CONCENTRATION PHENOMENA FOR A FOURTH ORDER EQUATION WITH EXPONENTIAL GROWTH: THE RADIAL CASE

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Abstract. We let Ω be a smooth bounded domain of $\mathbb{R}^4$ and a sequence of functions $(V_k)_{k\in\mathbb{N}} \in C^0(\Omega)$ such that $\lim_{k\to+\infty} V_k = 1$ in $C^0_{loc}(\Omega)$. We consider a sequence of functions $(u_k)_{k\in\mathbb{N}} \in C^4(\Omega)$ such that

$$\Delta^2 u_k = V_k e^{4u_k}$$

in Ω for all $k \in \mathbb{N}$. We address in this paper the question of the asymptotic behavior of the $(u_k)'s$ when $k \to +\infty$. The corresponding problem in dimension 2 was considered by Brézis-Merle and Li-Shafrir (among others), where a blow-up phenomenon was described and where a quantization of this blow-up was proved. Surprisingly, as shown by Adimurthi, Struwe and the author in [1], a similar quantization phenomenon does not hold for this fourth order problem. Assuming that the u_k's are radially symmetrical, we push further the analysis of [1]. We prove that there are exactly three types of blow-up and we describe each type in a very detailed way.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^4$. Let $(V_k)_{k\in\mathbb{N}} \in C^0(\Omega)$ be a sequence such that

$$\lim_{k\to+\infty} V_k = 1 \tag{1}$$

in $C^0_{loc}(\Omega)$. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of functions in $C^4(\Omega)$ such that

$$\Delta^2 u_k = V_k e^{4u_k} \tag{E}$$

in Ω for all $k \in \mathbb{N}$. Here and in the sequel, $\Delta = -\sum \partial_{ii}$ is the Laplacian with minus sign convention. In this paper, we address the question of the asymptotics of the u_k's when $k \to +\infty$. A natural (and simple) behavior is when there exists $u \in C^3(\Omega)$ such that, up to a subsequence,

$$\lim_{k\to+\infty} u_k = u \tag{2}$$

in $C^3_{loc}(\Omega)$. In this situation, we say that $(u_k)_{k\in\mathbb{N}}$ is relatively compact in $C^3_{loc}(\Omega)$. However, the structure of equation (E) is much richer due to its scaling invariance properties. The scaling invariance is as follows. Given $k \in \mathbb{N}$, $x_k \in \Omega$ and $\mu_k > 0$, we let

$$\tilde{u}_k(x) := u_k(x_k + \mu_k x) + \ln \mu_k \tag{3}$$

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for all $x \in \mu_k^{-1}(\Omega - x_k)$. Letting $\tilde{V}_k(x) = V_k(x_k + \mu_k x)$ for all $x \in \mu_k^{-1}(\Omega - x_k)$, we get that the rescaled function $\tilde{u}_k$ satisfies

$$\Delta^2 \tilde{u}_k = \tilde{V}_k e^{4\tilde{u}_k}$$

on $\mu_k^{-1}(\Omega - x_k)$ – an equation like (E). This scaling invariance forces some situations more subtle than (2) to happen. A very basic example is the following: we consider a sequence $(\mu_k)_{k\in\mathbb{N}} \in \mathbb{R}_{>0}$ such that $\lim_{k\to+\infty} \mu_k = 0$. Let a function $v \in C^4(\mathbb{R}^4)$ such that $e^{4v} \in L^1(\mathbb{R}^4)$ and

$$\Delta^2 v = e^{4v}. \tag{4}$$

The simplest example is the function $x \mapsto \ln \frac{\sqrt{96}}{\sqrt{96+|x|^2}}$. For any $k \in \mathbb{N}$, we define the function

$$f_k(x) = v\left(\mu_k^{-1} x\right) - \ln \mu_k$$

for all $x \in \mathbb{R}^4$. Then f_k satisfies (E) with $V_k \equiv 1$ for all $k \in \mathbb{N}$, but the sequence $(f_k)_{k\in\mathbb{N}}$ does not converge in $C^0_{loc}(\mathbb{R}^4)$: indeed, we have that

$$\lim_{k\to+\infty} f_k(0) = +\infty \text{ and } V_k e^{4f_k}\, dx \rightharpoonup \left(\int_{\mathbb{R}^4} e^{4v}\, dx\right) \delta_0$$

when $k \to +\infty$ weakly for the convergence of measures. Here and in the sequel, δ_0 denotes the Dirac mass at 0, and we say that the energy of the sequence (f_k) is $\int_{\mathbb{R}^4} e^{4v}\, dx$. Scaling as in (3), we get that

$$\lim_{k\to+\infty} f_k(\mu_k x) + \ln \mu_k = v(x)$$

for all $x \in \mathbb{R}^4$. In other words, (f_k) converges to v up to rescaling. Concerning terminology, we say that the sequence $(u_k)_{k\in\mathbb{N}}$ blows-up if it is not relatively compact in $C^3_{loc}(\Omega)$, so that, up to any subsequence, (2) does not hold. In the above example, the (f_k)'s blow up. In this paper, we are concerned with the blow-up behavior of solutions of (E).

In dimension two, the corresponding problem involves the Laplacian (and not the bi-Laplacian). This problem has been studied (among others) by Brézis-Merle [3] and Li-Shafrir [10]. We also refer to Druet [5] and Adimurthi-Struwe [2] for the description of equations with more intricate nonlinearities and to Tarantello [17] for equations with singularities. An important phenomenon that holds in dimension two is the quantization of the energy. Following standard terminology, we say that there is quantization if there exists a positive constant $C_m > 0$ such that the energy of any blowing-up sequence of solutions to the equation under consideration is (roughly speaking) asymptotically a multiple of C_m. In particular, when blow-up occurs, the sequence of solutions carries at least the energy C_m or carries no energy.

Surprisingly, such a quantization result is false when we come back to our initial four-dimensional problem (E). Let $\lambda \in (0, +\infty)$ arbitrary: in a joint work with Adimurthi and Michael Struwe [1], we exhibit a sequence of solutions to (E) that blows-up, carries the energy λ and develop singularities on a 3−dimensional hypersurface of $\mathbb{R}^4$. Still in [1], we described the behavior of arbitrary solutions to (E) and proved that any blowing-up sequence $(u_k)_{k\in\mathbb{N}}$ concentrates at the zero set of a nonpositive nontrivial bi-harmonic function, and that outside this set, $\lim_{k\to+\infty} u_k = -\infty$ uniformly. In view of the results of [1], giving a more precise description requires additional hypothesis on $(u_k)_{k\in\mathbb{N}}$.

A natural hypothesis is to impose a Navier boundary condition, (that is $u_k = \Delta u_k = 0$ on $\partial\Omega$) or a Dirichlet boundary condition (that is $u_k = \frac{\partial u_k}{\partial \nu} = 0$ on $\partial\Omega$): actually, in these cases, we get that there is no blow-up and we recover relative compactness, these claims are easy consequences of the result in [18]. Wei [18] also studied the case where $\Delta u_k = 0$ on $\partial\Omega$ and $u_k = c_k$ on $\partial\Omega$, where $(c_k)_{k\in\mathbb{N}} \in \mathbb{R}$ is a sequence of real numbers such that $\lim_{k\to+\infty} c_k = -\infty$: in this context, Wei described precisely the asymptotics and recovered a quantization result as in Li-Shafrir. In [14], we consider the case where the L^1-norm of Δu_k is uniformly bounded on a given subset of Ω: in this context, we also recover a quantization result (that is the energy of a blowing-up solution is a multiple of an explicit constant).

In the present paper, we consider the case when $\Omega = B$ is a ball and when the u_k's are radially symmetrical with respect to the center of the ball for all $k \in \mathbb{N}$. Without loss of generality, we assume that $B = B_1(0)$ is the unit ball of $\mathbb{R}^4$ centered at 0. In this rather natural situation, and contrary to the situation considered in [14], there is no quantization. This phenomenon is due to the abundance of solutions to equation (4) (see C.S.Lin [11]), contrary to the two-dimensional corresponding equation, where up to affine transformations, there is only one solution.

Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of blowing-up solutions to (E), with a sequence $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ such that (1) holds. Assuming that the u_k's are radially symmetrical, the first step in studying the blow-up behavior of the (u_k)'s is to prove that $V_k e^{4u_k}\,dx$ converges to the product of a real number (refered to as the energy) by a Dirac mass at 0 for the convergence of measures when $k \to +\infty$: it is much more tricky to have informations about the energy in front of the Dirac mass, and this is the object of Theorem 1.1. The intricate issue in this theorem concerns the localization of the energy at the microscopic level. More precisely, after rescaling as in (3), we prove (in general) that the (u_k)'s converge when $k \to +\infty$ to a solution $v \in C^4(\mathbb{R}^4)$ of (4) such that $e^{4v} \in L^1(\mathbb{R}^4)$: since the L^1-norm is invariant under the rescaling (3), we get that there exists a sequence $(r_k)_{k\in\mathbb{N}}$ of positive real numbers such that $\lim_{k\to+\infty} r_k = 0$ and such that the L^1-norm of e^{4u_k} in $B_{r_k}(0)$ converges to the L^1-norm of e^{4v} in $\mathbb{R}^4$. The difficult step is to prove that there is no energy left outside this ball of radius r_k when $k \to +\infty$, and so, in other words, that the L^1-norm of e^{4u_k} outside $B_{r_k}(0)$ goes to 0 when $k \to +\infty$. Refering to standard terminology, this corresponds to provinge that there is no energy lost in the necks. Our main result is the following.

Theorem 1.1. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ be a sequence of functions such that (1) holds. Let $(u_k)_{k\in\mathbb{N}}$ be a family of functions in $C^4(B)$ which are solutions to (E). We assume that there exists $\Lambda \in \mathbb{R}$ such that*

$$\int_B V_k e^{4u_k}\,dx \leq \Lambda$$

for all $k \in \mathbb{N}$ and that the (u_k)'s blow-up, that is (2) does not hold for any subsequence. In addition, we assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. Then, up to a subsequence, there exists $\alpha \in [0, 16\pi^2]$ such that

$$V_k e^{4u_k}\,dx \rightharpoonup \alpha\delta_0$$

when $k \to +\infty$ for the convergence of measures. More precisely,

(i) either there exists $C > 0$ such that, up to a subsequence, $u_k(0) \leq C$ for all $k \in \mathbb{N}$: then $\alpha = 0$ and $\lim_{k\to+\infty} u_k = -\infty$ uniformly locally on $B \setminus \{0\}$

(ii) or $\lim_{k\to+\infty} u_k(0)=+\infty$. *In this situation, for any* $\delta\in(0,1)$, *we have that*

$$\lim_{R\to+\infty}\lim_{k\to+\infty}\int_{B_\delta(0)\setminus B_{Re^{-u_k(0)}}(0)} V_k e^{4u_k}\,dx=0.$$

In addition, still in case (ii), the asymptotic behavior at the scale $e^{-u_k(0)}$ *is ruled as follows:*
(ii.a) if $\alpha=16\pi^2$, *then*

$$\lim_{k\to+\infty}\Big(u_k(e^{-u_k(0)}x)-u_k(0)\Big)=\ln\frac{\sqrt{96}}{\sqrt{96}+|x|^2}$$

for all $x\in\mathbb{R}^4$. *Moreover, this convergence holds in* $C^3_{loc}(\mathbb{R}^4)$.
(ii.b) if $\alpha\in(0,16\pi^2)$, *then there exists* $v\in C^4(\mathbb{R}^4)$ *such that* $e^{4v}\in L^1(\mathbb{R}^4)$ *and*

$$\lim_{k\to+\infty}\Big(u_k(e^{-u_k(0)}x)-u_k(0)\Big)=v(x)$$

when $k\to+\infty$ *for all* $x\in\mathbb{R}^4$. *Moreover, this convergence holds in* $C^3_{loc}(\mathbb{R}^4)$ *and there exists* $\lambda>0$ *such that* $\lim_{|x|\to+\infty}\frac{v(x)}{|x|^2}=-\lambda$.
(ii.c) If $\alpha=0$, *then* $\lim_{k\to+\infty} e^{-2u_k(0)}\Delta u_k(0)=+\infty$ *and we have that*

$$\lim_{k\to+\infty}\frac{u_k(e^{-u_k(0)}x)-u_k(0)}{e^{-2u_k(0)}\Delta u_k(0)}\to-\frac{|x|^2}{8}$$

when $k\to+\infty$ *for all* $x\in\mathbb{R}^4$. *Moreover, this convergence holds in* $C^3_{loc}(\mathbb{R}^4)$.

Note that this theorem is optimal: for any $\alpha\in[0,16\pi^2]$, we exhibit in section 2 examples of blowing-up solutions to (E) such that their energy converges to α. Note also that this theorem is specific to the radial case and does not hold in general for nonradial solutions (see for instance Adimurthi-Robert-Struwe [1]).

In a joint work with Olivier Druet [6], we studied the corresponding problem on four-dimensional Riemannian manifolds, where the bi-Laplacian is replaced by a fourth-order elliptic operator refered to as P: when the kernel of P is such that Ker $P=\{constants\}$, we get that blow-up occurs at finitely many isolated points, and that each point carries exactly the energy $16\pi^2$. Note that in the context of Theorem 1.1, the kernel of the bi-Laplacian contains more than the constant functions. Related references in the context of Riemannian manifolds are Malchiodi [12] and Malchiodi-Struwe [13]. As a remark, the corresponding question in dimension $n\geq 5$ was considered in Hebey-Robert [8], we refer also to Hebey-Robert-Wen [9].

This paper is organized as follows. In section 2, we exhibit examples of blowing-up solutions to (E) having any energy ranging in $[0,16\pi^2]$. In sections 3 to 6, we prove Theorem 1.1. More precisely, in section 3, we introduce the three types of convergence that correspond to the cases $\alpha=16\pi^2$, $0<\alpha<16\pi^2$ and $\alpha=0$ of Theorem 1.1. The case $\alpha=0$ of Theorem 1.1 is proved in section 4. The case $0<\alpha<16\pi^2$ of Theorem 1.1 is proved in section 5. The case $\alpha=16\pi^2$ of Theorem 1.1 is proved in section 6. In the sequel, C denotes a positive constant, with value allowed to change from one line to the other. Note also that all the convergence results are up to a subsequence, even when it is not precised.

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2. Examples of Log- and Quadratic-Convergences

We exhibit situations in which the three patterns *(ii.a)*, *.b)* and *.c)* occur.

2.1. **Log-Convergence.** We let $k \in \mathbb{N}^\star$ and define the function

$$u_k(x) := \ln \frac{k\sqrt{96}}{\sqrt{96} + k^2|x|^2}$$

for all $x \in \mathbb{R}^4$. We have that

$$u_k(e^{-u_k(0)}x) - u_k(0) = \ln \frac{\sqrt{96}}{\sqrt{96} + |x|^2}$$

for all $x \in \mathbb{R}^4$. As easily checked,

$$\Delta^2 u_k = e^{4u_k}$$

and

$$V_k e^{4u_k}\, dx \rightharpoonup 16\pi^2 \delta_0$$

in $B_1(0)$ in the sense of measures when $k \to +\infty$, and we are in the situation described in *(ii.a)*.

2.2. **Quadratic-Convergence (I).** We let $\alpha \in (0, 16\pi^2)$. It follows from [4] that there exists $v \in C^4(\mathbb{R}^4)$ such that $v \le v(0) = 0$ and $\Delta^2 v = e^{4v}$ in $\mathbb{R}^4$ and $\int_{\mathbb{R}^4} e^{4v}\, dx = \alpha$. For any $k \in \mathbb{N}^\star$, we define the function

$$u_k(x) := v\left(kx\right) + \ln k$$

for all $x \in \mathbb{R}^4$. We have that

$$\Delta^2 u_k = e^{4u_k}$$

in B for all $k \in \mathbb{N}^\star$ and we have that

$$u_k(e^{-u_k(0)}x) - u_k(0) = v(x)$$

for all $x \in \mathbb{R}^4$ and all $k \in \mathbb{N}^\star$. In addition, we have that

$$V_k e^{4u_k}\, dx \rightharpoonup \alpha \delta_0$$

in $B_1(0)$ in the sense of measures when $k \to +\infty$ and, using [11], we are in the situation described in *(ii.b)*.

2.3. **Quadratic-Convergence (II).** We let the unique radially symmetrical function $\varphi \in C^4(\mathbb{R}^4)$ such that $\Delta^2 \varphi = e^{-\frac{|x|^2}{2}}$ in $\mathbb{R}^4$, $\varphi(0) = \Delta\varphi(0) = 0$. We let

$$u_k(x) := \ln k - \frac{k^6|x|^2}{8} + k^{-8}\varphi\left(k^3 x\right)$$

for all $k \in \mathbb{N}^\star$ and $x \in \mathbb{R}^4$. We define

$$V_k = e^{-4u_k}\Delta^2 u_k$$

for all $k \in \mathbb{N}^\star$. All these functions are explicit (see [1]) and we get that

$$\lim_{k\to+\infty} V_k = 1 \text{ in } C^0_{loc}(\mathbb{R}^4).$$

Moreover, we have that

$$V_k e^{4u_k}\, dx \rightharpoonup 0$$

in $B_1(0)$ in the sense of measures when $k \to +\infty$ and we are in the situation described in *(ii.c)*. We refer to [1] for details about these assertions. A similar method permits to construct families (u_k) and (V_k) such that (1) and (E) hold and such that $u_k \leq u_k(0) = 0$ and $V_k e^{4u_k}\, dx \rightharpoonup 0$ when $k \to +\infty$, and we are in the situation described in *(i)*.

These three examples show that for any $\alpha \in [0, 16\pi^2]$, their exists a blowing-up sequence of solutions to (E) with energy α.

3. Preliminary estimates for (E)

We let B be the open unit ball of $\mathbb{R}^4$ and $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ a sequence such that

$$\lim_{k\to+\infty} V_k = 1 \text{ in } C^0_{loc}(B). \tag{5}$$

We let $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that for any $k \in \mathbb{N}$, we have that

$$\Delta^2 u_k = V_k e^{4u_k} \tag{6}$$

in B. We assume that is there exists $\Lambda \in \mathbb{R}$ such that for any $k \in \mathbb{N}$, we have that

$$\int_B e^{4u_k}\, dx \leq \Lambda. \tag{7}$$

We assume that the function u_k is radially symmetrical with respect to the center of the unit ball B, that is 0. For any radially symmetrical function h, there exists $\tilde{h}$ defined on an interval of $[0, +\infty)$ such that $h(x) = \tilde{h}(|x|)$ for all x such that this expression makes sense. With a standard abuse of notation, we write $h(r)$, $h'(r)$, etc for $\tilde{h}(r)$, $\tilde{h}'(r)$ respectively. This section is devoted to the proof of general estimates on the (u_k)'s and to the definition of the three types of convergence that will let us distinguish the three situations of blow-up in Theorem 1.1.

Step 3.1: We first deal with the behavior of u_k on subsets where it is bounded from above:

Lemma 3.1. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We let $\omega \subset\subset B$. Then, there exists $C(\omega) > 0$ such that*

$$|x| e^{u_k(x)} \leq C(\omega) \tag{8}$$

for all $x \in \omega$ and all $k \in \mathbb{N}$.

Proof of Lemma 3.1: We let $\delta_1 \in (0,1)$ such that $\omega \subset B_{\delta_1}(0)$. We let $\delta_2 \in (\delta_1, 1)$. Since (5) holds, we assume without loss of generality that

$$V_k(x) \geq \frac{1}{2} \tag{9}$$

for all $x \in B_{\delta_2}(0)$ and all $k \in \mathbb{N}$. With equation (6), we get that $\Delta(\Delta u_k) > 0$ on B, and then Δu_k (considered as a function of $r \in [0,1)$) is strictly decreasing on $[0, \delta_2]$. We distinguish three situations:

Case 3.1.1: We assume that $\Delta u_k \geq 0$ on $B_{\delta_2}(0)$. In this situation, we get that u_k is decreasing on $[0, \delta_2]$. We let $x \in B_{\delta_1}(0)$. With (7) and (9), we get that

$$\Lambda \geq \int_{B_{|x|}(0)} V_k e^{4u_k}\, dy \geq \frac{e^{4u_k(x)}}{2} \mathrm{Vol}(B_{|x|}(0)) \geq \frac{\pi^2 |x|^4 e^{4u_k(x)}}{4}.$$

In particular, (8) holds in Case 3.1.1.

Case 3.1.2: We assume that $\Delta u_k \leq 0$ on $B_{\delta_2}(0)$. In this situation, we get that u_k is increasing on $[0, \delta_2]$. We let $x \in B_{\delta_1}(0)$. With (7) and (9), we get that

$$\begin{aligned} \Lambda &\geq \int_{B_{\delta_2}(0)\setminus B_{|x|}(0)} V_k e^{4u_k}\, dy \geq \frac{e^{4u_k(x)}}{2} \mathrm{Vol}(B_{\delta_2}(0) \setminus B_{|x|}(0)) \\ &\geq \frac{\pi^2(\delta_2^4 - |x|^4) e^{4u_k(x)}}{4} \geq \frac{\pi^2(\delta_2^4 - \delta_1^4) e^{4u_k(x)}}{4}. \end{aligned}$$

In particular, (8) holds in Case 3.1.2.

Case 3.1.3: We assume that Δu_k takes some positive and some negative values in $B_{\delta_2}(0)$. Since Δu_k is decreasing, there exists $s_k \in (0, \delta_2)$ such that

$$\Delta u_k > 0 \text{ in } [0, s_k),\ \Delta u_k(s_k) = 0, \text{ and } \Delta u_k < 0 \text{ in } (s_k, \delta_2].$$

In particular, there exists $\tau_k \in [s_k, \delta_2]$ such that u_k is decreasing in $[0, \tau_k)$ and u_k is increasing in $[\tau_k, \delta_2]$ (note that the case $\tau_k = \delta_2$ is possible). We let $x \in B_{\delta_1}(0)$. If $|x| \leq \tau_k$, we proceed as in Case 3.1.1. If $|x| \geq \tau_k$, we proceed as in 3.1.2. In particular, (8) holds in Case 3.1.3.

These three cases prove Lemma 3.1. □

Step 3.2: The preceding step permits us to deal with the convergence outside 0. This is the object of the following Lemma:

Lemma 3.2. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. Then we are in one and only one of the following situations:*

(a) there exists $u \in C^4(B \setminus \{0\})$ such that, up to a subsequence,

$$\lim_{k\to+\infty} u_k = u \text{ in } C^3_{loc}(B \setminus \{0\}).$$

(b) there exists a sequence $(a_k)_{k\in\mathbb{N}} \in \mathbb{R}_{>0}$ such that $\lim_{k\to+\infty} a_k = +\infty$, there exists $\varphi \in C^4(B \setminus \{0\})$ such that $\Delta^2\varphi = 0$, $\varphi < 0$, and such that

$$\lim_{k\to+\infty} \frac{u_k}{a_k} = \varphi \text{ in } C^3_{loc}(B \setminus \{0\}).$$

In particular, $u_k \to -\infty$ uniformly on every compact subset of $B \setminus \{0\}$.

We omit the proof of the Lemma: it is a direct consequence of the results of [1] combined with Lemma 3.1. We refer to [1] for details.

Step 3.3: This short step is devoted to the case when u_k is bounded from above. More precisely we have:

Lemma 3.3. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that there exists $\delta_0 \in (0,1)$ and $C(\delta_0) > 0$ such that*

$$u_k(x) \leq C(\delta_0) \tag{10}$$

for all $x \in B_{\delta_0}(0)$. Then we are in one and only one of the following situations:

(a) there exists $u \in C^4(B)$ such that, up to a subsequence,

$$\lim_{k\to+\infty} u_k = u \text{ in } C^3_{loc}(B).$$

In particular,

$$V_k e^{4u_k}\, dx \rightharpoonup e^{4u}\, dx$$

when $k \to +\infty$ in the sense of measures.

(b) there exists a sequence $(a_k)_{k\in\mathbb{N}} \in \mathbb{R}_{>0}$ *such that* $\lim_{k\to+\infty} a_k = +\infty$*, there exists* $\varphi \in C^4(B)$ *such that* $\Delta^2\varphi = 0$*,* $\varphi < 0$ *in* $B \setminus \{0\}$*, and such that*

$$\lim_{k\to+\infty} \frac{u_k}{a_k} = \varphi \text{ in } C^3_{loc}(B).$$

In particular,

$$V_k e^{4u_k}\, dx \rightharpoonup 0$$

when $k \to +\infty$ *in the sense of measures.*

Proof of Lemma 3.3: it follows from (8) and (10) that for any $\delta \in (0,1)$, there exists $C(\delta) > 0$ such that $u_k(x) \le C(\delta)$ for all $x \in \overline{B}_\delta(0)$. We proceed as in [1] and we obtain that the function φ in Lemma 3.2 extends to the whole domain B, and is bi-harmonic in B. Since φ is radially symmetrical, $\varphi \le 0$ and $\varphi \not\equiv 0$, we get that $\varphi < 0$ in $B \setminus \{0\}$. This proves Lemma 3.3. □

Step 3.4:

Lemma 3.4. *Let* $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ *and* $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ *such that (5), (6) and (7) hold. We assume that* u_k *is radially symmetrical for all* $k \in \mathbb{N}$*. We assume that there exists* $\delta_0 \in (0,1)$ *such that*

$$\lim_{k\to+\infty} \sup_{B_{\delta_0}(0)} u_k = +\infty. \tag{11}$$

Then for all $\delta \in (0,1)$ *and for* $k > 0$ *large enough, we have that* $\sup_{B_\delta(0)} u_k = u_k(0)$*.*

Proof of Lemma 3.4: It follows from (8) and (11) that for any $\delta \in (0,1)$, we have that $\lim_{k\to+\infty} \sup_{B_\delta(0)} u_k = +\infty$. It follows from the study of the monotonicity carried out in Step 3.1 that $\sup_{B_\delta(0)} u_k \in \{u_k(0), u_k(\delta)\}$. With (8), we get that there exists $C(\delta) > 0$ such that $u_k(\delta) \le C(\delta)$ for all $k \in \mathbb{N}$. Since $\lim_{k\to+\infty} \sup_{B_\delta(0)} u_k = +\infty$, we get that the supremum is achieved at 0 for $k > 0$ large enough. This proves Lemma 3.4. □

From now on, we assume that the sequence (u_k) satisfies the hypothesis of Lemma 3.4. In particular, we assume that for any $\delta \in (0,1)$, we have that

$$\sup_{B_\delta(0)} u_k = u_k(0) \text{ and } \lim_{k\to+\infty} u_k(0) = +\infty. \tag{12}$$

Step 3.5: We now introduce the three fundamental types of convergence for (E). This is a specificity of the bi-harmonic operator, compared to the Laplacian:

Proposition-Definition 3.1. *Let* $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ *and* $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ *such that (5), (6) and (7) hold. We assume that* u_k *is radially symmetrical for all* $k \in \mathbb{N}$*. We assume that (12) holds. We let*

$$\mu_k := e^{-u_k(0)} \text{ and } v_k(x) := u_k(\mu_k x) - u_k(0) \tag{13}$$

for all $k \in \mathbb{N}$ *and all* $x \in B_{\mu_k^{-1}}(0)$*. Then one and only one of the following situations holds:*

(i: Log-convergence) For all $x \in \mathbb{R}^4$*,*

$$\lim_{k\to+\infty} v_k(x) = \ln \frac{\sqrt{96}}{\sqrt{96} + |x|^2}.$$

Moreover, this convergence holds in $C^3_{loc}(\mathbb{R}^4)$*.*

(ii: Quadratic-Convergence (I).) There exists $a>0$, $v\in C^4(\mathbb{R}^4)$ such that

$$\Delta^2 v = e^{4v} \text{ in } \mathbb{R}^4 \text{ and } \lim_{|x|\to+\infty}\frac{v(x)}{|x|^2}=-a$$

and such that

$$\lim_{k\to+\infty} v_k = v \text{ in } C^3_{loc}(\mathbb{R}^4).$$

(iii: Quadratic-Convergence (II).) We have that $\lim_{k\to+\infty}\Delta v_k(0)=+\infty$ and, for all $x\in\mathbb{R}^4$,

$$\lim_{k\to+\infty}\frac{v_k(x)}{\Delta v_k(0)}\to -\frac{|x|^2}{8}.$$

Moreover, this convergence holds in $C^3_{loc}(\mathbb{R}^4)$.

Proof of Lemma 3.1: We let $\tilde{V}_k(x):=V_k(\mu_k x)$ for all $x\in B_{\mu_k^{-1}}(0)$ and all $k\in\mathbb{N}$. In particular,

$$\lim_{k\to+\infty}\tilde{V}_k = 1 \text{ in } C^0_{loc}(\mathbb{R}^4).$$

Equation (6) rewrites as

$$\Delta^2 v_k = \tilde{V}_k e^{4v_k} \tag{14}$$

in $B_{\mu_k^{-1}}(0)$. Inequality (7) rewrites as

$$\int_{B_{\mu_k^{-1}}(0)} \tilde{V}_k e^{4v_k}\,dx \le \Lambda \tag{15}$$

for all $k\in\mathbb{N}$. Moreover, it follows from (12) and the definition (13) of v_k that

$$v_k(x)\le v_k(0)=0 \tag{16}$$

for all $x\in B_{\mu_k^{-1}}(0)$. We let $R>0$. We proceed as in [1] and let $w_k\in C^4(B_R(0))$ such that

$$\left\{\begin{array}{ll} \Delta^2 w_k = \tilde{V}_k e^{4v_k} & \text{in } B_R(0)\\ w_k=\Delta w_k = 0 & \text{on } \partial B_R(0)\end{array}\right\}.$$

It follows from (16) and standard elliptic theory that there exists $C(R)>0$ such that

$$\|w_k\|_{C^{3,1/2}(\overline{B}_R(0))}\le C(R) \tag{17}$$

for all $k\in\mathbb{N}$. We let $\varphi_k:=v_k-w_k$. It follows from (16) and (17) that there exists $C(R)>0$ such that

$$\varphi_k(x)\le C(R)$$

for all $x\in B_R(0)$ and all $k\in\mathbb{N}$. Since $\Delta^2\varphi_k=0$, proceeding as in [1], we get that either φ_k converges in $C^4_{loc}(B_R(0))$, or it converges in $C^4_{loc}(B_R(0))$ up to multiplication by a sequence of positive real numbers. Coming back to the function $v_k=w_k+\varphi_k$ and using arbitrarily large $R>0$, we get that we are in one and only one of the following cases:

Case 3.5.1: There exists $v\in C^3(\mathbb{R}^4)$ such that

$$\lim_{k\to+\infty} v_k = v \text{ in } C^3_{loc}(\mathbb{R}^4).$$

Pasing to the limit in (14), we get that $\Delta^2 v = e^{4v}$ in the distribution sense, and then $v\in C^4(\mathbb{R}^4)$ by elliptic theory. With (16), we get that $v(x)\le v(0)=0$ for all

$x \in \mathbb{R}^4$. Letting $k \to +\infty$ in (15), we get that $e^{4v} \in L^1(\mathbb{R}^4)$. It follows from [11] (Theorem 1.1 and 1.2) that

$$\text{either } v(x) = \ln \frac{\sqrt{96}}{\sqrt{96} + |x|^2} \text{ or there exists } a > 0 \text{ such that } \lim_{|x|\to+\infty} \frac{v(x)}{|x|^2} = -a.$$

We recover (i) and (ii) of Proposition-Definition 3.1. This ends Case 3.5.1.

Case 3.5.2: There exists $\varphi \in C^3(\mathbb{R}^4)$, there exists $(a_k)_{k\in\mathbb{N}} \in \mathbb{R}_{>0}$ such that $\lim_{k\to+\infty} a_k = +\infty$ and

$$\lim_{k\to+\infty} \frac{v_k}{a_k} = \varphi \text{ in } C^3_{loc}(\mathbb{R}^4).$$

Moreover, $\varphi \not\equiv 0$ and $\Delta^2\varphi = 0$ in the distribution sense, and then $\varphi \in C^4(\mathbb{R}^4)$ by elliptic theory. Passing to the limit in (16), we get that $\varphi(x) \le \varphi(0) = 0$ for all $x \in \mathbb{R}^4$. It follows that there exists $\alpha > 0$ such that $\varphi(x) = -\alpha|x|^2$ for all $x \in \mathbb{R}^4$. Estimating $\Delta v_k(0)$, we get that

$$\lim_{k\to+\infty} \Delta v_k(0) = +\infty \text{ and } \lim_{k\to+\infty} \frac{v_k(x)}{\Delta v_k(0)} = -\frac{|x|^2}{8}$$

for all $x \in \mathbb{R}^4$. Moreover, this convergence holds in $C^3_{loc}(\mathbb{R}^4)$. In this case, we recover (iii) of Proposition-Definition 3.1. This ends Case 3.5.2, and therefore the proof of Proposition-Definition 3.1.

□

Step 3.6: We state a very useful integral inequality. In the next section, this inequality will allow us to distinguish the three types of convergence above.

Lemma 3.5. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that (12) holds. Then, for any $0 < \delta < 1$, there exists $C(\delta) > 0$ such that*

$$\int_{B_R(0)} |\Delta v_k - \mu_k^2 \Delta u_k(\delta)|\, dx \le C(\delta) R^2 \tag{18}$$

for all $k \in \mathbb{N}$ and all $R < \delta\mu_k^{-1}$. In this expression, μ_k and v_k are as in (13).

Proof of Lemma 3.5: We follow the argument of Robert-Struwe [15]. We let G_δ be the Green's function for the Laplacian on $B_\delta(0)$ with Dirichlet boundary condition. Since Δu_k is radially symmetrical, we get that

$$\Delta u_k(z) = \int_{B_\delta(0)} G_\delta(z,y)\Delta^2 u_k(y)\, dy + \Delta u_k(\delta)$$

for all $z \in B_\delta(0)$. We choose $x \in \mathbb{R}^4$ such that $|x| < \delta\mu_k^{-1}$. Using (13), we get that

$$\Delta v_k(x) - \mu_k^2 \Delta u_k(\delta) = \int_{B_\delta(0)} \mu_k^2 G_\delta(\mu_k x, y)\Delta^2 u_k(y)\, dy. \tag{19}$$

Standard estimates on the Green's function (see for instance [7]) yield that there exists $C(\delta) > 0$ such that

$$|G_\delta(x,y)| \le \frac{C(\delta)}{|x-y|^2} \tag{20}$$

for all $x, y \in B_\delta(0)$. Integrating (19), using (20) and (6), we get that

$$\begin{aligned}
&\int_{B_R(0)} |\Delta v_k - \mu_k^2 \Delta u_k(\delta)|\, dx \\
&\leq \int_{x\in B_R(0)} \int_{y \in B_\delta(0)} \mu_k^2 G_\delta(\mu_k x, y) V_k(y) e^{4u_k(y)}\, dy\, dx \\
&\leq C(\delta) \int_{B_\delta(0)} V_k(y) e^{4u_k(y)} \left(\int_{B_R(0)} \frac{\mu_k^2}{|\mu_k x - y|^2}\, dx \right) dy \\
&\leq C(\delta) \int_{B_\delta(0)} V_k e^{4u_k(y)} \left(CR^2 \right) dy \leq C(\delta) \Lambda R^2,
\end{aligned}$$

where $C(\delta) > 0$ is independant of $k \in \mathbb{N}$ and $R \in (0, \delta\mu_k^{-1})$. In this last inequality, we have used (7). This proves Lemma 3.5. □

The key-quantity in Step 3.6 is the limit of $\mu_k^2 \Delta u_k(\delta)$ when $k \to +\infty$. We separate the study in three cases, each of the following three sections is devoted to one of these cases. Thanks to them, we will recover the three notions of convergence of Proposition 3.1.

4. The case $\lim_{k\to+\infty} \mu_k^2 \Delta u_k(\delta) = +\infty$

In this situation, we show that the second type of quadratic convergence of Proposition 3.1 holds and that $V_k e^{4u_k} \rightharpoonup 0$ when $k \to +\infty$ in the sense of measures.

Step 4.1: We prove that quadratic-convergence (II) of Proposition 3.1 holds in this case. More precisely,

Lemma 4.1. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that (12) holds. We assume that there exists $\delta_0 \in (0,1)$ such that*

$$\lim_{k\to+\infty} \mu_k^2 |\Delta u_k(\delta_0)| = +\infty, \tag{21}$$

where μ_k is as in (13). Then, the second type of quadratic convergence of Proposition 3.1 holds. In addition, for any $\delta \in (0,1)$,

$$\lim_{k\to+\infty} \frac{\Delta u_k}{\Delta u_k(\delta)} = 1$$

in $C^1_{loc}(B)$ when $k \to +\infty$.

Proof of Lemma 4.1: Let $\delta_0 \in (0,1)$ as in the Lemma. Let $R > 0$. It follows from (18) and (21) that

$$\left\| \Delta \left(\frac{v_k}{\mu_k^2 \Delta u_k(\delta_0)} \right) - 1 \right\|_{L^1(B_R(0))} = o(1) \tag{22}$$

when $k \to +\infty$. It follows from (14), (16) and (21) that

$$\Delta \left(\Delta \left(\frac{v_k}{\mu_k^2 \Delta u_k(\delta_0)} \right) - 1 \right) = \frac{\tilde{V}_k e^{4v_k}}{\mu_k^2 \Delta u_k(\delta_0)} = o(1)$$

where $o(1) \to 0$ in $C^0(B_R(0))$. It follows from (22) and standard elliptic theory that

$$\left\| \Delta \left(\frac{v_k}{\mu_k^2 \Delta u_k(\delta_0)} \right) - 1 \right\|_{L^\infty(B_{R/2}(0))} \to 0 \tag{23}$$

when $k \to +\infty$. With (14) and (16), we get that there exists $\psi_R \in C^4(B_{R/2}(0))$ such that $\psi_R \not\equiv 0$ and

$$\lim_{k\to+\infty} \frac{v_k}{\mu_k^2 \Delta u_k(\delta_0)} = \psi_R \text{ in } C^3(B_{R/4}(0)).$$

Moreover, (16) yields $\psi_R(x) \leq \psi_R(0) = 0$ for all $x \in B_{R/4}(0)$. With (23), we get that $\Delta\psi_R = 1$. Since the functions are radial, we get that $\psi_R(x) = -|x|^2/8$ for all $x \in B_{R/4}(0)$. In particular, taking R arbitrarily large, we get that

$$\lim_{k\to+\infty} \frac{v_k}{\mu_k^2 \Delta u_k(\delta_0)} = -\frac{|x|^2}{8} \text{ in } C^3_{loc}(\mathbb{R}^4). \tag{24}$$

Computing the Laplacian of v_k at 0, we get that

$$\lim_{k\to+\infty} \frac{\Delta u_k(0)}{\Delta u_k(\delta_0)} = 1.$$

In particular, $\Delta u_k(\delta_0) > 0$ for $k > 0$ large and $\lim_{k\to+\infty} \Delta u_k(0) = +\infty$. Combining this limit with (24), we obtain that the second type of quadratic convergence of Proposition 3.1 holds.

We let $\psi_k \in C^2(B)$ such that $\psi_k = \frac{\Delta u_k}{\Delta u_k(0)}$. With the equation (6) and the estimate (8), we get that

$$\lim_{k\to+\infty} \Delta\psi_k = 0 \text{ in } C^0_{loc}(B \setminus \{0\}).$$

Since Δu_k is decreasing, we have that $\psi_k(x) \leq 1$ for all $x \in B$. Noting that we have that $\lim_{k\to+\infty} \psi_k(\delta_0) = 1$, it follows from elliptic theory that there exists $\psi \in C^2(B \setminus \{0\})$ such that $\lim_{k\to+\infty} \psi_k = \psi$ in $C^1_{loc}(B \setminus \{0\})$ and $\Delta\psi = 0$. Letting $k \to +\infty$, we get that $\psi(\delta_0) = 1$. Since $\psi \leq 1$ in B and ψ is non-increasing, we get that $\psi \equiv 1$. In addition, since ψ_k is decreasing and achieves the value 1 at 0, we get that

$$\lim_{k\to+\infty} \frac{\Delta u_k}{\Delta u_k(0)} = 1 \text{ in } C^0_{loc}(B).$$

This ends the proof of Lemma 4.1. □

Step 4.2: In the case of quadratic convergence, the quadratic term happens to dominate the other ones asymptotically. More precisely, we have the following. Note that this Lemma does not use hypothesis (21).

Lemma 4.2. *[Pointwise estimate (I)] Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that (12) holds. Then for any $0 < \delta < 1$ there exists $C(\delta) > 0$ such that*

$$|x| \left| \nabla u_k(x) + \frac{\Delta u_k(\delta)}{4} x \right| \leq C(\delta)$$

for all $x \in B_\delta(0)$ and all $k \in \mathbb{N}$.

Proof of Lemma 4.2: We let $\delta \in (0,1)$. We let H_δ be the Green's function Δ^2 on $B_\delta(0)$ with Navier condition, that is for any $x \in B_\delta(0)$,

$$\begin{cases} \Delta^2 H_\delta(x,\cdot) = \delta_x & \text{in } \mathcal{D}'(B_\delta(0)) \\ H_\delta(x,\cdot) = \Delta H_\delta(x,\cdot) = 0 & \text{on } \partial B_\delta(0) \end{cases}$$

As easily checked, we have that $H_\delta = G_\delta * G_\delta$ where $*$ denotes the product of convolution and G_δ is the Green's function for Δ on $B_\delta(0)$ with Dirichlet boundary condition. Since u_k is radially symmetrical, we get that

$$u_k(x) = \int_{B_\delta(0)} H_\delta(x,y)\Delta^2 u_k(y)\,dy + u_k(\delta) + \frac{\delta^2 - |x|^2}{8}\Delta u_k(\delta)$$

for all $x \in B_\delta(0)$. Differentiating this identity, we get that

$$\nabla u_k(x) = \int_{B_\delta(0)} \nabla H_\delta(x,y)\Delta^2 u_k(y)\,dy - \frac{\Delta u_k(\delta)}{4}x \tag{25}$$

for all $x \in B_\delta(0)$. Standard estimates on the Green's function (see for instance [7]) yield that there exists $C(\delta) > 0$ such that

$$|\nabla H_\delta(x,y)| \le \frac{C(\delta)}{|x-y|} \tag{26}$$

for all $x, y \in B_\delta(0)$. Plugging (26) into (25), we get that

$$\left|\nabla u_k(x) + \frac{\Delta u_k(\delta)}{4}x\right| \le C(\delta)\int_{B_\delta(0)} \frac{e^{4u_k(y)}}{|x-y|}\,dy$$

for all $x \in B_\delta(0)$. Using the pointwise estimate (8), we get that

$$\begin{aligned}
\int_{B_\delta(0)} \frac{e^{4u_k(y)}}{|x-y|}\,dy &\le \int_{B_\delta(0)\cap B_{|x|/2}(0)} \frac{e^{4u_k(y)}}{|x-y|}\,dy + \int_{B_\delta(0)\setminus B_{|x|/2}(0)} \frac{e^{4u_k(y)}}{|x-y|}\,dy \\
&\le \frac{2}{|x|}\int_{B_\delta(0)} e^{4u_k(y)}\,dy + C\int_{B_\delta(0)\setminus B_{|x|/2}(0)} \frac{1}{|y|^4|x-y|}\,dy \\
&\le \frac{2\Lambda}{|x|} + \frac{C}{|x|}\int_{B_{\delta/|x|}(0)\setminus B_{1/2}(0)} \frac{1}{|y|^4\left|\frac{x}{|x|} - y\right|}\,dy \\
&\le \frac{C'(\delta,\Lambda)}{|x|}
\end{aligned}$$

for all $x \in B_\delta(0)\setminus\{0\}$ and all $k \in \mathbb{N}$. Here $C'(\delta,\Lambda)$ depends only on δ and Λ. This proves Lemma 4.2. □

Step 4.3: We are in position to describe precisely the asymptotics of the u_k's when $k \to +\infty$. This is the object of the following Lemma:

Lemma 4.3. *[Pointwise estimate (II)] Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that (12) holds. We assume that there exists $\delta_0 \in (0,1)$ such that*

$$\lim_{k\to+\infty} \mu_k^2|\Delta u_k(\delta_0)| = +\infty.$$

We let $0 < \delta < 1$. Then

$$u_k\left(\frac{x}{\sqrt{\Delta u_k(\delta)}}\right) - u_k(0) = -\frac{|x|^2}{8} + O(1)\ln(2+|x|^2) \tag{27}$$

for all $x \in B_{\delta\sqrt{\Delta u_k(0)}}(0)$ and all $k \in \mathbb{N}$, where $O(1)$ denotes a function such that there exists $C(\delta) > 0$ such that $|O(1)(x,k)| \leq C(\delta)$ for all $x \in B_{\delta\sqrt{\Delta u_k(0)}}(0)$ and all $k \in \mathbb{N}$.

Proof of Lemma 4.3: We let $x \in B_{\delta\sqrt{\Delta u_k(0)}}(0)$ such that $|x| > 1$. We let $x_0 = \frac{x}{|x|}$. With the pointwise estimate of Lemma 4.2, we get that

$$
\begin{aligned}
& u_k\left(\frac{x}{\sqrt{\Delta u_k(\delta)}}\right) - u_k\left(\frac{x_0}{\sqrt{\Delta u_k(\delta)}}\right) \\
&= \int_0^1 \frac{\partial}{\partial t}\left[u_k\left((1-t)\frac{x_0}{\sqrt{\Delta u_k(\delta)}} + t\frac{x}{\sqrt{\Delta u_k(\delta)}}\right)\right] dt \\
&= \frac{1}{\sqrt{\Delta u_k(\delta)}}\int_0^1 (x-x_0)^i \partial_i u_k\left(\frac{(1-t)x_0+tx}{\sqrt{\Delta u_k(\delta)}}\right) dt \\
&= -\frac{1}{4}\int_0^1 (x-x_0)^i((1-t)x_0+tx)_i\, dt \\
&+ \frac{1}{\sqrt{\Delta u_k(\delta)}}\int_0^1 (x-x_0)^i\left(\partial_i u_k\left(\frac{(1-t)x_0+tx}{\sqrt{\Delta u_k(\delta)}}\right) + \frac{\sqrt{\Delta u_k(\delta)}}{4}((1-t)x_0+tx)_i\right) dt \\
&= -\frac{|x|^2}{8} + \frac{|x_0|^2}{8} + O(1)\int_0^1 \frac{|x-x_0|}{|(1-t)x_0+tx|}\, dt \\
&= -\frac{|x|^2}{8} + \frac{|x_0|^2}{8} + O(1)\int_0^1 \frac{|x|-1}{t(|x|-1)+1}\, dt \\
&= -\frac{|x|^2}{8} + \frac{|x_0|^2}{8} + O(1)\ln|x| \qquad (28)
\end{aligned}
$$

where $O(1)$ is a function which is bounded with respect to both x and $k \in \mathbb{N}$. We claim that

$$\lim_{k\to+\infty}\left(u_k\left(\frac{x}{\sqrt{\Delta u_k(\delta)}}\right) - u_k(0)\right) = -\frac{|x|^2}{8} \qquad (29)$$

for all $x \in \mathbb{R}^4$, and that this convergence holds in $C^1_{loc}(\mathbb{R}^4)$. We prove the claim. We write that

$$\Delta\left(u_k\left(\frac{x}{\sqrt{\Delta u_k(\delta)}}\right) - u_k(0)\right) = \frac{\Delta u_k\left(\frac{x}{\sqrt{\Delta u_k(\delta)}}\right)}{\Delta u_k(\delta)},$$

for all $x \in B_\delta(0)$ and all $k \in \mathbb{N}$. It follows from Lemma 4.1, (12) and standard elliptic theory, that there exists $\varphi \in C^1(\mathbb{R}^4)$ such that

$$\lim_{k\to+\infty}\left(u_k\left(\frac{x}{\sqrt{\Delta u_k(\delta)}}\right) - u_k(0)\right) = \varphi(x)$$

for all $x \in \mathbb{R}^4$ when $k \to +\infty$. Moreover, $\varphi \in C^2(\mathbb{R}^4)$, $\Delta\varphi = 1$ and $\varphi \leq \varphi(0) = 0$. Since φ is radially symmetrical, we get that $\varphi(x) = -\frac{|x|^2}{8}$. This proves the claim.

The asymptotic (27) follows from (28) and (29). This proves Lemma 4.3. □

Step 4.4: We prove the vanishing of the L^1-norm of e^{4u_k} when $k \to +\infty$.

Lemma 4.4. *Let* $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ *and* $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ *such that (5), (6) and (7) hold. We assume that* u_k *is radially symmetrical for all* $k \in \mathbb{N}$*. We assume that (12) holds. We assume that there exists* $\delta_0 \in (0,1)$ *such that*

$$\lim_{k\to+\infty} \mu_k^2 |\Delta u_k(\delta_0)| = +\infty.$$

Then, for any $\delta \in (0,1)$*, we have that*

$$\int_{B_\delta(0)} e^{4u_k}\, dx \to 0$$

when $k \to +\infty$*. In particular* $V_k e^{4u_k}\, dx \rightharpoonup 0$ *when* $k \to +\infty$ *in the sense of measures.*

Proof of Lemma 4.4: We let $\delta \in (0,1)$. With the definition (13) of μ_k and a change of variables, we get that

$$\int_{B_\delta(0)} e^{4u_k}\, dx = \frac{1}{\mu_k^4 \Delta u_k(\delta)^2} \int_{B_{\delta\sqrt{\Delta u_k(\delta)}}(0)} e^{4\left(u_k\left(\frac{x}{\sqrt{\Delta u_k(\delta)}}\right) - u_k(0)\right)}\, dx.$$

Since $\lim_{k\to+\infty} \mu_k^2 |\Delta u_k(\delta_0)| = +\infty$, Lemma 4.1 yields that $\lim_{k\to+\infty} \mu_k^2 \Delta u_k(\delta) = +\infty$. The asymptotic (27) of Lemma 4.3 yield the conclusion of the Lemma 4.4. □

Point *(ii.c)* of Theorem 1.1 follows from Lemma 4.4.

5. The case $\lim_{k\to+\infty} \mu_k^2 |\Delta u_k(\delta)| = K_\delta > 0$

In this situation, we show that the first type of quadratic convergence of Proposition 3.1 holds. Moreover, we describe the asymptotics for u_k.

Step 5.1: We first prove that the quadratic-convergence (I) holds in this case. More precisely,

Lemma 5.1. *Let* $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ *and* $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ *such that (5), (6) and (7) hold. We assume that* u_k *is radially symmetrical for all* $k \in \mathbb{N}$*. We assume that (12) holds. We assume that there exists* $\delta_0 \in (0,1)$ *and* $K_{\delta_0} \in \mathbb{R}$ *such that*

$$\lim_{k\to+\infty} \mu_k^2 |\Delta u_k(\delta_0)| = K_{\delta_0} > 0.$$

Then the first type of quadratic convergence of Proposition-Definition 3.1 holds. In addition, we have that there exists $K > 0$ *such that*

$$\lim_{k\to+\infty} \frac{\Delta u_k}{\Delta u_k(0)} = K \text{ in } C^1_{loc}(B \setminus \{0\}).$$

Proof of Lemma 5.1: Let $R > 0$. Since, up to a subsequence,

$$\lim_{k\to+\infty} \mu_k^2 \Delta u_k(\delta_0) = K_{\delta_0} \neq 0. \tag{30}$$

It follows from (18) that

$$\|\Delta v_k\|_{L^1(B_R(0))} = O(1)$$

when $k \to +\infty$. It follows from equation (14), inequation (16) and elliptic theory that

$$\|\Delta v_k\|_{C^1(B_{R/2}(0))} = O(1) \tag{31}$$

when $k \to +\infty$. Inequation (16), equations (14) and (31), the Harnack inequality and standard elliptic theory yield that there exists $v \in C^3(\mathbb{R}^4)$ such that

$$\lim_{k\to+\infty} v_k = v \text{ in } C^3_{loc}(\mathbb{R}^4), \tag{32}$$

where $\Delta^2 v = e^{4v}$ in the distribution sense. Elliptic theory yields that $v \in C^4(\mathbb{R}^4)$. We are then in Case (i) or (ii) of Proposition-Definition 3.1.

We claim that we are in Case (ii) of Proposition-Definition 3.1. We proceed by contradiction and assume that Case (i) of Proposition-Definition 3.1 holds. We then get that $v = v_0$ where

$$v_0(x) = \ln \frac{\sqrt{96}}{\sqrt{96} + |x|^2}$$

for all $x \in \mathbb{R}^4$. We let $R > 0$. We let $k \to +\infty$ in (18) and get that

$$\int_{B_R(0)} |\Delta v(x) - K_{\delta_0}|\, dx \le CR^2.$$

Since $v = v_0$, using the explicit expression of v_0 above and letting $R \to +\infty$, we get that there exists a constant $C > 0$ independant of $R > 0$ such that

$$|K_{\delta_0}| \le CR^{-2}$$

for all $R > 0$. Letting $R \to +\infty$, we get that $K_{\delta_0} = 0$. A contradiction with our initial assumption (30). Then Case (i) does not hold and we are in Case (ii).

It follows from Case (ii) of Proposition-Definition 3.1 and Theorem 1.2 of [11] that there exists $a > 0$ such that

$$\lim_{|x|\to+\infty} \frac{v(x)}{|x|^2} = -a \text{ and } \lim_{|x|\to+\infty} \Delta v(x) = 8a. \tag{33}$$

We let $\delta \in (0,1)$. With (18), we get that there exists $C(\delta) > 0$ such that

$$\int_{B_R(0)} |\Delta v_k - \mu_k^2 \Delta u_k(\delta)|\, dx \le C(\delta) R^2 \tag{34}$$

for all $R \in (0, \delta\mu_k^{-1})$. It the follows from (32) that there exists $K_\delta \in \mathbb{R}$ such that $\lim_{k\to+\infty} \mu_k^2 \Delta u_k(\delta) = K_\delta$. Passing to the limit $k \to +\infty$ in (34), we get that

$$\int_{B_R(0)} |\Delta v - K_\delta|\, dx \le C(\delta) R^2$$

for all $R > 0$. Letting $R \to +\infty$ in this inequality and using (33), we get that $K_\delta = 8a > 0$ for all $\delta \in (0,1)$. In particular, with (30) and (32), we get that there exists $K > 0$ such that for any $\delta \in (0,1)$,

$$\lim_{k\to+\infty} \frac{\Delta u_k(\delta)}{\Delta u_k(0)} = K > 0.$$

The last assertion of Lemma 5.1 follows from this limit, equation (E), inequality (8) the decreasing of Δu_k and standard elliptic theory. □

Step 5.2: With some arguments very similar to the ones developed in the proof of Lemma 4.3, we get the following Lemma. We omit the proof:

Lemma 5.2. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that (12) holds. We assume that there exists $\delta_0 \in (0,1)$ such that*

$$\lim_{k\to+\infty} \mu_k^2 |\Delta u_k(\delta_0)| = K_{\delta_0} > 0.$$

We let $0 < \delta < 1$. Then there exists a sequence $(a_k)_{k\in\mathbb{N}} \in \mathbb{R}$ such that $\lim_{k\to+\infty} a_k = a_\infty > 0$ and such that

$$v_k(x) = -a_k|x|^2 + O(1)\ln(2+|x|^2)$$

for all $x \in B_{\delta\mu_k^{-1}}(0)$ and all $k \in \mathbb{N}$, where $O(1)$ denotes a function such that there exists $C(\delta) > 0$ such that $|O(1)(x,k)| \le C(\delta)$ for all $x \in B_{\delta\mu_k^{-1}}(0)$ and all $k \in \mathbb{N}$.

As a consequence of this pointwise estimate, we get the following quantization of the L^1-norm of e^{4u_k}:

Lemma 5.3. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that (12) holds. We assume that there exists $\delta_0 \in (0,1)$ such that*

$$\lim_{k\to+\infty} \mu_k^2 |\Delta u_k(\delta_0)| = K_{\delta_0} > 0.$$

Then for any $\delta \in (0,1)$, we have that

$$\lim_{k\to+\infty} \int_{B_\delta(0)} V_k e^{4u_k}\,dx = \int_{\mathbb{R}^4} e^{4v}\,dx < 16\pi^2.$$

In other words, $V_k e^{4u_k}\,dx \rightharpoonup (\int_{\mathbb{R}^4} e^{4v}\,dx)\delta_0$ when $k \to +\infty$ in the sense of the measures.

Proof of Lemma 5.3: It follows from Lemma 5.2 that there exists $C = C(\delta) > 0$ such that

$$v_k(x) \le -\frac{a_\infty}{2}|x|^2 + C$$

for all $x \in B_{\delta\mu_k^{-1}}(0)$ and all $k \in \mathbb{N}$. We let $R > 0$. With a change of variable, we get that

$$\int_{B_\delta(0)\setminus B_{R\mu_k}(0)} V_k e^{4u_k}\,dx = \int_{B_{\delta/\mu_k}(0)\setminus B_R(0)} \tilde{V}_k e^{4v_k}\,dx \le 2\int_{\mathbb{R}^4\setminus B_R(0)} e^{-2a_\infty|x|^2+4C}\,dx.$$

As a consequence,

$$\lim_{R\to+\infty}\lim_{k\to+\infty} \int_{B_\delta(0)\setminus B_{R\mu_k}(0)} V_k e^{4u_k}\,dx = 0. \tag{35}$$

On the other hand, with a change of variables and letting $k \to +\infty$, we get that

$$\int_{B_{R\mu_k}(0)} V_k e^{4u_k}\,dx = \int_{B_R(0)} \tilde{V}_k e^{4v_k}\,dx = \int_{B_R(0)} e^{4v}\,dx + o(1) \tag{36}$$

when $k \to +\infty$. Summing (35) and (36) and letting $k \to +\infty$ and then $R \to +\infty$, we get that

$$\lim_{k\to+\infty} \int_{B_\delta(0)} V_k e^{4u_k}\,dx = \int_{\mathbb{R}^4} e^{4v}\,dx.$$

Moreover, it follows from [11], Theorem 1.2, that

$$\int_{\mathbb{R}^4} e^{4v}\, dx < 16\pi^2.$$

This ends the proof of Lemma 5.3. □

Point *(ii.b)* of Theorem 1.1 follows from Lemma 5.3.

6. The case $\lim_{k\to+\infty} \mu_k^2 \Delta u_k(\delta) = 0$

In this case, the behavior of the u_k's is much more standard and is similar to the two-dimensional corresponding problem. We show that the Log-convergence of Proposition-Definition 3.1 holds. Moreover, we describe the asymptotics for u_k.

Step 6.1: We first prove that the Log-convergence holds in this case. More precisely,

Lemma 6.1. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that (12) holds. We assume that there exists $\delta_0 \in (0,1)$ such that*

$$\lim_{k\to+\infty} \mu_k^2 \Delta u_k(\delta_0) = 0.$$

Then for any $x \in \mathbb{R}^4$,

$$\lim_{k\to+\infty} v_k(x) = \ln \frac{\sqrt{96}}{\sqrt{96} + |x|^2},$$

where v_k is as in (13)*. Moreover, this convergence holds in $C^3_{loc}(\mathbb{R}^4)$.*

Proof of Lemma 6.1: With some arguments similar to the ones developed in the proof of Lemma 5.1, we get that there exists $v \in C^4(\mathbb{R}^4)$ such that $\lim_{k\to+\infty} v_k = v$ in $C^3_{loc}(\mathbb{R}^4)$. Moreover, $\Delta^2 v = e^{4v}$ and $e^v \in L^1(\mathbb{R}^4)$. We are then in Case (i) or (ii) of Proposition-Definition 3.1. We let $k \to +\infty$ in (18) and get for any $R > 0$ in $\mathbb{R}^4$ that

$$\int_{B_R(0)} |\Delta v(x)|\, dx \le CR^2. \tag{37}$$

We assume by contradiction that Case (ii) holds. It then follows from Lin [11] that $\lim_{|x|\to+\infty} \Delta v(x) = 8a > 0$. Letting $R \to +\infty$ in (37), we get that $8a = 0$. A contradiction. We are then in Case (i) of Proposition-Definition 3.1 and $v(x) = \ln \frac{\sqrt{96}}{\sqrt{96}+|x|^2}$ for all $x \in \mathbb{R}^4$, that is Log-Convergence holds. This proves Lemma 6.1. □

A consequence of this Lemma is the following. With a change of variable, we get that

$$\int_{B_{R\mu_k}(0)} V_k e^{4u_k}\, dx = \int_{B_R(0)} \tilde{V}_k e^{4v_k}\, dx = \int_{B_R(0)} e^{4v}\, dx + o(1)$$

when $k \to +\infty$. Passing to the limit $k \to +\infty$ and then $R \to +\infty$, we get that

$$\lim_{R\to+\infty} \lim_{k\to+\infty} \int_{B_{R\mu_k}(0)} V_k e^{4u_k}\, dx = 16\pi^2. \tag{38}$$

Step 6.2: We are in position to deal with the convergence outside 0. This is the object of the following Lemma:

Lemma 6.2. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that (12) holds. We assume that there exists $\delta_0 \in (0,1)$ such that*

$$\lim_{k\to+\infty} \mu_k^2 \Delta u_k(\delta_0) = 0.$$

Then $\lim_{k\to+\infty} u_k = -\infty$ uniformly on every compact subset of $B \setminus \{0\}$.

Proof of Lemma 6.2: Assume that the conclusion is false. It then follows from Lemma 3.2 that for any $K \subset\subset B \setminus \{0\}$, there exists $C(K) > 0$ such that

$$|u_k(z)| + |\Delta u_k(z)| \leq C(K) \tag{39}$$

for all $z \in K$. We let $\delta \in (0,1/2)$ and we let H_δ be the Green's function for Δ^2 in $B_\delta(0)$ with Navier condition on the boundary, that is for any $x \in B_\delta(0)$, we have that

$$\Delta^2 H_\delta(x,\cdot) = \delta_x$$

for all $x \in B_\delta(x)$ and $H_\delta(x,\cdot) = \Delta H_\delta(x,\cdot) = 0$ on $\partial B_\delta(x)$. We let $x \in B_\delta(0) \setminus \{0\}$. Since u_k is radially symmetrical, we have that

$$u_k(x) = \int_{B_\delta(0)} H_\delta(x,y) V_k(y) e^{4u_k(y)}\, dy + u_k(\delta) + \frac{\delta^2 - |x|^2}{8} \Delta u_k(\delta).$$

We let $\alpha > 0$ small. Since u_k is uniformly bounded in L^∞ outside 0 and since $H_\delta > 0$, we get with (5), (38) and (39) that there exists $C > 0$ independant of x and $\alpha > 0$ such that

$$\begin{aligned} u_k(x) &\geq \int_{B_{R\mu_k}(0)} H_\delta(x,y) V_k(y) e^{4u_k(y)}\, dy - C \\ &\geq \int_{B_R(0)} H_\delta(x,\mu_k y) \tilde{V}_k(y) e^{4v_k(y)}\, dy - C \\ &\geq \int_{B_R(0)} H_\delta(x,0) \lim_{k\to+\infty} \left(\tilde{V}_k(y) e^{4v_k(y)} \right) dy - C + o(1) \\ &\geq 16\pi^2 H_\delta(x,0) - C + o(1) \end{aligned}$$

for $x \in B$ such that $|x| \geq \alpha$ and for k large enough depending only on α. Since $H_\delta(x,0) = \frac{1}{8\pi^2} \ln \frac{\delta}{|x|} + \frac{|x|^2 - \delta^2}{32\pi^2\delta^2}$ for $x \in B_{\delta/2}(x_0)$. We then get that

$$u_k(x) \geq 2 \ln \frac{1}{|x|} - C' + o(1)$$

for $x \in B_\delta(0) \setminus B_\alpha(0)$ and k large depending only on $\alpha > 0$. We then get that for any $0 < \alpha < \beta$ small,

$$\Lambda \geq \int_{B_\beta(0)\setminus B_\alpha(0)} V_k e^{4u_k}\, dx \geq C \int_{B_\beta(0)\setminus B_\alpha(0)} \frac{1}{|x|^8}\, dx.$$

We get a contradiction by letting $\alpha \to 0$. Then $u_k \to -\infty$ on compact subsets of $B \setminus \{0\}$ when $k \to +\infty$ and Lemma 6.2 is proved. □

Step 6.3: We now prove that the whole L^1-norm of e^{4u_k} is actually $16\pi^2$. We borrow ideas from Schoen-Zhang [16], Druet [5] and Druet-Robert [6].

Lemma 6.3. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that (12) holds. We assume that there exists $\delta_0 \in (0,1)$ such that*

$$\lim_{k\to+\infty} \mu_k^2 \Delta u_k(\delta_0) = 0.$$

We let $\delta \in (0,1)$. Then there exists $(r_k)_{k\in\mathbb{N}} \in \mathbb{R}_{>0}$ such that $r_k \in [0,\delta]$ for all $k \in \mathbb{N}$ and

(i) $\lim_{k\to+\infty} \frac{r_k}{\mu_k} = +\infty$,
(ii) $r \mapsto re^{u_k(r)}$ is decreasing on $[4\mu_k, r_k]$,
(iii) $u_k \to -\infty$ uniformly on $\overline{B}_\delta(0) \setminus B_{r_k}(0)$.

Proof of Lemma 6.3: We let $\delta \in (0,1)$. Without loss of generality, we assume that $\lim_{k\to+\infty} \mu_k^2 \Delta u_k(\delta) = 0$ (otherwise, we are back to the previous cases).

Step 6.3.1: We claim that for any $R > 4$, we have that

$$r \mapsto re^{u_k(r)} \text{ is decreasing on } [4\mu_k, R\mu_k]$$

for k large enough. Indeed, we let $r \in [4\mu_k, R\mu_k]$ and we let $\rho_k := \frac{r}{\mu_k}$. With Lemma 6.1, we have that

$$\begin{aligned}(re^{u_k(r)})'(r) &= \mu_k^{-1}\frac{d}{dr}\left(r\mu_k e^{u_k(r\mu_k)}\right)(\rho_k) = \mu_k^{-1}\frac{d}{dr}\left(re^{v_k(r)}\right)(\rho_k)\\ &= \mu_k^{-1}\left(\frac{d}{dr}\left(re^{v(r)}\right) + o(1)\right)(\rho_k) = \frac{\sqrt{96}}{\mu_k}\left(\frac{\sqrt{96}-\rho_k^2}{(\sqrt{96}+\rho_k^2)^2} + o(1)\right)\end{aligned}$$

where $o(1) \to 0$ when $k \to +\infty$ uniformly for $r \in [4\mu_k, R\mu_k]$. Since $\rho_k \geq 4$, the right-hand-side is negative. Then $(re^{u_k(r)})' < 0$ and the function $r \mapsto re^{u_k(r)}$ is decreasing on $[4\mu_k, R\mu_k]$.

Step 6.3.2: We assume that $r \to re^{u_k(r)}$ is decreasing on $[4\mu_k, \delta]$ for all $k \in \mathbb{N}$. Then the conclusion of the Lemma holds with $r_k := \delta$, and Lemma 6.3 is proved.

From now on, we assume that

$$r \to re^{u_k(r)} \text{ is not decreasing on } [4\mu_k, \delta]. \tag{40}$$

We let

$$r_k := \inf\{\rho \in [4\mu_k, \delta] / \, (re^{u_k(r)})'(\rho) = 0.\}$$

Step 6.3.3: We claim that

$$\lim_{k\to+\infty} \frac{r_k}{\mu_k} = +\infty, \quad (re^{u_k(r)})'(r) < 0 \text{ when } 4\mu_k < r < r_k \text{ and } (re^{u_k(r)})'(r_k) = 0. \tag{41}$$

Indeed, it follows from Step 6.3.1 and (40) that r_k is defined and satisfies the two last statements of (41). The first statement is a consequence of Step 6.3.1.

Step 6.3.4: We claim that

$$\lim_{k\to+\infty} r_k e^{u_k(r_k)} = 0. \tag{42}$$

Indeed, we let $R \geq 4$. It follows from (41) that $re^{u_k(r)}$ is decreasing on $[R\mu_k, r_k]$. We then get that

$$\begin{aligned} r_k e^{u_k(r_k)} &\leq R\mu_k e^{u_k(R\mu_k)} \leq Re^{v_k(R)} \\ &\leq \left(Re^{v(R)} + o(1)\right) \leq \left(\frac{R\sqrt{96}}{\sqrt{96}+R^2} + o(1)\right) \end{aligned}$$

where $o(1) \to 0$ when $k \to +\infty$. Letting $k \to +\infty$ and then $R \to +\infty$, we get (42). This ends Step 6.3.4.

We let

$$\tilde{u}_k(x) = u_k(r_k x) - u_k(r_k) \tag{43}$$

for all $k \in \mathbb{N}$ and all $x \in B_{r_k^{-1}}(0)$.

Step 6.3.5: We claim that there exists $a \geq 1$ such that for any $x \in \mathbb{R}^4 \setminus \{0\}$, we have that

$$\lim_{k\to+\infty} u_k(r_k x) - u_k(r_k) = a \ln \frac{1}{|x|} + \frac{a-1}{2}(|x|^2 - 1). \tag{44}$$

Moreover, this convergence holds in $C^3_{loc}(\mathbb{R}^4 \setminus \{0\})$. Indeed, equation (6) rewrites as

$$\Delta^2 \tilde{u}_k(x) = V_k(r_k x) r_k^4 e^{4u_k(r_k x)} = V_k(r_k x) r_k^4 e^{4u_k(r_k)} e^{\tilde{u}_k(x)} \tag{45}$$

for all $k \in \mathbb{N}$ and all $x \in B_{r_k^{-1}}(0)$. The system (41) yields that

$$r\tilde{u}_k'(r) \leq -1 \text{ for } \frac{4\mu_k}{r_k} \leq r \leq 1 \text{ and } \tilde{u}_k'(1) = -1. \tag{46}$$

Proceeding as in Lemma 4.2 and using the definition (43), we get that there exists $C = C(\delta) > 0$ such that

$$|\tilde{u}_k'(r) + \frac{\Delta u_k(\delta)}{4} r_k^2 r| \leq \frac{C}{r} \tag{47}$$

for all $k \in \mathbb{N}$ and all $r \in (0, \delta r_k^{-1})$. Taking $r = 1$ in (47) and using (46), we then obtain that, up to a subsequence, there exists $\rho \in \mathbb{R}$ such that

$$\lim_{k\to+\infty} r_k^2 \Delta u_k(\delta) = \rho. \tag{48}$$

Since $\tilde{u}_k(1) = 0$, it follows from (47) and (48) that for any $U \subset\subset \mathbb{R}^4 \setminus \{0\}$, there exists $C'(U) > 0$ such that

$$|\tilde{u}_k(x)| \leq C'(U)$$

for all $x \in U$ and all $k \in \mathbb{N}$. It then follows from (45), (42) and standard elliptic theory that there exists $\tilde{u} \in C^4(\mathbb{R}^4 \setminus \{0\})$ such that $\Delta^2 \tilde{u} = 0$ and

$$\tilde{u}_k \to \tilde{u} \tag{49}$$

in $C^3_{loc}(\mathbb{R}^4 \setminus \{0\})$ when $k \to +\infty$. Since $\tilde{u}$ is radially symmetrical, we get that there exist $a, b, c, d \in \mathbb{R}$ such that

$$\tilde{u}(x) = a \ln \frac{1}{|x|} + \frac{b}{|x|^2} + c|x|^2 + d \tag{50}$$

for all $x \in \mathbb{R}^4 \setminus \{0\}$. Passing to the limit in (47) and using (48), we get that

$$|\tilde{u}'(r) + \frac{\rho}{4} r| \leq \frac{C}{r} \tag{51}$$

for all $r > 0$. It follows from (50) and (51) that $b = 2c + \frac{\rho}{4} = 0$, so that we can write

$$\tilde{u}(x) = a \ln \frac{1}{|x|} - \frac{\rho}{8}|x|^2 + d$$

for all $x \in \mathbb{R}^4 \setminus \{0\}$. Passing to the limit in (46), we get that

$$r\tilde{u}'(r) \leq -1 \text{ for } r < 1 \text{ and } \tilde{u}'(1) = -1.$$

With the explicit expression (50) of $\tilde{u}$, we get that

$$a - 1 = \frac{|\rho|}{4} \geq 0 \text{ and } \rho \leq 0.$$

Since $\tilde{u}(1) = 1$, the claim follows.

Step 6.3.6: We claim that

$$a \geq 2.$$

Indeed, integrating by parts, we get that

$$\begin{aligned} &\int_{B_{r_k}(0)} x^i \partial_i u_k \Delta^2 u_k \, dx \\ &= \int_{\partial B_{r_k}(0)} \left(\frac{(x,\nu)}{2}(\Delta u_k)^2 + \Delta u_k \frac{\partial (x, \nabla u_k)}{\partial \nu} - (x, \nabla u_k) \frac{\partial \Delta u_k}{\partial \nu} \right) d\sigma \end{aligned} \tag{52}$$

where ν denotes the outer normal vector at $\partial B_{r_k}(0)$. Using the change of variable $y = r_k x$ and the convergence (44), we get that

$$\begin{aligned} &\int_{\partial B_{r_k}(0)} \left(\frac{(x,\nu)}{2}(\Delta u_k)^2 + \Delta u_k \frac{\partial (x, \nabla u_k)}{\partial \nu} - (x, \nabla u_k) \frac{\partial \Delta u_k}{\partial \nu} \right) d\sigma \\ &= \int_{\partial B_1(0)} \left(\frac{(x,\nu)}{2}(\Delta \tilde{u}_k)^2 + \Delta \tilde{u}_k \frac{\partial (x, \nabla \tilde{u}_k)}{\partial \nu} - (x, \nabla \tilde{u}_k) \frac{\partial \Delta \tilde{u}_k}{\partial \nu} \right) d\sigma \\ &= \int_{\partial B_1(0)} \left(\frac{(x,\nu)}{2}(\Delta \tilde{u})^2 + \Delta \tilde{u} \frac{\partial (x, \nabla \tilde{u})}{\partial \nu} - (x, \nabla \tilde{u}) \frac{\partial \Delta \tilde{u}}{\partial \nu} \right) d\sigma + o(1) \\ &= -4\pi^2 a^2 + o(1) \end{aligned} \tag{53}$$

where $o(1) \to 0$ when $k \to +\infty$. On the other hand, using (6), we have that

$$\int_{B_{r_k}(0)} x^i \partial_i u_k \Delta^2 u_k \, dx = \int_{B_{r_k}(0)} x^i \partial_i u_k e^{4u_k} \, dx + \int_{B_{r_k}(0)} (V_k - 1) x^i \partial_i u_k e^{4u_k} \, dx. \tag{54}$$

It follows from Lemma 4.2 and (48) that there exists $C > 0$ such that

$$|x^i \partial_i u_k(x)| \leq C \tag{55}$$

for all $x \in B_{r_k}(0)$. Properties (5), (7) and (55) yield

$$\lim_{k \to +\infty} \int_{B_{r_k}(0)} (V_k - 1) x^i \partial_i u_k e^{4u_k} \, dx = 0. \tag{56}$$

Plugging (53) and (56) into (52) and (54), we get that

$$\int_{B_{r_k}(0)} x^i \partial_i u_k e^{4u_k} \, dx = -4\pi^2 a^2 + o(1)$$

when $k \to +\infty$. Integrating by parts, we get that

$$\begin{aligned} -4\pi^2 a^2 &= \int_{B_{r_k}(0)} x^i \partial_i \frac{e^{4u_k}}{4}\, dx + o(1) \\ &= -\int_{B_{r_k}(0)} e^{4u_k}\, dx + \int_{\partial B_{r_k}(0)} \frac{(x,\nu)}{4} e^{4u_k}\, d\sigma + o(1) \\ &= -\int_{B_{r_k}(0)} e^{4u_k}\, dx + r_k^4 e^{4u_k(r_k)} \int_{\partial B_1(0)} \frac{(x,\nu)}{4} e^{4\tilde{u}_k}\, d\sigma + o(1). \quad (57) \end{aligned}$$

With (42) and (49), we then get that

$$\int_{B_{r_k}(0)} e^{4u_k}\, dx = 4\pi^2 a^2 + o(1)$$

where $o(1) \to 0$ when $k \to +\infty$. Since (38) and (41) hold, we then get that $|a|^2 \geq 4$. Since $a \geq 1$, we get that $a \geq 2$, and the claim is proved.

Step 6.3.7: We let $\delta \in (0,1)$. We claim that

$$\lim_{k \to +\infty} \sup_{[r_k, \delta]} u_k = -\infty.$$

Indeed, it follows from (44) and (49) that $r\tilde{u}'(r) = -a + (a-1)r^2$. Since $a > 1$, we get that $\tilde{u}$ is decreasing on $\left(0, \sqrt{\frac{a}{a-1}}\right)$ and increasing on $\left(\sqrt{\frac{a}{a-1}}, +\infty\right)$. It follows from the study of the monotonicity of u_k provided in Case 3.1.3 of Step 3.1 that there exists $\tau_k \in (0,\delta)$ such that u_k decreases on $(0,\tau_k)$ and increases on (τ_k, δ). Since (44) and (49) hold and since the monotonicity of $\tilde{u}$ changes at $\sqrt{\frac{a}{a-1}}$, we get that

$$\lim_{k \to +\infty} \frac{\tau_k}{r_k} = \sqrt{\frac{a}{a-1}}. \quad (58)$$

We let $y_k \in \overline{B}_\delta(0) \setminus B_{r_k}(0)$ such that

$$\sup_{\overline{B}_\delta(0) \setminus B_{r_k}(0)} u_k = u_k(y_k).$$

We distinguish two cases:

Case 6.3.7.1: we assume that $\lim_{k \to +\infty} \frac{|y_k|}{r_k} = +\infty$. Then with (58), we get that u_k increases on $[\tau_k, \delta]$, and then $u_k(y_k) \leq u_k(\delta)$. With Lemma 6.2, we then get that $\lim_{k \to +\infty} u_k(y_k) = -\infty$.

Case 6.3.7.2: we assume that $|y_k| = O(r_k)$ when $k \to +\infty$. We let $z_k = \frac{y_k}{r_k}$. Since $|y_k| \geq r_k$, we get that, up to a subsequence, $\lim_{k \to +\infty} z_k = z_\infty \neq 0$. With (43), (49) and Case 6.3.7.1, we get that

$$\begin{aligned} u_k(y_k) &= u_k(y_k) - u_k(\tau_k) + u_k(\tau_k) \\ &\leq \tilde{u}_k(z_k) - \tilde{u}_k\left(\frac{\tau_k}{r_k}\right) + u_k(\tau_k) \leq O(1) + u_k(\delta) \end{aligned}$$

and then with Lemma 6.2, we get that $\lim_{k \to +\infty} u_k(y_k) = -\infty$. This proves the claim.

In particular, this proves Lemma 6.3. □

Step 6.4: With the same kind of arguments as above, the following monotonicity holds (we omit the proof):

Lemma 6.4. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that (12) holds. We assume that there exists $\delta_0 \in (0,1)$ such that*

$$\lim_{k\to+\infty} \mu_k^2 \Delta u_k(\delta_0) = 0.$$

We let $\delta \in (0,1)$ and $\eta \in (1,2)$. Then there exists $R_\eta > 0$, there exists $(r_k)_{k\in\mathbb{N}} \in \mathbb{R}_{>0}$ such that $r_k \in [0,\delta]$ for all $k \in \mathbb{N}$ and

(i) $\lim_{k\to+\infty} \frac{r_k}{\mu_k} = +\infty$,

(ii) $r \mapsto r^\eta e^{u_k(r)}$ *is decreasing on* $[R_\eta \mu_k, r_k]$,

(iii) $u_k \to -\infty$ *uniformly on* $\overline{B}_\delta(0) \setminus B_{r_k}(0)$.

Step 6.4 We are in position to get the energy estimate for e^{4u_k}.

Lemma 6.5. *Let $(V_k)_{k\in\mathbb{N}} \in C^0(B)$ and $(u_k)_{k\in\mathbb{N}} \in C^4(B)$ such that (5), (6) and (7) hold. We assume that u_k is radially symmetrical for all $k \in \mathbb{N}$. We assume that (12) holds. We assume that there exists $\delta_0 \in (0,1)$ such that*

$$\lim_{k\to+\infty} \mu_k^2 \Delta u_k(\delta_0) = 0.$$

Then for any $\delta \in (0,1)$, we have that:

$$\lim_{k\to+\infty} \int_{B_\delta(0)} V_k e^{4u_k}\, dx = 16\pi^2.$$

In particular, $V_k e^{4u_k} \rightharpoonup 16\pi^2 \delta_0$ when $k \to +\infty$ in the sense of measures.

Proof of Lemma 6.5: We prove the claim. We choose $\eta \in (1,2)$ and $R_\eta > 0$, $(r_k)_{k\in\mathbb{N}}$ as in Lemma 6.4. We let $R > R_\eta$. It follows from Lemmae 6.1 and 6.4 that

$$\int_{B_\delta(0)\setminus B_{R\mu_k}(0)} e^{4u_k}\, dx \leq \int_{B_{r_k}(0)\setminus B_{R\mu_k}(0)} \frac{(R\mu_k)^{4\eta} e^{4u_k(R\mu_k)}}{r^{4\eta}}\, dx + o(1)$$
$$\leq C (R\mu_k)^{4\eta} e^{4u_k(R\mu_k)} \int_{R\mu_k}^{\delta} r^{3-4\eta}\, dr + o(1) \leq C \frac{R^4 \mu_k^4 e^{4u_k(R\mu_k)}}{\eta - 1} + o(1)$$
$$\leq \frac{C}{\eta-1} \left(\frac{\sqrt{96} R}{\sqrt{96} + R^2} \right)^4 + o(1)$$

where $\lim_{k\to+\infty} o(1) = 0$. Summing this integral and (38), letting $k \to +\infty$ and then $R \to +\infty$, we get the result. This proves Lemma 6.5. □

Point *(ii.a)* of Theorem 1.1 follows from Lemma 6.5.

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