# QUANTIZATION EFFECTS FOR A FOURTH ORDER EQUATION OF EXPONENTIAL GROWTH IN DIMENSION FOUR

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Abstract. We investigate the asymptotic behavior as $k \to +\infty$ of sequences $(u_k)_{k\in\mathbb{N}} \in C^4(\Omega)$ of solutions of the equations $\Delta^2 u_k = V_k e^{4u_k}$ on $\Omega$, where $\Omega$ is a bounded domain of $\mathbb{R}^4$ and $\lim_{k\to+\infty} V_k = 1$ in $C^0_{loc}(\Omega)$. The corresponding 2-dimensional problem was studied by Brézis-Merle and Li-Shafrir who pointed out that there is a quantization of the energy when blow-up occurs. As shown by Adimurthi, Struwe and the author [1], such a quantization does not hold in dimension four for the problem in its full generality. We prove here that under natural hypothesis on $\Delta u_k$, we recover such a quantization as in dimension 2.

## 1. Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^4$. Let a sequence $(V_k)_{k\in\mathbb{N}} \in C^0(\Omega)$ be such that

$$\lim_{k\to+\infty} V_k = 1 \tag{1.1}$$

in $C^0_{loc}(\Omega)$. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of functions in $C^4(\Omega)$ such that

$$\Delta^2 u_k = V_k e^{4u_k} \tag{E}$$

in $\Omega$ for all $k \in \mathbb{N}$. Here and in the sequel, $\Delta = -\sum \partial_{ii}$ is the Laplacian with minus sign convention. In this paper, we address the question of the asymptotics of the $u_k$'s when $k \to +\infty$. A natural (and simple) behavior is when there exists $u \in C^4(\Omega)$ such that

$$\lim_{k\to+\infty} u_k = u \tag{1.2}$$

in $C^3_{loc}(\Omega)$. In this situation, we say that $(u_k)_{k\in\mathbb{N}}$ is relatively compact in $C^3_{loc}(\Omega)$. However, the structure of equation $(E)$ is much richer due to its scaling invariance properties. The scaling invariance is as follows. Given $k \in \mathbb{N}$, $x_k \in \Omega$ and $\mu_k > 0$ , we let

$$\tilde{u}_k(x) := u_k(x_k + \mu_k x) + \ln \mu_k \tag{1.3}$$

for all $x \in \mu_k^{-1}(\Omega - x_k)$. Letting $\tilde{V}_k(x) = V_k(x_k + \mu_k x)$ for all $x \in \mu_k^{-1}(\Omega - x_k)$, we get that the rescaled function $\tilde{u}_k$ satisfies

$$\Delta^2 \tilde{u}_k = \tilde{V}_k e^{4\tilde{u}_k}$$

on $\mu_k^{-1}(\Omega - x_k)$ – an equation like $(E)$. This scaling invariance forces some situations more subtle that (1.2) to happen. A very simple example is the following: we

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consider a sequence $(\mu_k)_{k\in\mathbb{N}} \in \mathbb{R}_+^\star$ such that $\lim_{k\to+\infty}\mu_k = 0$ and for any $k\in\mathbb{N}$, we define the function

$$f_k(x) = \ln \frac{\sqrt{96}\mu_k}{\sqrt{96}\mu_k^2 + |x|^2}$$

for all $x \in \mathbb{R}^4$. Then $f_k$ satisfies $(E)$ with $V_k \equiv 1$ for all $k\in\mathbb{N}$. The sequence $(f_k)_{k\in\mathbb{N}}$ does not converge in $C^0_{loc}(\mathbb{R}^4)$: we have that

$$\lim_{k\to+\infty} f_k(0) = +\infty \text{ and } \lim_{k\to+\infty} f_k = -\infty \text{ uniformly locally on } \mathbb{R}^4\setminus\{0\}.$$

In addition, we get that

$$V_k e^{4f_k}\,dx \rightharpoonup 16\pi^2\delta_0$$

when $k\to+\infty$ weakly for the convergence of measures. Scaling as in (1.3), we get that

$$\lim_{k\to+\infty} f_k(\mu_k x) - f_k(0) = \ln\frac{\sqrt{96}}{\sqrt{96}+|x|^2}$$

for all $x\in\mathbb{R}^4$. Concerning terminology, we say that the sequence $(u_k)_{k\in\mathbb{N}}$ blows-up if it is not relatively compact in $C^3_{loc}(\Omega)$, so that, up to any subsequence, (1.2) does not hold. In the above example, the $(f_k)$'s blow up. In this paper, we are mainly concerned with the blow-up behavior of solutions of $(E)$.

In dimension two, the corresponding problem has been studied (among others) by Brézis-Merle [3] and Li-Shafrir [8]. We also refer to Druet [5] and Adimurthi-Struwe [2] for the description of equations with more intricate nonlinearities and to Tarantello [14] for equations with singularities. Li and Shafrir proved the following:

**Theorem 1.1** (Li-Shafrir [8]). *Let $\Sigma$ be a bounded domain of $\mathbb{R}^2$, $(\bar{V}_k)_{k\in\mathbb{N}}\in C^0(\Sigma)$ be a sequence of functions such that $\lim_{k\to+\infty}\bar{V}_k = 1$ in $C^0_{loc}(\Sigma)$, and $(\bar{u}_k)_{k\in\mathbb{N}}\in C^2(\Sigma)$ be a sequence such that*

$$\Delta\bar{u}_k = \bar{V}_k e^{2\bar{u}_k}$$

*in $\Sigma$ for all $k\in\mathbb{N}$, and such that there exists $\Lambda\in\mathbb{R}$ such that $\int_\Sigma \bar{V}_k e^{2\bar{u}_k}\,dx \le \Lambda$ for all $k\in\mathbb{N}$. Then either (i) the sequence $(u_k)_{k\in\mathbb{N}}$ is relatively compact in $C^1(\Omega)$, or (ii) there exists $N\in\mathbb{N}$, there exist $\bar{x}_1,...,\bar{x}_N\in\Omega$, there exist $\bar{\alpha}_1,...,\bar{\alpha}_N\in\mathbb{N}^\star$ such that, up to a subsequence*

$$\bar{V}_k e^{2\bar{u}_k} \rightharpoonup \sum_{i=1}^N 4\pi\bar{\alpha}_i\delta_{\bar{x}_i}$$

*weakly for the convergence of measures when $k\to+\infty$. Moreover, $\lim_{k\to+\infty}\bar{u}_k = -\infty$ uniformly locally in $\Sigma\setminus\{\bar{x}_1,...,\bar{x}_N\}$.*

We refer to this statement as a quantization result. The justification of this terminology is as follows: if in Theorem 1.1 we have blow-up (that is case (i) does not hold), then for any $\omega\subset\subset\Sigma$ such that $\partial\omega\cap\{\bar{x}_1,...,\bar{x}_N\}=\emptyset$, we have that $\lim_{k\to+\infty}\int_\omega \bar{V}_k e^{2\bar{u}_k}\,dx \in 4\pi\mathbb{N}$. Moreover, the sequence $(u_k)_{k\in\mathbb{N}}$ develop singularities on a set at most finite, that is $\{\bar{x}_1,...,\bar{x}_N\}$.

Surprisingly, such a quantization result is false when we come back to our initial four-dimensional problem $(E)$. In a joint work with Adimurthi and Michael Struwe [1], we exhibit a sequence of solutions to $(E)$ that blows-up, carry a non-quantified energy and develop singularities on a hypersurface of $\mathbb{R}^4$. In [1], we described the behaviour of arbitrary solutions to $(E)$ and proved that any blowing-up sequence $(u_k)_{k\in\mathbb{N}}$ concentrates at the zero set of a nonpositive bi-harmonic function, and that outside this set, $\lim_{k\to+\infty} u_k = -\infty$ uniformly. In view of the examples provided in [1], this result is optimal. Therefore, giving a more precise description requires additional hypothesis on $(u_k)$.

A natural hypothesis is to impose a Navier boundary condition, (that is $u_k = \Delta u_k = 0$ on $\partial\Omega$) or a Dirichlet condition (that is $u_k = \frac{\partial u_k}{\partial \nu} = 0$ on $\partial\Omega$): actually, in these cases, we get that there is no blow-up and we recover relative compactness. Wei [15] studied a problem similar to $(E)$ assuming that $\Delta u_k = 0$ on $\partial\Omega$ and $u_k = c_k$ on $\partial\Omega$, where $(c_k)_{k\in\mathbb{N}} \in \mathbb{R}$ is a sequence of real numbers such that $\lim_{k\to+\infty} c_k = -\infty$: in this context, Wei describes precisely the asymptotics and recovers quantization. Another natural hypothesis is to assume that the functions $u_k$ are radially symmetrical: in this situation, we describe completely the asymptotics in [12]. In all these situations, the critical quantity to observe happens to be $\Delta u_k$ as shown in the following example. We let $\alpha \in (0, 16\pi^2)$. It follows from [4] that there exists $v \in C^4(\mathbb{R}^4)$ radially symmetrical such that $v \le v(0) = 0$ and

$$\Delta^2 v = e^{4v} \text{ in } \mathbb{R}^4 \text{ and } \int_{\mathbb{R}^4} e^{4v}\, dx = \alpha.$$

Contrary to the two-dimensional case, where the only solutions to the corresponding equation are of a type similar to $f_k$ with a quantization of the energy, we get in four dimensions many solutions with arbitrary small energy. More precisely, it follows from [9] that there exists $C > 0$ such that $\Delta v(x) \ge C$ for all $x \in \mathbb{R}^4$. For any $k \in \mathbb{N}^\star$, we define the function

$$g_k(x) = v(kx) + \ln k$$

for all $x \in \mathbb{R}^4$. As easily checked, due to the scaling invariance (1.3) of $(E)$, $g_k$ verifies $(E)$ with $V_k \equiv 1$. We also get that the sequence $(g_k)_{k\in\mathbb{N}}$ blows up. It follows from straightforward computations that

$$\lim_{k\to+\infty} \int_{B_1(0)} V_k e^{4g_k}\, dx = \alpha.$$

Moreover, for any $\omega \subset \mathbb{R}^4$, we have that

$$\lim_{k\to+\infty} \Delta g_k = +\infty$$

uniformly in $\omega$. Since $\alpha > 0$ can be chosen as small as we want, we then get blowing-up sequences with arbitrary positive small energy, and there is no quantization here. Note that concerning the sequence $(f_k)_{k\in\mathbb{N}}$ of the first example, we have that for any $\omega \subset\subset \mathbb{R}^4 \setminus \{0\}$, there exists $C(\omega) > 0$ such that

$$|\Delta f_k(x)| \le C(\omega)$$

for all $k \in \mathbb{N}^\star$ and all $x \in \omega$. The fundamental difference between the $(f_k)$'s and the $(g_k)$'s is that in the first case, the Laplacian is bounded outside the singularity, and in the second case, the Laplacian goes to $+\infty$ uniformly. This fact is actually general. The objective of this paper is to prove the following result:

**Theorem 1.2.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^4$, $(V_k)_{k\in\mathbb{N}} \in C^0(\Omega)$ be a sequence such that (1.1) holds, and $(u_k)_{k\in\mathbb{N}}$ be a sequence of functions in $C^4(\Omega)$ such that $(E)$ holds, and such that there exists $\Lambda > 0$ such that $\int_\Omega V_k e^{4u_k}\, dx \le \Lambda$ for all $k \in \mathbb{N}$. Assume there exist $C > 0$ and $\omega_0 \subset\subset \Omega$ such that $\|(\Delta u_k)_-\|_1 \le C$ and*

$$\|\Delta u_k\|_{L^1(\omega_0)} \le C$$

*for $k \in \mathbb{N}$. Then (i) either $(u_k)_{k\in\mathbb{N}}$ is relatively compact in $C^3_{loc}(\Omega)$, or (ii) there exists $N \in \mathbb{N}$, there exist $x_1, ..., x_N \in \Omega$, there exist $\alpha_1, ..., \alpha_N \in \mathbb{N}^\star$ such that*

$$V_k e^{4u_k} \rightharpoonup \sum_{i=1}^{N} 16\pi^2 \alpha_i \delta_{x_i}$$

*weakly in the sense of measures when $k \to +\infty$ up to a subsequence. Moreover, still in Case (ii), we have that $\lim_{k\to+\infty} u_k = -\infty$ uniformly locally in $\Omega \setminus \{x_1, ..., x_N\}$.*

As a remark, note that the control of the positive part of $\Delta u_k$ is only required on an arbitrary subdomain of $\Omega$. This result is optimal as shown in the preceding example involving the function $g_k$. In a joint work with Olivier Druet [6], we studied the corresponding problem on four-dimensional Riemannian manifolds, where the bi-Laplacian is replaced by a fourth-order elliptic operator refered to as $P$: when the kernel of $P$ is such that $Ker\, P = \{constants\}$, we get similar results as in Theorem 1.2 with the additional information that $\alpha_i = 1$ for all $i \in \{1, ..., N\}$. The techniques used in [6] are different from the techniques used here: the main reason is that for equation $(E)$, the kernel of the bi-Laplacian contains more than the constant functions. Related references in the context of Riemannian manifolds are Malchiodi [10] and Malchiodi-Struwe [11]. As a remark, the corresponding question in dimension $n \geq 5$ was considered in Hebey-Robert [7].

This paper is organized as follows. In section 2, we prove that under our hypothesis, concentration holds on finitely many points and not on a hypersurface. In section 3, we prove that, up to rescaling, the $u_k$'s converge to a generic pattern when $k \to +\infty$. In section 4, we analyse precisely the blow-up and we prove Theorem 1.2 in section 5. In the sequel, $C$ denotes a positive constant, with value allowed to change from one line to the other. Note also that all the convergence results are up to a subsequence, even when it is not precised.

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## 2. Construction of the concentration points

In the sequel, we let $\Omega$ be a bounded domain of $\mathbb{R}^4$. We consider a sequence $(V_k)_{k\in\mathbb{N}} \in C^0(\Omega)$ such that (1.1) holds. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of functions in $C^4(\Omega)$ such that $(E)$ holds. We assume that there exists $\Lambda > 0$ such that

$$\int_\Omega V_k e^{4u_k}\, dx \leq \Lambda \tag{2.1}$$

for all $k \in \mathbb{N}$. We assume that there exist $\omega_0 \subset\subset \Omega$ and $C > 0$ such that

$$\|\Delta u_k\|_{L^1(\omega_0)} \leq C \tag{2.2}$$

and

$$\|(\Delta u_k)_-\|_1 \leq C \tag{2.3}$$

for all $k \in \mathbb{N}$. The objective of this section is to prove that the $(u_k)$'s concentrate at a finite number of points. This is the object of the following proposition:

**Proposition 2.1.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^4$. Let $(V_k)_{k\in\mathbb{N}} \in C^0(\Omega)$ be a sequence such that (1.1) holds. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of functions in $C^4(\Omega)$*

*such that (E) holds. We assume that there exists $\Lambda > 0$ such that (2.1) holds. We assume that (2.3) and (2.2) hold. We define*

$$S_0 := \left\{ x \in \Omega / \liminf_{\delta \to 0} \liminf_{k \to +\infty} \int_{B_\delta(x)} V_k e^{4u_k}\, dy \geq 8\pi^2 \right\}. \tag{2.4}$$

*which is a finite set. Then for any $\omega \subset\subset \Omega \setminus S_0$, there exists $C(\omega) > 0$ such that*

$$|\Delta u_k(x)| \leq C(\omega) \text{ and } u_k(x) \leq C(\omega)$$

*for all $x \in \omega$ and all $k \in \mathbb{N}$. More precisely, we are in one and only one of the following situations:*

*(A1) there exists $u \in C^4(\Omega \setminus S_0)$ such that*

$$\lim_{k \to +\infty} u_k = u \text{ in } C^3_{loc}(B_\delta(x_0))$$

*(A2) $\lim_{k \to +\infty} u_k = -\infty$ uniformly locally on $\Omega \setminus S_0$.*

The proof of Proposition 2.1 proceeds in two steps. Note that it follows from (2.1) that $S_0$ is at most finite. We let $(u_k)_{k\in\mathbb{N}}$, $(V_k)_{k\in\mathbb{N}}$ and $\Lambda$ as in the statement of Proposition 2.1.

**Step 2.1:** We let $x_0 \in \Omega \setminus S_0$. We claim that there exists $\delta > 0$ such that we are in one and only one of the following situations:

(B1) there exists $u \in C^4(B_\delta(x_0))$ such that $\lim_{k\to+\infty} u_k = u$ in $C^3_{loc}(B_\delta(x_0))$.

(B2) there exists $\phi \in C^4(B_\delta(x_0))$ such that $\Delta^2\phi = 0$, $\phi \leq 0$, $\phi \not\equiv 0$ there exists a sequence $(\beta_k)_{k\in\mathbb{N}} \in \mathbb{R}_+^\star$ such that $\lim_{k\to+\infty} \beta_k = +\infty$ and

$$\lim_{k \to +\infty} \frac{u_k}{\beta_k} = \phi$$

in $C^3_{loc}\left(B_\delta(x_0) \cap \{\phi < 0\}\right)$.

*Proof of the claim:* This claim is a particular case of the Theorem obtained in [1]. As a preliminary remark, note that the two cases (B1) and (B2) are disjoint. Since $x_0 \in \Omega \setminus S_0$, we let $\delta > 0$ and $\alpha < 8\pi^2$ be such that

$$\int_{B_\delta(x_0)} V_k e^{4u_k}\, dx \leq \alpha < 8\pi^2$$

for all $k \in \mathbb{N}$. We let $w_k$ be such that

$$\Delta^2 w_k = V_k e^{4u_k} \text{ in } B_\delta(x_0), \quad w_k = \Delta w_k = 0 \text{ on } \partial B_\delta(x_0). \tag{2.5}$$

It follows from [9] (see also [1], [15]) that there exists $p > 1$ such that

$$\int_{B_\delta(x_0)} e^{4p|w_k|}\, dx \leq C \tag{2.6}$$

for all $k \in \mathbb{N}$. We let $h_k := u_k - w_k$ on $B_\delta(x_0)$. Clearly $\Delta^2 h_k = 0$. It follows from (2.1) and (2.6) that $\|(h_k)_+\|_{L^1(B_\delta(x_0))} = O(1)$ when $k \to +\infty$. We distinguish two situations:

*Case 2.1.1:* We assume that $\|h_k\|_{L^1(B_{\delta/2}(x_0))} = O(1)$ when $k \to +\infty$. Since $h_k$ is bi-harmonic, there exists $h_\infty \in C^4(B_\delta(x_0))$ such that

$$\lim_{k \to +\infty} h_k = h_\infty \tag{2.7}$$

in $C^4_{loc}(B_\delta(x_0))$. We refer to [1] for details about this assertion. Plugging (2.6) and (2.7) in (2.5), we get that $(w_k)_{k\in\mathbb{N}}$ is bounded in $C^0_{loc}(B_\delta(x_0))$, and so is $(u_k)_{k\in\mathbb{N}}$.

It then follows from standard elliptic theory that there exists $u \in C^4(B_\delta(x_0))$ such that $\lim_{k\to+\infty} u_k = u$ in $C^3_{loc}(B_\delta(x_0))$, and we recover Case (B1) of the claim. This proves the claim in Case 2.1.1.

*Case 2.1.2:* We assume that $\lim_{k\to+\infty} \|h_k\|_{L^1(B_{\delta/2}(x_0))} = +\infty$. Since $h_k$ is biharmonic, there exists $\phi \in C^4(B_\delta(x_0)) \setminus \{0\}$ such that $\Delta^2\phi = 0$, $\phi \leq 0$, there exists a sequence $(\beta_k)_{k\in\mathbb{N}} \in \mathbb{R}_+^\star$ such that $\lim_{k\to+\infty} \beta_k = +\infty$ and such that

$$\lim_{k\to+\infty} \frac{h_k}{\beta_k} = \phi \tag{2.8}$$

in $C^4_{loc}(B_\delta(x_0))$. We refer to [1] for details about this assertion. In particular, $h_k \to -\infty$ uniformly locally on $\phi < 0$. Arguing as in Case 2.1.1, we then obtain that $(w_k)_{k\in\mathbb{N}}$ converges in $C^3_{loc}(B_\delta(x_0) \cap \{\phi < 0\})$. It then follows from (2.8) that $\lim_{k\to+\infty} \frac{u_k}{\beta_k} = \phi$ in $C^3_{loc}(B_\delta(x_0) \cap \{\phi < 0\})$, and we recover Case (B2) of the claim. This proves the claim in Case 2.1.2. □

**Step 2.2:** We are in position to prove Proposition 2.1. Since $\Omega \setminus S_0$ is connected and harmonic functions are analytic, it follows from Step 2.1 that we are in one and only one of the following situations:

*Case 2.2.1:* There exists $u \in C^4(\Omega \setminus S_0)$ such that $\lim_{k\to+\infty} u_k = u$ in $C^3_{loc}(\Omega \setminus S_0)$. In this situation, we recover Case (A1) of Proposition 2.1.

*Case 2.2.2:* There exists $\phi \in C^4(\Omega \setminus S_0)$ such that $\Delta^2\phi = 0$, $\phi \leq 0$, $\phi \not\equiv 0$, there exists a sequence $(\beta_k)_{k\in\mathbb{N}} \in \mathbb{R}_+^\star$ such that $\lim_{k\to+\infty} \beta_k = +\infty$ and

$$\lim_{k\to+\infty} \frac{u_k}{\beta_k} = \phi \text{ in } C^3_{loc}(\Omega \cap \{\phi < 0\} \setminus S_0). \tag{2.9}$$

We claim that $\Delta\phi \equiv 0$. Indeed, there exists $x \in \omega_0$ ($\omega_0$ was defined in (2.2)) such that $\phi(x) < 0$ (otherwise $\phi \equiv 0$ on $\omega_0$ and then $\phi \equiv 0$ on $\Omega \setminus S_0$ since harmonic fonctions are analytic. A contradiction). We then get that (2.9) holds in a neighborhood of $x_0$. By (2.2), we then get that $\Delta\phi = 0$ in a neighborhood of $x$. Since $\Delta\phi$ is harmonic, and therefore analytic, we get that $\Delta\phi \equiv 0$ on $\Omega \setminus S_0$. This proves the claim.

Since $\phi \not\equiv 0$, $\phi \leq 0$ and $\Delta\phi = 0$, it follows from the maximum principle that $\phi < 0$ on $\Omega \setminus S_0$. Consequently,

$$\lim_{k\to+\infty} \frac{u_k}{\beta_k} = \phi \text{ in } C^3_{loc}(\Omega \setminus S_0).$$

In particular, we get that $\lim_{k\to+\infty} u_k = -\infty$ uniformly locally on $\Omega \setminus S_0$. From the equation $(E)$ and (2.3), it follows from elliptic theory that either $\lim_{k\to+\infty} \Delta u_k = +\infty$ uniformly locally in $\Omega \setminus S_0$, or $(\Delta u_k)_{k\in\mathbb{N}}$ is uniformly bounded when $k \to +\infty$ locally in $\Omega \setminus S_0$: it follows from hypothesis (2.2) that the first situation cannot hold, and we get that Case (A2) of Proposition 2.1 holds.

Clearly Proposition 2.1 is a consequence of Steps 2.1 and 2.2.

## 3. Pointwise estimates

This section is devoted to the proof of the following Proposition:

**Proposition 3.1.** *Let $\Omega$ be a bounded domain of $\mathbb{R}^4$. We let $(V_k)_{k\in\mathbb{N}} \in C^0(\Omega)$ be a sequence such that (1.1) holds. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence of functions in $C^4(\Omega)$ such that $(E)$ holds. We assume that there exists $\Lambda > 0$ such that (2.1) holds. We assume that (2.3) and (2.2) hold. We assume that*

$$S_0 \neq \emptyset.$$

*Then there exists $N \in \mathbb{N}^\star$, there exists families of points $(x_{1,k})_{k\in\mathbb{N}}, ..., (x_{N,k})_{k\in\mathbb{N}}$ in $\Omega$ such that for all $i \in \{1, ..., N\}$, we have that $\lim_{k\to+\infty} x_{i,k} = x_i \in S_0$ and such that for any $\omega \subset\subset \Omega$, there exists $C(\omega) > 0$ such that*

$$(\inf_{i\in\{1,...,N\}} |x - x_{i,k}|)e^{u_k(x)} \leq C(\omega) \text{ and } (\inf_{i\in\{1,...,N\}} |x - x_{i,k}|)^2|\Delta u_k(x)| \leq C(\omega)$$

*for all $k \in \mathbb{N}$ and all $x \in \omega$. Moreover,*

$$\lim_{k\to+\infty} \frac{|x_{i,k} - x_{j,k}|}{e^{-u_k(x_{i,k})}} = +\infty \text{ for all } i \neq j,$$

*and for any $i \in \{1, ..., N\}$ and any $x \in \mathbb{R}^4$, we have that*

$$\lim_{k\to+\infty} (u_k(x_{i,k} + e^{-u_k(x_{i,k})}x) - u_k(x_{i,k})) = \ln \frac{\sqrt{96}}{\sqrt{96} + |x|^2}.$$

*Moreover, this convergence holds in $C^3_{loc}(\mathbb{R}^4)$. In addition, $\lim_{k\to+\infty} u_k = -\infty$ uniformly on every compact subset of $\Omega \setminus S_0$.*

This section is devoted to the proof of Proposition 3.1. We consider $\omega \subset\subset \Omega$. Up to taking $\omega$ larger, we assume that $S_0 \subset \omega$. We follow the proof of [13]. We let $x_k \in \overline{\omega}$ be such that

$$u_k(x_k) = \sup_\omega u_k.$$

Since $S_0 \neq \emptyset$ and $S_0 \subset \omega$, we get that $\lim_{k\to+\infty} u_k(x_k) = +\infty$. In this situation, it follows from Proposition 2.1 that $\lim_{k\to+\infty} x_k = x_0 \in S_0 \cap \overline{\omega} = S_0 \cap \omega$. We let $\delta > 0$ small be such that

$$B_{2\delta}(x_k) \subset \omega \text{ and } B_{2\delta}(x_k) \cap S_0 = \{x_0\}$$

for all $k \in \mathbb{N}$. We define

$$\mu_k := e^{-u_k(x_k)} \text{ and } v_k(x) := u_k(x_k + \mu_k x) - u_k(x_k) \tag{3.1}$$

and $\tilde{V}_k(x) := V_k(x_k + \mu_k x)$ for $|x| < \frac{2\delta}{\mu_k}$ and all $k \in \mathbb{N}$. Equation $(E)$ yields

$$\Delta^2 v_k = \tilde{V}_k e^{4v_k}, \tag{3.2}$$

with $v_k(x) \leq v_k(0) = 0$.

**Step 3.1:** We claim that there existe $C > 0$ independant of $k$ and $R$ such that

$$\int_{B_R(0)} |\Delta v_k|\, dx \leq CR^2 + CR^4\mu_k^2 \tag{3.3}$$

for all $k \in \mathbb{N}$ and all $R < \delta\mu_k^{-1}$. We prove the claim. We let $G_{\delta,k}$ be the Green's function for the Laplacian on $B_\delta(x_k)$ with Dirichlet boundary condition. We get that

$$\Delta u_k(z) = \int_{B_\delta(x_k)} G_{\delta,k}(z, y)\Delta^2 u_k(y)\, dy + \varphi_k(z)$$

for all $z \in B_\delta(x_k)$, where $\varphi_k$ is the unique harmonic function on $B_\delta(x_k)$ such that $\varphi_k(y) = \Delta u_k(y)$ for all $y \in \partial B_\delta(x_k)$. From Proposition 2.1 and the comparison principle, we get that there exists $C > 0$ such that

$$|\varphi_k(z)| \leq C \tag{3.4}$$

for all $z \in B_\delta(x_k)$. We let $x \in \mathbb{R}^4$ be such that $|x| < \delta\mu_k^{-1}$. Using the definition (3.1) of $v_k$, we get that

$$\Delta v_k(x) = \int_{B_\delta(x_k)} \mu_k^2 G_{\delta,k}(x_k + \mu_k x, y)\Delta^2 u_k(y)\, dy + \mu_k^2 \varphi_k(x_k + \mu_k x).$$

Integrating this equation, using $(E)$, (2.1), (3.4) and standard estimates on the Green's function, we get that

$$\begin{aligned}\int_{B_R(0)} |\Delta v_k|\, dx &\leq C\int_{B_R(0)}\int_{B_\delta(x_k)} \mu_k^2 G_{\delta,k}(x_k+\mu_k x, y)e^{4u_k(y)}\, dy\, dx + CR^4\mu_k^2\\ &\leq C\int_{B_\delta(x_k)} e^{4u_k(y)}\left(\int_{B_R(0)} \frac{\mu_k^2}{|x_k+\mu_k x - y|^2}\, dx\right) dy + CR^4\mu_k^2\\ &\leq C\int_{B_\delta(x_k)} e^{4u_k(y)}\left(CR^2\right)\, dy \leq C\Lambda R^2 + CR^4\mu_k^2,\end{aligned}$$

for all $k \in \mathbb{N}$. This proves the claim.

**Step 3.2:** We claim that for any $x \in \mathbb{R}^4$, we have that

$$\lim_{k\to+\infty} v_k(x) = \ln\frac{\sqrt{96}}{\sqrt{96}+|x|^2} := U_0(x), \tag{3.5}$$

moreover this convergence holds in $C^3_{loc}(\mathbb{R}^4)$. We briefly prove the claim. By (3.3), we get that $\Delta v_k$ is bounded in $L^1_{loc}$ when $k \to +\infty$. Since $v_k \leq v_k(0) = 0$, it then follows from (3.2) and standard elliptic theory that, up to a subsequence, there exists $v \in C^4(\mathbb{R}^4)$ such that $\lim_{k\to+\infty} v_k = v$ in $C^3_{loc}(\mathbb{R}^4)$, with $\Delta^2 v = e^{4v}$ and $e^{4v} \in L^1(\mathbb{R}^4)$. Passing to the limit $k \to +\infty$ in (3.3) and using the classification of Lin [9], we get that $v \equiv U_0$. We refer to [13] for details about the proof. In particular, we get that

$$\lim_{R\to+\infty}\lim_{k\to+\infty}\int_{B_{R\mu_k}(x_k)} V_k e^{4u_k}\, dx = 16\pi^2.$$

**Step 3.3:** We claim that there exists $N \in \mathbb{N}^\star$, there exist $(x_{1,k})_{k\in\mathbb{N}}, ..., (x_{N,k})_{k\in\mathbb{N}}$ such that for all $\omega \subset\subset \Omega$, there exists $C(\omega) > 0$ such that

$$(\inf_{i\in\{1,...,N\}} |x - x_{i,k}|)e^{u_k(x)} \leq C(\omega) \tag{3.6}$$

for all $x \in \omega$ and all $k \in \mathbb{N}$. Here $x_{1,k} := x_k$.

*Proof of the claim:* Without loss of generality, we can assume that $S_0 \subset \omega$. If there exists $C(\omega) > 0$ such that $|x - x_k|e^{u_k(x)} \leq C(\omega)$ for all $k \in \mathbb{N}$ and all $x \in \omega$, then we are done. Otherwise, let $y_k \in \overline{\omega}$ be such that

$$\sup_{x\in\omega} |x - x_k|e^{u_k(x)} = |y_k - x_k|e^{u_k(y_k)} \to +\infty \tag{3.7}$$

when $k \to +\infty$. We define

$$\hat{u}_k(x) := u_k(y_k + \nu_k x) - u_k(y_k)$$

for all $x \in \nu_k^{-1}(\omega - y_k)$, where $\nu_k := e^{-u_k(y_k)}$ for all $k \in \mathbb{N}$. It follows from (3.7) that $\hat{u}_k$ is bounded from above uniformly locally on $\mathbb{R}^4$ independantly of $k$. We proceed as in Steps 3.1 and 3.2 and prove that $\hat{u}_k$ converges to $U_0$ in $C^3_{loc}(\mathbb{R}^4)$, and that these two rescaled functions do not interact one with the other. We then add another level of energy $16\pi^2$. If (3.6) holds with $x_{1,k} = x_k$ and $x_{2,k} = y_k$, then we are done. Otherwise, the process goes on and must cease, because when we have constructed $N$ points, we have that the energy $16\pi^2 N$ and by (2.1) we must have $16\pi^2 N \leq \Lambda$. We refer to [13] for the details. □

**Step 3.4:** We claim that

$$\lim_{k\to+\infty} u_k = -\infty$$

uniformly on every compact subset of $\Omega \setminus S_0$.

*Proof of the claim:* We prove the claim by contradiction and assume that the conclusion is false. It then follows that point (A1) of Proposition 2.1 holds, and then that $u_k$ is uniformly bounded in $C^3_{loc}(\Omega \setminus S_0)$. We let $x_0 \in S_0$ and $\delta > 0$ be such that $B_{2\delta}(x_0) \subset \Omega$ and $B_{2\delta}(x_0) \cap S_0 = \{x_0\}$. We let $x_k \in \Omega$ be such that $u_k(x_k) = \sup_{B_\delta(x_0)} u_k$ and we define $v_k$ and $\mu_k$ as in (3.1). As in the proof of Proposition 3.1, we get that $\lim_{k\to+\infty} x_k = x_0 \in \omega \cap S_0$ and that (3.5) holds. We let $H_\delta$ be the Green's function for $\Delta^2$ in $B_\delta(x_0)$ with Navier condition on the boundary, that is for any $x \in B_\delta(x_0)$, we have that $\Delta^2 H_\delta(x, \cdot) = \delta_x$ in $\mathcal{D}'(B_\delta(x_0))$ and $H_\delta(x, \cdot) = \Delta H_\delta(x, \cdot) = 0$ on $\partial B_\delta(x_0)$. For any $x \in B_\delta(x_0) \setminus \{x_0\}$, we then have that

$$u_k(x) = \int_{B_\delta(x_0)} H_\delta(x, y) V_k(y) e^{4u_k(y)}\, dy + \varphi_k(x)$$

where $\Delta^2 \varphi_k = 0$, $\varphi_k(y) = u_k(y)$ and $\Delta\varphi_k(y) = \Delta u_k(y)$ for $y \in \partial B_\delta(x_0)$. It follows from point (A1) of Proposition 2.1 and the comparison principle that $\varphi_k$ is uniformly bounded when $k \to +\infty$. We let $x \in B_\delta(x_0)$ such that $|x - x_0| > \alpha$. Since $H_\delta > 0$, we get with (1.1), $(E)$, a change of variable and (3.2) that

$$\begin{aligned} u_k(x) &\geq \int_{B_{R\mu_k}(x_k)} H_\delta(x, y) V_k(y) e^{4u_k(y)}\, dy - C \\ &\geq \int_{B_R(0)} H_\delta(x, x_k + \mu_k y) \tilde{V}_k(y) e^{4v_k(y)}\, dy - C \\ &\geq \int_{B_R(0)} H_\delta(x, x_0) \lim_{k\to+\infty} \left(\tilde{V}_k(y) e^{4v_k(y)}\right) dy - C \end{aligned}$$

for all $x \in B_{\delta/2}(x_0) \setminus \overline{B}_\alpha(x_0)$. With standard properties of $H_\delta$, we get that $H_\delta(x, x_0) \geq \frac{1}{8\pi^2} \ln \frac{1}{|x - x_0|} - C$ for $x \in B_{\delta/2}(x_0)$. From (3.5), we then get that for any $\epsilon \in (0, 2)$,

$$u_k(x) \geq (2 - \epsilon) \ln \frac{1}{|x - x_0|} - C'$$

for $x \in B_{\delta/2}(x_0) \setminus \overline{B}_\alpha(x_0)$, $x \neq x_0$ and $k$ large depending on $\alpha$ and $\epsilon$. We then get that for any $0 < \alpha < \beta$ small,

$$\Lambda \geq \int_{B_\beta(x_0)\setminus B_\alpha(x_0)} V_k e^{4u_k}\, dx \geq C \int_{B_\beta(x_0)\setminus B_\alpha(x_0)} \frac{1}{|x - x_0|^{8-4\epsilon}}\, dx,$$

for $k$ large depending on $\alpha$. We then get a contradiction by letting $\alpha \to 0$. Then Case (A1) of Proposition 2.1 does not hold and Case (A2) holds. We then get that

$\lim_{k\to+\infty} u_k = -\infty$ on compact subsets of $\Omega \setminus S_0$ when $k \to +\infty$. This proves the claim. □

**Step 3.5:** We claim that for any $\omega \subset\subset \Omega$, there exists $C(\omega) > 0$ such that

$$(\inf_{i\in\{1,\dots,N\}} |x - x_{i,k}|)^2 |\Delta u_k(x)| \leq C(\omega) \tag{3.8}$$

for all $x \in \omega$ and all $k \in \mathbb{N}$.

*Proof of the claim:* We let $x_0 \in S_0$ and $\delta > 0$ be such that $B_{3\delta}(x_0) \subset \Omega$ and $B_{3\delta}(x_0) \cap S_0 = \{x_0\}$. We denote $H_\delta$ the Green's function for $\Delta$ on $B_{2\delta}(x_0)$ with Dirichlet boundary condition. It follows from Green's representation formula that

$$\Delta u_k(x) = \int_{B_{2\delta}(x_0)} H_\delta(x, y)\Delta^2 u_k(y)\, dy + \psi_k(x) \tag{3.9}$$

for all $x \in B_{2\delta}(x_0)$. In this expression, $\psi_k$ is such that $\Delta\psi_k = 0$ in $B_{2\delta}(x_0)$ and $\psi_k(x) = \Delta u_k(x)$ on $\partial B_{2\delta}(x_0)$. It follows from Proposition 2.1 and the comparison principle that there exists $C_\delta > 0$ such that

$$|\psi_k(x)| \leq C_\delta \tag{3.10}$$

for all $x \in B_{2\delta}(x_0)$. We consider a sequence $(y_k)_{k\in\mathbb{N}} \in B_\delta(x_0)$ that converges. We assume that $\lim_{k\to+\infty} y_k = x_0$ when $k \to +\infty$. With standard properties of the Green's function, (3.9) and (3.10), we get that there exists $C > 0$ such that

$$|\Delta u_k(y_k)| \leq C \int_{B_{2\delta}(x_0)} \frac{e^{4u_k(y)}}{|y_k - y|^2}\, dy + C.$$

We define $R_k(x) = \inf_{i\in\{1,\dots,N\}} |x - x_{i,k}|$ for all $x \in \Omega$, we define $\theta_{i,k} = \frac{y_k - x_{i,k}}{|y_k - x_{i,k}|}$ and $\Omega_{i,k} = \{y \in B_{2\delta}(x_0)/R_k(y) = |y - x_{i,k}|\}$. By (2.1) and the pointwise estimate (3.6), we then get that

$$\begin{aligned}
|\Delta u_k(y_k)| &\leq C \int_{B_{2\delta}(x_0)\setminus\cup B_{\frac{|y_k - x_{i,k}|}{2}}(x_{i,k})} + C \int_{\cup B_{\frac{|y_k - x_{i,k}|}{2}}(x_{i,k})} + C\\
&\leq C \sum_{i=1}^N \int_{\Omega_{i,k}\setminus B_{\frac{|y_k - x_{i,k}|}{2}}(x_{i,k})} + C \sum_{i=1}^N \int_{B_{\frac{|y_k - x_{i,k}|}{2}}(x_{i,k})} + C\\
&\leq C \sum_{i=1}^N \int_{B_{2\delta}(x_0)\setminus B_{\frac{|y_k - x_{i,k}|}{2}}(x_{i,k})} \frac{1}{|y_k - y|^2 |x_{i,k} - y|^4}\, dy\\
&\quad + C \sum_{i=1}^N \int_{B_{\frac{|y_k - x_{i,k}|}{2}}(x_{i,k})} \frac{e^{4u_k(y)}}{R_k(y_k)^2}\, dy + C\\
&\leq C \sum_{i=1}^N \int_{\mathbb{R}^4\setminus B_{\frac{1}{2}}(0)} \frac{1}{R_k(y_k)^2 |\theta_{i,k} - z|^2 |z|^4}\, dz + \frac{C}{R_k(y_k)^2} + C
\end{aligned}$$

for all $k \in \mathbb{N}$ large enough, and then

$$R_k(y_k)^2 |\Delta u_k(y_k)| = O(1) \tag{3.11}$$

when $k \to +\infty$ in case $\lim_{k\to+\infty} y_k = x_0$. When $\lim_{k\to+\infty} y_k \neq x_0$, inequality (3.11) is a consequence of Proposition 2.1. Since the sequence $y_k$ is arbitrary, this proves (3.8) on $B_\delta(x_0)$. As easily checked, (3.8) follows from this estimate taken in the neighborhood of each of the points in $S_0$ and Proposition 2.1. □

Proposition 3.1 is a consequence of Steps 3.1 to 3.5.

## 4. Blow-Up analysis

The proof of Theorem 1.2 goes through an induction that will use the following proposition. The paper of Li-Shafrir [8] was a source of inspiration.

**Proposition 4.1.** *Let $x_0 \in \mathbb{R}^4$, $\delta > 0$ and $\Lambda > 0$. We let $V_k \in C^0(B_{4\delta}(x_0))$ be such that $\lim_{k\to+\infty} V_k = 1$ in $C^0(B_{4\delta}(x_0))$. We let $u_k \in C^4(B_{4\delta}(x_0))$ be such that*

$$\Delta^2 u_k = V_k e^{4u_k} \tag{4.1}$$

*in $B_{4\delta}(x_0)$. We assume that*

$$\int_{B_{4\delta}(x_0)} e^{4u_k}\, dx \le \Lambda \tag{4.2}$$

*for all $k \in \mathbb{N}$. We let $\rho_k \ge 0$ be such that $\lim_{k\to+\infty} \rho_k = 0$. We assume that there exists $(x_k = x_{1,k})_{k\in\mathbb{N}}, ..., (x_{N,k})_{k\in\mathbb{N}} \in B_{4\delta}(x_0)$ such that for any $i \in \{1, ..., N\}$, we have that*

$$\lim_{k\to+\infty} x_{i,k} = x_0 \text{ and } \lim_{k\to+\infty} u_k(x_{i,k}) = +\infty. \tag{4.3}$$

*Moreover, we assume that there exists $C > 0$ such that*

$$\inf_{i\in\{1,...,N\}} |x - x_{i,k}|e^{u_k(x)} \le C \text{ and } \inf_{i\in\{1,...,N\}} |x - x_{i,k}|^2|\Delta u_k(x)| \le C \tag{4.4}$$

*for all $k \in \mathbb{N}$ and all $x \in B_{2\delta}(x_k) \setminus \overline{B}_{\rho_k}(x_k)$. We assume that*

$$\lim_{k\to+\infty} \frac{|x_{i,k} - x_{j,k}|}{\mu_{i,k}} = +\infty \tag{4.5}$$

*for all $i \ne j$, $i, j \in \{1, ..., N\}$. In this expression, we have let $\mu_{i,k} := e^{-u_k(x_{i,k})}$. We assume that*

$$\lim_{k\to+\infty} (u_k(x_{i,k} + \mu_{i,k}x) - u_k(x_{i,k})) = \ln \frac{\sqrt{96}}{\sqrt{96} + |x|^2} \tag{4.6}$$

*for all $x \in \mathbb{R}^4$, and that this convergence holds in $C^3_{loc}(\mathbb{R}^4)$. We let $(r_k)_{k\in\mathbb{N}}$ be such that $r_k > 0$ for all $k \in \mathbb{N}$ and that $\lim_{k\to+\infty} r_k = r \in [0, \delta]$. We define*

$$I := \left\{ i \in \{2, ..., N\} \,/\, \frac{x_{i,k} - x_k}{r_k} = O(1) \text{ when } k \to +\infty \right\}. \tag{4.7}$$

*Note that $I$ may be empty. We define $\tilde{x}_i = \lim_{k\to+\infty} \frac{x_{i,k}-x_k}{r_k}$ for $i \in I$. We assume that $\tilde{x}_i \ne 0$ for all $i \in I$ and that*

$$\rho_k = o(r_k) \tag{4.8}$$

*and that $\mu_k = \mu_{1,k} = o(r_k)$ when $k \to +\infty$. We let $\nu, R$ be such that*

$$0 < \nu < \frac{1}{10} \min \{\{|\tilde{x}_i| / i \in I\} \cup \{|\tilde{x}_i - \tilde{x}_j| / i, j \in I,\, \tilde{x}_i \ne \tilde{x}_j\}\} \tag{4.9}$$

*and*

$$3 \max \{|\tilde{x}_i| / i \in I\} < R < \frac{\delta}{2r}. \tag{4.10}$$

*In case $r=0$, we define $\frac{\delta}{2r}=+\infty$. We define*

$$D_k:=B_{Rr_k}(x_k)\setminus\bigcup_{i\in I}\overline{B}_{\nu r_k}(x_{i,k}).$$

*Then, if $\mu_k=o(\rho_k)$, we have that*

$$\lim_{k\to+\infty}\int_{D_k\setminus\overline{B}_{3\rho_k}(x_k)}e^{4u_k(x)}\,dx=0.$$

*If $\rho_k=O(\mu_k)$, we have that*

$$\lim_{\tilde{R}\to+\infty}\lim_{k\to+\infty}\int_{D_k\setminus\overline{B}_{\tilde{R}\mu_k}(x_k)}e^{4u_k(x)}\,dx=0.$$

This section is devoted to the proof of the proposition. Up to relabelling the $\tilde{x}_i$'s, we assume that there exists $\phi:\{1,...,l\}\to\{1,...,N\}$ such that $\tilde{x}_{\phi(i)}\neq\tilde{x}_{\phi(j)}$ for all $i\neq j$, and

$$\{\tilde{x}_i/\,i\in I\}=\left\{\tilde{x}_{\phi(i)}/\,i\in\{1,..,l\}\right\},\text{ and } I=\{2,...,\phi(l)\}.\tag{4.11}$$

Moreover, we assume that $\phi$ is increasing and $\tilde{x}_j=\tilde{x}_{\phi(i)}$ for all $j$ such that $\phi(i)\leq j<\phi(i+1)$. Note that $1\notin I$ and that $\phi(1)\neq 1$. For all $i\in\{1,...,N\}$, we define

$$\tilde{x}_{i,k}:=\frac{x_{i,k}-x_k}{r_k}.\tag{4.12}$$

**Step 4.1 (Rescaling):** We define

$$\Omega_k=\left(B_{3R}(0)\setminus\bigcup_{i=1}^{l}\overline{B}_\nu(\tilde{x}_{\phi(i),k})\right)\setminus\overline{B}_{\frac{\rho_k}{r_k}}(0).$$

With the choice (4.10) of $R$, we have that $x_k+r_kx\in B_{2\delta}(x_k)\subset B_{4\delta}(x_0)$ for all $x\in\Omega_k$. We let $x\in\Omega_k$ and $j\in\{1,...,N\}$. We distinguish three cases:

*Case 4.1.1:* We assume that $j\in I$. We let $i\in\{1,...,l\}$ be such that $\phi(i)\leq j<\phi(i+1)$. Then, from (4.9), (4.10), the definition (4.12) and the choice of the numbering of the $\tilde{x}_j$'s, we have that

$$\begin{aligned}|x_k+r_kx-x_{j,k}| &= r_k|x-\tilde{x}_{j,k}|\geq r_k\left(|x-\tilde{x}_{\phi(i),k}|-|\tilde{x}_{\phi(i),k}-\tilde{x}_{j,k}|\right)\\ &\geq r_k(\nu+o(1))\geq r_k\frac{\nu}{2}\geq r_k\frac{\nu}{6R}|x|.\end{aligned}\tag{4.13}$$

*Case 4.1.2:* We assume that $j\in\{2,...,N\}$ is such that $j\notin I$. Then with (4.10), the definition (4.12) and the definition (4.7) of $I$, we get that

$$|x_k+r_kx-x_{j,k}|=r_k|x-\tilde{x}_{j,k}|\geq r_k\left(|\tilde{x}_{j,k}|-3R\right)\geq r_k\geq\frac{r_k}{3R}|x|.\tag{4.14}$$

*Case 4.1.3:* If $j=1$, we get that

$$|x_k+r_kx-x_k|=r_k|x|>\rho_k.\tag{4.15}$$

It follows from (4.13)-(4.15) that

$$\inf_{i\in\{1,...,N\}}|x_k+r_kx-x_{i,k}|\geq C(\nu,R)r_k|x|\tag{4.16}$$

$$\text{and } x_k+r_kx\in B_{2\delta}(x_k)\setminus\overline{B}_{\rho_k}(x_k).$$

for all $x\in\Omega_k$. For $x\in B_{3R}(0)$, we define

$$\tilde{u}_k(x):=u_k(x_k+r_kx)+\ln r_k.\tag{4.17}$$

It follows from (4.16), (4.1) and (4.4) that there exists $C > 0$ such that

$$\Delta^2 \tilde{u}_k = \tilde{V}_k e^{4\tilde{u}_k} \text{ in } B_{3R}(0) \tag{4.18}$$

and

$$|x| e^{\tilde{u}_k(x)} \leq C \text{ and } |x|^2 |\Delta \tilde{u}_k(x)| \leq C \tag{4.19}$$

for all $x \in \Omega_k$. Here, we have defined $\tilde{V}_k(x) := V_k(x_k + r_k x)$ for all $x \in B_{3R}(0)$ and all $k \in \mathbb{N}$.

**Step 4.2 (Harnack inequality):** We claim that there exists $C = C(\nu, R)$, there exists $\beta = \beta(\nu, R) > 0$ such that

$$\beta \sup_{\partial\left(B_r(0) \setminus \bigcup_{i=1}^l \overline{B}_{2\nu}(\tilde{x}_{\phi(i),k})\right)} \tilde{u}_k \leq \inf_{\partial\left(B_r(0) \setminus \bigcup_{i=1}^l \overline{B}_{2\nu}(\tilde{x}_{\phi(i),k})\right)} \tilde{u}_k + (1-\beta) \ln r + C \tag{4.20}$$

for all $r > 0$ such that

$$\frac{3\rho_k}{r_k} \leq r \leq 2R.$$

*Proof of the claim:* We let $s_k > 0$ be such that $\frac{3\rho_k}{r_k} \leq s_k \leq 2R$ and we prove the claim for $r = s_k$. Up to a subsequence, we assume that $\lim_{k \to +\infty} s_k = s \geq 0$. We distinguish two cases:

*Case 4.2.1:* We assume that

$$0 \leq s < \frac{4}{5}\nu. \tag{4.21}$$

With (4.9) and (4.10), we get that

$$B_{\frac{5}{4}}(0) \setminus \overline{B}_{\frac{1}{2}}(0) \subset \frac{\Omega_k}{s_k}.$$

For any $x \in B_{\frac{5}{4}}(0) \setminus \overline{B}_{\frac{1}{2}}(0)$, we define

$$U_k(x) = \tilde{u}_k(s_k x) + \ln s_k. \tag{4.22}$$

It follows from (4.19) that there exists $C > 0$ such that

$$U_k(x) \leq C \text{ and } |\Delta U_k(x)| \leq C \tag{4.23}$$

for all $k \in \mathbb{N}$ and all $x \in B_{\frac{5}{4}}(0) \setminus \overline{B}_{\frac{1}{2}}(0)$. It then follows from the Harnack inequality that there exists $\beta, C > 0$ such that

$$\beta \sup_{\partial B_1(0)} U_k \leq \inf_{\partial B_1(0)} U_k + C \tag{4.24}$$

for all $k \in \mathbb{N}$. Coming back to $\tilde{u}_k$ with (4.22), using the assumption (4.21), (4.9) and (4.10) we get that

$$\partial \left( B_{s_k}(0) \setminus \bigcup_{i=1}^l \overline{B}_{2\nu}(\tilde{x}_{\phi(i),k}) \right) = \partial B_{s_k}(0)$$

and then (4.20) follows from (4.24). This ends Case 4.2.1.

*Case 4.2.2:* We assume that

$$\frac{4}{5}\nu \leq s \leq 2R. \tag{4.25}$$

We define

$$\mathcal{A} = \left( B_{3R}(0) \setminus \bigcup_{i=1}^l \overline{B}_{\frac{5}{4}\nu}(\tilde{x}_{\phi(i)}) \right) \setminus \overline{B}_{\frac{\nu}{5}}(0).$$

It follows from (4.9) and (4.8) that

$$\mathcal{A} \subset \Omega_k$$

for $k > 0$ large enough. Moreover, it follows from (4.9) and (4.10) that $\mathcal{A}$ is connected. It follows from (4.19) that there exists $C > 0$ such that

$$\tilde{u}_k(x) \leq C \text{ and } |\Delta \tilde{u}_k(x)| \leq C$$

for all $x$ in a neighborhood of $\mathcal{A}$. With Harnack's inequality, we get that there exists $\beta, C > 0$ such that

$$\beta \sup_{\mathcal{A}} \tilde{u}_k \leq \inf_{\mathcal{A}} \tilde{u}_k + C \tag{4.26}$$

for all $k \in \mathbb{N}$. With (4.9), (4.10) and (4.25), we get that

$$\partial \left( B_{s_k}(0) \setminus \bigcup_{i=1}^{l} \overline{B}_{2\nu}(\tilde{x}_{\phi(i),k}) \right) \subset \mathcal{A}.$$

From (4.25) and (4.26), we get that there exists $\beta = \beta(\nu, R) > 0$, $C = C(\nu, R) > 0$ such that

$$\beta \sup_{\partial\left(B_{s_k}(0) \setminus \bigcup_{i=1}^{l} \overline{B}_{2\nu}(\tilde{x}_{\phi(i),k})\right)} \tilde{u}_k \leq \inf_{\partial\left(B_{s_k}(0) \setminus \bigcup_{i=1}^{l} \overline{B}_{2\nu}(\tilde{x}_{\phi(i),k})\right)} \tilde{u}_k + (1-\beta) \ln s_k + C$$

for all $k \in \mathbb{N}$. This ends Case 4.2.2, and the proof of the claim is complete. □

**Step 4.3 (Upper bound):** We claim that there exists $\theta > -1$, there exists $R_0 > 0$ such that

$$\sup_{\partial\left(B_{s_k}(0) \setminus \bigcup_{i=1}^{l} \overline{B}_{2\nu}(\tilde{x}_{\phi(i),k})\right)} \tilde{u}_k \leq -\left(1 + \frac{1+\theta}{\beta}\right) \ln s_k - \frac{1+\theta}{\beta} \ln \frac{r_k}{\mu_k} + C \tag{4.27}$$

for all $k \in \mathbb{N}$ where $s_k > 0$ is such that

$$s_k \in \left[ \frac{3\rho_k}{r_k}, 2R \right] \text{ if } \mu_k = o(\rho_k) \tag{4.28}$$

and

$$s_k \in \left[ \frac{R_0 \mu_k}{r_k}, 2R \right] \text{ if } \rho_k = O(\mu_k), \tag{4.29}$$

where $\mu_k$ and $\rho_k$ are as in Proposition 4.1.

*Proof of the claim:* We let $U_k$ be defined as in (4.22) on $B_{\frac{3R}{s_k}}(0)$. We assume that

$$0 \leq s < 8\nu. \tag{4.30}$$

Let $H_k$ be the Green's function of $\Delta^2$ on

$$\mathcal{D}_k := B_1(0) \setminus \bigcup_{i=1}^{l} \overline{B}_{\frac{2\nu}{s_k}} \left( \frac{\tilde{x}_{\phi(i),k}}{s_k} \right) = B_1(0) \tag{4.31}$$

with Navier condition on the boundary, that is for any $x \in \mathcal{D}_k$, we have that $\Delta^2 H_\delta(x, \cdot) = \delta_x$ in the distribution sense, $H_\delta(x, \cdot) = \Delta H_\delta(x, \cdot) = 0$ on $\partial B_1(0)$. Note that equality (4.31) is a consequence of (4.9), (4.10) and (4.30). It follows from Green's representation formula that

$$U_k(0) = \int_{\mathcal{D}_k} H_k(0, y) \Delta^2 U_k(y) \, dy + \varphi_k(0) + \psi_k(0) \tag{4.32}$$

where

$$\left\{\begin{array}{ll}\Delta\varphi_k = 0 & \text{in } \mathcal{D}_k\\ \varphi_k = U_k & \text{on } \partial\mathcal{D}_k\end{array}\right\} \text{ and } \left\{\begin{array}{ll}\Delta^2\psi_k = 0 & \text{in } \mathcal{D}_k\\ \Delta\psi_k = \Delta U_k & \text{on } \partial\mathcal{D}_k\\ \psi_k = 0 & \text{on } \partial\mathcal{D}_k\end{array}\right\}. \tag{4.33}$$

It follows from (4.8), (4.9) and (4.10) that $\partial\mathcal{D}_k = \partial B_1(0) \subset s_k^{-1}\Omega_k$. We then get from (4.23), the maximum principle and (4.33) that there exists $C > 0$ such that

$$|\psi_k(0)| \leq C \tag{4.34}$$

for all $k \in \mathbb{N}$. It follows from the comparison principle and (4.33) that

$$\varphi_k(0) \geq \inf_{\partial\mathcal{D}_k} U_k. \tag{4.35}$$

We take $\tilde{R} > 0$. Moreover, by (4.28), (4.29) we get that

$$B_{\frac{\tilde{R}\mu_k}{s_k r_k}}(0) \subset B_{1/2}(0) \subset \mathcal{D}_k \tag{4.36}$$

with $R_0 > 2\tilde{R}$. Noting that $H_k \geq 0$, we get from (4.32), (4.34), (4.35) and (4.36) that

$$U_k(0) \geq \int_{B_{\frac{\tilde{R}\mu_k}{s_k r_k}}(0)} H_k(0,y)\Delta^2 U_k(y)\,dy + \inf_{\partial\mathcal{D}_k} U_k - C.$$

It follows from standard elliptic estimates that there exists $C > 0$ such that

$$H_k(0,y) \geq \frac{1}{8\pi^2}\ln\frac{1}{|y|} - C$$

for $y \in B_{1/2}(0) \setminus \{0\}$. We then get that

$$U_k(0) \geq \int_{B_{\frac{\tilde{R}\mu_k}{s_k r_k}}(0)} \left(\frac{1}{8\pi^2}\ln\frac{1}{|y|} - C\right)\Delta^2 U_k(y)\,dy + \inf_{\partial\mathcal{D}_k} U_k - C.$$

With the change of variable $y = \frac{\mu_k}{s_k r_k} z$ and coming back to the definitions (4.17) and (4.22), we get that

$$\begin{aligned}\ln\frac{r_k s_k}{\mu_k} \quad &\geq \int_{B_{\tilde{R}}(0)} \left(\frac{1}{8\pi^2}\ln\frac{s_k r_k}{\mu_k} + \frac{1}{8\pi^2}\ln\frac{1}{|z|} - C\right)\tilde{V}_k(z)e^{4(u_k(x_k+\mu_k z)-u_k(x_k))}\,dz\\ &\quad + \inf_{\partial\mathcal{D}_k} U_k - C,\end{aligned}$$

where $\tilde{V}_k(z) := V_k(x_k + \mu_k z)$. By (4.6), we then get that

$$C(\tilde{R}) \geq \left(1 + \frac{\theta_k(\tilde{R})}{8\pi^2}\right)\ln\frac{s_k r_k}{\mu_k} + \inf_{\partial\mathcal{D}_k} U_k,$$

with $\lim_{\tilde{R}\to+\infty}\lim_{k\to+\infty}\theta_k(\tilde{R}) = 0$. Choosing $\tilde{R}$ large enough, and then choosing $R_0 > 2\tilde{R}$ large, we get that there exists $\theta > -1$ such that

$$C \geq (1+\theta)\ln\frac{s_k r_k}{\mu_k} + \inf_{\partial\mathcal{D}_k} U_k,$$

for all $k \in \mathbb{N}$. Coming back to $\tilde{u}_k$ and using (4.20), we get the inequality of the Lemma. This ends the proof of the claim when (4.30) holds. In case (4.30) does not hold, the claim follows from the case $s_k = 7\nu$ and the Harnack inequality (4.20).□

**Step 4.4 (Proof of Proposition 4.1):** We let $y_k \in B_{2R}(0) \setminus \bigcup_{i=1}^l \overline{B}_{2\nu}(\tilde{x}_{\phi(i),k})$ be such that

$$|y_k| \geq \frac{3\rho_k}{r_k} \text{ if } \mu_k = o(\rho_k) \text{ or } |y_k| \geq \frac{R_0\mu_k}{r_k} \text{ if } \rho_k = O(\mu_k),$$

where $R_0$ is as in (4.27). Defining $s_k = |y_k|$, we get that

$$y_k \in \partial\left(B_{s_k}(0) \setminus \bigcup_{i=1}^l \overline{B}_{2\nu}(\tilde{x}_{\phi(i),k})\right).$$

It follows from (4.27) that

$$\tilde{u}_k(y_k) \leq -\left(1 + \frac{1+\theta}{\beta}\right)\ln|y_k| - \frac{1+\theta}{\beta}\ln\frac{r_k}{\mu_k} + C. \tag{4.37}$$

We distinguish two cases:

*Case 4.4.1:* We assume that $\mu_k = o(\rho_k)$. We then get by (4.37) that

$$\int_{\left(B_{2R}(0)\setminus\bigcup_{i=1}^l \overline{B}_{2\nu}(\tilde{x}_{\phi(i),k})\right)\setminus\overline{B}_{\frac{3\rho_k}{r_k}}(0)} e^{4\tilde{u}_k(y)}\,dy$$

$$\leq C\int_{B_{2R}(0)\setminus\overline{B}_{\frac{3\rho_k}{r_k}}(0)} \left(\frac{\mu_k}{r_k}\right)^{4\frac{1+\theta}{\beta}} \frac{1}{|y|^{4+4\frac{1+\theta}{\beta}}}\,dy \leq C\left(\frac{\mu_k}{\rho_k}\right)^{4\frac{1+\theta}{\beta}} = o(1)$$

when $k \to +\infty$. Coming back to the definition of $\tilde{u}_k$ and the relabelling (4.11), this proves Proposition 4.1 in Case 4.4.1.

*Case 4.4.2:* We assume that $\rho_k = O(\mu_k)$ when $k \to +\infty$. We take $\tilde{R} > R_0$. We then get by (4.37) that

$$\int_{\left(B_{2R}(0)\setminus\bigcup_{i=1}^l \overline{B}_{2\nu}(\tilde{x}_{\phi(i),k})\right)\setminus\overline{B}_{\frac{\tilde{R}\mu_k}{r_k}}(0)} e^{4\tilde{u}_k(y)}\,dy$$

$$\leq C\int_{B_{2R}(0)\setminus\overline{B}_{\frac{\tilde{R}\mu_k}{r_k}}(0)} \left(\frac{\mu_k}{r_k}\right)^{4\frac{1+\theta}{\beta}} \frac{1}{|y|^{4+4\frac{1+\theta}{\beta}}}\,dy \leq \frac{C}{\tilde{R}^{4\frac{1+\theta}{\beta}}}$$

for all $k \in \mathbb{N}$. Coming back to the definition of $\tilde{u}_k$ and the relabelling (4.11), this proves Proposition 4.1 in Case 4.4.2.

## 5. Proof of Theorem 1.2

We prove Theorem 1.2 by induction. We consider $N \in \mathbb{N}^\star$. We say that $(\mathbf{H_N})$ holds if the following Proposition holds:

**Proposition $(\mathbf{H_N})$:** *Let $x_0 \in \mathbb{R}^4$, $\delta > 0$ and $\lambda > 0$. Let $u_k \in C^4(B_{4\delta}(x_0))$ and $V_k \in C^0(B_{4\delta}(x_0))$ be such that $\lim_{k\to+\infty} V_k = 1$ in $C^0_{loc}(B_{4\delta}(x_0))$ and*

$$\Delta^2 u_k = V_k e^{4u_k} \tag{5.1}$$

*in $B_{4\delta}(x_0)$ and*

$$\int_{B_{4\delta}(x_0)} e^{4u_k}\,dx \leq \Lambda.$$

*We assume that there exists $1 \leq K \leq N$, $x_k = x_{1,k}, ..., x_{K,k} \in B_{4\delta}(x_0)$ such that for any $i \in \{1, ..., K\}$, we have that*

$$\lim_{k \to +\infty} x_{i,k} = x_0 \text{ and } \lim_{k \to +\infty} u_k(x_{i,k}) = +\infty.$$

*Moreover, we assume that there exists $C > 0$ such that*

$$\inf_{i \in \{1,...,K\}} |x - x_{i,k}| e^{u_k(x)} \leq C \text{ and } \inf_{i \in \{1,...,K\}} |x - x_{i,k}|^2 |\Delta u_k(x)| \leq C \tag{5.2}$$

*for all $k \in \mathbb{N}$ and all $x \in B_{2\delta}(x_k)$. We assume that*

$$\lim_{k \to +\infty} \frac{|x_{i,k} - x_{j,k}|}{\mu_{i,k}} = +\infty \tag{5.3}$$

*for all $i \neq j$, $i, j \in \{1, ..., K\}$. In this expression, we have let $\mu_{i,k} = e^{-u_k(x_{i,k})}$. We assume that*

$$\lim_{k \to +\infty} (u_k(x_{i,k} + \mu_{i,k} x) - u_k(x_{i,k})) = \ln \frac{\sqrt{96}}{\sqrt{96} + |x|^2} \tag{5.4}$$

*for all $x \in \mathbb{R}^4$, and that this convergence holds in $C^3_{loc}(\mathbb{R}^4)$. Then, we have that*

$$\int_{B_\delta(x_0)} V_k e^{4u_k(x)}\, dx = 16\pi^2 K + o(1)$$

*when $k \to +\infty$.*

We prove by induction that $(\mathbf{H_N})$ holds for all $N \geq 1$.

**Step 5.1 (Proof of $(\mathbf{H_1})$):** We claim that $(\mathbf{H_1})$ holds. We prove the claim. We apply Proposition 4.1 with $r_k = \delta$ and $\rho_k = 0$. We then get that

$$\lim_{R \to +\infty} \lim_{k \to +\infty} \int_{B_{\frac{\delta}{2}}(x_0) \setminus B_{R\mu_k}(x_k)} V_k e^{4u_k(x)}\, dx = 0. \tag{5.5}$$

Plugging (5.4), Proposition 3.1 and (5.5) together yields

$$\int_{B_\delta(x_0)} V_k e^{4u_k(x)}\, dx = 16\pi^2 + o(1)$$

when $k \to +\infty$. This proves the claim, and therefore $(\mathbf{H_1})$.

**Step 5.2 (Induction):** We let $N \geq 2$. We assume that $(\mathbf{H_{N-1}})$ holds. We let $(u_k)_{k \in \mathbb{N}} \in C^4(B_{4\delta}(x_0))$. We assume that $u_k$ verifies the hypothesis of $(\mathbf{H_N})$. Without loss of generality, we assume that $K = N$. Up to renumbering, we define

$$r_{1,k} = \inf_{i \neq j} \{|x_{i,k} - x_{j,k}|\} = \inf_{i \neq 1} \{|x_{1,k} - x_{i,k}|\}.$$

From (5.3), we get that

$$\lim_{k \to +\infty} \frac{r_{1,k}}{\mu_{1,k}} = +\infty.$$

We define

$$I_1 = \left\{ i \in \{2, ..., N\} / \frac{x_{1,k} - x_{i,k}}{r_{1,k}} = O(1) \text{ as } k \to +\infty \right\}.$$

Note here that $I_1 \neq \emptyset$. We define by induction:

$$r_{q+1,k} = \inf\{|x_{1,k} - x_{j,k}| / j \notin \{1\} \cup I_1 \cup ... \cup I_q\} \tag{5.6}$$

$$I_{q+1} = \left\{ i \notin \{1\} \cup I_1 \cup ... \cup I_q / \frac{x_{1,k} - x_{i,k}}{r_{q+1,k}} = O(1) \text{ as } k \to +\infty \right\}, \tag{5.7}$$

when these quantities are defined. Since we have a finite number of points, this process must end. We let $q_0 \in \mathbb{N}$ be such that $r_{q,k}$ is defined for $q \in \{1, ..., q_0\}$ and not afterwards. Moreover, for any $q < q_0$, we have that

$$\lim_{k\to+\infty} \frac{r_{q+1,k}}{r_{q,k}} = +\infty.$$

*Step 5.2.1:* We claim that

$$\lim_{k\to+\infty} \int_{B_{Rr_{1,k}}(x_k)} V_k e^{4u_k(x)}\, dx = 16\pi^2\, \mathrm{Card}(\{1\} \cup I_1),$$

where $x_k = x_{1,k}$. We prove the claim. We apply Proposition 4.1 with $u_k$, $\rho_k = 0$ and $r_k = r_{1,k}$. For $R$, $\nu$ and $\phi$ as in the proof of Proposition 4.1, similarly to what was done for the proof of $(\mathbf{H_1})$ we get that

$$\lim_{k\to+\infty} \int_{B_{Rr_k}(x_k)\setminus \bigcup_{i=1}^{l} \overline{B}_{\nu r_k}(x_{\phi(i),k})} e^{4u_k(x)}\, dx = 16\pi^2. \tag{5.8}$$

We fix $i \in \{1, ..., l\}$. We define

$$v_k(x) := u_k(x_{\phi(i),k} + r_k x) + \ln r_k$$

and $\tilde{V}_k(x) := V_k(x_{\phi(i),k} + r_k x)$ for all $x \in B_R(0)$ and all $k \in \mathbb{N}$. From (5.1), we have that

$$\Delta^2 v_k = \tilde{V}_k e^{4v_k} \text{ in } B_{4\nu}(0) \text{ and } \int_{B_{4\nu}(0)} e^{4v_k}\, dx \le \Lambda.$$

For any $j$ such that $\phi(i) \le j < \phi(i+1)$, we define

$$X_{j,k} = \frac{x_{j,k} - x_{\phi(i),k}}{r_k}.$$

It follows from the definition of $\phi$ that $\lim_{k\to+\infty} X_{j,k} = 0$ for all $j \in \{\phi(i), ..., \phi(i+1) - 1\}$. Arguing as in Step 4.1, and letting $U_i := \{\phi(i), ..., \phi(i+1) - 1\}$, we get that

$$\inf_{j\in U_i} |x - X_{j,k}| e^{v_k(x)} \le C \text{ and } \inf_{j\in U_i} |x - X_{j,k}|^2 |\Delta v_k(x)| \le C$$

for all $x \in B_{4\nu}(0)$. For any $j, m \in \{\phi(i), ..., \phi(i+1) - 1\}$, $j \ne m$, we have by (5.3) that

$$\frac{|X_{j,k} - X_{m,k}|}{e^{-v_k(X_{j,k})}} = \frac{|x_{j,k} - x_{m,k}|}{e^{-u_k(x_{j,k})}} \to +\infty$$

when $k \to +\infty$. By the definition of $I_1$ and (5.3), we get that $\lim_{k\to+\infty} v_k(X_{j,k}) = +\infty$ for all $j$. By (5.4), a straightforward computation shows that for any $j \in \{\phi(i), ..., \phi(i+1) - 1\}$, we have that for any $x \in \mathbb{R}^4$,

$$v_k(X_{j,k} + e^{-v_k(X_{j,k})} x) - v_k(X_{j,k}) \to \ln \frac{\sqrt{96}}{\sqrt{96} + |x|^2}$$

when $k \to +\infty$. Moreover, this convergence holds in $C^3_{loc}(\mathbb{R}^4)$. We then apply the induction hypothesis $(\mathbf{H_{N-1}})$ with $v_k$ (which has at most $N-1$ concentration points) and we get that

$$\int_{B_{\nu r_k}(x_{\phi(i),k})} V_k e^{4u_k(x)}\, dx = \int_{B_\nu(0)} \tilde{V}_k e^{4v_k(x)}\, dx = (\phi(i+1) - \phi(i)) 16\pi^2 + o(1)$$

when $k \to +\infty$. Since this inequality is valid for all $i$, we get from (5.8) that

$$\lim_{k\to+\infty} \int_{B_{Rr_k}(x_k)} V_k e^{4u_k(x)}\, dx = 16\pi^2 \operatorname{Card}(\{1\} \cup I_1).$$

This proves the claim, and then Step 5.2.1.

*Step 5.2.2:* We take $q < q_0$ and assume that for any $R > 0$ large enough, we have that

$$\lim_{k\to+\infty} \int_{B_{Rr_{q,k}}(x_k)} V_k e^{4u_k(x)}\, dx = 16\pi^2 \operatorname{Card}(\{1\} \cup I_1 \cup ... \cup I_q). \tag{5.9}$$

We claim that for any $R > 0$ large enough, we have that

$$\lim_{k\to+\infty} \int_{B_{Rr_{q+1,k}}(x_k)} V_k e^{4u_k(x)}\, dx = 16\pi^2 \operatorname{Card}(\{1\} \cup I_1 \cup ... \cup I_{q+1}).$$

We prove the claim. We define

$$R_0 = \max \left\{ \frac{|x_k - x_{i,k}|}{r_{q,k}} / \, i \in I_q \right\}.$$

We consider $\rho_k = R_1 r_{q,k}/3$ with $R_1 > 6R_0$ and $r_k = r_{q+1,k}$. We let $i \in \{1\} \cup I_1 \cup ... \cup I_q$ and $x \in B_{2\delta}(x_k) \setminus \overline{B}_{\rho_k}(x_k)$. We assume that $i \in I_p$, $p \leq q$. From the definitions (5.6), (5.7) and from the estimate (5.2), we get that there exists $C > 0$ such that

$$\inf_{i \notin I_1 \cup ... \cup I_q} |x - x_{i,k}| e^{u_k(x)} \leq C \text{ and } \inf_{i \notin I_1 \cup ... \cup I_q} |x - x_{i,k}|^2 |\Delta u_k(x)| \leq C$$

for all $k \in \mathbb{N}$ and all $x \in B_{2\delta}(x_k) \setminus \overline{B}_{\rho_k}(x_k)$. Note that $1 \notin I_1 \cup ... \cup I_q$. We apply Proposition 4.1 with $u_k$, $\rho_k$ and $r_k$. Similarly to what was done in Step 5.2.1, we get, using our induction hypothesis, that for $R >> 1$ large

$$\lim_{k\to+\infty} \int_{B_{Rr_{q+1,k}}(x_k) \setminus \overline{B}_{R_1 r_{q,k}}(x_k)} V_k e^{4u_k(x)}\, dx = 16\pi^2 \operatorname{Card}(I_{q+1}).$$

The claim then follows from this last equality and (5.9).

*Step 5.2.3:* From Step 5.2.2, we get that

$$\lim_{k\to+\infty} \int_{B_{Rr_{q_0,k}}(x_k)} V_k e^{4u_k(x)}\, dx = 16\pi^2 N, \tag{5.10}$$

for all $R > 0$ large enough. Similarly to what was done in Step 5.2.2, there exists $R_0 > 0$ such that for all $x \in B_{4\delta}(x_k) \setminus \overline{B}_{R_0 r_{q_0,k}}(x_k)$, we have that

$$|x - x_k| e^{u_k(x)} \leq C \text{ and } |x - x_k|^2 |\Delta u_k(x)| \leq C.$$

We apply Proposition 4.1 with $r_k = \frac{\delta}{2}$ and $\rho_k = R_0 r_{q_0,k}$. We then get that

$$\lim_{k\to+\infty} \int_{B_{\frac{\delta}{2}}(x_k) \setminus B_{3R_0 r_{q_0,k}}(x_k)} V_k e^{4u_k(x)}\, dx = 0.$$

This limit, (5.10) and Proposition 3.1 yield

$$\lim_{k\to+\infty} \int_{B_\delta(x_0)} V_k e^{4u_k(x)}\, dx = 16\pi^2 N.$$

This proves the quantification with $N$ points. We have then proved that $(\mathbf{H_N})$ holds.

In particular, we have proved by induction that $(\mathbf{H_N})$ holds for all $N$.

**Step 5.3 (Proof of Theorem 1.2):** We let $u_k$ be as in the statement of the Theorem 1.2. It follows from Proposition 3.1 that the hypothesis of $(\mathbf{H_N})$ holds in the neighborhood of each of the points of $S_0$. As a consequence, we apply locally $(\mathbf{H_N})$. It then follows that

$$V_k e^{4u_k} \rightharpoonup \sum_{i=1}^{N} 16\pi^2 \alpha_i \delta_{x_i}$$

weakly in the sense of measures when $k \to +\infty$. And the proof of Theorem 1.2 is complete.

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