CRITICAL FUNCTIONS AND OPTIMAL SOBOLEV INEQUALITIES

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1. Introduction and statement of the results

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$. We denote by $H_1^2(M)$ the standard Sobolev space of functions in $L^2(M)$ with one derivative in $L^2(M)$. As is well known, it follows from Sobolev's embedding theorem that there exist $A, B \in \mathbb{R}$ such that

$$\left(\int_M |u|^{2^\star}\, dv_g\right)^{\frac{2}{2^\star}} \leq A \int_M |\nabla u|_g^2\, dv_g + B \int_M u^2\, dv_g \tag{1}$$

for all $u \in H_1^2(M)$, where $2^\star = \frac{2n}{n-2}$ is critical. The sharp constant A in (1) is $A_{opt}(M) = K(n,2)^{-2}$, where

$$\frac{1}{K(n,2)^2} = \frac{4}{n(n-2)\omega_n^{4/n}}$$

and ω_n is the volume of the unit n-sphere. In 1995, using intricate developments from blow-up analysis, Hebey and Vaugon [HeVa2, HeVa3] proved that the sharp constant $A_{opt}(M)$ is attained in (1). It follows that there exists $B_0(g) > 0$ such that

$$\left(\int_M |u|^{2^\star}\, dv_g\right)^{\frac{2}{2^\star}} \leq K(n,2)^2 \left(\int_M |\nabla u|_g^2\, dv_g + B_0(g) \int_M u^2\, dv_g\right) \tag{$I_{opt,g}$}$$

for all $u \in H_1^2(M)$. Following Hebey [Heb2], we choose $B_0(g)$ such that it cannot be lowered. Natural questions with respect to $(I_{opt,g})$, see Druet and Hebey [DrHe] for a presentation in book form, are to compute $B_0(g)$, and to know wether or not $(I_{opt,g})$ possesses extremal functions. Important notions with respect to these questions are the notions of weakly critical and critical functions introduced by Hebey and Vaugon [HeVa4]. These notions have independant applications and were used by Druet [Dru] to prove the Brézis-Nirenberg conjecture on low dimensions. For f a smooth function on M, we let

$$\mu_{g,f} = \inf \frac{\int_M \left(|\nabla u|_g^2 + f u^2\right)\, dv_g}{\left(\int_M |u|^{2^\star}\, dv_g\right)^{\frac{2}{2^\star}}},$$

where the infimum is taken over the nonzero functions of $H_1^2(M)$, and dv_g is the Riemannian volume element. It is by now standard that $\mu_{g,f} \leq \frac{1}{K(n,2)^2}$. By definition, following Hebey and Vaugon [HeVa4], we say that f is a weakly critical function for g if $\mu_{g,f} = \frac{1}{K(n,2)^2}$, and we say that f is a critical function if it is

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weakly critical and such that $\mu_{g,\tilde{f}} < \frac{1}{K(n,2)^2}$ for all smooth functions $\tilde{f}$ satisfying that $\tilde{f} \leq f$ and $\tilde{f} \not\equiv f$. Let

$$B_0(g)_{ext} = \frac{n-2}{4(n-1)} \max_M S_g,$$

where S_g denotes the scalar curvature of g. It is easily checked that when $n \geq 4$, we always have that $B_0(g) \geq B_0(g)_{ext}$. Moreover, as proved by Djadli and Druet [DjDr], one of the two following conditions hold:
(i) either $B_0(g) = B_0(g)_{ext}$,
(ii) or $(I_{opt,g})$ possesses extremal functions.
We know since Hebey [Heb1] that there are examples of manifolds for which (i) is true and (ii) is false. The examples in [Heb1] are in the conformal class of the unit n-sphere, and we are left with the question (question (Q1) in [HeVa4]) of describing the class of manifolds for which (i) is true and (ii) is false. Such a question is closely related to the following conjecture of Hebey and Vaugon [HeVa4]: if $n \geq 6$, the Weyl curvature tensor $Weyl_g$ of g vanishes up to the order 1 at some point x_0, and f is a smooth nonegative function vanishing at x_0, then there exists $g_f \in [g]$ a conformal metric to g such that the scalar curvature S_{g_f} is maximal at x_0 and such that $\alpha = B_0(g_f)_{ext} - f$ is a weakly critical function for g_f. We prove in this article that this conjecture is true. We also treat the case of the dimension $n = 5$. Concerning terminology, we say that the Weyl tensor $Weyl_g$ of g vanishes up to order $p \geq 1$ at x_0 if $Weyl_g(x_0) = 0$ and $\nabla^j Weyl_g(x_0) = 0$ for all $1 \leq j \leq p$. Our main result states as follows:

Theorem 1.1. *Let (M, g) be a smooth compact Riemannian n-manifold, $n \geq 5$, x_0 a point in M, and f a smooth nonnegative nonzero function on M with the property that $f(x_0) = 0$. We assume that $Weyl_g$ is null up to the order 2 at x_0 if $n = 5$, and up to the order 1 at x_0 if $n \geq 6$. Then there exists $g_f \in [g]$ a conformal metric to g such that S_{g_f} is maximal at x_0 and such that $\alpha = \frac{n-2}{4(n-1)} \max_M S_{g_f} - f$ is a weakly critical function for g_f. In particular, for any smooth compact Riemannian manifold (M, g) of dimension $n \geq 5$, whose Weyl curvature tensor satisfies the above conditions, there exists a conformal metric $\tilde{g}$ to g such that $B_0(\tilde{g}) = B_0(\tilde{g})_{ext}$ and $(I_{opt,\tilde{g}})$ does not possess extremal functions.*

Hebey and Vaugon [HeVa4] also conjectured that the following surprising complementary result to Theorem 1.1 should hold: if (M, g) is a smooth compact Riemannian manifold of dimension $n \geq 6$ such that its Weyl tensor is null up to the order 2 at some point, then there exists $\tilde{g} \in [g]$ a conformal metric to g such that $B_0(\tilde{g}) = B_0(\tilde{g})_{ext} = Vol_{\tilde{g}}^{-2/n}$ and such that $(I_{opt,\tilde{g}})$ possesses constant extremal functions. In this statement, $Vol_{\tilde{g}}$ denotes the volume of M wrt the metric $\tilde{g}$. As a remark, it easily follows from the approach we use to prove Theorem 1.1 that this conjecture is also true. With respect to the mathematics developed in Hebey and Vaugon [HeVa4], both conjectures reduce to the proof that for a blowing-up sequence (u_ϵ) of solutions of the equations attached to the above problems, if x_0 is the geometric concentration point of the sequence (u_ϵ), then $d_g(x_0, x_\epsilon) = O(\mu_\epsilon)$ where the x_ϵ's are the centers and the μ_ϵ's are the weights of the bubbles developed by the u_ϵ's.

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2. Proof of theorem 1.1

When possible, we follow the approach developed in [HeVa4]. Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. We first assume that $n \geq 6$ and that there exists $x_0 \in M$ such that $Weyl_g(x_0) = 0$ and $\nabla Weyl_g(x_0) = 0$. Up to a conformal change of metric (see for instance Lee-Parker [LePa] and [HeVa1]), we can assume that

$$Rm_g(x_0) = 0 \text{ and } \nabla Rm_g(x_0) = 0. \tag{2}$$

Here Rm_g and ∇Rm_g denote the Riemann tensor and its covariant derivative. Note that in this case, $\nabla^i S_g(x_0) = 0$ for $i = 0...2$. We denote by $[g]$ the conformal class of the metric g and

$$[g]_s = \left\{ \tilde{g} \in [g] \text{ s.t. } S_{\tilde{g}}(x_0) = \max_M S_{\tilde{g}} \right\}.$$

We let $f \in C^\infty(M) - \{0\}$ such that for any $x \in M$,

$$f(x) \geq f(x_0) = 0. \tag{3}$$

In this section, we prove that there exists $\tilde{g} \in [g]_s$ such that $\alpha_{\tilde{g}} = c_n \max_M S_{\tilde{g}} - f$ is a weakly critical function. Here and in the sequel, we let $c_n = \frac{n-2}{4(n-1)}$. We argue by contradiction, and assume that for any $\tilde{g} \in [g]_s$,

$$\inf_{u \in H_1^2(M)\setminus\{0\}} \frac{\int_M \left(|\nabla u|_{\tilde{g}}^2 + \alpha_{\tilde{g}} u^2 \right) dv_{\tilde{g}}}{\left(\int_M |u|^{2^\star} dv_{\tilde{g}} \right)^{\frac{2}{2^\star}}} < \frac{1}{K(n,2)^2}.$$

Letting $\varphi \in C^\infty(M)$ positive such that $\tilde{g} = \varphi^{\frac{4}{n-2}} g$, the preceding inequality is equivalent to

$$\inf_{u \in H_1^2(M)\setminus\{0\}} \frac{\int_M \left(|\nabla u|_g^2 + h_\varphi u^2 \right) dv_g}{\left(\int_M |u|^{2^\star} dv_g \right)^{\frac{2}{2^\star}}} < \frac{1}{K(n,2)^2}, \tag{4}$$

where

$$h_\varphi = c_n S_g + \varphi^{2^\star - 2} \left(c_n (\max_M S_{\tilde{g}} - S_{\tilde{g}}) - f \right) \tag{5}$$

for all positive functions φ. Let $\delta \in (0, i_g(M)/2)$, where $i_g(M)$ is the injectivity radius of (M,g), and $\eta \in C^\infty(M)$ such that $\eta \equiv 1$ in $B_{x_0}(\delta)$ and $\eta \equiv 0$ in $M \backslash B_{x_0}(2\delta)$. We also let $c > 0$ and consider the function

$$\varphi_\epsilon(x) = \frac{\eta(x)}{(\epsilon^2 + d_g(x,x_0)^2)^{\frac{n-2}{8}}} + (1 - \eta(x))c.$$

Clearly, φ_ϵ is smooth and positive.

Step 1: It follows from [HeVa4] that the metric $g_\epsilon = \varphi_\epsilon^{\frac{4}{n-2}} g$ belongs to $[g]_s$ for small ϵ. Moreover, noting $h_\epsilon = h_{\varphi_\epsilon}$, we get that

$$h_\epsilon(x) \geq a(\epsilon) d_g(x,x_0)^2 \tag{6}$$

for all $x \in M$ and all $\epsilon > 0$, with

$$\lim_{\epsilon \to 0} a(\epsilon) = +\infty. \tag{7}$$

It is by now classical that it follows from (4) that there exists $u_\epsilon \in C^\infty(M)$ positive such that

$$\Delta_g u_\epsilon + h_\epsilon u_\epsilon = \lambda_\epsilon u_\epsilon^{2^\star - 1} \text{ and } \int_M u_\epsilon^{2^\star} \, dv_g = 1 \tag{E_ϵ}$$

where

$$\lambda_\epsilon = \mu_{g,h_\epsilon} < \frac{1}{K(n,2)^2} \tag{8}$$

It then follows that $u_\epsilon \rightharpoonup 0$ weakly in $H_1^2(M)$ when $\epsilon \to 0$. We refer to [HeVa4] for the details. We let $x_\epsilon \in M$ and $\mu_\epsilon > 0$ such that

$$\max_M u_\epsilon = u_\epsilon(x_\epsilon) = \mu_\epsilon^{-\frac{n-2}{2}}.$$

Since $u_\epsilon \rightharpoonup 0$ weakly in $H_1^2(M)$, it follows from (E_ϵ) that $\lim_{\epsilon\to 0}\mu_\epsilon = 0$. We let $y_\epsilon \in M$ such that

$$\frac{d_g(x_\epsilon, y_\epsilon)}{\mu_\epsilon} = O(1)$$

when $\epsilon \to 0$. With (6), it follows from [DrRo], see also [DrHe], that there exists $C > 0$ such that

$$d_g(y_\epsilon, x)^{\frac{n-2}{2}} u_\epsilon(x) \le C \text{ and } u_\epsilon(x) \le C\left(\frac{\mu_\epsilon}{\mu_\epsilon^2 + d_g(x, y_\epsilon)^2}\right)^{\frac{n-2}{2}} \tag{9}$$

for all $x \in M$ and $\epsilon > 0$. Moreover, it follows from [HeVa4] that $y_\epsilon \to x_0$ when $\epsilon \to 0$.

Step 2: We denote by exp_{y_ϵ} the exponential map at y_ϵ with respect to the metric g. We let $\tilde{u}_\epsilon(x) = u_\epsilon(exp_{y_\epsilon}(x))$ for all $x \in B_0(3\delta)$ and $g_\epsilon = exp_{y_\epsilon}^\star g$ the pullback metric of g via the exponential chart at y_ϵ. In the sequel, we denote by g_ϵ^{ij} the coordinates of g^{-1}, the inverse of the metric tensor, via the chart exp_{y_ϵ}. We let $\delta \in (0, i_g(M)/3)$. We let $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ in $B_0(\delta)$ and $\eta \equiv 0$ in $\mathbb{R}^n \backslash B_0(2\delta)$. Th optimal Euclidean Sobolev inequality asserts that

$$\left(\int_{\mathbb{R}^n} |u|^{2^\star}\, dx\right)^{\frac{2}{2^\star}} \le K(n,2)^2 \int_{\mathbb{R}^n} |\nabla u|_\xi^2\, dx,$$

for all smooth compactly supported function u on $\mathbb{R}^n$. It follows from this inequality that

$$\left(\int_{\mathbb{R}^n} (\eta\tilde{u}_\epsilon)^{2^\star}\, dx\right)^{\frac{2}{2^\star}} \le K(n,2)^2 \int_{\mathbb{R}^n} |\nabla(\eta\tilde{u}_\epsilon)|_\xi^2\, dx. \tag{10}$$

We denote the volume element by $dv_{g_\epsilon} = \sqrt{|g_\epsilon|}\, dx$. We then get that

$$\begin{aligned} &\int_{\mathbb{R}^n} (\eta\tilde{u}_\epsilon)^{2^\star}\, dx \qquad (11)\\ &= 1 + \int_{B_0(\delta)} \tilde{u}_\epsilon^{2^\star}(1 - \sqrt{|g_\epsilon|})\, dx + \int_{B_0(2\delta)\backslash B_0(\delta)} (\eta\tilde{u}_\epsilon)^{2^\star}\, dx - \int_{M\backslash B_{y_\epsilon}(\delta)} u_\epsilon^{2^\star}\, dv_g. \end{aligned}$$

It follows from Cartan's expansion of the metric that

$$g_\epsilon^{ij}(x) = \delta^{ij} + O\left(|Rm_g(y_\epsilon)|_g |x|^2 + |\nabla Rm_g(y_\epsilon)|_g |x|^3 + |x|^4\right)$$

uniformly for $x \in B_0(3\delta)$ when $\epsilon \to 0$. Plugging this expansion in (11), using (E_ϵ) and (9), we get that

$$\left| \left(\int_{\mathbb{R}^n} (\eta \tilde{u}_\epsilon)^{2^\star} \, dv_\xi \right)^{\frac{2}{2^\star}} - 1 \right| \leq C |Rm_g(y_\epsilon)|_g \int_{B_0(\delta)} \tilde{u}_\epsilon^2 \, dx \tag{12}$$

$$+ C |\nabla Rm_g(y_\epsilon)|_g \int_{B_0(\delta)} |x| \tilde{u}_\epsilon^2 \, dx + C \int_{B_0(\delta)} |x|^2 \tilde{u}_\epsilon^2 \, dx + C \int_{M \setminus B_{y_\epsilon}(\delta)} \tilde{u}_\epsilon^2 \, dx.$$

We now deal with the LHS of (10). Clearly,

$$\int_{\mathbb{R}^n} |\nabla(\eta \tilde{u}_\epsilon)|_\xi^2 \, dv_\xi = \int_{\mathbb{R}^n} g_\epsilon^{ij} \partial_i(\eta \tilde{u}_\epsilon) \partial_j(\eta \tilde{u}_\epsilon) \sqrt{|g_\epsilon|} \, dx$$

$$+ \int_{\mathbb{R}^n} g_\epsilon^{ij} \partial_i(\eta \tilde{u}_\epsilon) \partial_j(\eta \tilde{u}_\epsilon) \left(1 - \sqrt{|g_\epsilon|} \right) \, dx + \int_{\mathbb{R}^n} \left(\delta^{ij} - g_\epsilon^{ij} \right) \partial_i(\eta \tilde{u}_\epsilon) \partial_j(\eta \tilde{u}_\epsilon) \, dx$$

Noting $\eta_\epsilon = \eta \circ exp_{y_\epsilon}^{-1}$, integrating the first term of the RHS by parts and using Cartan's expansion of the metric, we then get that

$$\int_{\mathbb{R}^n} |\nabla(\eta \tilde{u}_\epsilon)|_\xi^2 \, dv_\xi = \int_M \eta_\epsilon^2 u_\epsilon \Delta_g u_\epsilon \, dv_g + \int_M |\nabla \eta_\epsilon|_g^2 u_\epsilon^2 \, dv_g$$

$$+ O \left(\int_M \left(|Rm_g(y_\epsilon)|_g r^2 + |\nabla Rm_g(y_\epsilon)|_g r^3 + r^4 \right) |\nabla(\eta_\epsilon u_\epsilon)|_g^2 \, dv_g \right)$$

where $r = d_g(x, y_\epsilon)$. Integrating by parts the last term of the RHS, using equation (E_ϵ), the estimate (9) and that $h_\epsilon \geq 0$, we get that

$$\int_{\mathbb{R}^n} |\nabla(\eta \tilde{u}_\epsilon)|_\xi^2 \, dv_\xi \leq \lambda_\epsilon - \lambda_\epsilon \int_M \eta_\epsilon^2 h_\epsilon u_\epsilon^2 \, dv_g \tag{13}$$

$$+ C |Rm_g(y_\epsilon)|_g \int_{B_{y_\epsilon}(\delta)} u_\epsilon^2 \, dv_g + C |\nabla Rm_g(y_\epsilon)|_g \int_{B_{y_\epsilon}(\delta)} r u_\epsilon^2 \, dv_g$$

$$+ C \int_{B_{y_\epsilon}(\delta)} r^2 u_\epsilon^2 \, dv_g + C \int_{M \setminus B_{y_\epsilon}(\delta)} u_\epsilon^2 \, dv_g,$$

in this expression, $r = d_g(x, y_\epsilon)$. Plugging (12) and (13) into (10) and using (8), we get that

$$\int_{B_{y_\epsilon}(\delta)} h_\epsilon u_\epsilon^2 \, dv_g \leq C |Rm_g(y_\epsilon)|_g \int_{B_{y_\epsilon}(\delta)} u_\epsilon^2 \, dv_g + C |\nabla Rm_g(y_\epsilon)|_g \int_{B_{y_\epsilon}(\delta)} r u_\epsilon^2 \, dv_g$$

$$+ C \int_{B_{y_\epsilon}(\delta)} r^2 u_\epsilon^2 \, dv_g + C \int_{M \setminus B_{y_\epsilon}(\delta)} u_\epsilon^2 \, dv_g$$

in this expression, $r = d_g(x, y_\epsilon)$. It follows from [HeVa4] that

$$\int_{M \setminus B_{y_\epsilon}(\delta)} u_\epsilon^2 \, dv_g = o \left(\int_{B_{y_\epsilon}(\delta)} h_\epsilon u_\epsilon^2 \, dv_g \right).$$

It then follows from these last two estimates that

$$\int_{B_{y_\epsilon}(\delta)} h_\epsilon u_\epsilon^2 \, dv_g \leq C |Rm_g(y_\epsilon)|_g \int_{B_{y_\epsilon}(\delta)} u_\epsilon^2 \, dv_g$$

$$+ C |\nabla Rm_g(y_\epsilon)|_g \int_{B_{y_\epsilon}(\delta)} d_g(x, y_\epsilon) u_\epsilon^2 \, dv_g + C \int_{B_{y_\epsilon}(\delta)} d_g(x, y_\epsilon)^2 u_\epsilon^2 \, dv_g.$$

Since $d_g(x,y_\epsilon)^2 \leq 2d_g(x,x_0)^2 + 2d_g(x_0,y_\epsilon)^2$, we get with (6) and (7) that

$$\begin{aligned}\int_{B_{y_\epsilon}(\delta)} h_\epsilon u_\epsilon^2\, dv_g &\leq C\cdot\left(d_g(x_0,y_\epsilon)^2 + |Rm_g(y_\epsilon)|_g\right)\cdot \int_{B_{y_\epsilon}(\delta)} u_\epsilon^2\, dv_g\\ &+C|\nabla Rm_g(y_\epsilon)|_g \int_{B_{y_\epsilon}(\delta)} d_g(x,y_\epsilon) u_\epsilon^2\, dv_g. \qquad (14)\end{aligned}$$

With (2), we then get that

$$\begin{aligned}\int_{B_{y_\epsilon}(\delta)} h_\epsilon u_\epsilon^2\, dv_g &\leq Cd_g(x_0,y_\epsilon)^2 \int_{B_{y_\epsilon}(\delta)} u_\epsilon^2\, dv_g\\ &+Cd_g(x_0,y_\epsilon) \int_{B_{y_\epsilon}(\delta)} d_g(x,y_\epsilon) u_\epsilon^2\, dv_g \qquad (15)\end{aligned}$$

as soon as $\frac{d_g(x_\epsilon,y_\epsilon)}{\mu_\epsilon} = O(1)$ when $\epsilon\to 0$.

Step 3: We claim that

$$\frac{d_g(x_\epsilon,x_0)}{\mu_\epsilon} = O(1) \qquad (16)$$

when $\epsilon\to 0$. We prove the claim by contradiction and assume that

$$\lim_{\epsilon\to 0}\frac{d_g(x_\epsilon,x_0)}{\mu_\epsilon} = +\infty. \qquad (17)$$

Inequality (15) is obviously verified with $y_\epsilon = x_\epsilon$. Since $B_{x_\epsilon}(\mu_\epsilon)\subset B_{x_\epsilon}(\delta)$ when $\epsilon\to 0$, we get with a change of variable that

$$\begin{aligned}&\mu_\epsilon^2\int_{B_0(1)} h_\epsilon(exp_{x_\epsilon}(\mu_\epsilon x))v_\epsilon^2\, dv_{\tilde{g}_\epsilon} \leq Cd_g(x_0,x_\epsilon)^2\mu_\epsilon^2\int_{B_0(\delta\mu_\epsilon^{-1})} v_\epsilon^2\, dv_{\tilde{g}_\epsilon}\\ &+Cd_g(x_0,x_\epsilon)\mu_\epsilon^3\int_{B_0(\delta\mu_\epsilon^{-1})}|x|v_\epsilon^2\, dv_{\tilde{g}_\epsilon},\end{aligned}$$

where $v_\epsilon(x) = \mu_\epsilon^{\frac{n-2}{2}} u_\epsilon(exp_{y_\epsilon}(\mu_\epsilon x))$ for all $x\in B_0(\delta/\mu_\epsilon)$ and $\tilde{g}_\epsilon = exp_{x_\epsilon}^\star g(\mu_\epsilon x) = g_\epsilon(\mu_\epsilon x)$. For any $x\in B_0(1)$, we get with (17) that

$$d_g(x_0, exp_{x_\epsilon}(\mu_\epsilon x)) \geq d_g(x_0,x_\epsilon) - \mu_\epsilon \geq \frac{1}{2}d_g(x_0,x_\epsilon).$$

With the lower bound (6) of h_ϵ, we then get that

$$\begin{aligned}&a(\epsilon)d_g(x_0,x_\epsilon)^2\mu_\epsilon^2\int_{B_0(1)} v_\epsilon^2\, dv_{\tilde{g}_\epsilon} \leq Cd_g(x_0,x_\epsilon)^2\mu_\epsilon^2\int_{B_0(\delta\mu_\epsilon^{-1})} v_\epsilon^2\, dv_{\tilde{g}_\epsilon}\\ &+Cd_g(x_0,x_\epsilon)\mu_\epsilon^3\int_{B_0(\delta\mu_\epsilon^{-1})}|x|v_\epsilon^2\, dv_{\tilde{g}_\epsilon}.\end{aligned}$$

It follows from Moser's iterative scheme (see for instance [DrRo]) that

$$\int_{B_0(1)} v_\epsilon^2\, dv_{\tilde{g}_\epsilon} \geq Cv_\epsilon(0) = C > 0.$$

Using the estimates (9) and assuming that $n\geq 6$, we get that

$$a(\epsilon)d_g(x_0,x_\epsilon)^2\mu_\epsilon^2 \leq Cd_g(x_0,y_\epsilon)^2\mu_\epsilon^2 + Cd_g(x_0,y_\epsilon)\mu_\epsilon^3.$$

A contradiction with (7) and (17). Then (16) holds. This proves the claim.

It follows from (16) that (15) holds with $y_\epsilon = x_0$. A contradiction, since $h_\epsilon\geq 0$. We have then contradicted our initial assumption (4). Then there exists $\tilde{g}$

conformal to g such that $c_n \max_M S_{\tilde{g}} - f$ is a weakly critical function. Now we claim that $B_0(\tilde{g})$ has no extremal function. We prove the claim by contradiction and assume that $B_0(\tilde{g})$ has an extremal function. By definition, $B_0(\tilde{g})$ is weakly critical: since it has an extremal function, it is a critical function. Since $B_0(\tilde{g}) \geq c_n \max_M S_{\tilde{g}} \geq c_n \max_M S_{\tilde{g}} - f$ (see for instance [HeVa4]) and $c_n \max_M S_{\tilde{g}} - f$ is weakly critical, we get that $B_0(\tilde{g}) = c_n \max_M S_{\tilde{g}} = c_n \max_M S_{\tilde{g}} - f$ and then $f \equiv 0$. A contradiction with the choice of f in (3). Then $B_0(\tilde{g})$ has no extremal function. It then follows from [DjDr] that $B_0(\tilde{g}) = B_0(\tilde{g})_{ext}$. This proves Theorem 1.1 when $n \geq 6$. Concerning the case of dimension $n = 5$, the proof is similar and uses inequality (14). Note also that when $\nabla^i Weyl_g(x_0) = 0$ for $i = 0...2$, it follows from [HeVa1] that we can assume that $\nabla^i Rm_g(x_0) = 0$ for $i = 0...2$ up to a conformal change of metric. This ends the proof of Theorem 1.1.

References

[DjDr] Djadli, Z.; Druet, O. Extremal functions for optimal Sobolev inequalities on compact manifolds. *Calc. Var. Partial Differential Equations* **12** (2001), 59-84.

[Dru] Druet, O. Elliptic equations with critical Sobolev exponents in dimension 3. *Ann. Inst. H. Poincaré. Anal. Non Linéaire*, **19**, (2002), 125-142.

[DrHe] Druet, O.; and Hebey, E. *The AB program in geometric analysis. Sharp Sobolev inequalities and related problems*, Memoirs of the American Mathematical Society, MEMO/160/761, 2002.

[DrRo] Druet, O; Robert, F. Asymptotic profile and blow-up estimates on compact Riemannian manifolds. *Preprint* (2000). Reproduced in [DrHe].

[Heb1] Hebey, E. Fonctions extrémales pour une inégalité de Sobolev optimale dans la classe conforme de la sphère. *J. Math. Pures Appl.* **77** (1998), 721-733.

[Heb2] Hebey, E. *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, CIMS Lecture Notes, Courant Institute of Mathematical Sciences, Vol. 5, 1999. Second edition published by the American Mathematical Society, 2000.

[HeVa1] Hebey, E.; Vaugon, M. Le problème de Yamabe équivariant. *Bull. Sci. Math.* **117** (1993), 241-286.

[HeVa2] ——— The best constant problem in the Sobolev embedding theorem for complete Riemannian manifolds. *Duke Math. J.* **79** (1995), 235-279.

[HeVa3] ——— Meilleures constantes dans le théorème d'inclusion de Sobolev, *Ann. Inst. H. Poincaré. Anal. Non Linéaire* **13** (1996), 57-93.

[HeVa4] ——— From best constants to critical functions. *Math. Z.* **237** (2001), 737-767.

[LePa] Lee, J. M.; Parker, T.H. The Yamabe problem. *Bull. Amer. Math. Soc.* **17** (1987), 37-91.

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