STRUWE'S COMPACTNESS FOR FREE FUNCTIONALS INVOLVING THE BI-HARMONIC OPERATOR

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Abstract. In 1984, Struwe gave a complete description of Palais-Smale sequences for a functional arising in the study of nonlinear elliptic equations with critical Sobolev growth. Hebey and the author gave a similar description in the Riemannian context for a functional involving the bi-harmonic operator. We extend this result to more general functionals with nearly critical Sobolev growth.

1. Introduction and statement of the results

Let (M,g) be a compact Riemannian manifold of dimension $n \geq 5$. Let $H_2^2(M)$ be the standard Sobolev space defined as the completion of $C^\infty(M)$ w.r.t. the norm

$$\|u\|_{H_2^2(M)} = \sqrt{\int_M (\Delta_g u)^2\, dv_g + \int_M |\nabla u|_g^2\, dv_g + \int_M u^2\, dv_g},$$

where $\Delta_g = -div_g(\nabla)$ is the Riemannian Laplacian and dv_g is the Riemannian volume element on M. We denote by $2^\sharp = \frac{2n}{n-4}$ the critical exponent for the Sobolev embeddings, that is $H_2^2(M) \hookrightarrow L^q(M)$ for $q \leq 2^\sharp$ is continuous, and compact if and only if $q < 2^\sharp$. A classical question is to find conditions to obtain positive smooth solutions for the problem

$$\Delta_g u + au = fu^q \quad \text{ in } M$$

where a, f are functions on M. This problem is well understood when $q < \frac{n+2}{n-2}$, but the critical case $q = \frac{n+2}{n-2}$ is quite intricate and has been intensively studied in the past years. We now generalize this equation to the bi-harmonic operator and investigate for solutions $u \in H_2^2(M)$ satisfying

$$\Delta_g^2 u - div_g(A\nabla u) + au = f|u|^{2^\sharp-2}u \text{ in } M \tag{1}$$

where $A \in \Lambda^0_{(2,0)}(M)$ is a continuous symmetrical $(2,0)-$tensor field, $a, f \in C^0(M)$. Such a solution to our problem will be smooth at the cost of slightly further assumptions on A, a and f. As easily checked, the problem of finding H_2^2-solutions to (1) is precisely that of finding critical points for the functional

$$I(u) = \frac{1}{2}\int_M (\Delta_g u)^2\, dv_g + \frac{1}{2}\int_M A(\nabla u, \nabla u)\, dv_g + \frac{1}{2}\int_M au^2\, dv_g - \frac{1}{2^\sharp}\int_M f|u|^{2^\sharp}\, dv_g$$

In their celebrated paper [AmRa], Ambrosetti and Rabinowitz introduced the mountain pass lemma and constructed some Palais-Smale sequences for the functional I. We say that $u_n \in H_2^2(M)$ for

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all $n \in \mathbb{N}$ is a Palais-Smale sequence for I if

$$\begin{cases} I(u_n) & \text{is bounded} \\ dI(u_n) \to 0 & \text{strongly in } H_2^2(M)' \end{cases}$$

It is natural to inquire whether u_n converges, and in which sense, to a function u solution of (1). The lack of compactness due to the critical exponent $2^\sharp$ leads to serious difficulties. Therefore the study of Palais-Smale sequences attempted to be crucial for the study of equation (1).

In 1984, Struwe studied Palais-Smale sequences for the following functional:

$$J(u) = \int_\Omega |\nabla u|_\xi^2 \, dv_\xi - \frac{n-2}{2n} \int_\Omega |u|^{\frac{2n}{n-2}} \, dv_\xi,$$

where Ω is an open bounded subset of $\mathbb{R}^n$, ξ is the Euclidean metric and $u \in H_{1,0}^2(\Omega)$, the completion of smooth functions with compact support in Ω w.r.t. the norm

$$\|u\| = \sqrt{\int_\Omega |\nabla u|^2 \, dv_\xi}.$$

In [Str], he gave a complete description of Palais-Smale sequences for the functional J.

In [HeRo], Hebey and the author rewrote this result for the functional I with f a positive constant function. Our aim here is to generalize this result to the more general functional

$$\begin{aligned} I_\varepsilon(u) \quad = \quad & \frac{1}{2} \int_M (\Delta_g u)^2 \, dv_g + \frac{1}{2} \int_M A(\nabla u, \nabla u) \, dv_g + \frac{1}{2} \int_M a u^2 \, dv_g \\ & - \frac{1}{2^\sharp - \varepsilon} \int_M f |u|^{2^\sharp - \varepsilon} \, dv_g, \end{aligned}$$

where $u \in H_2^2(M)$ and $0 \leq \varepsilon < 2^\sharp - 2$. It turns out that finding sequences verifying that

$$\begin{cases} I_\varepsilon(u_\epsilon) & \text{is bounded} \\ dI_\varepsilon(u_\epsilon) \to 0 & \text{strongly in } H_2^2(M)' \end{cases}$$

is easy through the mountain-pass lemma applied to the functional I_ε. We say that such a sequence is a Palais-Smale sequence for I_ε.

To describe these sequences, we need some definitions. Let $f \in C^0(M)$. For $p \in M$, we define

$$\mathcal{E}_p(v) = \frac{1}{2} \int_{\mathbb{R}^n} (\Delta_\xi v)^2 \, dv_\xi - \frac{1}{2^\sharp} f(p) \int_{\mathbb{R}^n} |v|^{2^\sharp} \, dv_\xi$$

for all $v \in D_2^2(\mathbb{R}^n)$, where $D_2^2(\mathbb{R}^n)$ is the completion of $C_c^\infty(\mathbb{R}^n)$, the set of smooth functions with compact support in $\mathbb{R}^n$, w.r.t. the norm

$$\|u\| = \sqrt{\int_{\mathbb{R}^n} (\Delta_\xi u)^2 \, dv_\xi}.$$

We denote by $i_g(M) > 0$ the injectivity radius of (M, g), and take $\delta \in]0, \frac{i_g(M)}{2}[$. We choose $\tilde{\eta} \in C^\infty(\mathbb{R}^n)$ such that $\tilde{\eta}(x) = 1$ if $|x| \leq \delta$ and $\tilde{\eta}(x) = 0$ if $|x| \geq 2\delta$. We then define, for $p \in M$, $\eta_p(x) = \tilde{\eta}(exp_p^{-1}(x))$ for $d_g(x, p) < i_g(M)$ and 0 elsewhere.

Our result concerning Palais-Smale sequences is the following:

Theorem 1. *Let $(u_\epsilon)_{\varepsilon>0} \in H_2^2(M)$ be a Palais-Smale sequence for I_ε, where $a, f \in C^0(M)$ and $A \in \Lambda^0_{(2,0)}(M)$ is a continuous symmetrical $(2,0)-$tensor field. We assume that (u_ϵ) is bounded in $H_2^2(M)$ (this occurs if $f > 0$ or if $\Delta_g^2 - div_g(A\nabla) + a$ is coercive). Then*

(i) $\exists u_0 \in H_2^2(M)$ a weak solution of (1)
(ii) there exists $p \in \mathbb{N}$, there exist $x_{\varepsilon,1} \to x_1 \in M$, ..., $x_{\varepsilon,p} \to x_p \in M$ such that $f(x_i) > 0$ for all $i = 1, ..., p$,
(iii) there exist $k_{\varepsilon,i} > 0$ such that $k_{\varepsilon,i} \to 0$ and $k^\varepsilon_{\varepsilon,i} \to c_i \in]0,1]$, $i = 1, ..., p$,
(iv) there exist $v_i \in D_2^2(\mathbb{R}^n)$, $i = 1, ..., p$, weak nonzero solutions of

$$\Delta_\xi^2 v_i = f(x_i)|v_i|^{2^\sharp-2} v_i,$$

verifying that, up to a subsequence,

$$\|u_\epsilon - u_0 - \sum_{i=1}^p \eta_{x_{\varepsilon,i}} u_{\varepsilon,i}\|_{H_2^2(M)} \to 0,$$

where

$$u_{\varepsilon,i}(x) = \mu_{\varepsilon,i}^{-\frac{n-4}{2}} v_i \left(\frac{exp^{-1}_{x_{\varepsilon,i}}(x)}{k_{\varepsilon,i}} \right)$$

for $d_g(x, x_{\varepsilon,i}) < i_g(M)$, and

$$k_{\varepsilon,i} = \mu_{\varepsilon,i}^{1-\varepsilon\frac{n-4}{8}}.$$

Moreover, we have the following:

$$I_\varepsilon(u_\epsilon) = I_0(u_0) + \sum_{i=1}^p c_i^{-\frac{(n-4)^2}{8}} \mathcal{E}_{x_i}(v_i) + o(1),$$

where the c_i's, given by point (iii) above, are positive constants in $]0,1]$.

Let us make a few remarks:

Remark 1: If $f_\varepsilon \in C^0(M)$ converges to $f \in C^0(M)$ in C^0-norm, let

$$\begin{aligned} I_\varepsilon(u) &= \frac{1}{2}\int_M (\Delta_g u)^2\, dv_g + \frac{1}{2}\int_M A(\nabla u, \nabla u)\, dv_g + \frac{1}{2}\int_M a u^2\, dv_g \\ &\quad - \frac{1}{2^\sharp - \varepsilon}\int_M f_\varepsilon |u|^{2^\sharp-\varepsilon}\, dv_g, \end{aligned}$$

and

$$\begin{aligned} \tilde{I}_\varepsilon(u) &= \frac{1}{2}\int_M (\Delta_g u)^2\, dv_g + \frac{1}{2}\int_M A(\nabla u, \nabla u)\, dv_g + \frac{1}{2}\int_M a u^2\, dv_g \\ &\quad - \frac{1}{2^\sharp - \varepsilon}\int_M f |u|^{2^\sharp-\varepsilon}\, dv_g, \end{aligned}$$

for $u \in H_2^2(M)$. Then an $H_2^2(M)-$bounded Palais-Smale sequence for I_ε is an $H_2^2(M)-$bounded Palais-Smale sequence for $\tilde{I}_\varepsilon$ and we can apply the theorem.

Remark 2: It is natural to inquire whether $c_i = 1$ for all $i = 1, ..., p$, that is $k^\varepsilon_{\varepsilon,i} \to 1$. Actually, c_i can assume any value in $]0,1]$, as shown in the following example. Let $\delta \in]0, \frac{i_g(M)}{2}[$, $c \in]0,1[$,

$x_0 \in M$ and $v \in C^\infty(\mathbb{R}^n)$ a positive solution of $\Delta_\xi^2 v = v^{2^\sharp - 1}$ (see [Lin] for the explicit form of these solutions). We set

$$u_\epsilon(x) = \mu_\varepsilon^{-\frac{n-4}{2}} v\left(\frac{exp_{x_0}^{-1}(x)}{k_\varepsilon}\right)\eta(exp_{x_0}^{-1}(x))$$

with

$$\mu_\varepsilon = c^{\frac{1}{\varepsilon}} \text{ and } k_\varepsilon = \mu_\varepsilon^{1-\varepsilon\frac{n-4}{8}}.$$

As easily checked, (u_ϵ) is a Palais-Smale sequence for the functional

$$u \to \frac{1}{2}\int_M (\Delta_g u)^2 \, dv_g - \frac{1}{2^\sharp - \varepsilon}\int_M |u|^{2^\sharp - \varepsilon} \, dv_g.$$

However $k_\varepsilon^\varepsilon = c \in]0,1[$. For the case $c = 1$, we can take $k_\varepsilon = \varepsilon$.

The proof of Theorem 1 follows closely the proof of Theorem 2.1 in [HeRo]. Here, the difficulty is that the function f is allowed to change sign and that the exponent $2^\sharp - \varepsilon$ is subcritical.

2. Weak convergence of u_ϵ

Let $(u_\epsilon) \in H_2^2(M)$ be an H_2^2-bounded Palais-Smale sequence for the functional I_ε. There exists $u^0 \in H_2^2(M)$ verifying that, up to a subsequence

$$u_\epsilon \rightharpoonup u^0 \text{ weakly in } H_2^2(M)$$

$$u_\epsilon \to u^0 \text{ strongly in } H_1^2(M)$$

$$u_\epsilon(x) \to u^0(x) \text{ for almost every } x \in M$$

Let $\varphi \in C^\infty(M)$. We observe that

$$\langle dI_\varepsilon(u_\epsilon), \varphi\rangle = \int_M \Delta_g u_\epsilon \Delta_g \varphi + \int_M A(\nabla u_\epsilon, \nabla\varphi)\, dv_g + \int_M a u_\epsilon \varphi \, dv_g - \int_M f|u_\epsilon|^{2^\sharp - 2 - \varepsilon} u_\epsilon \varphi \, dv_g.$$

Through classical arguments, u^0 is a weak solution of

$$\Delta_g^2 u^0 - div_g(A\nabla u^0) + au^0 = f|u^0|^{2^\sharp - 2}u^0.$$

Moreover, if we set $v_\varepsilon = u_\epsilon - u^0$, then (v_ε) is an H_2^2-bounded Palais-Smale sequence for the functional

$$J_\varepsilon(v) = \frac{1}{2}\int_M (\Delta_g v)^2 \, dv_g - \frac{1}{2^\sharp - \varepsilon}\int_M f|v|^{2^\sharp - \varepsilon} \, dv_g$$

and

$$J_\varepsilon(v_\varepsilon) = I_\varepsilon(u_\epsilon) - I_0(u^0) + o(1).$$

3. Critical energy

Assume that $\text{Sup}_M f > 0$. We define

$$\beta^{\#} = \frac{2}{n} K_0^{-\frac{n}{4}} (\text{Sup}_M f)^{-\frac{n-4}{4}}$$

where

$$\frac{1}{K_0} = \text{Inf}_{u \in D_2^2(\mathbb{R}^n) - \{0\}} \frac{\int_{\mathbb{R}^n} (\Delta_\xi u)^2 \, dv_\xi}{\left(\int_{\mathbb{R}^n} |u|^{2^\sharp} \, dv_\xi\right)^{\frac{2}{2^\sharp}}} > 0. \tag{2}$$

Its value has been explicitly computed in [EFJ], [Lie],[Lio]. We assume that $J_\varepsilon(v_\varepsilon) = \beta + o(1)$ with $\beta < \beta^{\#}$. The fact that v_ε is a Palais-Smale sequence for J_ε implies that

$$\int_M (\Delta_g v_\varepsilon)^2 \, dv_g = \int_M f |v_\varepsilon|^{2^\sharp - \varepsilon} \, dv_g + o(1) = \frac{n}{2}\beta + o(1).$$

We then have that $\beta \geq 0$ and

$$\int_M f |v_\varepsilon|^{2^\sharp - \varepsilon} \, dv_g \leq (\text{Sup}_M f) \; Vol_g(M)^{\frac{\varepsilon}{2^\sharp}} \left(\int_M |v_\varepsilon|^{2^\sharp - \varepsilon} \, dv_g\right)^{\frac{2^\sharp - \varepsilon}{2^\sharp}}$$

Now, with [DHL], we know that for all $\nu > 0$, there exists $B_\nu > 0$ such that the following Sobolev inequality holds:

$$\left(\int_M |v|^{2^\sharp} \, dv_g\right)^{\frac{2}{2^\sharp}} \leq (K_0 + \nu) \int_M (\Delta_g v)^2 \, dv_g + B_\nu \int_M v^2 \, dv_g$$

for all $v \in H_2^2(M)$. We then obtain that

$$\frac{n}{2}\beta + o(1) \leq (\text{Sup}_M f) \; (1 + o(1)) \left((K_0 + \nu)(\frac{n}{2}\beta + o(1)) + o(1)\right)^{\frac{2^\sharp - \varepsilon}{2}}.$$

Letting ε go to zero, and then ν to zero, the preceding inequality becomes

$$\frac{n}{2}\beta \leq (\text{Sup}_M f) \; \left(K_0 \frac{n}{2}\beta\right)^{\frac{2^\sharp}{2}},$$

if $\beta > 0$, then

$$\beta \geq \frac{2}{n} K_0^{-\frac{n}{4}} (\text{Sup}_M f)^{-\frac{n-4}{4}} = \beta^{\#}.$$

A contradiction. Thus $\beta = 0$ and v_ε goes to zero strongly in $H_2^2(M)$.

If $f \leq 0$, similar arguments show that (v_ε) goes to 0 strongly in $H_2^2(M)$. We then have proved the proposition

Proposition 1. *If $f \leq 0$, or if $Sup_M f > 0$ and $\beta < \beta^{\#}$, then v_ε goes to zero strongly in $H_2^2(M)$.*

4. Fundamental lemma

The next lemma is the main step in proving Theorem 1.

Lemma 1. *Let (v_ε) a Palais-Smale sequence for J_ε such that $v_\varepsilon \rightharpoonup 0$ weakly in $H_2^2(M)$, but not strongly. Then there exist $x_\varepsilon \to x_0 \in M$ such that $f(x_0) > 0$, $\mu_\varepsilon > 0$ such that $\mu_\varepsilon \to 0$ and $\mu_\varepsilon^\varepsilon \to c \in]0,1]$, and $v^0 \in D_2^2(\mathbb{R}^n)$ a weak nonzero solution of*

$$\Delta_\xi^2 v^0 = f(x_0) |v^0|^{2^\sharp - 2} v^0,$$

such that the following holds: if we define

$$\tilde{v}_\varepsilon(x) = \mu_\varepsilon^{-\frac{n-4}{2}} v^0\left(\frac{exp_{x_\varepsilon}^{-1}(x)}{k_\varepsilon}\right)$$

for all $x \in M$ *such that* $d_g(x, x_\varepsilon) < i_g(M)$, *and* 0 *elsewhere, where* $k_\varepsilon = \mu_\varepsilon^{1-\varepsilon\frac{n-4}{8}}$, *then, for all* $\delta \in]0, \frac{i_g(M)}{2}[$,

$$w_\varepsilon = v_\varepsilon - \eta_{x_\varepsilon}\tilde{v}_\varepsilon$$

is a Palais-Smale sequence for J_ε *and*

$$J_\varepsilon(w_\varepsilon) = J_\varepsilon(v_\varepsilon) - c^{-\frac{(n-4)^2}{8}}\mathcal{E}_{x_0}(v^0) + o(1),$$

where $o(1) \to 0$ *as* $\varepsilon \to 0$.

Remark: Observe that, since $v^0 \not\equiv 0$, the optimal Euclidean Sobolev inequality (2) leads us to

$$c^{-\frac{(n-4)^2}{8}}\mathcal{E}_{x_0}(v^0) \geq \beta^\#.$$

Proof of the lemma:

Since (v_ε) does not go to zero strongly, with section 3, we get that $\mathrm{Sup}_M f > 0$ and $J_\varepsilon(v_\varepsilon) \geq \beta^\# + o(1)$. Therefore

$$\int_M f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g \geq \frac{n}{2}\beta^\# + o(1).$$

We will need the following lemma. It is proved in detail in [HeRo].

Lemma 2. *Let* (M, g) *a smooth compact Riemannian* $n-$*manifold. Then, there exist* $r \in]0, i_g(M)[$, $(\Omega_i)_{i\in J}$ *an open covering of* M, *and* $C(M, r) > 1$ *such that the following holds:* $\forall R \geq 1$, $\forall y \in M$, *if we note* $\tilde{g}_{y,R}(x) = exp_y^\star g\left(\frac{x}{R}\right)$, *then*

$$\frac{1}{C(M,r)}\int_{\mathbb{R}^n}(\Delta_\xi u)^2\,dv_\xi \leq \int_{\mathbb{R}^n}(\Delta_{\tilde{g}_{y,R}}u)^2\,dv_{\tilde{g}_{y,R}} \leq C(M,r)\int_{\mathbb{R}^n}(\Delta_\xi u)^2\,dv_\xi$$

for all $u \in D_2^2(\mathbb{R}^n)$ *having the property that* $Supp\, u \in B_\xi(0, rR)$, *and*

$$\frac{1}{C(M,r)}\int_{\mathbb{R}^n}|u|\,dv_\xi \leq \int_{\mathbb{R}^n}|u|\,dv_{\tilde{g}_{y,R}} \leq C(M,r)\int_{\mathbb{R}^n}|u|\,dv_\xi$$

for all $u \in L^1(\mathbb{R}^n)$ *having the property that* $Supp\, u \in B_\xi(0, rR)$.

4.1. **Blow-up of** v_ε**.** For $0 < k_\varepsilon, \lambda_\varepsilon \leq 1$, $x_\varepsilon \in M$ and $|x| < \frac{i_g(M)}{k_\varepsilon}$, we set

$$\tilde{v}_\varepsilon(x) = \lambda_\varepsilon^{\frac{n-4}{2}} v_\varepsilon(exp_{x_\varepsilon}(k_\varepsilon x)),$$

$$\tilde{g}_\varepsilon(x) = (exp_{x_\varepsilon}^\star g)(k_\varepsilon x).$$

Let $0 < C_0 < 2$, $0 < \varepsilon_0 < i_g(M)$, $(\Omega_i)_{i\in J}$ an open covering of M such that for all $i \in J$

$$d_g(exp_x u, exp_x v) \leq C_0|u - v|,$$

for all $x \in \Omega_j$, and $u, v \in T_x M$ such that $|u|, |v| < \varepsilon_0$. Now, let $z \in \mathbb{R}^n$ and $\delta > 0$ such that $|z| + \delta < \frac{i_g(M)}{k_\varepsilon}$,

$$\int_{B_\xi(z,\delta)}(\Delta_{\tilde{g}_\varepsilon}\tilde{v}_\varepsilon)^2\,dv_{\tilde{g}_\varepsilon} = \left(\frac{\lambda_\varepsilon}{k_\varepsilon}\right)^{n-4}\int_{exp_{x_\varepsilon}(k_\varepsilon B_\xi(z,\delta))}(\Delta_g v_\varepsilon)^2\,dv_g$$

and

$$\int_{B_\xi(z,\delta)} f(exp_{x_\varepsilon})(k_\varepsilon x))|\tilde{v}_\varepsilon|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon} = \left(\frac{\lambda_\varepsilon}{k_\varepsilon}\right)^n \lambda_\varepsilon^{-\varepsilon\frac{n-4}{2}} \int_{exp_{x_\varepsilon}(k_\varepsilon B_\xi(z,\delta))} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g.$$

For $|z|+\delta<\frac{\varepsilon_0}{k_\varepsilon}$, we observe that

$$exp_{x_\varepsilon}(k_\varepsilon B_\xi(z,\delta)) \subset B_g(exp_{x_\varepsilon}(k_\varepsilon z), C_0\delta k_\varepsilon),$$

and that

$$exp_{x_\varepsilon}(k_\varepsilon B_\xi(0,C_0\delta)) = B_g(x_\varepsilon, C_0\delta k_\varepsilon)$$

with $\delta<\frac{i_g(M)}{2}$. For $0<\mu\leq 1$, we now set

$$M_\varepsilon(\mu) = \mathrm{Sup}_{x\in M}\int_{B_g(x,C_0\delta\mu)} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g,$$

and $\mathcal{V}=\limsup_{\varepsilon\to 0}\int_M |v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g$. We claim that there exist $x_1\in M$, $\bar\lambda>0$ such that

$$\limsup_{\varepsilon\to 0}\int_{B_g(x_1,C_0\delta\mu)} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g = \bar\lambda.$$

Otherwise, for all $x\in M$,

$$\limsup_{\varepsilon\to 0}\int_{B_g(x,C_0\delta\mu)} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g \leq 0.$$

Let $M_+=\{x\in M/f(x)\geq 0\}\subset\bigcup_{i=1}^q B_g(z_i,C_0\delta\mu)$ with $f(z_i)\geq 0$ (compactness of M_+). Then,

$$\begin{aligned}\int_M f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g &\leq \int_{M_+} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g \leq \int_{\bigcup_{i=1}^q B_g(z_i,C_0\delta\mu)\cap M_+} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g\\ &\leq \sum_{\substack{i=1\dots q\\ B_g(z_i,C_0\delta\mu)}} \int_{B_g(z_i,C_0\delta\mu)} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g\\ &\quad+\int_{\bigcup_{\substack{i=1\dots q\\ B_g(z_i,C_0\delta\mu)\cap M_-^*\neq\emptyset}} B_g(z_i,C_0\delta\mu)\cap M_+} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g\end{aligned}$$

where $M_-^*=M-M_+$. Let $0<\alpha<\frac{n\beta^\#}{4\mathcal{V}}$ and $\beta>0$ such that

$$d_g(x,y)\leq\beta\Rightarrow|f(x)-f(y)|\leq\alpha.$$

As one easily checks, with $\delta<\frac{\beta}{2}$, we obtain that for all $x\in B_g(z_i,C_0\delta\mu)$ such that $B_g(z_i,C_0\delta\mu)\cap M_-^*\neq\emptyset$, $|f(x)|\leq 2\alpha$. Then,

$$\begin{aligned}\int_M f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g &\leq \sum_{\substack{i=1\dots q\\ B_g(z_i,C_0\delta\mu)}} \int_{B_g(z_i,C_0\delta\mu)} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g\\ &\quad+2\alpha\int_{\bigcup_{\substack{i=1\dots q\\ B_g(z_i,C_0\delta\mu)\cap M_-^*\neq\emptyset}} B_g(z_i,C_0\delta\mu)\cap M_+} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g.\end{aligned}$$

Now, letting $\varepsilon\to 0$, one obtains that $\frac{n}{2}\beta^\#\leq 2\alpha\mathcal{V}$. A contradiction. The claim is proved.

Then, for all $0 < \mu \leq 1$, there exists $x_1 \in M$ and $\bar{\lambda} > 0$ such that

$$\limsup_{\varepsilon \to 0} \int_{B_g(x_1, C_0\delta\mu)} f|v_\varepsilon|^{2^\sharp - \varepsilon}\, dv_g = \bar{\lambda}.$$

Up to a subsequence, $M_\varepsilon(\mu) \geq \frac{\bar{\lambda}}{2}$ for all $\varepsilon > 0$ and $M_\varepsilon(0) = 0$. Now let $0 < \lambda < \frac{\bar{\lambda}}{2}$, λ will be fixed later. The continuity of M_ε yields the existence of $0 < k_\varepsilon \leq 1$ such that $M_\varepsilon(k_\varepsilon) = \lambda$. The compactness of M allows us to choose $x_\varepsilon \in M$ satisfying

$$\lambda = \int_{B_g(x_\varepsilon, C_0\delta k_\varepsilon)} f|v_\varepsilon|^{2^\sharp - \varepsilon}\, dv_g = \mathrm{Sup}_{x \in M} \int_{B_g(x, C_0\delta k_\varepsilon)} f|v_\varepsilon|^{2^\sharp - \varepsilon}\, dv_g. \tag{3}$$

4.2. H_2^2-**bound for** $\tilde{v}_\varepsilon$**.** Let $r > 0$ as in Lemma 2. Let $\Omega_\varepsilon = B_\xi(0, \frac{r}{k_\varepsilon})$. We now choose $\eta_r \in C^\infty(\mathbb{R}^n)$ such that $\eta_r \equiv 1$ on $B_\xi(0, r/4)$ and $\eta_r \equiv 0$ on $\mathbb{R}^n - B_\xi(0, r/2)$. We set $\tilde{\eta}_\varepsilon(x) = \eta_r(k_\varepsilon x)$. As easily checked,

$$\begin{aligned}
\int_{\Omega_\varepsilon} (\Delta_{\tilde{g}_\varepsilon} \tilde{v}_\varepsilon)^2\, dv_{\tilde{g}_\varepsilon} &= \left(\frac{\lambda_\varepsilon}{k_\varepsilon}\right)^{n-4} \int_{B_g(x_\varepsilon, r)} (\Delta_g v_\varepsilon)^2\, dv_g \\
&\leq C \left(\frac{\lambda_\varepsilon}{k_\varepsilon}\right)^{n-4} \\
\int_{\Omega_\varepsilon} |\nabla \tilde{v}_\varepsilon|^2_{\tilde{g}_\varepsilon}\, dv_{\tilde{g}_\varepsilon} &= \frac{1}{k_\varepsilon^2} \left(\frac{\lambda_\varepsilon}{k_\varepsilon}\right)^{n-4} \int_{B_g(x_\varepsilon, r)} |\nabla v_\varepsilon|^2_g\, dv_g \\
&\leq C \frac{1}{k_\varepsilon^2} \left(\frac{\lambda_\varepsilon}{k_\varepsilon}\right)^{n-4} \\
\int_{\Omega_\varepsilon} \tilde{v}_\varepsilon^2\, dv_{\tilde{g}_\varepsilon} &= \frac{1}{k_\varepsilon^4} \left(\frac{\lambda_\varepsilon}{k_\varepsilon}\right)^{n-4} \int_{B_g(x_\varepsilon, r)} v_\varepsilon^2\, dv_g.
\end{aligned}$$

Thus $\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon \in D_2^2(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} (\Delta_{\tilde{g}_\varepsilon} \tilde{\eta}_\varepsilon \tilde{v}_\varepsilon)^2\, dv_{\tilde{g}_\varepsilon} \leq C \left(\frac{\lambda_\varepsilon}{k_\varepsilon}\right)^{n-4}.$$

If we choose $\lambda_\varepsilon = O(k_\varepsilon)$, then, with Lemma 2, the preceding inequality becomes

$$\|\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon\|_{D_2^2(\mathbb{R}^n)} = O(1),$$

so, up to a subsequence, there exists $v^0 \in D_2^2(\mathbb{R}^n)$ having the property that

$$\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon \rightharpoonup v^0$$

weakly in $D_2^2(\mathbb{R}^n)$.

4.3. **An estimate on** $k_\varepsilon^\varepsilon$**.** In this subsection, we rule out the case when $k_\varepsilon^\varepsilon$ goes to zero. We first observe that

$$\int_{B_\xi(0, C_0\delta)} f(exp_{x_\varepsilon}(k_\varepsilon x)) |\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon|^{2^\sharp - \varepsilon}\, dv_{\tilde{g}_\varepsilon} = \left(\frac{\lambda_\varepsilon}{k_\varepsilon}\right)^n \left(\lambda_\varepsilon^{-\varepsilon}\right)^{\frac{n-4}{2}} \lambda.$$

Moreover, thanks to Lemma 2,

$$\int_{B_\xi(0, C_0\delta)} |\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon|^{2^\sharp - \varepsilon}\, dv_{\tilde{g}_\varepsilon} \leq C \left(1 + \int_{B_\xi(0, C_0\delta)} |\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon|^{2^\sharp}\, dv_\xi\right).$$

We take $\lambda_\varepsilon = k_\varepsilon$. In view of (2) and subsection 4.2,

$$\int_{B_\xi(0,C_0\delta)} |\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon} = O(1).$$

Consequently, $\lambda\,(\lambda_\varepsilon^{-\varepsilon})^{\frac{n-4}{2}} = O(1)$, then $k_\varepsilon^\varepsilon \not\to 0$. We now let $c \in]0,1]$ such that $k_\varepsilon^\varepsilon \to c$ (up to a subsequence, of course).

4.4. **Strong convergence for $\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon$.** We now define $\mu_\varepsilon > 0$ chosen such that $k_\varepsilon = \mu_\varepsilon^{1-\frac{n-4}{8}\varepsilon}$. As easily checked, $\frac{k_\varepsilon}{\mu_\varepsilon} \to c^{-\frac{n-4}{8}} \neq 0$. We can apply the preceding results with $\lambda_\varepsilon = \mu_\varepsilon$. Without loss of generality, we can assume that $v_\varepsilon \in C^\infty(M)$. Let $y_0 \in \mathbb{R}^n$. Since the embedding $H_2^2(B_\xi(y_0,\rho)) \hookrightarrow H_{3/2}^2(\partial B_\xi(y_0,\rho))$ is compact, there exists $\rho \in [\delta, 2\delta]$ such that

$$\tilde{\eta}_\varepsilon \tilde{v}_{\varepsilon|\partial B_\xi(y_0,\rho)} \to v^0_{|\partial B_\xi(y_0,\rho)} \text{ strongly in } H_{3/2}^2(\partial B_\xi(y_0,\rho)).$$

Let $z_\varepsilon \in H_2^2(B_\xi(y_0,3\delta) - B_\xi(y_0,\rho))$ such that

$$\left\{ \begin{array}{rcl} \Delta_\xi^2 z_\varepsilon &=& 0 \text{ in } B_\xi(y_0,3\delta) - B_\xi(y_0,\rho) \\ z_\varepsilon &=& 0 \text{ on } \partial B_\xi(y_0,3\delta) \\ z_\varepsilon &=& \tilde{\eta}_\varepsilon \tilde{v}_\varepsilon - v^0 \text{ on } \partial B_\xi(y_0,\rho), \end{array} \right.$$

and

$$\psi_\varepsilon(x) = \left\{ \begin{array}{ll} \tilde{\eta}_\varepsilon \tilde{v}_\varepsilon - v^0 & \text{in } B_\xi(y_0,\rho) \\ z_\varepsilon & \text{in } B_\xi(y_0,3\delta) \\ 0 & \text{elsewhere.} \end{array} \right.$$

Clearly $\|z_\varepsilon\|_{H_2^2(\mathbb{R}^n - B_\xi(y_0,\rho))} = o(1)$ and $\psi_\varepsilon \in D_2^2(\mathbb{R}^n)$. We define

$$\begin{array}{rl} \tilde{\psi}_\varepsilon(x) & = \mu_\varepsilon^{-\frac{n-4}{2}} \psi_\varepsilon\left(\frac{exp_{x_\varepsilon}^{-1}(x)}{k_\varepsilon}\right) \text{ if } d_g(x,x_\varepsilon) < 6\delta, \\ & = 0 \text{ elsewhere.} \end{array}$$

Under the assumption that $|y_0| < \frac{\delta}{k_\varepsilon}$, we have $\tilde{\psi}_\varepsilon \in H_2^2(M)$. Moreover, if $\delta < \frac{r}{24}$, then $\eta_r(exp_{x_\varepsilon}^{-1}(x)) = 1$ as soon as $d_g(x,x_\varepsilon) < 6\delta$. Some computations yield

$$\langle dJ_\varepsilon(v_\varepsilon), \tilde{\psi}_\varepsilon \rangle = (\mu_\varepsilon^\varepsilon)^{-\frac{(n-4)^2}{8}} \left(\int_{\mathbb{R}^n} \Delta_{\tilde{g}_\varepsilon} \tilde{\eta}_\varepsilon \tilde{v}_\varepsilon \Delta_{\tilde{g}_\varepsilon} \psi_\varepsilon \, dv_{\tilde{g}_\varepsilon} - \int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x)) |\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon|^{2^\sharp-2-\varepsilon} \tilde{\eta}_\varepsilon \tilde{v}_\varepsilon \tilde{\psi}_\varepsilon dv_{\tilde{g}_\varepsilon} \right).$$

In view of the fact that $\|\tilde{\psi}_\varepsilon\|_{H_2^2(M)} = O(\|\psi_\varepsilon\|_{D_2^2(\mathbb{R}^n)})$, that (v_ε) is a Palais-Smale sequence for J_ε, and that $\mu_\varepsilon^\varepsilon \to c \neq 0$, the equation becomes

$$\int_{\mathbb{R}^n} \Delta_{\tilde{g}_\varepsilon} \tilde{\eta}_\varepsilon \tilde{v}_\varepsilon \Delta_{\tilde{g}_\varepsilon} \psi_\varepsilon \, dv_{\tilde{g}_\varepsilon} = \int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x)) |\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon|^{2^\sharp-2-\varepsilon} \tilde{\eta}_\varepsilon \tilde{v}_\varepsilon \tilde{\psi}_\varepsilon dv_{\tilde{g}_\varepsilon} + o(1).$$

With the definition of ψ_ε,

$$\int_{\mathbb{R}^n} (\Delta_{\tilde{g}_\varepsilon} \psi_\varepsilon)^2 \, dv_{\tilde{g}_\varepsilon} = \int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x)) |\psi_\varepsilon|^{2^\sharp-\varepsilon} \, dv_{\tilde{g}_\varepsilon} + o(1). \tag{4}$$

Basically,

$$\int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x)) |\psi_\varepsilon|^{2^\sharp-\varepsilon} \, dv_{\tilde{g}_\varepsilon} \le (1+o(1))\,(\text{Sup}_M f) \left(\int_{\mathbb{R}^n} |\psi_\varepsilon|^{2^\sharp} \, dv_{\tilde{g}_\varepsilon} \right)^{1-\frac{\varepsilon}{2^\sharp}}.$$

Since $|y_0| + 3\delta < \frac{r}{k_\varepsilon}$, we have $Supp\, \psi_\varepsilon \subset B_\xi(0, \frac{r}{k_\varepsilon})$. Therefore, Lemma 2 and (2) yield

$$(5)\quad \int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon} \leq (\mathrm{Sup}_M f)\, C(M,r)^{1+\frac{2^\sharp}{2}} K_0^{\frac{2^\sharp}{2}} \left(\int_{\mathbb{R}^n} (\Delta_{\tilde{g}_\varepsilon}\psi_\varepsilon)^2\, dv_{\tilde{g}_\varepsilon}\right)^{1-\frac{\varepsilon}{2^\sharp}}.$$

Independently, (4) and (5) together give

$$\left(\int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon}\right)$$

$$(6)\times \left(1 - (\mathrm{Sup}_M f)\, C(M,r)^{1+\frac{2^\sharp}{2}} K_0^{\frac{2^\sharp}{2}} \left(\int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon}\right)^{\frac{2^\sharp}{2}(1-\frac{\varepsilon}{2^\sharp})-1}\right) \leq o(1).$$

Recall that we have $\int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon} = \int_{B_\xi(y_0,\rho)} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon} + o(1)$.

Three different cases arise considering the sign of f:

- *First case:* $f(exp_{x_\varepsilon}(k_\varepsilon x)) \leq 0$ for all $x \in B_\xi(y_0, 3\delta)$. Since $\rho < 3\delta$, one easily gets that $\int_{\mathbb{R}^n} (\Delta_{\tilde{g}_\varepsilon}\psi_\varepsilon)^2\, dv_{\tilde{g}_\varepsilon} = o(1)$, and then, with Lemma 2,
$$\psi_\varepsilon \to 0 \text{ in } D_2^2(\mathbb{R}^n).$$
- *Second case:* $f(exp_{x_\varepsilon}(k_\varepsilon x))$ changes sign in $B_\xi(y_0, 3\delta)$.

  Let $\alpha > 0$ that will be chosen later and $\beta > 0$ such that $d_g(x,y) < \beta \Rightarrow |f(x) - f(y)| < \alpha$. As in the beginning of subsection 4.1, with $\delta < \frac{\varepsilon_0}{4}$ and $\delta < \frac{\beta}{6}$, we clearly obtain that
$$|f(exp_{x_\varepsilon}(k_\varepsilon x))| \leq 2\alpha\ \forall x \in B_\xi(y_0, 3\delta).$$
  Then, with Lemma 2
$$\left|\int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon}\right| \leq 2\alpha C(M,r) \int_{B_\xi(y_0,\rho)} |\psi_\varepsilon|^{2^\sharp-\varepsilon}\, dv_\xi + o(1).$$
  But
$$\begin{aligned}\|\psi_\varepsilon\|_{L^{2^\sharp}(B_\xi(y_0,\rho))} &= \|\tilde{\eta}_\varepsilon\tilde{v}_\varepsilon - v^0\|_{L^{2^\sharp}(B_\xi(y_0,\rho))}\\ &\leq \|\tilde{\eta}_\varepsilon\tilde{v}_\varepsilon\|_{L^{2^\sharp}(B_\xi(y_0,\rho))} + \|v^0\|_{L^{2^\sharp}(B_\xi(y_0,\rho))}\end{aligned}$$
  then
$$\liminf_{\varepsilon\to 0} \|\psi_\varepsilon\|_{L^{2^\sharp}(B_\xi(y_0,\rho))} \leq 2 \liminf_{\varepsilon\to 0} \|\tilde{\eta}_\varepsilon\tilde{v}_\varepsilon\|_{L^{2^\sharp}(B_\xi(y_0,\rho))}.$$
  Note that we have that
$$\int_{B_\xi(y_0,\delta)} |\tilde{\eta}_\varepsilon\tilde{v}_\varepsilon|^{2^\sharp}\, dv_\xi \leq C(M,r)\left(\frac{\mu_\varepsilon}{k_\varepsilon}\right)^n \int_M |v_\varepsilon|^{2^\sharp}\, dv_g.$$
  In view of the fact that $\mu_\varepsilon \leq k_\varepsilon$, the inequality becomes
$$\liminf_{\varepsilon\to 0} \int_{B_\xi(y_0,\rho)} |\psi_\varepsilon|^{2^\sharp-\varepsilon}\, dv_\xi \leq C(M,r)^2 2^{2^\sharp} (\liminf_{\varepsilon\to 0} \|v_\varepsilon\|_{L^{2^\sharp}(M)})^{2^\sharp}.$$
  Then,
$$1 - (\mathrm{Sup}_M f)\, C(M,r)^{1+\frac{2^\sharp}{2}} K_0^{\frac{2^\sharp}{2}} \left(\int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon}\right)^{\frac{2^\sharp}{2}(1-\frac{\varepsilon}{2^\sharp})-1}$$
$$\geq 1 - (\mathrm{Sup}_M f)\, 2^{\frac{(2^\sharp+1)(2^\sharp-2)}{2}} C(M,r)^{2\times 2^\sharp - 2} K_0^{\frac{2^\sharp}{2}} \liminf_{\varepsilon\to 0} \|v_\varepsilon\|_{L^{2^\sharp}(M)}^{2^\sharp\frac{2^\sharp-2}{2}} \alpha + o(1) \geq \frac{1}{2}$$

with α small enough. Then (6) yields

$$\int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon} \le o(1)$$

and

$$\int_{\mathbb{R}^n} (\Delta_{\tilde{g}_\varepsilon}\psi_\varepsilon)^2\,dv_{\tilde{g}_\varepsilon} = o(1),$$

with Lemma 2, we get that

$$\psi_\varepsilon \to 0 \text{ in } D_2^2(\mathbb{R}^n).$$

- *Third case:* We now assume that $f(exp_{x_\varepsilon}(k_\varepsilon x)) \ge 0$ for all $x \in B_\xi(y_0, 3\delta)$. Let $L \in \mathbb{N}^\star$ such that there exists $\tilde{y}_1, ..., \tilde{y}_L \in B_\xi(0,2)$ having the property that

$$B_\xi(0,2) \subset \bigcup_{i=1}^{L} B_\xi(\tilde{y}_i, 1).$$

Then, there exist $y_1, ..., y_L \in B_\xi(y_0, 2\delta)$ such that

$$B_\xi(y_0, 2\delta) \subset \bigcup_{i=1}^{L} B_\xi(y_i, \delta).$$

Standard integration theory yields

$$\int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon} = \int_{B_\xi(y_0,\rho)} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon} + o(1)$$

$$= \int_{B_\xi(y_0,\rho)} f(exp_{x_\varepsilon}(k_\varepsilon x))|\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon} - \int_{B_\xi(y_0,\rho)} f(exp_{x_\varepsilon}(k_\varepsilon x))|v^0|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon} + o(1)$$

$$\le \int_{B_\xi(y_0,2\delta)} f(exp_{x_\varepsilon}(k_\varepsilon x))|\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon} + o(1)$$

$$\le \sum_{i=1}^{L} \int_{B_\xi(y_i,2\delta)} f(exp_{x_\varepsilon}(k_\varepsilon x))|\tilde{\eta}_\varepsilon \tilde{v}_\varepsilon|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon} + o(1)$$

$$\le \sum_{i=1}^{L} \left(\frac{\mu_\varepsilon}{k_\varepsilon}\right)^n (\mu_\varepsilon^{-\varepsilon})^{\frac{n-4}{2}} \int_{B_g(exp_{x_\varepsilon}(k_\varepsilon y_i), C_0\delta k_\varepsilon)} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g + o(1)$$

(these computations are valid provided $\delta < \varepsilon_0/4$ and $\delta < r/16$). But $\left(\frac{\mu_\varepsilon}{k_\varepsilon}\right)^n (\mu_\varepsilon^{-\varepsilon})^{(\frac{n-4}{2})} = (\mu_\varepsilon^{\varepsilon})^{\frac{(n-4)^2}{8}} \le 1$. The definition of λ in (3) yields

$$\int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon} \le \lambda L + o(1).$$

As a consequence,

$$1 - (\text{Sup}_M f)\, C(M,r)^{1+\frac{2^\sharp}{2}} K_0^{\frac{2^\sharp}{2}} \left(\int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon}\right)^{\frac{2^\sharp}{2}(1-\frac{\varepsilon}{2^\sharp})-1}$$

$$\ge 1 - (\text{Sup}_M f)\, C(M,r)^{1+\frac{2^\sharp}{2}} K_0^{\frac{2^\sharp}{2}} (\lambda L)^{\frac{2^\sharp-2}{2}} + o(1).$$

Choosing λ such that

$$0<\lambda<\frac{1}{\left(2\left(\mathrm{Sup}_M f\right)C(M,r)^{1+\frac{2^\sharp}{2}}K_0^{\frac{2^\sharp}{2}}\right)^{\frac{2}{2^\sharp-2}}},$$

we obtain, as in the second case, that $\int_{\mathbb{R}^n} f(exp_{x_\varepsilon}(k_\varepsilon x))|\psi_\varepsilon|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon}\leq o(1)$ and that

$$\psi_\varepsilon\to 0 \text{ in } D_2^2(\mathbb{R}^n).$$

We have proved that

$$\tilde{\eta}_\varepsilon\tilde{v}_\varepsilon\to v^0$$

strongly in $H_2^2(B_\xi(y_0,\delta))$ for all $y_0\in\mathbb{R}^n$ such that $|y_0|<\frac{\delta}{k_\varepsilon}$ for $\varepsilon\to 0$. But $k_\varepsilon\leq 1$, $C_0<2$ and $B_\xi(0,C_0\delta)$ is covered by some balls of radius δ and having their center in $B_\xi(0,\delta)$. Then

$$\tilde{\eta}_\varepsilon\tilde{v}_\varepsilon\to v^0 \text{ strongly in } H_2^2(B_\xi(0,C_0\delta)).$$

Observe that

$$\lambda=\int_{B_g(x_\varepsilon,C_0\delta k_\varepsilon)} f|v_\varepsilon|^{2^\sharp-\varepsilon}\,dv_g=\left(\frac{k_\varepsilon}{\mu_\varepsilon}\right)^n\mu_\varepsilon^{\varepsilon\frac{n-4}{2}}\int_{B_\xi(0,C_0\delta)} f(exp_{x_\varepsilon}(k_\varepsilon x))|\tilde{\eta}_\varepsilon\tilde{v}_\varepsilon|^{2^\sharp-\varepsilon}\,dv_{\tilde{g}_\varepsilon}.$$

Noting that $\tilde{f}(x)=\lim_{\varepsilon\to 0} f(exp_{x_\varepsilon}(k_\varepsilon x))$ and $\tilde{g}(x)=\lim_{\varepsilon\to 0}\tilde{g}_\varepsilon(x)$, we get

$$\lambda=c^{-\frac{(n-4)^2}{8}}\int_{B_\xi(0,C_0\delta)}\tilde{f}|v^0|^{2^\sharp}\,dv_{\tilde{g}}$$

so $v^0\not\equiv 0$. As a consequence, $k_\varepsilon\to 0$. If not, since v_ε goes to 0 weakly, then $\tilde{v}_\varepsilon$ would also go to 0 weakly. But $v^0\not\equiv 0$, a contradiction. Thus $\tilde{\eta}_\varepsilon\tilde{v}_\varepsilon$ goes to v^0 strongly in $H_2^2(B_\xi(y_0,\delta))$ for all $y_0\in\mathbb{R}^n$. This evidently shows that

$$\tilde{v}_\varepsilon\to v^0 \text{ strongly in } H_{2,loc}^2(\mathbb{R}^n).$$

Now let $\varphi\in C_c^\infty(\mathbb{R}^n)$ and $R>0$ such that $Supp\,\varphi\subset B_\xi(0,R)$. We define φ_ε as follows:

$$\varphi_\varepsilon(x)=\mu_\varepsilon^{-\frac{n-4}{2}}\varphi\left(\frac{exp_{x_\varepsilon}^{-1}(x)}{k_\varepsilon}\right) \text{ if } d_g(x,x_\varepsilon)<k_\varepsilon R$$

and 0 otherwise. Then $\varphi_\varepsilon\in C^\infty(M)$ and $\|\varphi_\varepsilon\|_{H_2^2(M)}=O(1)$. Since v_ε is a Palais-Smale sequence for J_ε,

$$\begin{aligned} o(1) &= \langle dJ_\varepsilon(v_\varepsilon),\varphi_\varepsilon\rangle\\ &= \left(\tfrac{k_\varepsilon}{\mu_\varepsilon}\right)^{n-4}\Big(\textstyle\int_{B(0,R)}\Delta_{\tilde{g}_\varepsilon}\tilde{v}_\varepsilon\Delta_{\tilde{g}_\varepsilon}\varphi\,dv_{\tilde{g}_\varepsilon}\\ &\quad -\left(\tfrac{k_\varepsilon}{\mu_\varepsilon}\right)^4(\mu_\varepsilon^\varepsilon)^{\frac{n-4}{2}}\textstyle\int_{B(0,R)} f(exp_{x_\varepsilon}(k_\varepsilon x))|\tilde{v}_\varepsilon|^{2^\sharp-2-\varepsilon}\tilde{v}_\varepsilon\varphi\,dv_{\tilde{g}_\varepsilon}\Big). \end{aligned}$$

Letting $\varepsilon\to 0$ and noting that $k_\varepsilon=\mu_\varepsilon^{1-\varepsilon\frac{n-4}{8}}$, the preceding equation becomes

$$\int_{\mathbb{R}^n}\Delta_\xi v^0\Delta_\xi\varphi\,dv_\xi=\int_{\mathbb{R}^n} f(x_0)|v^0|^{2^\sharp-2}v^0\varphi\,dv_\xi$$

where $\lim_{\varepsilon\to 0}x_\varepsilon=x_0$. Then $v^0\in D_2^2(\mathbb{R}^n)$ attempts to be a weak solution of

$$\Delta_\xi^2 v^0=f(x_0)|v^0|^{2^\sharp-2}v^0. \tag{7}$$

Multiplying by v^0 and integrating, we remark that $f(x_0)>0$.

Now, let $\eta_\varepsilon(x) = \eta_r(\exp_{x_\varepsilon}^{-1}(x))$ for $|x| < i_g(M)$ and 0 elsewhere. We define

$$V_\epsilon(x) = \mu_\varepsilon^{-\frac{n-4}{2}} v^0 \left(\frac{exp_{x_\varepsilon}^{-1}(x)}{k_\varepsilon} \right) \eta_\varepsilon(x)$$

and

$$w_\varepsilon = v_\varepsilon - V_\epsilon.$$

The function V_ϵ is usually called a bubble.

4.5. **Weak limit of V_ϵ.** We briefly prove the weak convergence of V_ϵ. Let $\varphi \in C^\infty(M)$. For all $R > 0$,

$$\begin{aligned}
\left| \int_{M - B_g(x_\varepsilon, Rk_\varepsilon)} V_\epsilon \varphi \, dv_g \right| &\leq C \|\varphi\|_\infty \left(\frac{k_\varepsilon}{\mu_\varepsilon} \right)^{\frac{n-4}{2}} \left(\int_{B_\xi(0, \frac{r}{k_\varepsilon}) - B_\xi(0,R)} |v^0|^{2^\sharp} \, dv_\xi \right)^{\frac{1}{2^\sharp}} \\
\left| \int_{B_g(x_\varepsilon, Rk_\varepsilon)} V_\epsilon \varphi \, dv_g \right| &\leq C \|\varphi\|_\infty \left(\frac{k_\varepsilon}{\mu_\varepsilon} \right)^{\frac{n-4}{2}} k_\varepsilon^{\frac{n+4}{2}} \int_{B_\xi(0,R)} |v^0| \, dv_\xi
\end{aligned}$$

With similar estimates for $\int_M (\nabla V_\epsilon, \nabla \varphi)_g \, dv_g$ and $\int_M \Delta_g V_\epsilon \Delta_g \varphi \, dv_g$, we prove that V_ϵ goes to zero weakly, and then

$$w_\varepsilon \rightharpoonup 0 \text{ weakly in } H_2^2(M).$$

4.6. **Strong convergence of $dJ_\varepsilon(w_\varepsilon)$.** We now estimate $\langle dJ_\varepsilon(w_\varepsilon), \varphi \rangle$.

$$\begin{aligned}
\langle dJ_\varepsilon(w_\varepsilon), \varphi \rangle &= \int_M \Delta_g w_\varepsilon \Delta_g \varphi \, dv_g - \int_M f |w_\varepsilon|^{2^\sharp - 2 - \varepsilon} w_\varepsilon \varphi \, dv_g \\
&= \int_M \Delta_g v_\varepsilon \Delta_g \varphi \, dv_g - \int_M \Delta_g V_\epsilon \Delta_g \varphi \, dv_g - \int_M f |v_\varepsilon - V_\epsilon|^{2^\sharp - 2 - \varepsilon} (v_\varepsilon - V_\epsilon) \varphi \, dv_g
\end{aligned}$$

$$\begin{aligned}
\int_M \Delta_g V_\epsilon \Delta_g \varphi \, dv_g &= \int_{B_g(x_\varepsilon, r)} \Delta_g V_\epsilon \Delta_g \varphi \, dv_g \\
&= \int_{B_\xi(0,\alpha)} \Delta_{exp^\star_{x_\varepsilon} g} V_\epsilon \circ exp_{x_\varepsilon} \Delta_{exp^\star_{x_\varepsilon} g} \varphi \circ exp_{x_\varepsilon} \, dv_{exp^\star_{x_\varepsilon} g} \\
& \quad \int_{B_g(x_\varepsilon, r) - B_g(x_\varepsilon, \alpha)} \Delta_g V_\epsilon \Delta_g \varphi \, dv_g
\end{aligned}$$

for all $0 < \alpha < r$. We have that

$$\begin{aligned}
\left| \int_{B_g(x_\varepsilon, r) - B_g(x_\varepsilon, \alpha)} \Delta_g V_\epsilon \Delta_g \varphi \, dv_g \right| &\leq C \|\varphi\|_{H_2^2(M)} \left(\int_{B_g(x_\varepsilon, r) - B_g(x_\varepsilon, \alpha)} (\Delta_g V_\epsilon \Delta_g \varphi) \, dv_g \right)^{\frac{1}{2}} \\
&\leq C \|\varphi\|_{H_2^2(M)} \left(\frac{k_\varepsilon}{\mu_\varepsilon} \right)^{n-4} \left(\int_{\mathbb{R}^n - B_\xi(0, \frac{\alpha}{k_\varepsilon})} |\nabla_\xi^2 v^0|^2 \, dv_\xi \right)^{\frac{1}{2}} \\
&\leq o(\|\varphi\|_{H_2^2(M)}).
\end{aligned}$$

The fact that the exponential map is a normal chart at 0 yields

$$\int_M \Delta_g V_\epsilon \Delta_g \varphi \, dv_g = \int_{B_\xi(0,\alpha)} \Delta_\xi V_\epsilon \circ exp_{x_\varepsilon} \Delta_\xi \varphi \circ exp_{x_\varepsilon} \, dv_\xi + O(\alpha \|\varphi\|_{H_2^2(M)}) + o(\|\varphi\|_{H_2^2(M)}).$$

Now, let $\nu_\alpha \in C^\infty(\mathbb{R}^n)$ such that $\nu_\alpha \equiv 1$ in $B_\xi(0, \alpha/2)$ and $\nu_\alpha \equiv 0$ in $\mathbb{R}^n - B_\xi(0, 3\alpha/4)$. We define $\overline{\varphi}_\varepsilon \in C^\infty(\mathbb{R}^n)$ such that

$$\overline{\varphi}_\varepsilon(x) = \mu_\varepsilon^{\frac{n-4}{2}} \nu_\alpha(k_\varepsilon x) \varphi \circ exp_{x_\varepsilon}(k_\varepsilon x)$$

if $d_g(x, x_\varepsilon) \leq \frac{i_g(M)}{k_\varepsilon}$ and 0 elsewhere. We obtain that

$$\begin{aligned}\int_M \Delta_g V_\epsilon \Delta_g \varphi \, dv_g &= \int_{B_\xi(0,\alpha)} \Delta_\xi V_\epsilon \circ exp_{x_\varepsilon} \Delta_\xi \nu_\alpha \varphi \circ exp_{x_\varepsilon} \, dv_\xi + O(\alpha \|\varphi\|_{H_2^2(M)}) + o(\|\varphi\|_{H_2^2(M)}). \\ &= \left(\frac{k_\varepsilon}{\mu_\varepsilon}\right)^{n-4} \int_{\mathbb{R}^n} \Delta_\xi v^0 \Delta_\xi \overline{\varphi}_\varepsilon \, dv_\xi + O(\alpha \|\varphi\|_{H_2^2(M)}) + o(\|\varphi\|_{H_2^2(M)}).\end{aligned}$$

Classical integration arguments assert that

$$\begin{aligned}\int_M f|v_\varepsilon - V_\epsilon|^{2^\sharp-2-\varepsilon}(v_\varepsilon - V_\epsilon)\varphi \, dv_g &= \left(\frac{k_\varepsilon}{\mu_\varepsilon}\right)^{n-4} \left(\frac{k_\varepsilon}{\mu_\varepsilon}\right)^4 (\mu_\varepsilon^\varepsilon)^{\frac{n-4}{2}} f(x_0) \int_{\mathbb{R}^n} |v^0|^{2^\sharp-2} v^0 \overline{\varphi}_\varepsilon \, dv_\xi \\ &\quad + O(\alpha \|\varphi\|_{H_2^2(M)}) + O(\varepsilon(R) \|\varphi\|_{H_2^2(M)}) + o(\|\varphi\|_{H_2^2(M)}),\end{aligned}$$

where $\varepsilon(R)$ goes to zero when R goes to $+\infty$. Then

$$\begin{aligned}\langle dJ_\varepsilon(w_\varepsilon), \varphi \rangle &= \int_M \Delta_g v_\varepsilon \Delta_g \varphi \, dv_g - \int_M f|v_\varepsilon|^{2^\sharp-2-\varepsilon} v_\varepsilon \varphi \, dv_g \\ &\quad - \left(\frac{k_\varepsilon}{\mu_\varepsilon}\right)^{n-4} \left(\int_{\mathbb{R}^n} \Delta_\xi v^0 \Delta_\xi \overline{\varphi}_\varepsilon \, dv_\xi - (\mu_\varepsilon^\varepsilon)^{\frac{n-4}{2}} f(x_0) \int_{\mathbb{R}^n} |v^0|^{2^\sharp-2} v^0 \overline{\varphi}_\varepsilon \, dv_\xi \right) \\ &\quad + O(\alpha \|\varphi\|_{H_2^2(M)}) + O(\varepsilon(R) \|\varphi\|_{H_2^2(M)}) + o(\|\varphi\|_{H_2^2(M)}).\end{aligned}$$

In view of $k_\varepsilon = \mu_\varepsilon^{1-\varepsilon\frac{n-4}{8}}$ and (7), we obtain that

$$\langle dJ_\varepsilon(w_\varepsilon), \varphi \rangle = O(\alpha \|\varphi\|_{H_2^2(M)}) + O(\varepsilon(R) \|\varphi\|_{H_2^2(M)}) + o(\|\varphi\|_{H_2^2(M)}).$$

Taking $\alpha > 0$ small and R large enough, the preceding formula can be written as

$$dJ_\varepsilon(w_\varepsilon) \to 0 \text{ strongly in } H_2^2(M)'.$$

4.7. **Convergence of $J_\varepsilon(w_\varepsilon)$.** Concerning the energy $J_\varepsilon(w_\varepsilon)$, we similarly get that

$$\int_M (\Delta_g w_\varepsilon)^2 \, dv_g = \int_M (\Delta_g v_\varepsilon)^2 \, dv_g - \left(\frac{k_\varepsilon}{\mu_\varepsilon}\right)^{n-4} \int_{\mathbb{R}^n} (\Delta_\xi v^0)^2 \, dv_\xi + o(1),$$

$$\int_{M - B_g(x_\varepsilon, r/4)} f|w_\varepsilon|^{2^\sharp-\varepsilon} \, dv_g = \int_{M - B_g(x_\varepsilon, r/4)} f|v_\varepsilon|^{2^\sharp-\varepsilon} \, dv_g + o(1)$$

and that

$$\int_{B_g(x_\varepsilon, r/4)} f|w_\varepsilon|^{2^\sharp-\varepsilon} \, dv_g = \left(\frac{k_\varepsilon}{\mu_\varepsilon}\right)^{n-4} \left(\int_{B_\xi(0, \frac{r}{4k_\varepsilon}) - B_\xi(0,R)} f(exp_{x_\varepsilon}(k_\varepsilon x)) |\tilde{v}_\varepsilon - v^0|^{2^\sharp-\varepsilon} \, dv_{\tilde{g}_\varepsilon} \right) + o(1).$$

Let estimate the following:

$$\begin{array}{rl} & \int_{B_\xi(0,\frac{r}{4k_\varepsilon})-B_\xi(0,R)} |v^0|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon} \\ \leq & C(M,r)\left(\int_{B_\xi(0,\frac{r}{4k_\varepsilon})-B_\xi(0,R)} dv_\xi\right)^{\frac{\varepsilon}{2^\sharp}} \times \left(\int_{B_\xi(0,\frac{r}{4k_\varepsilon})-B_\xi(0,R)} |v^0|^{2^\sharp}\, dv_\xi\right)^{1-\frac{\varepsilon}{2^\sharp}} \\ \leq & C\frac{1}{(k_\xi^\varepsilon)^{1/2^\sharp}}\left(\int_{\mathbb{R}^n-B_\xi(0,R)} |v^0|^{2^\sharp}\, dv_\xi\right)^{1-\frac{\varepsilon}{2^\sharp}} \end{array}$$

The fact that $k_\varepsilon^\varepsilon \to c \neq 0$ and $v^0 \in L^{2^\sharp}(\mathbb{R}^n)$ imply that

$$\int_{B_\xi(0,\frac{r}{4k_\varepsilon})-B_\xi(0,R)} |v^0|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon} \leq \varepsilon(R),$$

where $\lim_{R\to+\infty}\varepsilon(R) = 0$. There exists $C > 0$ such that

$$\left| |\tilde{v}_\varepsilon|^{2^\sharp-\varepsilon} - |\tilde{v}_\varepsilon - v^0|^{2^\sharp-\varepsilon} \right| \leq C\left(|\tilde{v}_\varepsilon|^{2^\sharp-1-\varepsilon}|v^0| + |v^0|^{2^\sharp-1-\varepsilon}|\tilde{v}_\varepsilon| + |v^0|^{2^\sharp-\varepsilon} \right).$$

The same kind of computations as before yield

$$\begin{array}{rcl} \int_{B_g(x_\varepsilon,r/4)} f|w_\varepsilon|^{2^\sharp-\varepsilon}\, dv_g & = & \left(\frac{k_\varepsilon}{\mu_\varepsilon}\right)^{n-4}\left(\int_{B_\xi(0,\frac{r}{4k_\varepsilon})-B_\xi(0,R)} f(exp_{x_\varepsilon}(k_\varepsilon x))|\tilde{v}_\varepsilon|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon}\right) \\ & & +o(1) + O(\varepsilon(R)) \\ & = & \int_{B_g(x_\varepsilon,r/4)} f|v_\varepsilon|^{2^\sharp-\varepsilon}\, dv_g \\ & & -\left(\frac{k_\varepsilon}{\mu_\varepsilon}\right)^{n-4}\int_{B_\xi(0,R)} f(exp_{x_\varepsilon}(k_\varepsilon x))|\tilde{v}_\varepsilon|^{2^\sharp-\varepsilon}\, dv_{\tilde{g}_\varepsilon} + o(1) + O(\varepsilon(R)) \end{array}$$

Thus, considering the limit of $k_\varepsilon/\mu_\varepsilon$, we obtain that

$$J_\varepsilon(w_\varepsilon) = J_\varepsilon(w_\varepsilon) - c^{-\frac{(n-4)^2}{8}}\mathcal{E}_{x_0}(v^0) + o(1),$$

which ends the proof of the lemma.

We apply the result of Lemma 1 to prove Theorem 1. Since $c^{-\frac{(n-4)^2}{8}}\mathcal{E}_{x_0}(v^0) \geq \beta^\#$, we inductively remove some bubbles from u_ϵ. In a finite number of times, we obtain a Palais-Smale sequence of energy strictly less than $\beta^\#$. With section 3, this last sequence goes to zero strongly, and the theorem is proved.

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