta_k}\nabla_{\alpha_2...\alpha_k\beta_1\alpha_1}u\nabla_{\beta_2...\beta_k}v + \nabla^{k-1}u\star\nabla^{k-1}v\star R\\ &\equiv -g^{\alpha_2\beta_2}...g^{\alpha_k\beta_k}\nabla_{\alpha_2...\alpha_k}g^{\alpha_1\beta_1}\nabla_{\beta_1\alpha_1}u\nabla_{\beta_2...\beta_k}v + \nabla^{k-1}u\star\nabla^{k-1}v\\ &\equiv g^{\alpha_2\beta_2}...g^{\alpha_k\beta_k}\nabla_{\alpha_2...\alpha_k}\Delta_g u\nabla_{\beta_2...\beta_k}v + \nabla^{k-1}u\star\nabla^{k-1}v\star R\\ &\equiv (\nabla^{k-1}\Delta_g u, \nabla^{k-1}v)_g + \nabla^{k-1}u\star\nabla^{k-1}v\star R.\end{aligned}$$

the same procedure applied to $(\nabla^{k-1}v, \nabla^{k-1}\Delta_g u)_g$ yields

$$\begin{aligned}(\nabla^k u, \nabla^k v)_g &\equiv (\nabla^{k-2}\Delta_g u, \nabla^{k-2}\Delta_g v)_g\\ &\quad +\nabla^{k-1}u\star\nabla^{k-1}v\star R + \nabla^{k-2}\Delta_g u\star\nabla^{k-2}v\star R.\end{aligned}$$

Taking $u = v = u_i$, integrating over M and using (7) yields

$$\int_M |\nabla^k u_i|_g^2\, dv_g = \int_M |\nabla^{k-2}\Delta_g u_i|_g^2\, dv_g + o(1)$$

when $i \to +\infty$. Iterating this process and considering separately the cases k odd and k even, we get that

$$\int_M |\nabla^k u_i|_g^2\, dv_g = \int_M (\Delta_g^{\frac{k}{2}} u_i)^2\, dv_g + o(1)$$

when $i \to +\infty$. This is a contradiction with (8) and Proposition 2 is proved. □

3. General considerations on the equivariant Yamabe invariant

We let $(M, \mathcal{C})$ be a conformal Riemannian manifold. We let $G \subset Diff(M)$ be a subgroup of diffeomorphisms of M. We define

$$\mathcal{C}_G := \{g \in \mathcal{C} / \, G \subset Isom_g(M)\},$$

and we assume that $\mathcal{C}_G \neq \emptyset$. In particular, G is contained in a compact group. Therefore, without loss of generality, we assume that G is a compact group. As easily checked, for any $g \in \mathcal{C}_G$, we have that

$$\mathcal{C}_G = \{e^{2\omega} g / \, \omega \in C^\infty_G(M)\}$$

where $C^\infty_G(\Omega) = \{\omega \in C^\infty(M) / \, \omega \circ \sigma = \omega \text{ for all } \sigma \in G\}$ is the set of $G-$invariant smooth functions on M. We assume that $n > 2k$: in this context, it is more convenient to write a metric $\hat{g} \in \mathcal{C}$ as $\hat{g} = u^{\frac{4}{n-2k}} g$ with $u \in C^\infty_+(M)$. The relation between P_g and $P_{\hat{g}}$ is given by (2). With the new parametrization, we have that

$$\mathcal{C}_G = \{u^{\frac{4}{n-2k}} g / \, u \in C^\infty_{G,+}(M)\},$$

where $C^\infty_{G,+}(M) := \{u \in C^\infty_G(M) / \, u > 0\}$. Let $f \in C^\infty_{G,+}(M)$ be a smooth positive $G-$invariant function. By analogy with the Yamabe invariant, we define

$$\mu_f(\mathcal{C}_G) := \inf_{g \in \mathcal{C}_G} \frac{\int_M Q_g\, dv_g}{V_f(M,g)^{\frac{2}{2^\star}}}$$

where $V_f(M, g)$ is the $f-$volume defined in the introduction and $2^\star := \frac{2n}{n-2k}$. We fix $g \in \mathcal{C}_G$: as easily checked, we have that

$$\mu_f(\mathcal{C}_G) = \frac{2}{n-2k} \inf_{u \in C^\infty_{G,+}(M)} I_g(u)$$

where

$$I_g(u) := \frac{\int_M uP_g u\, dv_g}{\left(\int_M f|u|^{2^\star}\, dv_g\right)^{\frac{2}{2^\star}}}$$

for all $u \in H_k^2(M) \setminus \{0\}$.

Proposition 3. *A metric $g \in \mathcal{C}_G$ is a critical point of the functional $g \mapsto \frac{\int_M Q_g\, dv_g}{V_f(M,g)^{\frac{2}{2^\star}}}$ if and only if there exists $\lambda \in \mathbb{R}$ such that $Q_g = \lambda f$.*

Proof. We fix $g \in \mathcal{C}_G$ and $t \mapsto g(t) \in \mathcal{C}_G$ a differentiable family of metrics conformal to g such that $g(0) = g$. In particular, there exists a differentiable family $t \mapsto u(t) \in C^\infty_{G,+}(M)$ such that $g(t) = u(t)^{\frac{4}{n-2k}} g$ and $u(0) = 1$. We define $\dot{u} := u'(0)$. Using the self-adjointness of P_g, straightforward computations yield

$$\frac{d}{dt}\left(\frac{\int_M Q_{g(t)}\, dv_{g(t)}}{V_f(M,g(t))^{\frac{2}{2^\star}}}\right)_{t=0} = 2\frac{\int_M \dot{u}\left(Q_g - f\bar{Q}_g^f\right)\, dv_g}{V_f(M,g(t))^{\frac{2}{2^\star}}}$$

where

$$\bar{Q}_g^f = \frac{\int_M Q_g\, dv_g}{V_f(M,g)}.$$

Since u is $G-$invariant, the function $\dot{u}$ ranges $C^\infty_G(M)$. Fix $v \in C^\infty(M)$ and let v_G be its symmetrization via the Haar measure (which is well-defined since G is compact). We then define $u(t) := 1 + tv_G$ for all $t \in \mathbb{R}$: since f and Q_g are $G-$invariant (this is a consequence of point (ii) of the characterization of P_g and of the definition of Q_g), we get that

$$\int_M \dot{u}\left(Q_g - f\bar{Q}_g^f\right)\, dv_g = \int_M v_G\left(Q_g - f\bar{Q}_g^f\right)\, dv_g = \int_M v\left(Q_g - f\bar{Q}_g^f\right)\, dv_g.$$

Therefore, g is a critical point if and only if $Q_g = f\bar{Q}_g^f$. This proves Proposition 3. □

To carry out the analysis, coercivity and positivity preserving property are required. More precisely, we assume that there exists $g \in \mathcal{C}$ such that

$$\left\{\begin{array}{ll} (C) & \text{the operator } P_g \text{ is coercive} \\ (PPP) & \text{for any } u \in C^\infty(M) \text{ such that } P_g \geq 0 \text{ then either } u > 0 \text{ or } u \equiv 0 \end{array}\right\}.$$

Note that (C) and (PPP) are conformally invariant: they hold for some $g \in \mathcal{C}$ iff they hold for all $g \in \mathcal{C}$.

Proposition 4. *Assume that the metric g is Einstein with positive scalar curvature and $n > 2k$, then P_g satisfies (C) and (PPP).*

Proof. This relies essentially on the the explicit expression of the GJMS operator in the Einstein case: see Proposition 7.9 of Fefferman-Graham [13] and also Gover [14] for a proof via tractors. Indeed, for an Einstein metric g, P_g expresses as an explicit product of second-order operators with constant coefficients depending only on the scalar curvature. For positive curvature, a direct consequence is that P_g satisfies (PPP) by k applications of the second-order comparison principle. Moreover, still in this case, since $P_g = S(\Delta_g)$ with S a polynomial with positive constant coefficients, it follows from Hebey-Robert [19] that the first eigenvalue of P_g is $S(0) > 0$ (0 is the first eigenvalue of Δ_g), and then P_g satisfies (C). □

Due to the lack of compactness of the embedding $H_k^2(M)\hookrightarrow L^{2^\star}(M)$, it is standard to use the subcritical method. Given $q\in(2,2^\star]$, we define

$$I_{g,q}(u):=\frac{\int_M uP_gu\,dv_g}{\left(\int_M f|u|^q\,dv_g\right)^{\frac{2}{q}}}$$

for all $u\in H_k^2(M)\setminus\{0\}$, and

$$\mu_q:=\inf_{u\in H_{k,G}^2(M)\setminus\{0\}}I_{g,q}(u),$$

where $H_{k,G}^2(M):=\{u\in H_k^2(M)/\,u\circ\sigma=u\text{ a.e. for all }\sigma\in G\}$. The first result is that μ_q is achieved at a smooth positive minimizer when $q<2^\star$:

Proposition 5. *We fix $q\in(2,2^\star)$, we assume that (C) and (PPP) hold and that $\mathcal{C}_G\neq\emptyset$. Then $\mu_q>0$ is achieved. Moreover, there exists $u_q\in C^\infty_{G,+}(M)$ a smooth positive function such that $\mu_q=I_{g,q}(u_q)$ and*

$$P_gu_q=\mu_qfu_q^{q-1}\text{ in }M\text{ with }\int_M fu_q^q\,dv_g=1. \tag{9}$$

Proof. Since P_g is coercive, the norms $\|\cdot\|_{H_k^2}$ and $\|\cdot\|_{P_g}$ are equivalent, and then, it follows from Hölder's and Sobolev's inequality that

$$\begin{aligned}\left(\int_M f|u|^q\,dv_g\right)^{\frac{2}{q}}&\leq\left(\int_M f\,dv_g\right)^{\frac{2}{q}-\frac{2}{2^\star}}\left(\int_M f|u|^{2^\star}\,dv_g\right)^{\frac{2}{2^\star}}\\&\leq C\left(\int_M f\,dv_g\right)^{\frac{2}{q}-\frac{2}{2^\star}}\|u\|_{H_k^2}^2\leq C'\left(\int_M f\,dv_g\right)^{\frac{2}{q}-\frac{2}{2^\star}}\|u\|_{P_g}^2,\end{aligned}\tag{10}$$

and then $I_{g,q}(u)\geq(C')^{-1}\left(\int_M f\,dv_g\right)^{-\frac{2}{q}+\frac{2}{2^\star}}$ for all $u\in H_k^2(M)\setminus\{0\}$, and therefore $\mu_q>0$. The existence of a minimizer is standard and we omit it. Let us take then $u\in H_{k,G}^2(M)\setminus\{0\}$ be a mimimizer. Without loss of generality, we can assume that $\int_M f|u|^q\,dv_g=1$.

The Euler-Lagrange equation for $I_{g,q}$ yields $I'_{g,q}(u)\varphi=0$ for all $\varphi\in H_{k,G}^2(M)$. Using the Haar measure and arguing as in the proof of Proposition 3 (see also [18]), we get that this equality holds for all $\varphi\in H_k^2(M)$. Since the exponent q is subcritical, we get with standard bootstrap arguments that $u\in C_G^{2k}(M)$ and $P_gu=\mu_qf|u|^{q-2}u$. We are left with proving that $u>0$ or $u<0$. We let $v\in C_G^{2k}(M)$ be such that $P_gv=|P_gu|$ in M. Since $u\not\equiv0$, it follows from (PPP) that $v\geq|u|$ and $v>0$. Using again the definition of μ_q, we have that

$$\begin{aligned}\mu_q&\leq\frac{\int_M vP_gv\,dv_g}{\left(\int_M fv^q\,dv_g\right)^{\frac{2}{q}}}=\mu_q\frac{\int_M fv|u|^{q-1}\,dv_g}{\left(\int_M fv^q\,dv_g\right)^{\frac{2}{q}}}\\&\leq\mu_q\frac{\left(\int_M fv^q\,dv_g\right)^{\frac{1}{q}}\left(\int_M f|u|^q\,dv_g\right)^{\frac{q-1}{q}}}{\left(\int_M fv^q\,dv_g\right)^{\frac{2}{q}}}\\&\leq\mu_q\left(\int_M f|u|^q\,dv_g\right)^{\frac{q-2}{q}}=\mu_q\text{ since }v\geq|u|\end{aligned}$$

Therefore equality holds everywhere and $|u|=v>0$. In particular u does not change sign, and we can assume that it is positive. Bootstrap and regularity theory (see [1]) then yield $u\in C^\infty_{G,+}(M)$, and Proposition 5 is proved with $u_q:=u$. □

Proposition 6. *We claim that* $\lim_{q\to 2^\star}\mu_q=\mu_{2^\star}=\frac{n-2k}{2}\mu_f(\mathcal{C}_G)$.

Proof. Using the Hölder's inequality (10), we get that

$$I_{g,2^\star}(u)\leq I_{g,q}(u)V_f(M,g)^{\frac{2}{q}-\frac{2}{2^\star}}$$

for all $u\in H_k^2(M)\setminus\{0\}$, and then $\mu_{2^\star}\leq \mu_q V_f(M,g)^{\frac{2}{q}-\frac{2}{2^\star}}$, which yields $\mu_{2^\star}\leq \liminf_{q\to 2^\star}\mu_q$. Conversely, fix $\epsilon>0$ and let $u\in H_{k,G}^2(M)\setminus\{0\}$ be such that $I_{g,2^\star}(u)<\mu_{2^\star}+\epsilon$. Since $\lim_{q\to 2^\star}I_{g,q}(u)=I_{g,2^\star}(u)$, we then get that there exists $q_0<2^\star$ such that $\mu_q<\mu_{2^\star}+\epsilon$ for $q\in(q_0,2^\star)$, and then $\limsup_{q\to 2^\star}\mu_q\leq\mu_{2^\star}$. Therefore, $\lim_{q\to 2^\star}\mu_q=\mu_{2^\star}$.

For $q\in(2,2^\star]$, we define $\mu_{q,+}:=\inf\{I_{g,q}(u)/\, u\in H_{k,G}^2(M)\setminus\{0\}\text{ and } u\geq 0\text{ a.e.}\}$. Arguing as above, we get that $\lim_{q\to 2^\star}\mu_{q,+}=\mu_{2^\star,+}$. Since $\mu_{q,+}=\mu_q$ for all $q<2^\star$ with Proposition 5, we then get that $\mu_{2^\star}=\mu_{2^\star,+}$.

We claim that $\mu_{2^\star,+}=\frac{n-2k}{2}\mu_f(\mathcal{C}_G)$. Indeed, via local convolutions with a positive kernel, we get that $C_+^\infty(M)$ is dense in $H_{k,+}^2(M)$ for the H_k^2-norm. A symmetrization via the Haar measure then yields that $C_{G,+}^\infty(M)$ is dense in $H_{k,G,+}^2(M)$: clearly this yields $\mu_{2^\star,+}=\frac{n-2k}{2}\mu_f(\mathcal{C}_G)$, and the claim is proved. □

We define $D_k^2(\mathbb{R}^n)$ as the completion of $C_c^\infty(\mathbb{R}^n)$ for the norm $u\mapsto\|\Delta_\xi^{\frac{k}{2}}u\|_2$ and we define

$$\frac{1}{K(n,k)}:=\inf_{u\in D_k^2(\mathbb{R}^n)\setminus\{0\}}\frac{\int_{\mathbb{R}^n}(\Delta_\xi^{\frac{k}{2}}u)^2\,dv_\xi}{\left(\int_{\mathbb{R}^n}|u|^{2^\star}\,dv_\xi\right)^{\frac{2}{2^\star}}}. \tag{11}$$

It follows from Sobolev's embedding theorem that $K(n,k)>0$. Moreover, it follows from Lions [23] that the infimum is achieved by $U:x\mapsto(1+|x|^2)^{k-\frac{n}{2}}$, and that all minimizers are compositions of U by translations and homotheties.

Proposition 7. *We have that*

$$\mu_f(\mathcal{C}_G)\leq\frac{2}{n-2k}\cdot\frac{|O_G(x)|^{\frac{2k}{n}}}{f(x)^{\frac{2}{2^\star}}K(n,k)} \tag{12}$$

for all $x\in M$, where $|O_G(x)|$ denotes the cardinal (possibly ∞) of the orbit $O_G(x)$.

Proof. We fix $x\in M$. Without loss of generality, we assume that $m:=|O_G(x)|<+\infty$ (otherwise (12) is clear). We let $\sigma_1=Id_M,...,\sigma_m\in G$ be such that $O_G(x)=\{x_1,...,x_m\}$ where $x_i=\sigma_i(x)$ for all $i\in\{1,...,m\}$ are distinct. We let $u\in C_c^\infty(\mathbb{R}^n)$ be a radially symmetrical smooth function and we define for $\epsilon>0$ small the function

$$u_{\epsilon,i}(z):=u\left(\frac{1}{\epsilon}\exp_{x_i}^{-1}(z)\right)\text{ if } d_g(z,x_i)<i_g(M)\text{ and 0 otherwise.}$$

Clearly, $u_{\epsilon,i}\in C^\infty(M)$ for $\epsilon>0$ small enough. We now define

$$u_\epsilon:=\sum_{i=1}^m u_{\epsilon,i}.$$

As one checks, since u is radially symmetrical, we have that $u_\epsilon\in C_G^\infty(M)$ is $G-$invariant for $\epsilon>0$ small enough.

Let us compute $I_{g,2^\star}(u_\epsilon)$. We fix $\delta \in (0, i_g(M))$ and we define the metric $g_\epsilon := (\exp_g^\star)(\epsilon \cdot)$: since the elements of G are isometries (and then $P_g = P_{\sigma^\star g} = \sigma^\star P_g$ for all $\sigma \in G$) and the $u_{\epsilon,i}$'s have disjoint supports, we get that

$$\begin{aligned}\int_M u_\epsilon P_g u_\epsilon \, dv_g &= \sum_{i,j=1}^m \int_M u_{\epsilon,i} P_g u_{\epsilon,j} \, dv_g = \sum_{i=1}^m \int_M u_{\epsilon,i} P_g u_{\epsilon,i} \, dv_g \\ &= \sum_{i=1}^m \int_{B_\delta(x_i)} u_{\epsilon,1} \circ \sigma_i^{-1} P_g (u_{\epsilon,1} \circ \sigma_i^{-1}) \, dv_g \\ &= m \int_{B_\delta(x)} u_{\epsilon,1} P_g u_{\epsilon,1} \, dv_g = m\epsilon^{n-2k} \int_{B_{\epsilon^{-1}\delta}(0)} u P_{g_\epsilon} u \, dv_{g_\epsilon}\end{aligned}$$

since $\lim_{\epsilon \to 0} g_\epsilon = \xi$, the Euclidean metric, we get that

$$\int_M u_\epsilon P_g u_\epsilon \, dv_g = \epsilon^{n-2k} \left(m \int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} u)^2 \, dv_\xi + o(1) \right)$$

when $\epsilon \to 0$. Similarly, using the $G-$invariance of f, we get that

$$\int_M f |u_\epsilon|^{2^\star} \, dv_g = \epsilon^n \left(m f(x) \int_{\mathbb{R}^n} |u|^{2^\star} \, dv_\xi + o(1) \right)$$

when $\epsilon \to 0$, and then

$$I_{g,2^\star}(u_\epsilon) = \frac{m^{\frac{2k}{n}}}{f(x)^{\frac{2}{2^\star}}} \cdot \frac{\int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} u)^2 \, dv_\xi}{\left(\int_{\mathbb{R}^n} |u|^{2^\star} \, dv_\xi \right)^{\frac{2}{2^\star}}} + o(1)$$

when $\epsilon \to 0$. Therefore, since $\mu_f(\mathcal{C}_G) = \mu_{2^\star}$, taking the limit $\epsilon \to 0$ and taking the infimum on the u's, we get that

$$\mu_{2^\star} \leq \frac{|O_G(x)|^{\frac{2k}{n}}}{f(x)^{\frac{2}{2^\star}}} \cdot \inf_{u \in C_c^\infty(\mathbb{R}^n) \setminus \{0\} \text{ radial}} \frac{\int_{\mathbb{R}^n} (\Delta_\xi^{\frac{k}{2}} u)^2 \, dv_\xi}{\left(\int_{\mathbb{R}^n} |u|^{2^\star} \, dv_\xi \right)^{\frac{2}{2^\star}}}$$

It follows from Lions [23] that the infimum $K(n,k)^{-1}$ is achieved at smooth radially symmetrical functions, therefore we obtain (12). □

4. The quantization of the formation of singularities

The objective of this section is to prove the following result:

Theorem 2. *Let $(M, \mathcal{C})$ be a conformal Riemannian manifold of dimension $n \geq 3$ and let $k \in \mathbb{N}^\star$ be such that $2k < n$. Let G be a group of diffeomorphisms such that $\mathcal{C}_G \neq \emptyset$ and let $f \in C_{G,+}^\infty(M)$ be a positive $G-$invariant function. Assume that there exists $g \in \mathcal{C}$ such that P_g satisfies (C) and (PPP). For any $q \in (2, 2^\star)$, we let $u_q \in C_{G,+}^\infty(M)$ as in Proposition 5. Then:*

(i) either $\limsup_{q \to +\infty} \|u_q\|_\infty = +\infty$*, and there exists $x \in M$ such that $\nabla f(x) = 0$ and*

$$\mu_f(\mathcal{C}_G) = \frac{2}{n-2k} \cdot \frac{|O_G(x)|^{\frac{2k}{n}}}{f(x)^{\frac{2}{2^\star}} K(n,k)},$$

(ii) or $\|u_q\|_\infty \leq C$ for all $q < 2^\star$, and there exists $u \in C_{G,+}^\infty(M)$ such that $\lim_{q \to 2^\star} u_q = u$ in $C^{2k}(M)$ and $P_g u = \frac{n-2k}{2} \mu_f(\mathcal{C}_G) f u^{2^\star - 1}$ in M. In particular, there exists $\hat{g} \in \mathcal{C}_G$ such that $Q_{\hat{g}} = f$ and the infimum $\mu_f(\mathcal{C}_G)$ is achieved.

This type of result is classical. The proof of Theorem 2 goes through nine steps. For $q \in (2, 2^\star)$, we let $u_q \in C^\infty_{G,+}(M)$ be as in Proposition 5 (this is relevant since (C) and (PPP) hold).

Step 1: We assume that there exists $C > 0$ such that $\|u_q\|_\infty \leq C$ for all $q < 2^\star$. We claim that (ii) of Theorem 2 holds.

We prove the claim. Indeed, it follows from (9), Proposition 6, the uniform bound of $(u_q)_q$ in L^∞ and standard elliptic (see for instance [1]), that, up to a subsequence, there exists $u \in C^{2k}(M)$ nonnegative such that $\lim_{q\to 2^\star} u_q = u$ in $C^{2k}(M)$: therefore, $P_g u = \mu_{2^\star} f u^{2^\star-1}$ in M and $\int_M f u^{2^\star}\, dv_g = 1$. In particular, $P_g u \geq 0$ and $u \not\equiv 0$, and then it follows from (PPP) that $u > 0$. Since u_q is $G-$invariant for all $q \in (2, 2^\star)$, we get that $u \in C^\infty_{G,+}(M)$. Moreover, $I_g(u) = \mu_{2^\star}$, and then the metric $u^{\frac{4}{n-2k}} g$ is extremal for $\mu_f(\mathcal{C}_G)$: it then follows from Proposition 3 that $\hat{g} := (\mu_f(\mathcal{C}_G))^{1/k} u^{\frac{4}{n-2k}} g$ is also an extremal for $\mu_f(\mathcal{C}_G)$ and $Q_{\hat{g}} = f$. This ends Step 1.

From now on, we assume that $\limsup_{q\to 2^\star} \|u_q\|_\infty = +\infty$. For the sake of clearness, we will write (u_q) even for a subsequence of (u_q). For any $q \in (2, 2^\star)$, we let $x_q \in M$ be such that

$$u_q(x_q) = \max_M u_q \text{ and } \lim_{q\to 2^\star} u_q(x_q) = +\infty. \tag{13}$$

We define

$$\alpha_q := u_q(x_q)^{-\frac{2}{n-2k}} \text{ and } \beta_q := \alpha_q^{\frac{q-2}{2^\star-2}}$$

for all $q \in (2, 2^\star)$. It follows from (13) that

$$\lim_{q\to 2^\star} \alpha_q = 0 \text{ and } \beta_q \geq \alpha_q \text{ for } q \to 2^\star. \tag{14}$$

We define

$$\tilde{u}_q(x) := \alpha_q^{\frac{n-2k}{2}} u_q(\exp_{x_q}(\beta_q x)) \tag{15}$$

for all $x \in B_{\beta_q^{-1}\delta}(0)$, where $\delta \in (0, i_g(M))$.

Step 2: We claim that there exists $\tilde{u} \in C^{2k}(\mathbb{R}^n)$ such that $\lim_{q\to 2^\star} \tilde{u}_q = \tilde{u}$ in $C^{2k}_{loc}(\mathbb{R}^n)$ where

$$0 \leq \tilde{u} \leq \tilde{u}(0) = 1 \text{ and } \Delta_\xi^k \tilde{u} = \mu_{2^\star} f(x_\infty) \tilde{u}^{2^\star-1} \text{ in } \mathbb{R}^n, \tag{16}$$

and $x_\infty := \lim_{q\to 2^\star} x_q$.

We prove the claim. It follows of the naturality of the geometric operator P_g and of (9) that

$$P_{g_q} \tilde{u}_q = \mu_q f(\exp_{x_q}(\beta_q \cdot)) \tilde{u}_q^q \text{ in } B_{\beta_q^{-1}\delta}(0) \tag{17}$$

for all $q \in (2, 2^\star)$, where $g_q := (\exp_{x_q}^\star g)(\beta_q \cdot)$. In particular, since the exponential is a normal chart at x_q, we have that $\lim_{q\to 2^\star} g_q = \xi$ in $C^{2k}_{loc}(\mathbb{R}^n)$. Since $0 \leq \tilde{u}_q \leq \tilde{u}_q(0) = 1$, it follows from standard elliptic theory (see for instance [1]) that there exists $\tilde{u} \in C^{2k}(\mathbb{R}^n)$ such that $\lim_{q\to 2^\star} \tilde{u}_q = \tilde{u}$ in $C^{2k}_{loc}(\mathbb{R}^n)$. In addition, using that $P_\xi = \Delta_\xi^k$, passing to the limit in (17) yields (16). This proves the claim.

Step 3: We claim that there exists $C > 0$ such that

$$\alpha_q \leq \beta_q \leq C\alpha_q \tag{18}$$

when $q \to 2^\star$.

We prove the claim. We fix $R>0$ and we let q be in $(2,2^\star)$: a change of variable and Sobolev's embedding yields

$$\int_{B_R(0)} \tilde{u}_q^{2^\star}\, dv_{g_q} = \left(\frac{\alpha_q}{\beta_q}\right)^n \int_{B_{R\beta_q}(x_q)} u_q^{2^\star}\, dv_g \leq C\left(\frac{\alpha_q}{\beta_q}\right)^n \|u_q\|_{P_g}^{2^\star}$$

for all $q\in(2,2^\star)$. Using (9) and Proposition 6, letting $q\to 2^\star$, we get that

$$\left(\frac{\beta_q}{\alpha_q}\right)^n \leq \frac{C'}{\int_{B_R(0)} \tilde{u}^{2^\star}\, dv_\xi} + o(1)$$

when $q\to 2^\star$. Since $\tilde{u}(0)>0$, we the get that $\beta_q = O(\alpha_q)$ when $q\to 2^\star$. This inequality combined with (14) yields (18). This proves the claim.

Step 4: We claim that $\tilde{u}\in D_k^2(\mathbb{R}^n)$.

We prove the claim. Indeed, for all $i\in\{0,...,k\}$, it follows from (18) and a change of variable that $\|\nabla^i \tilde{u}_q\|_{L^{p_i}(B_R(0))} \leq C\|\nabla^i u_q\|_{L^{p_i}(B_{R\beta_q}(x_q))} \leq \|\nabla^i \tilde{u}_q\|_{L^{p_i}(M)}$ for all $q\in(2,2^\star)$, all $R>0$ and where $p_i := \frac{2n}{n-2k+2i}$. It follows from Sobolev's inequalities that the right-hand-side is dominated by $\|u_q\|_{H_k^2}$, and therefore, letting $q\to 2^\star$ and $R\to+\infty$ yields $\nabla^i\tilde{u}\in L^{p_i}(\mathbb{R}^n)$ for all $i\in\{0,...,k\}$. We let $\eta\in C_c^\infty(\mathbb{R}^n)$ be such that $\eta_{|B_1(0)}\equiv 1$: as easily checked, $(\eta(m^{-1}\cdot)\tilde{u})_m\in C_c^\infty(\mathbb{R}^n)$ is a Cauchy sequence for the D_k^2-norm, and therefore $\tilde{u}\in D_k^2(\mathbb{R}^n)$. This proves the claim.

Step 5: We claim that

$$\mu_{2^\star} = \frac{|O_G(x_\infty)|^{\frac{2k}{n}}}{f(x_\infty)^{\frac{2}{2^\star}} K(n,k)} \text{ and } \lim_{\alpha\to+\infty}\frac{\beta_q}{\alpha_q} = 1 \tag{19}$$

We prove the claim. Since $\tilde{u}\in D_k^2(\mathbb{R}^n)$, we multiply (16) by $\tilde{u}$ and integrate to get $\int_{\mathbb{R}^n}(\Delta_\xi^{\frac{k}{2}}\tilde{u})^2\, dv_\xi = \mu_{2^\star} f(x_\infty)\int_{\mathbb{R}^n}\tilde{u}^{2^\star}\, dv_\xi$. Since $\tilde{u}\not\equiv 0$, plugging this identity in the Sobolev inequality (11) yields

$$\int_{\mathbb{R}^n}\tilde{u}^{2^\star}\, dv_\xi \geq \left(\frac{1}{\mu_{2^\star} f(x_\infty) K(n,k)}\right)^{\frac{2^\star}{2^\star-2}} \tag{20}$$

We let $m := |O_G(x_\infty)|$ if $|O_G(x_\infty)|<\infty$, and any $m\in\mathbb{N}\setminus\{0\}$ otherwise. We let $\sigma_1,...,\sigma_m\in G$ be such that $\sigma_i(x_\infty)\neq\sigma_j(x_\infty)$ for all $i,j\in\{1,...,m\}$, $i\neq j$. We fix $\delta<\min_{i\neq j}\{d_g(z,z')/\, z\neq z'\in O_G(x_\infty)\}$. The $G-$invariance yields

$$\begin{aligned} 1 &= \int_M f u_q^q\, dv_g \geq \sum_{i=1}^m \int_{B_\delta(\sigma_i(x_\infty))} f u_q^q\, dv_g = m\int_{B_\delta(x_\infty)} f u_q^q\, dv_g \\ &\geq m\int_{B_{R\beta_q}(x_q)} f u_q^q\, dv_g = m\left(\frac{\beta_q}{\alpha_q}\right)^{n-2k}\int_{B_R(0)} f(\exp_{x_q}(\beta_q\cdot))\tilde{u}_q^q\, dv_{g_q} \end{aligned} \tag{21}$$

for all $q\in(2,2^\star)$ and all $R>0$. Letting $q\to+\infty$, and then $R\to+\infty$ and using (20), we get that

$$1 \geq \left(\lim_{q\to 2^\star}\frac{\beta_q}{\alpha_q}\right)^{n-2k}\frac{m f(x_\infty)}{(\mu_{2^\star} f(x_\infty) K(n,k))^{\frac{2^\star}{2^\star-2}}}.$$

In particular, since $\beta_q \geq \alpha_q$ with (18), we get an upper-bound for m, and therefore $|O_G(x)| < \infty$, and we take $m = |O_G(x)|$. The inequality rewrites

$$\mu_f(\mathcal{C}_G) \geq \frac{2}{n-2k} \cdot \frac{|O_G(x_\infty)|^{\frac{2k}{n}}}{f(x_\infty)^{\frac{2}{2^\star}} K(n,k)} \cdot \left(\lim_{q\to 2^\star} \frac{\beta_q}{\alpha_q} \right)^{\frac{2k(n-2k)}{n}}.$$

It then follows from (12) and (18) that (19) holds. Moreover, we also get that equality holds in (20) and that $\tilde{u}$ is an extremal for the Sobolev inequality (11). This proves the claim.

Step 6: We claim that

$$f u_q^q \, dv_g \rightharpoonup \frac{1}{|O_G(x)|} \delta_{O_G(x)} \text{ in the sense of measure when } q \to 2^\star. \tag{22}$$

We prove the claim. Since equality holds in (20), that $\lim_{q\to 2^\star} \frac{\alpha_q}{\beta_q} = 1$ and that (19) holds, we get with a change of variables that

$$\lim_{R\to+\infty} \lim_{q\to 2^\star} \int_{B_{R\beta_q}(x_q)} f u_q^q \, dv_g = f(x_\infty) \int_{\mathbb{R}^n} \tilde{u}^{2^\star} \, dv_\xi = \frac{1}{m}. \tag{23}$$

For $\delta > 0$, we let $B_\delta(O_G(x_\infty))$ be the union of balls of radius δ centered at the orbit. Therefore, since $\int_M f u_q^q \, dv_g = 1$, (21), (23) and the $G-$invariance yield

$$\lim_{q\to 2^\star} \int_{M\setminus B_\delta(O_G(x_\infty))} f u_q^q \, dv_g = 0 \tag{24}$$

for all $\delta > 0$. Consequently, $\lim_{q\to 2^\star} \int_{B_\delta(z)} f u_q^q \, dv_g = \frac{1}{m}$ for all $\delta > 0$ small enough and all $z \in O_G(x)$. Assertion (22) then follows. This proves the claim.

Step 7: We claim that there exists $C > 0$ such that

$$d(x, O_G(x_q))^{\frac{n-2k}{2}} u_q(x) \leq C \tag{25}$$

for all $x \in M$ and all $q \in (2, 2^\star)$.

We prove the claim. This pointwise inequality has its origins in Druet [10]. We define $w_q(x) := d(x, O_G(x_q))^{\frac{n-2k}{2}} u_q(x)$ for all $q \in (2, 2^\star)$ and all $x \in M$. We argue by contradiction and assume that $\lim_{q\to 2^\star} \|w_q\|_\infty = +\infty$. We define $(y_q)_{q\in(2,2^\star)} \in M$ such that

$$\max_{y\in M} w_q(y) = w_q(y_q) \to +\infty \tag{26}$$

when $q \to 2^\star$. We define $\gamma_q := u_q(y_q)^{-\frac{2}{n-2k}}$ for all $q \in (2, 2^\star)$. It follows from (26) that

$$\lim_{q\to 2^\star} u_q(y_q) = +\infty \text{ and } \lim_{q\to 2^\star} \gamma_q = 0. \tag{27}$$

As easily checked, coming back to the definitions of α_q and β_q, it follows from (19) that $\lim_{q\to 2^\star} u_q(x_q)^{2^\star - q} = 1$. Therefore, since $u_q(y_q) \leq u_q(x_q)$ for all q and (27) holds, we get that $\lim_{q\to 2^\star} \gamma_q^{2^\star - q} = 1$. We define

$$\bar{u}_q(x) := \gamma_q^{\frac{n-2k}{2}} u_q(\exp_{y_q}(\gamma_q x))$$

for all $q \in (2, 2^\star)$ and all $x \in B_{\delta\gamma_q^{-1}}(0)$ where $\delta \in (0, i_g(M))$. Arguing as in Step 2 and using that $\lim_{q\to 2^\star} \gamma_q^{2^\star - q} = 1$, we get that

$$P_{\bar{g}_q} \bar{u}_q = \mu_q(1 + o(1)) f(\exp_{y_q}(\gamma_q \cdot)) \bar{u}_q^q \text{ in } B_{\delta\gamma_q^{-1}}(0) \tag{28}$$

for all $q\in(2,2^\star)$, where $\lim_{q\to 2^\star} o(1)=0$ uniformly. We fix $R>0$. It follows from the definition (26) of w_q and y_q that

$$d(\exp_{y_q}(\gamma_q x), O_G(x_q))^{\frac{n-2k}{2}}\bar{u}_q(x)\leq d(y_q,O_G(x_q))^{\frac{n-2k}{2}} \tag{29}$$

for all $x\in B_R(0)$ and $q\in(2,2^\star)$. The limit $w_q(y_q)\to+\infty$ when $q\to 2^\star$ rewrites $\lim_{q\to 2^\star}\gamma_q^{-1}d_g(y_q,O_G(x_q))=+\infty$: therefore, there exists $q_0\in(2,2^\star)$ such that $d(\exp_{y_q}(\gamma_q x),O_G(x_q))\geq d(y_q,O_G(x_q))/2$ for all $x\in B_R(0)$ and all $q\in(q_0,2^\star)$, and it follows from (29) that $0\leq\bar{u}_q(x)\leq 2^{\frac{n-2k}{2}}$ for all $x\in B_R(0)$ and all $q\in(q_0,2^\star)$. It then follows from (28) and standard elliptic theory (see for instance [1]) that there exists $\bar{u}\in C^{2k}(\mathbb{R}^n)$ such that $\lim_{q\to 2^\star}\bar{u}_q=\bar{u}$ in $C^{2k}_{loc}(\mathbb{R}^n)$. Moreover, $\bar{u}\geq 0$ and $\bar{u}(0)=\lim_{q\to 2^\star}\bar{u}_q(0)=1$, and then $\bar{u}\not\equiv 0$. In particular,

$$\lim_{R\to+\infty}\lim_{q\to 2^\star}\int_{B_{R\gamma_q}(y_q)} f u_q^q\, dv_g = f(y_\infty)\int_{\mathbb{R}^n}\bar{u}^{2^\star}\,dv_\xi \tag{30}$$

where $y_\infty:=\lim_{q\to 2^\star}y_q$. Since $\lim_{q\to 2^\star}\gamma_q^{-1}d_g(y_q,O_G(x_q))=+\infty$ and $\gamma_q\geq\alpha_q=(1+o(1))\beta_q$ when $q\to 2^\star$, we get that for any $R,R'>0$

$$B_{R\gamma_q}(y_q)\cap B_{R'\beta_q}(O_G(x_q))=\emptyset$$

where $q\to 2^\star$. We let $\sigma_1,...,\sigma_m\in G$ be such that $O_G(x_\infty)=\{\sigma_1(x_\infty),...,\sigma_m(x_\infty)\}$ and these points are distinct: as easily checked, we have that $\cup_{i=1}^m B_{R'\beta_q}(\sigma_i(x_q))\subset B_{R'\beta_q}(O_G(x_q))$ and the balls are distinct. Therefore

$$1=\int_M f u_q^q\,dv_g\geq\int_{B_{R\gamma_q}(y_q)} f u_q^q\,dv_g+\sum_{i=1}^m\int_{B_{R'\beta_q}(\sigma_i(x_q))} f u_q^q\,dv_g$$

for all $q\in(2,2^\star)$ and $R,R'>0$. Letting $q\to 2^\star$, then $R,R'\to+\infty$ and using (23) and (30), we get that

$$1\geq f(y_\infty)\int_{\mathbb{R}^n}\bar{u}^{2^\star}\,dv_\xi+1,$$

a contradiction since $\bar{u}\not\equiv 0$. Then (26) does not hold and therefore (25) holds. This proves the claim.

Step 8: We claim that

$$\lim_{q\to 2^\star}u_q=0 \text{ in } C^{2k}_{loc}(M\setminus O_G(x_\infty)). \tag{31}$$

We prove the claim. We fix $\Omega\subset\subset M\setminus O_G(x_\infty)$ a relatively compact subset. It follows from Step 7 that there exists $C(\Omega)>0$ such that $u_q(x)\leq C(\Omega)$ for all $x\in\Omega$ and all $q\in(2,2^\star)$. It then follows from (9) and standard elliptic theory (see for instance [1]) that there exists $u_\infty\in C^\infty(M\setminus O_G(x_\infty))$ such that $\lim_{q\to 2^\star}u_q=u_\infty$ in $C^{2k}_{loc}(\Omega)$. It then follows from (24) that $u_\infty\equiv 0$, and then (31) holds. This proves the claim.

The following remark will be useful in the sequel: since $\|u_q\|_{P_g}^2=\mu_q\to\mu_{2^\star}$ when $q\to 2^\star$ and $u_q\to 0$ in C^{2k} outside the orbit, we get from the compact embedding $H_k^2\hookrightarrow H_{k-1}^2$ that

$$\lim_{q\to 2^\star}u_q=0 \text{ strongly in } H_{k-1}^2(M) \tag{32}$$

Step 9: We claim that $\nabla f(x_\infty)=0$.

We prove the claim. Indeed, this is equivalent to proving that $X(f)(x_\infty)=0$ for all vector field X on M. With no loss of generality, we assume that $\nabla X(x_\infty)=0$ (this is always possible by modifying X in a normal chart at x_∞) and that X has its support in $B_\delta(x_\infty)$, where $\delta<\min\{d_g(z,z')/\, z\neq z'\in O_G(x_\infty)\}$. We are going to estimate $\int_M X(u_q)\Delta_g^k u_q\, dv_g$ with two different methods. We detail here the case $k=2l$ even and we leave the odd case to the reader.

Integrating by parts, we have that

$$\begin{aligned}&\int_M X(u_q)\Delta_g^{2l}u_q\, dv_g=\int_M \Delta_g^l(X(u_q))\Delta_g^l u_q\, dv_g=\int_M X(\Delta_g^l u_q)\Delta_g^l u_q\, dv_g\\ &+\sum_{i=1}^l\int_M \Delta_g^l u_q\Delta_g^{l-i}\left(\Delta_g(X(\Delta_g^{i-1}u_q))-X(\Delta_g^i u_q)\right)\, dv_g.\end{aligned}$$

Using the explicit contraction in (5), we get that

$$\Delta_g(X(v))-X(\Delta_g v)=(\Delta_g X)(\nabla v)-2(\nabla X,\nabla^2 v)-Ric_g(X,\nabla v),$$

where $v\in C^\infty(M)$ and $\Delta_g X$ is the rough Laplacian, that is $(\Delta_g X)^\alpha=-g^{ij}\nabla_{ij}X^\alpha$. Therefore, we have that (for convenience, we omit the curvature tensor R)

$$\Delta_g(X(\Delta_g^{i-1}u_q))-X(\Delta_g^i u_q)=\nabla^2 X\star\nabla^{2i-1}u_q+\nabla X\star\nabla^{2i}u_q+X\star\nabla^{2i-1}u_q$$

for all $i\in\{1,...,l\}$, and then, denoting as $\nabla^{\{m\}}T$ any linear combination of covariant derivatives of T up to order m, we get that

$$\begin{aligned}&\Delta_g^{l-i}\left(\Delta_g(X(\Delta_g^{i-1}u_q))-X(\Delta_g^i u_q)\right)\\ &=\Delta_g^{l-i}(\nabla^2 X\star\nabla^{2i-1}u_q+\nabla X\star\nabla^{2i}u_q+X\star\nabla^{2i-1}u_q)\\ &=\nabla^{\{2l-2i+2\}}X\star\nabla^{\{2l-1\}}u_q+\nabla X\star\nabla^{2l}u_q,\end{aligned}$$

and then

$$\begin{aligned}&\int_M X(u_q)\Delta_g^{2l}u_q\, dv_g=\int_M X(\Delta_g^l u_q)\Delta_g^l u_q\, dv_g\\ &+\int_M \Delta_g^l u_q\left(\nabla^{\{2+2l\}}X\star\nabla^{\{2l-1\}}u_q+\nabla X\star\nabla^{2l}u_q\right)\, dv_g\\ =\ &\int_M X(\Delta_g^l u_q)\Delta_g^l u_q\, dv_g+\int_M \Delta_g^l u_q\star\nabla^{\{2l+2\}}X\star\nabla^{\{2l-1\}}u_q\, dv_g\\ &+\int_M \Delta_g^l u_q\nabla X\star\nabla^{2l}u_q\, dv_g\end{aligned}$$

Since $k=2l$, it follows from (32) and the Cauchy-Schwarz inequality that

$$\int_M \Delta_g^l u_q\star\nabla^{\{2l+2\}}X\star\nabla^{\{2l-1\}}u_q\, dv_g=O\left(\|u_q\|_{H_k^2}\|u_q\|_{H_{k-1}^2}\right)=o(1)$$

when $q\to 2^\star$. Moreover, since $\nabla X(x_\infty)=0$ and (31) holds, we get that

$$\int_M \Delta_g^l u_q\nabla X\star\nabla^{2l}u_q\, dv_g=o(\|u_q\|_{H_k^2})=o(1)$$

when $q \to 2^\star$. Therefore, integrating by parts, we get that

$$\int_M X(u_q)\Delta_g^{2l} u_q\, dv_g = \int_M X(\Delta_g^l u_q)\Delta_g^l u_q\, dv_g + o(1)$$
$$= \int_M X\left(\frac{(\Delta_g^l u_q)^2}{2}\right) dv_g + o(1) = -\int_M \frac{\operatorname{div}_g(X)}{2}(\Delta^l u_q)^2 + o(1)$$

when $q \to 2^\star$ and where $\operatorname{div}_g(X) = \nabla_i X^i$. Since $\nabla X(x_\infty) = 0$, (31) holds and $\|u_q\|_{H_k^2} \leq C$ for all $q \to 2^\star$, we get that the right-hand-side above goes to zero, and then

$$\lim_{q\to 2^\star} \int_M X(u_q)\Delta_g^{2l} u_q\, dv_g = 0. \tag{33}$$

We now estimate $\int_M X(u_q)\Delta_g^{2l} u_q\, dv_g$ using equation (9). It follows from (4) that

$$\int_M X(u_q)P_g u_q\, dv_g = \int_M \Delta_g^l X(u_q)\Delta_g^l u_q\, dv_g + \sum_{l=0}^{k-1}\int_M A_{(l)}(\nabla^l X(u_q), \nabla^l u_q)\, dv_g$$

It then follows from (32) and an integration by parts that

$$\int_M X(u_q)\Delta_g^{2l} u_q\, dv_g = \int_M X(u_q)P_g u_q\, dv_g + o(1)$$

when $q \to 2^\star$. We now use equation (9) to get that

$$\int_M X(u_q)\Delta_g^{2l} u_q\, dv_g = \mu_q \int_M f X(u_q) u_q^{q-1}\, dv_g + o(1)$$
$$= \mu_q \int_M f X\left(\frac{u_q^q}{q}\right) dv_g = -\frac{\mu_q}{q}\int_M (X(f) + f\operatorname{div}_g(X))u_q^q\, dv_g + o(1)$$

when $q \to 2^\star$. It now follows from Proposition 6, (22) and $\nabla X(x_\infty) = 0$ that

$$\lim_{q\to 2^\star} \int_M X(u_q)\Delta_g^{2l} u_q\, dv_g = -\frac{\mu_{2^\star} X(f)(x_\infty)}{2^\star |O_G(x_\infty)| f(x_\infty)}.$$

This limit combined with (33) yields $X(f)(x_\infty) = 0$, which, as already mentioned, proves that $\nabla f(x_\infty) = 0$. This ends Step 9.

Theorem 2 is a direct consequence of Steps 1 to 9.

As a direct byproduct of Theorem 2, we have the following proposition:

Proposition 8. *Let $(M, \mathcal{C})$ be a conformal Riemannian manifold of dimension $n \geq 3$ and let $k \in \mathbb{N}^\star$ be such that $2k < n$. Let G be a group of diffeomorphisms such that $\mathcal{C}_G \neq \emptyset$ and let $f \in C^\infty_{G,+}(M)$ be a positive $G-$invariant function. Assume that there exists $g \in \mathcal{C}_G$ such that P_g satisfies (C) and (PPP). We assume that*

$$\mu_f(\mathcal{C}_G) < \frac{2}{n-2k}\cdot\frac{|O_G(x)|^{\frac{2k}{n}}}{f(x)^{\frac{2}{2^\star}} K(n,k)},$$

for all $x \in M$. Then there exists $\hat{g} \in \mathcal{C}_G$ such that $Q_{\hat{g}} = f$ and the infimum $\mu_f(\mathcal{C}_G)$ is achieved.

A similar result was proved in [18] for $k = 1$ and in [2] when $n = 2k$.

5. The case of the sphere

We consider here the standard unit $n-$sphere $\mathbb{S}^n$ endowed with its standard round metric h and the associated conformal class $\mathcal{C} := [h]$.

Proposition 9. *Let G be a subgroup of $Isom_h(\mathbb{S}^n)$ and let $f \in C^\infty_{G,+}(\mathbb{S}^n)$ be a smooth positive function. Let $p \in \mathbb{S}^n$ be such that $\nabla^i f(p) = 0$ for all $i \in \{1, ..., n-2k\}$ and $|O_G(p)| \geq 2$. Then*

$$\mu_f(\mathcal{C}_G) < \frac{2}{n-2k} \cdot \frac{|O_G(p)|^{\frac{2k}{n}}}{K(n,k) f(p)^{\frac{2}{2^\star}}}.$$

Proof. Given $\lambda > 1$ and $x_0 \in \mathbb{S}^n$, we let $\phi_\lambda : \mathbb{S}^n \to \mathbb{S}^n$ be such that $\phi_\lambda(x) = \pi_{x_0}^{-1}(\lambda^{-1}\pi_{x_0}(x))$ if $x \neq x_0$ and $\phi_\lambda(x_0) = x_0$, where π_{x_0} is the stereographic projection of pole x_0. Up to a rotation, we can assume that $x_0 := (0, ..., 0, 1)$ is the north pole: then we have that $(\pi_N^{-1})^\star h = U_1^{\frac{4}{n-2k}}\xi$, where $U_1(x) := \left((1+|x|^2)/2\right)^{k-n/2}$. As easily checked, ϕ_λ is a conformal diffeomorphism and standard computations yield $\phi_\lambda^\star h = u_{x_0,\beta}^{\frac{4}{n-2k}} h$ where $\beta := (\lambda^2+1)(\lambda^2-1)^{-1}$ and

$$u_{x_0,\beta}(x) := \left(\frac{\sqrt{\beta^2-1}}{\beta - \cos d_h(x,x_0)}\right)^{\frac{n-2k}{2}}$$

for all $x \in \mathbb{S}^n$ and $\beta > 1$. In particular, we have that

$$\int_{\mathbb{S}^n} u_{p,\beta}^{2^\star}\, dv_h = \omega_n \tag{34}$$

where $\omega_n > 0$ is the volume of $(\mathbb{S}^n, h)$. It follows from the conformal law (2) that

$$P_h u_{x_0,\beta} = c_{n,k} Q_h u_{x_0,\beta}^{2^\star-1} \text{ in } \mathbb{S}^n \text{ with } c_{n,k} := \frac{n-2k}{2}. \tag{35}$$

We now fix $p \in \mathbb{S}^n$ as in the statement of Proposition 9 and we let $\sigma_1, ...\sigma_m \in G$ be such that $O_G(p) = \{\sigma_1(p), ..., \sigma_m(p)\}$ and $|O_G(p)| = m \geq 2$. We define

$$u_\beta := \sum_{i=1}^m u_{\sigma_i(p),\beta}$$

for all $\beta > 1$. One checks that u_β is positive and $G-$invariant. Let us estimate

$$I_h(u_\beta) := \frac{\int_{\mathbb{S}^n} u_\beta P_h u_\beta\, dv_h}{\left(\int_{\mathbb{S}^n} f u_\beta^{2^\star}\, dv_h\right)^{\frac{2}{2^\star}}}.$$

The $G-$invariance and (35) yield

$$\int_{\mathbb{S}^n} u_\beta P_h u_\beta\, dv_h \quad = \quad c_{n,k} Q_h \sum_{i,j=1}^m \int_{\mathbb{S}^n} u_{\sigma_i(p),\beta} u_{\sigma_j(p),\beta}^{2^\star-1}\, dv_h = m c_{n,k} Q_h \left(\omega_n + d_\beta\right)$$

where we have used (34) and where

$$d_\beta := \sum_{i=2}^m \int_{\mathbb{S}^n} u_{\beta,p} u_{\beta,\sigma_i(p)}^{2^\star-1}\, dv_h$$

for all $\beta > 1$. Standard computations yield

$$d_\beta = (1+o(1))\Lambda_{p,G}(\beta^2-1)^{\frac{n-2k}{2}}$$

when $\beta\to 1$, where

$$\Lambda_{p,G}:=\left(\int_{\mathbb{S}^n}(1-\cos d_h(x,p))^{k-n/2}\,dv_h\right)\cdot\sum_{i=2}^m(1-\cos d_h(p,\sigma_i(p)))^{k-n/2}\,\,dv_h>0.$$

Concerning the denominator, it follows from the cancelation hypothesis on the derivatives of f that $|f(x)-f(p)|\leq C d_h(x,O_G(p))^{n-2k+1}$ for all $x\in\mathbb{S}^n$. Therefore, rough estimates yield

$$\left|\int_{\mathbb{S}^n}(f-f(p))u_\beta^{2^\star}\,dv_h\right|\leq C(\beta^2-1)^{\frac{n-2k+1}{2}}$$

for all $\beta>1$. A convexity inequality yields

$$\begin{aligned}\int_{\mathbb{S}^n}u_\beta^{2^\star}\,dv_h &\geq \sum_{i=1}^m\int_{\mathbb{S}^n}u_{\beta,\sigma_i(p)}^{2^\star}\,dv_h+2^\star\sum_{i\neq j}\int_{\mathbb{S}^n}u_{\sigma_i(p),\beta}u_{\sigma_j(p),\beta}^{2^\star-1}\,dv_h\\ &\geq m\left(\omega_n+2^\star d_\beta\right)\end{aligned}$$

Noting $\Lambda_{p,G}>0$ and that $c_{n,k}Q_h\omega_n^{\frac{2^\star-2}{2^\star}}=K(n,k)^{-1}$ (since pulling back $u_{\beta,p}$ by the stereographic projections gives U_1, an extremal for (11)), these estimates yield

$$\begin{aligned}I_h(u_\beta) &\leq \frac{|O_G(p)|^{\frac{2k}{n}}}{f(p)^{\frac{2}{2^\star}}K(n,k)}\cdot\left(1-\frac{\Lambda_{p,G}}{\omega_n}(\beta^2-1)^{\frac{n-2k}{2}}+o((\beta^2-1)^{\frac{n-2k}{2}})\right)\\ &< \frac{|O_G(p)|^{\frac{2k}{n}}}{f(p)^{\frac{2}{2^\star}}K(n,k)}.\end{aligned}$$

Coming back to the definition of $\mu_f(\mathcal{C}_G)$, this proves Proposition 9. □

Proof of Theorem 1: In the case $n=2k+1$, it follows from Proposition 4 and 9 that Case (i) of Theorem 2 cannot hold. Therefore Case (ii) holds, and Theorem 1 is proved.

More generally, Propositions 4 and 7, Theorem 2 and Proposition 9, yield:

Theorem 3. *Let $k\geq 1$ and let G be a subgroup of isometries of $(\mathbb{S}^n,h)$, $n>2k$. Let $f\in C^\infty(M)$ be a positive $G-$invariant function and assume that G acts without fixed point (that is $|O_G(x)|\geq 2$ for all $x\in\mathbb{S}^n$). Assume that there exists $p\in\mathbb{S}^n$ such that*

$$\frac{|O_G(p)|^{\frac{2k}{n}}}{f(p)^{\frac{2}{2^\star}}}\leq\frac{|O_G(x)|^{\frac{2k}{n}}}{f(x)^{\frac{2}{2^\star}}}$$

for all $x\in\mathbb{S}^n$ and that $\nabla^i f(p)=0$ for all $i\in\{1,...,n-2k\}$. Then there exists $g\in[h]$ such that $Q_g=f$ and $G\subset Isom_g(\mathbb{S}^n)$.

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