ON THE INFLUENCE OF THE KERNEL OF THE BI-HARMONIC OPERATOR ON FOURTH ORDER EQUATIONS WITH EXPONENTIAL GROWTH

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Abstract. Continuing the analysis of [1, 9, 10], we discuss in this note the influence of the Kernel of the bi-harmonic operator Δ^2 on the behavior of families of solutions to $\Delta^2 u = e^{4u}$ on a four-dimensional domain of the Euclidean space. We also make a remark on the Paneitz-type equation in the context of compact Riemannian manifolds.

1. **Introduction.** Let Ω be a nonempty open domain of $\mathbb{R}^4$. Consider a sequence $(V_i)_{i\in\mathbb{N}} \in C^0(\Omega)$ such that $\lim_{i\to+\infty} V_i = 1$ in $C^0_{loc}(\Omega)$. Given $\Lambda > 0$, we consider a sequence of functions $(u_i)_{i\in\mathbb{N}} \in C^4(\Omega)$ which are solutions to the fourth order equation

$$\Delta^2 u_i = V_i e^{4u_i} \text{ in } \Omega \qquad (E)$$

$$\int_\Omega V_i e^{4u_i}\,dx \le \Lambda \qquad (1)$$

where $\Delta := -\sum_i \partial_{ii}$ is the Laplacian with minus sign convention. Continuing the analysis of [1, 9, 10], we aim at describing asymptotics for u_i when $i \to \infty$. This equation has its origins in conformal geometry: we refer the interested reader to [3] and [8]. A fairly natural and simple behavior would be that, up to a subsequence, there exists $u \in C^3(\Omega)$ such that

$$\lim_{i\to+\infty} u_i = u \text{ in } C^3_{loc}(\Omega). \qquad (2)$$

When (2) holds for a subsequence, we say that $(u_i)_{i\in\mathbb{N}}$ is relatively compact. Equation (E) enjoys a scaling invariance. Indeed, let $(\mu_i)_{i\in\mathbb{N}} \in \mathbb{R}_{>0}$ and $(x_i)_{i\in\mathbb{N}} \in \Omega$ and define

$$\tilde{u}_i(x) := u_i(x_i + \mu_i x) + \ln \mu_i$$

for all $x \in \mu_i^{-1}(\Omega - x_i)$. It is straightforward that $\Delta^2 \tilde{u}_i = V_i(x_i + \mu_i x)e^{4\tilde{u}_i}$, an equation like (E). This scaling invariance forces situations more subtle than (2) to occur: consider a sequence $(\mu_i)_{i\in\mathbb{N}} \in \mathbb{R}_{>0}$ such that $\lim_{i\to+\infty} \mu_i = 0$ and the functions

$$f_i(x) := \ln \frac{\sqrt{96}\mu_i}{\sqrt{96}\mu_i^2 + |x|^2}$$

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for all $i \in \mathbb{N}$ and all $x \in \mathbb{R}^4$. Then, we get that f_i satisfies (E) with $V_i \equiv 1$ and

$$\lim_{i \to +\infty} f_i(0) = +\infty \text{ and } \lim_{i \to +\infty} f_i(x) = -\infty \text{ for all } x \neq 0.$$

Clearly, (2) does not hold for any subsequence of (f_i). Concerning terminology, we say that the u_i's blow-up if, up to any subsequence, (2) does not hold. In particular, the f_i's above blow-up. This fourth order problem corresponds to the pde $\Delta u = e^{2u}$ in dimension two: this equation has been studied, among others, by Brézis-Merle [2], Li-Shafrir [6] and Tarantello [12]. In particular, one has the following result:

Theorem 1. *[Li-Shafrir [6]] Let Σ be a bounded domain of $\mathbb{R}^2$, $(\bar{V}_i)_{i \in \mathbb{N}} \in C^0(\Sigma)$ be a sequence of functions such that $\lim_{i \to +\infty} \bar{V}_i = 1$ in $C^0_{loc}(\Sigma)$, and $(\bar{u}_i)_{i \in \mathbb{N}} \in C^2(\Sigma)$ be a sequence such that*

$$\Delta \bar{u}_i = \bar{V}_i e^{2\bar{u}_i}$$

in Σ for all $i \in \mathbb{N}$, and such that there exists $\Lambda \in \mathbb{R}$ such that $\int_\Sigma \bar{V}_i e^{2\bar{u}_i} \, dx \leq \Lambda$ for all $i \in \mathbb{N}$. Then either (i) the sequence $(\bar{u}_i)_{i \in \mathbb{N}}$ is relatively compact in $C^1(\Omega)$, or (ii) there exists $N \in \mathbb{N}$, there exist $\bar{x}_1, ..., \bar{x}_N \in \Omega$, there exists $\bar{\alpha}_1, ... \bar{\alpha}_N \in \mathbb{N}^\star$ such that

$$\lim_{i \to +\infty} \bar{V}_i e^{2\bar{u}_i} \, dx = \sum_{p=1}^N 4\pi \bar{\alpha}_p \delta_{\bar{x}_p}$$

in the sense of measures. Moreover,

$$\lim_{i \to +\infty} \bar{u}_i = -\infty \text{ uniformly locally in } \Sigma \setminus \{\bar{x}_1, ... \bar{x}_N\}.$$

In particular, when $N = 0$, $\lim_{i \to +\infty} \bar{u}_i = -\infty$ uniformly locally in Σ.

This result can be seen as a quantization result since blow-up implies that the energy is quantified (in general, it is a multiple of 4π). Concerning the proof of this result, it is important to note that it uses a sup+inf inequality due to Shafrir. More precisely, Given Σ an open subset of $\mathbb{R}^2$, Shafrir [11] proved that any solution $u \in C^2(\Sigma)$ to $\Delta u = V e^{2u}$ with $a \leq V \leq b$ satisfies

$$\sup_\omega u + C_1 \inf_\Sigma u \leq C_2$$

for all $\omega \subset \Sigma$ compact, where C_1, C_2 depend only on a, b, ω and Σ (when $\|\nabla V\|_\infty \leq A$, it is possible to take $C_1 = 1$). This inequality is crucial in the proof of Theorem 1.

Naturally, one is tempted to prove a similar result for our fourth order problem (E). Let us say right now that this is impossible: indeed, it was proved in [1] that any positive value of the energy can be assumed. More precisely, we have that

Proposition 1. *[Adimurthi-Robert-Struwe [1]] Let $\lambda \in (0, +\infty)$. Let B be the unit ball of $\mathbb{R}^4$. Then there exists $(v_i)_{i \in \mathbb{N}} \in C^4(B)$ such that $\Delta^2 v_i = e^{4v_i}$ in B and $\lim_{i \to +\infty} \int_B e^{4u_i} \, dx = \lambda$. Moreover, the v_i's blow-up.*

The fundamental reason for this difference of behavior is due to the different structures of the Kernel of the linear operators that are considered in 2D or in 4D. As is well known,

$$\text{Ker } \Delta = \{ \text{ harmonic functions } \}$$

and harmonic functions enjoy two important properties. First, Hopf's comparison principle asserts that a nonnegative nontrivial harmonic function is positive. Second, and this can be considered as a consequence of the first point, nonnegative

harmonic functions satisfy the Harnack inequality: more precisely, for any $\omega \subset\subset \Sigma$, there exists $c(\omega) > 0$ depending only on ω such that

$$\sup_\omega u \leq c(\omega) \inf_\omega u \tag{3}$$

for all $u \in C^2(\Sigma)$ such that $\Delta u = 0$ and $u \geq 0$. These two properties are of great importance in the proof of Theorem 1. When one considers the Kernel of the bi-Laplacian (the dimension is pointless here), that is

$$\text{Ker } \Delta^2 = \{ \text{ bi-harmonic functions } \}$$

the two properties above have no equivalent. For instance, a nonnegative nontrivial bi-harmonic function is not necessary positive, as shown in the following example: we have that

$$\Delta^2 |x|^2 = 0, \ |x|^2 \geq 0 \text{ and } |0|^2 = 0.$$

The Harnack inequality is also not satisfied in general by bi-harmonic function: inequality (3) is clearly not satisfied by the function $x \mapsto |x|^2$ above when $0 \in \omega$. Despite it is quite naive, this example is generic for fourth order equations and perfectly illustrates the difficulties one has to face in this context. Note that this is not just a technical point: as already mentioned, the expected quantization theorem is false in dimension four (see Proposition 1).

To overcome this difficulties, one can search out for natural hypothesis to carry out the asymptotic study. A first idea is to consider radial functions: in this situation, Ker Δ^2 is explicit and Ker $\Delta^2 = \{a+b|x|^2 /\, a, b \in \mathbb{R}\}$, and a complete study of radial solutions to (E) was carried out in [9]. Another idea is to find a context in which the considered bi-harmonic functions are bound to satisfy a Harnack-type inequality. Indeed, consider the family of functions $\mathcal{A}_M := \left(\text{Ker } \Delta^2\right) \cap \{u \in C^2(\Omega)/\, \|\Delta u\|_1 \leq M\}$: it follows from standard elliptic theory that for any $\omega \subset\subset \Omega$, there exists $C(\omega, M) > 0$ depending only on ω and M such that

$$\sup_\omega v \leq C(\omega, M) \inf_\omega v + C(\omega, M)$$

for all $v \in \mathcal{A}_M$ such that $v \geq 0$. Therefore, we have a kind of Harnack inequality, since a control on the infimum yields a control on the maximum. In this spirit, we prove the following result in [10]

Theorem 2. *[Robert [10]] Let Ω be a domain of $\mathbb{R}^4$, $(V_i)_{i\in\mathbb{N}} \in C^0(\Omega)$ be a sequence such that $\lim_{i\to+\infty} V_i = 1$ in $C^0_{loc}(\Omega)$, and $(u_i)_{i\in\mathbb{N}}$ be a sequence of functions in $C^4(\Omega)$ such that (E) holds, and such that there exists $\Lambda > 0$ such that (1) holds. Assume there exist $C > 0$ and $\omega_0 \subset\subset \Omega$ a nonempty open set such that $\|(\Delta u_i)_-\|_1 \leq C$ and*

$$\|\Delta u_i\|_{L^1(\omega_0)} \leq C$$

for all $i \in \mathbb{N}$. Then (i) either $(u_i)_{i\in\mathbb{N}}$ is relatively compact in $C^3_{loc}(\Omega)$, or (ii) there exists $N \in \mathbb{N}$, there exist $x_1, ..., x_N \in \Omega$, there exists $\alpha_1, ..., \alpha_N \in \mathbb{N}^\star$ such that

$$\lim_{i\to+\infty} V_i e^{4u_i}\, dx = \sum_{p=1}^N 16\pi^2 \alpha_p \delta_{x_p}$$

in the sense of measures. Moreover,

$$\lim_{i\to+\infty} u_i = -\infty \text{ uniformly locally in } \Omega \setminus \{x_1, ...x_N\}.$$

In particular, when $N = 0$, $\lim_{i\to+\infty} u_i = -\infty$ uniformly locally in Ω.

This result can be seen as a fourth order analogous to Theorem 1. However, in the hypothesis of Theorem 1, one gets that $\int_\Sigma |\Delta v_i|\, dx = \int_\Omega e^{2v_i}\, dx$ is uniformly bounded. Therefore, concerning the fourth order problem, we have two natural possibilities for the hypothesis:

(i) either assume that there exists $\Lambda > 0$ such that $\int_\Omega e^{4u_i}\, dx \le \Lambda$ for all $i \in \mathbb{N}$,

(ii) or assume that there exists $\Lambda > 0$ such that $\int_\Omega |\Delta u_i|\, dx \le \Lambda$ for all $i \in \mathbb{N}$.

In Theorem 2, we have chosen to assume that (i) was satisfied. X.X.Chen and G.Tarantello suggested that the quantization result of Theorem 2 should also be true and that a sup+inf inequality was possible under assumption (ii) and not (i). Indeed, by using the analysis in [10], we prove here that this is the case:

Theorem 3. *Let Ω be a bounded domain of $\mathbb{R}^4$. Let a sequence $(V_i)_{i \in \mathbb{N}} \in C^0(\Omega)$ such that $\lim_{i \to +\infty} V_i = 1$ in $C^0_{loc}(\Omega)$. Let $(u_i)_{i \in \mathbb{N}}$ be a sequence of functions in $C^4(\Omega)$ such that*

$$\Delta^2 u_i = V_i e^{4u_i} \qquad (E)$$

in Ω for all $i \in \mathbb{N}$. We assume that there exists $C_1 > 0$ such that

$$\int_\Omega |\Delta u_i|\, dx \le C_1 \qquad (4)$$

for all $i \in \mathbb{N}$. Then, there exists a set $J \subset \mathbb{N}$, there exists $(x_p)_{p \in J} \in \Omega$ such that for all $\omega \subset\subset \Omega$, we have that $\omega \cap \{x_p / p \in J\}$ is finite, there exists $(\alpha_p)_{p \in J} \in \mathbb{N}^\star$ such that

$$\lim_{i \to \infty} V_i e^{4u_i}\, dx = \sum_{p \in J} 16\pi^2 \alpha_p \delta_{x_p}$$

in the sense of measures. Moreover

$$\lim_{i \to +\infty} u_i = -\infty \text{ uniformly locally in } \Omega \setminus \{x_1, ... x_N\}.$$

In particular, when $N = 0$, $\lim_{i \to +\infty} u_i = -\infty$ uniformly locally in Ω.

In addition, for all $\omega \subset\subset \Omega$, there exists $C_2 = C_2(\omega, \Omega, \Lambda, C_1) > 0$ depending only on $\omega, \Omega, \Lambda, C_1$ such that

$$(1 + o(1)) \sup_\omega u_i + \inf_\Omega u_i \le C_2$$

for all $i \in \mathbb{N}$, where $\lim_{i \to +\infty} o(1) = 0$.

Section 2 is devoted to the proof of Theorem 3. In Section 3, we discuss the influence of the nontrivial Kernel in the Riemannian context.

2. **Proof of Theorem 3.** We prove Theorem 3 through the two following claims.

Claim 1: we claim that for any $\omega \subset\subset \Omega$, there exists $C_3(\omega) > 0$ such that

$$\int_\omega e^{4u_i}\, dx \le C_3(\omega) \qquad (5)$$

for all $i \in \mathbb{N}$.

Proof of Claim 1: We let $x_0 \in \Omega$ and $\delta > 0$ such that $B_\delta(x_0) \subset\subset \Omega$. We let $\delta_2 > \delta_1 > \delta$ such that $B_{\delta_2}(x_0) \subset\subset \Omega$. It follows from (4) that

$$\int_{\delta_1}^{\delta_2} \left(\int_{\partial B_r(x_0)} |\Delta u_i|\, d\sigma_r \right) dr = \int_{B_{\delta_2}(x_0) \setminus B_{\delta_1}(x_0)} |\Delta u_i|\, dx \le \int_\Omega |\Delta u_i|\, dx \le C_1,$$

where $d\sigma_r$ is the volume element on $\partial B_{r_i}(x_0)$. In particular, there exists a sequence $(r_i)_{i\in\mathbb{N}} \in [\delta_1, \delta_2]$ such that

$$\int_{\partial B_{r_i}(x_0)} |\Delta u_i| \, d\sigma_{r_i} \leq \frac{C_1}{\delta_2 - \delta_1} \tag{6}$$

for all $i \in \mathbb{N}$. We let $\varphi_i \in C^2(\overline{B_{r_i}(x_0)})$ such that

$$\left\{ \begin{array}{ll} \Delta\varphi_i = 0 & \text{in } B_{r_i}(x_0) \\ \varphi_i = \Delta u_i & \text{in } \partial B_{r_i}(x_0) \end{array} \right\}$$

Since $\delta < \delta_1 \leq r_i$ for all $i \in \mathbb{N}$, it follows from elliptic theory and (6) that there exists $C_4(\delta) > 0$ such that

$$|\varphi_i(x)| \leq C_4(\delta) \tag{7}$$

for all $x \in B_\delta(x_0)$ and all $i \in \mathbb{N}$. For any $i \in \mathbb{N}$, we let H_i be the Green's function for Δ on $B_{r_i}(x_0)$ with Dirichlet boundary condition. Green's representation formula yields

$$\Delta u_i(x) = \int_{B_{r_i}(x_0)} H_i(x,y) V_i(y) e^{4u_i(y)} \, dy + \varphi_i(x)$$

for all $x \in B_{r_i}(x_0)$ and all $i \in \mathbb{N}$. Integrating the above equality on $B_\delta(x_0)$ and using Fubini's theorem, we get that

$$\int_{B_{r_i}(x_0)} \left(\int_{B_\delta(x_0)} H_i(x,y) V_i(y) e^{4u_i(y)} \, dy \right) dx \leq \int_{B_\delta(x_0)} (|\Delta u_i|(x) + |\varphi_i(x)|) \, dx.$$

Since $r_i > \delta$, we get with (4) and (7) that

$$\int_{B_\delta(x_0)} \left(\int_{B_\delta(x_0)} H_i(x,y) V_i(y) e^{4u_i(y)} \, dy \right) dx \leq C_5(\delta).$$

Since $\delta < \delta_1 \leq r_i$, there exists $C_6(\delta) > 0$ such that $H_i(x,y) \geq C_6(\delta)$ for all $x, y \in B_\delta(x_0)$, $x \neq y$. Therefore, since $\lim_{i\to+\infty} V_i = 1$ in $C^0_{loc}(\Omega)$, we get that there exists $C_7(\delta) > 0$ such that

$$\int_{B_\delta(x_0)} e^{4u_i(y)} \, dy \leq C_7(\delta)$$

for all $i \in \mathbb{N}$. The claim follows from a covering argument. $\qquad\square$

Claim 2: We prove the theorem. The first part of the theorem is a simple consequence of Theorem 2 applied on compact subsets of Ω exhausting Ω. Note that in case $\#J = \infty$, $\lim_{p\to+\infty, p\in J} d(x_p, \partial\Omega) = 0$, and $\partial\omega \cap \{x_p/p \in J\}$ is at most finite when $\omega \subset\subset \Omega$.

We prove the sup+inf inequality. We let $\omega \subset\subset \Omega$. It follows from Claim 1 that there exists $C > 0$ such that $\int_\omega e^{4u_i} \, dx \leq C$ for all $i \in \mathbb{N}$. It then follows from Robert ([10], Proposition 3.1) that, up to taking $\omega \subset\subset \Omega$ larger,

(i) either $(u_i)_{i\in\mathbb{N}}$ is uniformly bounded in $C^3_{loc}(\omega)$,

(ii) or there exist $N \in \mathbb{N}$, $x_1, ..., x_N \in \omega$ such that $\lim_{i\to+\infty} u_i = -\infty$ uniformly locally in $\omega \setminus \{x_1, ..., x_N\}$.

Moreover, in addition to case (ii), we have that

(ii.a) Δu_i is uniformly bounded in $L^\infty_{loc}(\omega \setminus \{x_1, ..., x_N\})$ for $i \to +\infty$,

and

(ii.b) in case $N > 0$, there exists a sequence $(x_i)_{i \in \mathbb{N}} \in \omega$ such that $\lim_{i \to +\infty} x_i = x_\infty \in \omega$, $\sup_\omega u_i = u_i(x_i) \to +\infty$ when $i \to +\infty$ and

$$\lim_{i \to +\infty} u_i(x_i + e^{-u_i(x_i)}x) - u_i(x_i) = \ln \frac{\sqrt{96}}{\sqrt{96} + |x|^2}$$

for all $x \in \mathbb{R}^4$. Moreover, this convergence holds in $C^3_{loc}(\mathbb{R}^4)$. This is always possible, up to taking ω larger.

In case (i) or when $N = 0$ in (ii), the conclusion of the theorem holds. We assume now that we are in case (ii) and that $N > 0$. Let $\delta > 0$ such that $B_{2\delta}(y_1) \subset\subset \omega$, and let G_i be the Green's function for Δ^2 on $B_\delta(x_i)$ with Navier condition on the boundary, that is

$$\begin{cases} \Delta^2 G_i(x, \cdot) = \delta_x & \text{in } \mathcal{D}'(B_\delta(x_i)) \\ G_i(x, \cdot) = \Delta G_i(x, \cdot) = 0 & \text{on } \partial B_\delta(x_i). \end{cases}$$

It follows from Green's representation formula that

$$u_i(x_i) = \int_{B_\delta(x_i)} G_i(x_i, y)V_i(y)e^{4u_i(y)}\, dy + \psi_i(x_i),$$

where $\psi_i \in C^4(\overline{B_\delta(x_i)})$ such that

$$\Delta^2 \psi_i = 0 \text{ in } B_\delta(x_i) \text{ and } \psi_i = u_i, \Delta\psi_i = \Delta u_i \text{ on } \partial B_\delta(x_i).$$

It follows from points (ii) and (ii.a) above and from the maximum principle that there exists $C_8(\delta) > 0$ such that $\psi_i(x_i) \geq \inf_{\partial B_\delta(x_i)} u_i - C_8(\delta)$ for all $i \in \mathbb{N}$. We let $R > 0$. Since $G_i > 0$ and $\lim_{i \to +\infty} u_i(x_i) = +\infty$, we get that

$$u_i(x_i) \geq \int_{B_{Re^{-u_i(x_i)}}(x_i)} G_i(x_i, y)V_i(y)e^{4u_i(y)}\, dy + \inf_{\partial B_\delta(x_i)} u_i - C_8(\delta)$$

for all $i \in \mathbb{N}$. It follows from standard properties of the Green's function that there exists $C_9(\delta) > 0$ such that $G_i(x_i, y) \geq \frac{1}{8\pi^2} \ln \frac{1}{|x_i - y|} - C_9(\delta)$ for all $x \in B_{\delta/2}(x_i)$. With the change of variable $y = x_i + e^{-u_i(x_i)}z$, we get that

$$\begin{aligned} u_i(x_i) \geq & \int_{B_R(0)} \left(\frac{u_i(x_i)}{8\pi^2} + \frac{1}{8\pi^2} \ln \frac{1}{|z|} - C_9(\delta) \right) V_i(x_i + e^{-u_i(x_i)}z)e^{4\tilde{u}_i(z)}\, dz \\ & + \inf_{\partial B_\delta(x_i)} u_i - C_8(\delta) \end{aligned}$$

for all $i \in \mathbb{N}$, where

$$\tilde{u}_i(x) := u_i(x_i + e^{-u_i(x_i)}x) - u_i(x_i)$$

for all $x \in B_{\delta e^{u_i(x_i)}}$ and all $i \in \mathbb{N}$. Letting $i \to +\infty$ and then $R \to +\infty$ and using (ii.b), one gets that

$$(1 + o(1))u_i(x_i) + \inf_{\partial B_\delta(x_i)} u_i \leq C_{10}(\delta)$$

where $\lim_{i \to +\infty} o(1) = 1$. Since $u_i(x_i) = \sup_\omega u_i$ we get that

$$(1 + o(1)) \sup_\omega u_i + \inf_\Omega u_i \leq C_{10}(\delta)$$

and the claim is proved. $\square$

3. **The Kernel in the Riemannian context.** Let (M, g) be a compact Riemannian manifold of dimension four without boundary. Given two sequence $(a_i), (b_i)_{i \in \mathbb{N}} \in C^{0,\theta}(M)$, $\theta \in (0, 1)$, converging respectively to a_∞, b_∞ in $C^{0,\theta}(M)$, we consider a sequence of functions $(u_i)_{i \in \mathbb{N}} \in C^4(M)$ solutions to the problem:

$$(S_\Lambda) \begin{cases} Pu + a_i = b_i e^{4u} & \text{in } M \\ \int_M e^{4u_i} \, dv_g \leq \Lambda \end{cases}$$

Here, $P : C^4(M) \to C^0(M)$ is defined as

$$Pu := \Delta_g^2 u - \operatorname{div}_g(A(\nabla u)^\sharp)$$

for all $u \in C^4(M)$, where $\Delta_g := -\operatorname{div}_g(\nabla)$ is the Laplace-Beltrami operator and $\operatorname{div}_g(A(\nabla u)^\sharp) := g^{ij} \nabla_i (A_{j\alpha} \nabla^\alpha u)$, given $A \in \Lambda_{(2,0)}(M)$ a smooth symmetric $(2,0)$−tensor on M (we have raised the indices with the metric and we have adopted Einstein's summation convention). As already mentioned, systems like (S_Λ) have their origin in conformal geometry (see [3] or [8]). Here again, we focus on the Kernel of the operator P, a Kernel that contains the constant functions. Since M is compact without boundary, integrating by parts, we get that Ker $\Delta_g^2 = \mathbb{R}$, the constant functions. A natural hypothesis on P is then to assume that Ker $P = \mathbb{R}$ (this holds for the Paneitz operator when the Yamabe invariant is positive, see Gursky [5]). Under this hypothesis, it is possible to carry out the asymptotic study of solutions to $(\mathcal{S}_\Lambda)$: in [4] and [7], it was proved that when there is no relative compactness of solutions, then $\lim_{i \to +\infty} \int_M b e^{4u_i} \, dv_g \in 16\pi^2 \mathbb{N}$. In the analysis provided in [4], we prove and use use the following proposition:

Proposition 2. *Let* $(a_i)_{i \in \mathbb{N}}, (b_i)_{i \in \mathbb{N}} \in C^{0,\theta}(M)$ *as above. Let* $(u_i)_{i \in \mathbb{N}} \in C^4(M)$ *satisfying* (S_Λ). *We assume that*

$$Ker \, P = \mathbb{R},$$

and that there exists $C_0 > 0$ *such that*

$$u_i(x) \leq C_0$$

for all $i \in \mathbb{N}$ *and all* $x \in M$. *Then there exists* $C_1 > 0$ *such that* $|\nabla u_i|_g(x) \leq C_1$ *for all* $x \in M$. *In particular, we are in one an only one of the following situation:*

(i) $\lim_{i \to +\infty} u_i = -\infty$ *uniformly in* M, *or*
(ii) there exists $u_\infty \in C^4(M)$ *such that* $\lim_{i \to +\infty} u_i = u_\infty$ *in* $C^4(M)$.

In this section, we show that the hypothesis on the Kernel of P is not removable:

Proposition 3. *Let* (M, g) *be a compact Riemannian manifold without boundary. We assume that Ker* $P \neq \mathbb{R}$. *Let* $a_\infty \in C^{0,\theta}(M)$ *such that* $a_\infty \in (Ker \, P)^\perp$ *(for the* L^2−*product). Then there exists* $(a_i)_{i \in \mathbb{N}}, (b_i)_{i \in \mathbb{N}} \in C^{0,\theta}(M)$ *such that* $\lim_{i \to +\infty} a_i = a_\infty$, $\lim_{i \to +\infty} b_i = b_\infty > 0$ *in* $C^{0,\theta}(M)$, *there exists* $(u_i)_{i \in \mathbb{N}} \in C^{4,\theta}(M)$ *satisfying* (S_Λ) *for some* $\Lambda > 0$, *there exists exists* Ω *an open subset of* M *such that* $\Omega \neq \emptyset$, $\Omega \neq M$ *and such that*

(i) there exists $C_0 > 0$ *such that* $u_i(x) \leq C_0$ *for all* $i \in \mathbb{N}$ *and all* $x \in M$,
(ii) $\lim_{i \to +\infty} u_i = -\infty$ *uniformly locally in* Ω,
(iii) there exists $C_1 > 0$ *such that* $|u_i(x)| \leq C_1$ *for all* $x \in M \setminus \Omega$ *and all* $i \in \mathbb{N}$.

In particular, we do not have the dichotomy of Proposition 2. The rest of this section is devoted to the proof of Proposition 3. Let $\varphi_0 \in (Ker \, P) \setminus \mathbb{R}$ and let $\varphi := \varphi_0 - \min_M \varphi_0$. We then have that $P\varphi \equiv 0$, $\varphi \geq 0$, $\varphi \not\equiv 0$ and there exists $x_0 \in M$ such that $\varphi(x_0) = 0$. We denote by $\pi : C^4(M) \to C^4(M)$ the L^2−projection

on Ker P (this is relevant since weak solutions to $Pu = 0$ are smooth). We let $\psi_i \in C^4(M)$ be the unique solution to the problem

$$\left\{ \begin{array}{l} P\psi_i + a_\infty = e^{-4i\varphi} - \pi\left(e^{-4i\varphi}\right) \\ \psi_i \in (\text{Ker } P)^\perp. \end{array} \right\}$$

It follows from standard elliptic theory that $\lim_{i \to +\infty} \psi_i = \psi_\infty$ in $C^3(M)$, where $\psi_\infty \in C^4(M)$ is the only solution to

$$\left\{ \begin{array}{l} P\psi_\infty + a_\infty = 0 \\ \psi_\infty \in (\text{Ker } P)^\perp. \end{array} \right\}$$

We let $a_i := a_\infty + \pi\left(e^{-4i\varphi}\right)$ and $b_i := e^{-4\psi_i}$ for all $i \in \mathbb{N}$. Letting

$$u_i := -i\varphi + \psi_i,$$

we get that

$$Pu_i + a_i = b_i e^{4u_i} \text{ in } M.$$

Clearly, we have that $\lim_{i \to +\infty} a_i = a_\infty$ and $\lim_{i \to +\infty} b_i = b_\infty := e^{-4\psi_\infty}$ in $C^{0,\theta}(M)$ for all $\theta \in (0,1)$. Since $(\psi_i)_{i \in \mathbb{N}}$ is uniformly bounded in $C^3(M)$, we get the conclusion of the proposition by taking $\Omega = \varphi^{-1}((0, +\infty))$.

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