Asymptotic behaviour of a nonlinear elliptic equation with critical Sobolev exponent
The radial case II

by

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Abstract

Let B be the unit ball of $\mathbb{R}^N$, $N \geq 3$, and $a, f : \mathbb{R} \to \mathbb{R}$ be two smooth functions. For $\epsilon > 0$ small, the equation

$$\Delta u + a(|x|)u = N(N-2)f(|x|)u^{\frac{N+2}{N-2}-\epsilon} \text{ in } B, \text{ on } \partial B \qquad (1)$$

has a positive radially symmetrical solution u_ϵ. We describe the asymptotic behaviour of (u_ϵ) as $\epsilon \to 0$. We also recover existence results for the critical equation.

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1 Introduction and statement of the results

Let B be the unit ball in $\mathbb{R}^N$, $N \geq 3$, and $a, f : \mathbb{R} \to \mathbb{R}$ two smooth functions. We regard $x \mapsto a(|x|)$ and $x \mapsto f(|x|)$ as functions of the variable $x \in \mathbb{R}^N$. As easily seen, these functions are locally Lipschitz. In particular, they are locally in $C^{0,\alpha}$ for all $\alpha \in]0,1[$. In order to fix ideas, we suppose that $f > 0$ and that $f(0) = 1$. Then we consider the following problem:

$$(I) \begin{cases} \Delta u + a(|x|)u = N(N-2)f(|x|)u^p \text{ in } B \\ \\ u > 0 \text{ in } B, \quad u = 0 \text{ on } \partial B \end{cases}$$

where $\Delta = -\sum \frac{\partial^2}{\partial x_i^2}$ is the Laplacian with the minus sign convention, and $p = \frac{N+2}{N-2}$ is critical from the view point of Sobolev embeddings. We let $H_0^1(B)$ be the standard Sobolev space, defined as the completion of $\mathcal{D}(B)$, the set of smooth functions with compact support in B, with respect to the norm

$$\|u\| = \sqrt{\int_B |\nabla u|^2 dx} \ .$$

In the sequel, we suppose that the operator $u \mapsto \Delta u + a(|x|)$ is coercive on $H_0^1(B)$. This is the case when $a > -\lambda_1$, where λ_1 is the first eigenvalue of Δ for the Dirichlet problem.

Situations where (I) does not have a solution are in Pohozaev [Poh]. In particular, (I) does not possess a solution if $a \equiv 0$ and $f \equiv 1$. However, as it is subcritical from the view point of Sobolev embeddings, the problem

$$(I_\epsilon) \begin{cases} \Delta u_\epsilon + a(|x|)u_\epsilon = N(N-2)f(|x|)u_\epsilon^{p-\epsilon} \text{ in } B \\ \\ u_\epsilon > 0 \text{ in } B, \ \ u_\epsilon = 0 \text{ on } \partial B \end{cases}$$

has a solution $u_\epsilon \in C^2(\overline{B})$ for all $\epsilon \in]0, p-1[$. This solution can be assumed to be minimizing and radially symmetrical (MRS), where u_ϵ is said to be MRS if u_ϵ is radially symmetrical and

$$\frac{\int_B \left(|\nabla u_\epsilon|^2 + a(|x|)u_\epsilon^2\right) dx}{\left(\int_B f(|x|)u_\epsilon^{p-\epsilon+1} dx\right)^{\frac{2}{p-\epsilon+1}}} = \inf_{v \in \mathcal{D}(B)_R \backslash \{0\}} \frac{\int_B \left(|\nabla v|^2 + a(|x|)v^2\right) dx}{\left(\int_B f(|x|)|v|^{p-\epsilon+1} dx\right)^{\frac{2}{p-\epsilon+1}}}$$

where $\mathcal{D}(B)_R$ denotes the set of smooth radially symmetrical functions with compact support in B. The arguments required for the proof of this result are by now classical. We refer to Hebey [Heb1], proposition 1, for the proof of this result.

On the one hand, we are concerned in this article with the existence of conditions on a and f for (I) to have a solution. On the other hand, we are concerned with the asymptotic behaviour of u_ϵ as $\epsilon \to 0$ when (I) does not have a solution. The existence of solutions for (I) has been studied by various authors. In particular, when $f \equiv 1$ and $a \equiv \lambda$, $\lambda \in \mathbb{R}$, Brézis and Nirenberg [BrNi] got that (I) has a solution if and only if $\lambda \in]0, \lambda_1[$ when $N \geq 4$, and $\lambda \in]\frac{1}{4}\lambda_1, \lambda_1[$ when $N = 3$. Independently, asymptotic type studies were first developed by Atkinson and Peletier [AtPe]. With arguments from ODE's theory, and assuming that $a \equiv 0$ and $f \equiv 1$, they got that

$$\lim_{\epsilon \to 0} \epsilon u_\epsilon^2(0) = \frac{4\Gamma(N)}{(N-2)\Gamma(\frac{N}{2})^2} \ ,$$

and that, for all $x \in B \backslash \{0\}$,

$$\lim_{\epsilon \to 0} \epsilon^{-1/2} u_\epsilon(x) = \frac{\sqrt{N-2}\,\Gamma(\frac{N}{2})}{2\sqrt{\Gamma(N)}} \left(\frac{1}{|x|^{N-2}} - 1\right) \ .$$

Brézis and Peletier [BrPe] returned to this problem, but with arguments from PDE's theory, and they conjectured that a similar behaviour should occur in the non radial case. This was proved to be true independently by Han [Han] and Rey [Rey]. When $a \equiv 0$ and f is non-constant, our problem was studied by Hebey [Heb2],[Heb4]. Existence results for (I) and the asymptotic behaviour of the u_ϵ's were given in these articles. A closely related problem is studied in Y.Y. Li [Li].

We generalize in the present work what was done in [Heb2]. In particular, we do not assume anymore that $a \equiv 0$. As one may easily check, the linear term au, and more precisely its negative part $a_- u$, leads to serious difficulties. These difficulties where overcome in [Rob1] under the assumption that the $L^{N/2}$-norm of the negative part a_- of a is small. We prove here that this condition can in turn be removed, the only condition to be required being that $\Delta + a$ is coercive.

In what follows, we set

$$k_a \overset{\text{def}}{=} \inf\{l \geq 0 / a^{(l)}(0) \neq 0\}$$

$$k_f \overset{\text{def}}{=} \inf\{l \geq 1 / f^{(l)}(0) \neq 0\}$$

with the convention that $k_a = \infty$ (respectively $k_f = \infty$) if $a^{(l)}(0) = 0$ for all $l \geq 0$ (respectively $f^{(l)}(0) = 0$ for all $l \geq 1$). We denote by G the Green's function of the operator $\Delta + a$, so that G is such that

$$\Delta_y G(x,y) + a(|y|)G(x,y) = \delta_x$$

on $B \times B$ minus its diagonal, and $G(x,y) = 0$ for $y \in \partial B$ and $x \in B$. (As already mentioned, $\Delta + a$ is supposed to be coercive). If $y \notin \partial B$, $G(x,y) > 0$, while $(x,y) \mapsto G(x,y)$ is symmetrical in (x,y). Moreover, $G(x,0)$ is radially symmetrical. We let $g(r) = G(x,0)$ where $r = |x|$. This function is defined on $]0,1]$. If $a \equiv 0$,

$$g(r) = \frac{1}{(N-2)\omega_{N-1}}\left(\frac{1}{r^{N-2}} - 1\right)$$

where ω_{N-1} denotes the volume of the standard sphere of $\mathbb{R}^N$. For $k \in \mathbb{N}$ and $q > 0$, we let

$$I_{k,q} = \int_0^\infty \frac{r^k}{(1+r^2)^{\frac{(N-2)q}{2}}} dr$$

when this integral makes sense, and we let ω_k be the volume of the standard sphere of $\mathbb{R}^{k+1}$. We also let

$$\alpha_k(N) = \frac{(k+1)(k+2)I_{k+N-1,2}}{(N-2)^2 I_{k+N+1,p+1}} \quad , \quad \alpha(N) = \frac{I_{2N-3,p+1}}{(N-3)!\omega_{N-1}^2}$$

and

$$\Phi(a) = \int_0^1 \left(a(r) + \frac{1}{2}ra'(r)\right) g(r)^2 r^{N-1} dr \ .$$

As easily checked, Φ is defined as soon as $k_a > N-4$. Our first result is concerned with the existence of solutions to (I). This result generalizes previous results obtained by Demengel and Hebey [DeHe] with another method.

Theorem 1 *If we are in one of the following cases:*

1. $k_a < N-4$,

  (a) $k_f < k_a + 2$, *and* $f^{(k_f)}(0) > 0$

  (b) $k_f = k_a + 2$, *and* $\alpha_{k_a}(N) a^{(k_a)}(0) < f^{(k_a+2)}(0)$

  (c) $k_f > k_a + 2$, *and* $a^{(k_a)}(0) < 0$

2. $k_a = N-4$,

  (a) $k_f < N-2$, *and* $f^{(k_f)}(0) > 0$

  (b) $k_f \geq N-2$, *and* $a^{(k_a)}(0) < 0$

3. $k_a > N-4$,

  (a) $k_f < N-2$, *and* $f^{(k_f)}(0) > 0$

  (b) $k_f \geq N-2$, *and* $g'(1)^2 + 2\Phi(a) < \alpha(N) f^{(N-2)}(0)$

then (I) *possesses a MRS solution, obtained as the limit of a subsequence of* u_ϵ *in* $C^2(\overline{B})$.

As already mentioned, there are situations in which the u_ϵ's do not converge, but develop a concentration. The concentration is characterized by one of the following properties: a subsequence of (u_ϵ) which converges almost everywhere converges to 0, or $u_\epsilon \to 0$ in $L^q(B)$ as soon as $q < p+1$. Such a situation occurs when $a \equiv 0$ and $f \equiv 1$. This follows from Hopf's maximum principle and the Pohozaev identity applied to (I),

$$\frac{(N-2)^2}{2} \int_B |x| f'(|x|) u^{p+1} dx - \int_B \left(a(|x|) + \frac{1}{2} |x| a'(|x|) \right) u^2 dx = \frac{1}{2} \int_{\partial B} |\nabla u|^2 d\sigma \quad .$$

Still according to this identity, the u_ϵ's also develop a concentration when f is decreasing and $a + \frac{1}{2} r a' \geq 0$. As a first step, the concentration is ruled by the following classical result:

Theorem 2 *If the* u_ϵ*'s develop a concentration, then*

1. $\lim_{\epsilon \to 0} u_\epsilon = 0$ *in* $C^2_{loc}(\overline{B} \backslash \{0\})$ *and* $\lim_{\epsilon \to 0} ||u_\epsilon||_\infty = +\infty$

2. $\lim_{\epsilon \to 0} {u_\epsilon}^{p+1-\epsilon} = \frac{\omega_N}{2^N} \delta_0$ *in the sense of distributions*

3. $\lim_{\epsilon \to 0} \frac{u_\epsilon(0)}{||u_\epsilon||_\infty} = 1$

where $||u_\epsilon||_\infty$ *is the* L^∞*-norm of* u_ϵ*.*

Given $k \in \mathbb{N}$, we now set

$$\alpha_{k,N}^{(1)} = \frac{2^{N+1}(k+2)\omega_{N-1}I_{k+N-1,2}}{(N-2)^3 k!\omega_N} \quad , \quad \alpha_{k,N}^{(2)} = \frac{2^{N+1}\omega_{N-1}I_{k+N-1,p+1}}{(N-2)(k-1)!\omega_N}$$

and

$$\alpha_N^{(1)} = \frac{2^{N+2}\omega_{N-1}}{(N-4)!(N-2)^3\omega_N} \quad , \quad \alpha_N^{(2)} = \frac{2^{N+1}\omega_{N-1}^3}{(N-2)\omega_N} \quad .$$

Generalizing the results of [AtPe], [BrPe], and [Heb2], the asymptotic behaviour of the u_ϵ's is ruled by the following result:

Theorem 3 *If the* u_ϵ*'s develop a concentration, then*

$$\lim_{\epsilon\to 0} u_\epsilon(0)u_\epsilon(x) = (N-2)\omega_{N-1}G(x,0)$$

in $C^2_{loc}(\overline{B}\backslash\{0\})$*, and:*

1. *If* $k_a < N-4$ *and*

(a) $k_f < k_a + 2$, *then* $\epsilon u_\epsilon(0)^{\frac{2k_f}{N-2}} \to -\alpha_{k_f,N}^{(2)} f^{(k_f)}(0)$

(b) $k_f = k_a + 2$, *then* $\epsilon u_\epsilon(0)^{\frac{2k_f}{N-2}} \to \alpha_{k_a,N}^{(1)} a^{(k_a)}(0) - \alpha_{k_a+2,N}^{(2)} f^{(k_a+2)}(0)$

(c) $k_f > k_a + 2$, *then* $\epsilon u_\epsilon(0)^{\frac{2(k_a+2)}{N-2}} \to \alpha_{k_a,N}^{(1)} a^{(k_a)}(0)$

2. *If* $k_a = N-4$ *and*

(a) $k_f < N-2$, *then* $\epsilon u_\epsilon(0)^{\frac{2k_f}{N-2}} \to -\alpha_{k_f,N}^{(2)} f^{(k_f)}(0)$

(b) $k_f \geq N-2$, *then* $\epsilon \dfrac{u_\epsilon(0)^2}{\ln u_\epsilon(0)} \to \alpha_N^{(1)} a^{(k_a)}(0)$

3. *If* $k_a > N-4$ *and*

(a) $k_f < N-2$, *then* $\epsilon u_\epsilon(0)^{\frac{2k_f}{N-2}} \to -\alpha_{k_f,N}^{(2)} f^{(k_f)}(0)$

(b) $k_f = N-2$, *then* $\epsilon u_\epsilon(0)^2 \to -\alpha_{N-2,N}^{(2)} f^{(N-2)}(0) + \alpha_N^{(2)} g'(1)^2 + 2\alpha_N^{(2)}\Phi(a)$

(c) $k_f > N-2$, *then* $\epsilon u_\epsilon(0)^2 \to \alpha_N^{(2)} g'(1)^2 + 2\alpha_N^{(2)}\Phi(a)$

where $\alpha_{k,N}^{(1)}$, $\alpha_{k,N}^{(2)}$, $\alpha_N^{(1)}$, $\alpha_N^{(2)}$, *are* $\Phi(a)$ *as above.*

The following sections are devoted to the proofs of these three theorems.

2 Elements from concentration theory

We let (u_ϵ) be a sequence of MRS solutions to (I_ϵ). In what follows, we suppose that

$$\lim_{\epsilon\to 0} \frac{\int_B \left(|\nabla u_\epsilon|^2 + a{u_\epsilon}^2\right) dx}{\left(\int_B f u_\epsilon^{p+1-\epsilon} dx\right)^{\frac{2}{p+1-\epsilon}}} = \frac{N(N-2)\omega_N^{\frac{2}{N}}}{4} \quad . \tag{2}$$

Note that the right hand side in this relation is the inverse of the square of the best constant $K(N,2)$ for the Sobolev inequality corresponding to the embedding of $H^1(\mathbb{R}^N)$ in $L^{p+1}(\mathbb{R}^N)$. We say that $x_0 \in \overline{B}$ is a concentration point of the u_ϵ's if for all $\delta > 0$,

$$\limsup_{\epsilon\to 0} \int_{B\cap B(x_0,\delta)} f(|x|) u_\epsilon^{p+1-\epsilon} dx > 0 \quad .$$

We suppose here that any subsequence of (u_ϵ) which converges almost everywhere converges to 0. Then, the u_ϵ's develop a concentration. Multiplying (I_ϵ) by u_ϵ and integrating by parts, we get that

$$\lim_{\epsilon\to 0} \int_B \left(|\nabla u_\epsilon|^2 + a{u_\epsilon}^2\right) dx = \frac{N(N-2)\omega_N}{2^N}$$

and

$$\lim_{\epsilon\to 0} \int_B f u_\epsilon^{p+1-\epsilon} dx = \frac{\omega_N}{2^N} \quad .$$

Since the operator $\Delta + a$ is coercive, the sequence (u_ϵ) is bounded in $H^1(B)$.

Given $x_0 \in \overline{B}$ and $\delta > 0$, we let $\eta \in C^\infty(\mathbb{R}^N)$ be a cut-off function such that $0 \le \eta \le 1$, $\eta = 1$ in $B(x_0, \delta/2)$, and $\eta = 0$ in $\mathbb{R}^N \backslash B(x_0,\delta)$. Multiplying (I_ϵ) by $\eta^2 {u_\epsilon}^k$, where $k \ge 1$, we easily obtain that

$$\begin{aligned} &\frac{4k}{(k+1)^2} \int_B \left|\nabla(\eta {u_\epsilon}^{\frac{k+1}{2}})\right|^2 dx - \frac{2(k-1)}{(k+1)^2} \int_B \eta(\Delta\eta) u_\epsilon^{k+1} dx \\ &- \frac{2}{k+1} \int_B |\nabla\eta|^2 u_\epsilon^{k+1} dx + \int_B a\eta^2 u_\epsilon^{k+1} dx = N(N-2) \int_B f(|x|)\eta^2 u_\epsilon^{k+p-\epsilon} dx \quad . \end{aligned}$$

The following result follows from this relation and our original assumption. It is by now classical, and we refer to [Heb2] or [Heb3] for its proof.

Lemma 2.1 *The following properties hold:*

1. $u_\epsilon \to 0$ *in* $L^q(B)$ *for all* $1 < q < p+1$, *in particular for* $q = 2$
2. *If* $x_0 \in \overline{B}$ *is a concentration point, then for all* $\delta > 0$,

$$f(x_0)^{1-\frac{2}{N}} \left(\limsup_{\epsilon\to 0} \int_{B\cap B(x_0,\delta)} f(|x|) u_\epsilon^{p+1-\epsilon} dx\right)^{\frac{2}{N}} \ge \frac{{\omega_N}^{\frac{2}{N}}}{4}$$

3. (u_ϵ) *possesses one and only one concentration point, the point* $x_0 = 0$

4. $\lim_{\epsilon\to 0} ||u_\epsilon||_{L^\infty(B)} = +\infty$

5. $\lim_{\epsilon\to 0} u_\epsilon = 0$ *in* $C^2_{loc}(\overline{B}\backslash\{0\})$

6. $\lim_{\epsilon\to 0} {u_\epsilon}^{p+1-\epsilon} = \frac{\omega_N}{2^N}\delta_0$ *in the sense of distributions*

In particular, if $x_\epsilon \in B$ *is such that* $u_\epsilon(x_\epsilon) = ||u_\epsilon||_{L^\infty(B)}$*, then* $\lim_{\epsilon\to 0} x_\epsilon = 0$.

Now we let $\mu_\epsilon^{-\frac{N-2}{2}} = ||u_\epsilon||_{L^\infty(B)}$, and, for $x \in B_\epsilon$, we set

$$V_\epsilon(x) = \mu_\epsilon^{\frac{N-2}{2}} u_\epsilon(x_\epsilon + k_\epsilon x)$$

where $k_\epsilon = \mu_\epsilon^{1-\frac{N-2}{4}\epsilon}$ and $B_\epsilon = B\left(\frac{-x_\epsilon}{k_\epsilon}, \frac{1}{k_\epsilon}\right)$. Clearly, $0 \le V_\epsilon \le 1$, $V_\epsilon(0) = 1$ and $\cup B_\epsilon = \mathbb{R}^N$. Moreover, V_ϵ is such that

$$\Delta V_\epsilon(x) + k_\epsilon^2 a(x_\epsilon + k_\epsilon x)V_\epsilon(x) = N(N-2)f(x_\epsilon + k_\epsilon x)V_\epsilon(x)^{p-\epsilon} \tag{3}$$

where $x \in B_\epsilon$ and $a(x) = a(|x|)$, $f(x) = f(|x|)$. An easy claim, see [Rob1] for details, is that the V_ϵ's converge C^2 to a function v on any compact subset, and that

$$\begin{cases} \Delta v = N(N-2)v^p \quad \text{in } \mathbb{R}^N \\ \\ 0 \le v \le 1\ , \quad v(0) = 1\ \ . \end{cases}$$

By Caffarelli, Gidas and Spruck [CGS], it follows that

$$v(x) = \left(\frac{1}{1+|x|^2}\right)^{\frac{N-2}{2}} \quad .$$

Then we have the following result. We refer to [Rob1] for its proof.

Lemma 2.2 *The two following properties hold:*

1. $\lim_{\epsilon\to 0} V_\epsilon = v$ *in* $L^{p+1}\left(\mathbb{R}^N\right)$

2. $\lim_{\epsilon\to 0} \mu_\epsilon^\epsilon = 1$

where μ_ϵ, V_ϵ, v *are as above, and* V_ϵ *is extended by* 0 *outside* B_ϵ.

3 An asymptotic estimate

As in section 2, we assume that the u_ϵ's develop a concentration. Our main goal here is to establish the following fondamental estimate:

Proposition 1 *For all x in B,*

$$u_\epsilon(x) \leq A \left(\frac{\mu_\epsilon}{{\mu_\epsilon}^2 + |x - x_\epsilon|^2} \right)^{\frac{N-2}{2}} \tag{4}$$

where $A > 0$ is a constant independent of x and ϵ.

Such an estimate was obtained by Han [Han] and Hebey [Heb2] when $a \equiv 0$, and by [Rob1] under the assumption that $||a_-||_{L^{\frac{N}{2}}(B)}$ is small enough. As already mentioned, the linear part au, and more precisely the negative part $a_- u$ of au, makes that we have to face a much more critical situation than in [Han] and [Heb2]. Several steps that we detail in this section are involved in the proof of this result.

3.1 A first estimate

As a first step in the proof of the proposition, we prove the following:

Lemma 3.1 *Given (c_ϵ) a sequence of real numbers which has a limit as $\epsilon \to 0$,*

$$|x - x_\epsilon|^{\frac{N-2}{2} + c_\epsilon \epsilon} u_\epsilon(x) \leq A \tag{5}$$

for all $\epsilon > 0$, and all $x \in B$, where $A > 0$ is a constant which does not depend on ϵ and x. Moreover, for all $R > 0$, there exists $\epsilon(R) > 0$ such that

$$\forall x \in B,\ |x - x_\epsilon| \geq R\mu_\epsilon \Rightarrow |x - x_\epsilon|^{\frac{N-2}{2} + c_\epsilon \epsilon} u_\epsilon(x) \leq \epsilon(R)$$

where $\lim_{R \to +\infty} \epsilon(R) = 0$.

Proof: We use arguments that were developed by Druet [Dru]. For $x \in B$, we set

$$w_\epsilon(x) = |x - x_\epsilon|^{\frac{N-2}{2} + c_\epsilon \epsilon} u_\epsilon(x)$$

and let y_ϵ be a point such that $w_\epsilon(y_\epsilon) = ||w_\epsilon||_\infty$. We assume by contradiction that $w_\epsilon(y_\epsilon) \to \infty$. Then $y_\epsilon \to 0$. We write

$$\begin{aligned} w_\epsilon(y_\epsilon) &= |y_\epsilon - x_\epsilon|^{\frac{N-2}{2} + c_\epsilon \epsilon} u_\epsilon(y_\epsilon) \\ &\leq |y_\epsilon - x_\epsilon|^{\frac{N-2}{2} + c_\epsilon \epsilon} u_\epsilon(x_\epsilon) \\ &\leq |y_\epsilon - x_\epsilon|^{\frac{N-2}{2} + c_\epsilon \epsilon} {\mu_\epsilon}^{-\left(\frac{N-2}{2} + c_\epsilon \epsilon\right)} {\mu_\epsilon}^{c_\epsilon \epsilon} \quad . \end{aligned}$$

It follows that

$$\frac{|y_\epsilon - x_\epsilon|}{\mu_\epsilon} \to +\infty \ .$$

Let $k'_\epsilon = u_\epsilon(y_\epsilon)^{-\frac{2}{N-2}+\frac{\epsilon}{2}}$. Since $u_\epsilon(y_\epsilon) \to +\infty$, we get that $k'_\epsilon \to 0$. For $x \in B\left(-\frac{y_\epsilon}{k'_\epsilon}, \frac{1}{k'_\epsilon}\right)$, we set

$$\overline{u}_\epsilon(x) = u_\epsilon(y_\epsilon)^{-1} u_\epsilon(y_\epsilon + k'_\epsilon x) \ .$$

As one easily checks,

$$\Delta \overline{u}_\epsilon(x) + {k'_\epsilon}^2 a(y_\epsilon + k'_\epsilon x)\overline{u}_\epsilon(x) = N(N-2) f(y_\epsilon + k'_\epsilon x)\bar{u}_\epsilon(x)^{p-\epsilon}$$

for all $x \in B\left(-\frac{y_\epsilon}{k'_\epsilon}, \frac{1}{k'_\epsilon}\right)$. For ϵ small, $1 \le u_\epsilon(y_\epsilon) \le \mu_\epsilon^{-\frac{N-2}{2}}$, and then $u_\epsilon(y_\epsilon)^\epsilon \to 1$. Now, take $x \in B(0,2)$. For ϵ sufficiently small, $B(0,2) \subset B\left(-\frac{y_\epsilon}{k'_\epsilon}, \frac{1}{k'_\epsilon}\right)$, and

$$\begin{aligned} |x_\epsilon - y_\epsilon - k'_\epsilon x| &\ge |x_\epsilon - y_\epsilon| - |k'_\epsilon x| \\ &\ge |x_\epsilon - y_\epsilon| \left(1 - 2\frac{k'_\epsilon}{|x_\epsilon - y_\epsilon|}\right) \ge \frac{1}{2}|x_\epsilon - y_\epsilon| \end{aligned}$$

since $\frac{k'_\epsilon}{|x_\epsilon - y_\epsilon|} \to 0$. Taking $x \in B(0,2)$,

$$\begin{aligned} u_\epsilon(y_\epsilon + k'_\epsilon x) &= \frac{w_\epsilon(y_\epsilon + k'_\epsilon x)}{|x_\epsilon - y_\epsilon - k'_\epsilon x|^{\frac{N-2}{2}+c_\epsilon \epsilon}} \\ &\le 2^{\frac{N-2}{2}+c_\epsilon \epsilon} \frac{w_\epsilon(y_\epsilon)}{|x_\epsilon - y_\epsilon|^{\frac{N-2}{2}+c_\epsilon \epsilon}} \\ &= 2^{\frac{N-2}{2}+c_\epsilon \epsilon} u_\epsilon(y_\epsilon) \ . \end{aligned}$$

As a consequence, $\bar{u}_\epsilon(x) \le 2^{\frac{N}{2}}$ for ϵ small and all $x \in B(0,2)$. Independently,

$$\int_{B(0,2)} \bar{u}_\epsilon^{p+1} dx = u_\epsilon(y_\epsilon)^{-\epsilon\frac{N}{2}} \int_{B(y_\epsilon, 2k'_\epsilon)} u_\epsilon^{p+1} dx$$

while

$$B(y_\epsilon, 2k'_\epsilon) \cap B(x_\epsilon, R\mu_\epsilon) = \emptyset$$

for all $R > 0$, as soon as ϵ is small enough. From lemma 2.2, we easily get that

$$\int_{B(x_\epsilon, R\mu_\epsilon)^c} u_\epsilon^{p+1} dx \to \int_{B(0,R)^c} v^{p+1} dx \ .$$

It follows that for all $R > 0$,

$$\limsup \int_{B(0,2)} \bar{u}_\epsilon^{p+1} dx \le \int_{B(0,R)^c} v^{p+1} dx$$

and then

$$\int_{B(0,2)} \bar{u}_\epsilon^{p+1} dx \to 0 \ .$$

In other words, $\bar{u}_\epsilon \to 0$ in $L^{p+1}(B(0,2))$, and $(\bar{u}_\epsilon)$ is bounded. Coming back to the equation satisfied by $\bar{u}_\epsilon$, and by standard elliptic theory, it follows that $\bar{u}_\epsilon \to 0$ in $C^0(B(0,1))$, a contradiction with the relation $\bar{u}_\epsilon(0) = 1$. This ends the proof of the first part of the lemma.

To prove the second part of the lemma, we assume that there exists $R_0, \epsilon_0 > 0$ such that

$$\sup_{|x-x_\epsilon| \geq R\mu_\epsilon} w_\epsilon \geq \epsilon_0$$

for all $R > R_0$. Taking a smaller ball than $B(0,2)$, one obtains the result following the preceding proof, and the lemma is proved. □

Note that one of the consequences of lemma 3.1 is that $V_\epsilon(x) \leq A|x|^{-\frac{N-2}{2}}$ for all $x \in B_\epsilon \backslash \{0\}$.

3.2 An estimate for x_ϵ

We prove in this subsection the following result:

Lemma 3.2 $|x_\epsilon| = o(k_\epsilon)$.

Proof: Since u_ϵ is radially symmetrical, $\int_B x^i {u_\epsilon}^k dx = 0$ for all $i = 1, \ldots, N$ and all $k \in \mathbb{N}$. Noting that

$$\int_B x^i {u_\epsilon}^k dx = \frac{k_\epsilon^N}{{\mu_\epsilon}^{k\frac{N-2}{2}}} \int_{B_\epsilon} (x_\epsilon^i + k_\epsilon z^i) V_\epsilon^k dz$$

this leads to

$$\frac{x_\epsilon^i}{k_\epsilon} \int_{B_\epsilon} {V_\epsilon}^k dz + \int_{B_\epsilon} z^i {V_\epsilon}^k dz = 0 \ .$$

By lemma 3.1, $V_\epsilon(x) \leq A|x|^{-\frac{N-2}{2}}$ for all $x \in B_\epsilon \backslash \{0\}$. Choosing k such that $k > \frac{2(N+1)}{N-2}$, and since v is radially symmetrical, we get with Lebesgue's dominated convergence theorem that

$$\begin{aligned} \int_{B_\epsilon} {V_\epsilon}^k dz \ &\to \ \int_{\mathbb{R}^N} v^k dz > 0 \\ \int_{B_\epsilon} z^i {V_\epsilon}^k dz \ &\to \ \int_{\mathbb{R}^N} z^i v^k dz = 0 \ . \end{aligned}$$

It follows that $x_\epsilon^i = o(k_\epsilon)$ for all i, a relation from which the lemma easily follows. □

3.3 A second estimate

We let v_ϵ be defined by

$$v_\epsilon(x) = \mu_\epsilon^{\frac{N-2}{2}} u_\epsilon(k_\epsilon x) \quad .$$

Clearly, v_ϵ is radially symmetrical. A priori, and contrary to V_ϵ, $v_\epsilon(0)$ does not equal 1. On the other hand, writing $v_\epsilon(x) = V_\epsilon(x - \frac{x_\epsilon}{k_\epsilon})$, and according to lemma 3.2, we see that $v_\epsilon(0) \to 1$. In particular, this proves the third part of theorem 2:

Lemma 3.3 $\lim_{\epsilon\to 0} \frac{u_\epsilon(0)}{||u_\epsilon||_\infty} = 1$.

More generally, $v_\epsilon \to v$ in $C^2(K)$ for all compact K in $\mathbb{R}^N$, where v_ϵ is extended by 0 outside $B\left(0, \frac{1}{k_\epsilon}\right)$. Moreover, v_ϵ satisfies in $B\left(0, \frac{1}{k_\epsilon}\right)$ the equation

$$\Delta v_\epsilon + k_\epsilon^2 a(k_\epsilon x) v_\epsilon = N(N-2) f(k_\epsilon x) v_\epsilon^{p-\epsilon} \quad .$$

As easily seen, V_ϵ has the same properties than v_ϵ. In particular, $v_\epsilon(x) \le A|x|^{-\frac{N-2}{2}}$ for all x in $B\left(0, \frac{1}{k_\epsilon}\right) \backslash \{0\}$. We prove here the following result:

Lemma 3.4 *For all $\nu \in]0, N-2[$, there exists $C(\nu) > 0$ such that:*

$$|x|^{n-2-\nu} \mu_\epsilon^{-\frac{N-2}{2}+\nu} u_\epsilon(x) \le C(\nu), \ \forall x \in B \tag{6}$$

in other words, $\forall \nu \in]0, N-2[$, there exists $C(\nu) > 0$ such that:

$$v_\epsilon(x) \le \frac{C(\nu)}{|x|^{N-2-\nu}}, \quad \forall x \in B\left(0, \frac{1}{k_\epsilon}\right) \backslash \{0\} \quad .$$

In particular, $\int_{\mathbb{R}^N} v_\epsilon^{p-\epsilon}\, dx$ is bounded if we take $\nu \le \frac{2}{p}$.

Proof: We basically follow the proof of [DrRo]. It is clear that (6) is true on any ball $B(0, R\mu_\epsilon)$ when ϵ goes to 0. We prove the lemma by comparing u_ϵ with another function throughout with a maximum principle. Choose $\epsilon_0 > 0$ and $0 < \eta < 1$ and set $\tilde{a} = \frac{a-\epsilon_0}{1-\eta}$ such that the operator $u \mapsto \Delta u + \tilde{a} u$ is coercive on B. We denote by $\tilde{G}$ the Green's function for this operator with Dirichlet condition on the boundary. We now let L_ϵ be the operator

$$L_\epsilon u = \Delta u + au - N(N-2) f u_\epsilon^{p-1-\epsilon} u \quad .$$

Some computations lead to

$$L_\epsilon \tilde{G}(0,x)^{1-\eta} = \tilde{G}(0,x)^{1-\eta} \left(\epsilon_0 - N(N-2) f u_\epsilon^{p-1-\epsilon} + \eta(1-\eta) \left(\frac{|\nabla \tilde{G}|}{\tilde{G}} \right) (0,x)^2 \right)$$

for all $x \in B\backslash\{0\}$. Standard properties of the Green's function assert that there exists $C_1 > 0$, $\delta \in]0,1[$ such that

$$\frac{|\nabla \tilde{G}|}{\tilde{G}}(0,x) \geq \frac{C_1}{|x|}, \quad \forall x \in B(0,\delta)\backslash\{0\} \ .$$

Such a property, and the following ones, follows from the construction of the Green's function. A possible reference where such properties are proved in details is Robert [Rob2]. We also refer to Aubin [Aub].

With lemma 3.1, we know that there exists $\epsilon(R) > 0$ such that

$$|x|^2 u_\epsilon^{p-1-\epsilon}(x) \leq \epsilon(R), \quad \forall |x| \geq R\mu_\epsilon$$

with $\lim_{R\to+\infty} \epsilon(R) = 0$. Then, on $B(0,\delta)\backslash B(0,R\mu_\epsilon)$, we have

$$\begin{aligned} L_\epsilon \tilde{G}(0,x)^{1-\eta} &\geq \tilde{G}(0,x)^{1-\eta}\left(\epsilon_0 - N(N-2)f\tfrac{\epsilon(R)}{|x|^2} + \eta(1-\eta)\tfrac{C_1^2}{|x|^2}\right) \\ &\geq \left(\tilde{G}(0,x)^{1-\eta}\right)\tfrac{\eta(1-\eta)C_1^2 - N(N-2)||f||_\infty \epsilon(R)}{|x|^2} \geq 0 \end{aligned}$$

with R large enough. Now, on $B\backslash B(0,\delta)$,

$$L_\epsilon \tilde{G}(0,x)^{1-\eta} \geq \tilde{G}(0,x)^{1-\eta}\left(\epsilon_0 - N(N-2)f||u_\epsilon||^{p-1-\epsilon}_{L^\infty(\overline{B}-B(0,\delta))}\right) \ .$$

Since $u_\epsilon \to 0$ in $C^0_{loc}(B - \backslash\{0\})$, we get

$$L_\epsilon \tilde{G}(0,x)^{1-\eta} \geq 0 \text{ in } B\backslash B(0,\delta) \ .$$

We finally obtain that

$$L_\epsilon \tilde{G}(0,x)^{1-\eta} \geq 0 \text{ in } B\backslash B(0,R\mu_\epsilon) \ .$$

Now, there exists $C_2 > 0$ and $\delta' \in]0,1[$ such that

$$\tilde{G}(0,x) \geq \frac{C_2}{|x|^{N-2}}, \quad \forall x \in B(0,\delta')$$

if we set

$$C_\epsilon = \frac{C_2^{1-\eta}}{R^{(N-2)(1-\eta)}}\mu_\epsilon^{-\frac{N-2}{2}+(N-2)\eta},$$

we get

$$\begin{array}{ll} L_\epsilon C_\epsilon \tilde{G}(0,x)^{1-\eta} \geq L_\epsilon u_\epsilon & \text{in } B\backslash B(0,R\mu_\epsilon) \\ C_\epsilon \tilde{G}(0,x)^{1-\eta} \geq u_\epsilon & \text{on } \partial(B\backslash B(0,R\mu_\epsilon)) \ . \end{array}$$

Since there exists $C_3 > 0$ such that $|x|^{n-2}\tilde{G}(0,x) \leq C_3$, $\forall x \in \overline{B}\backslash\{0\}$, we obtain the lemma with $\nu = (N-2)\eta$ if

$$C_\epsilon \tilde{G}(0,x)^{1-\eta} \geq u_\epsilon \text{ in } B\backslash B(0,R\mu_\epsilon).$$

To obtain this inequality, we just have to prove the following maximum principle for L_ϵ on $\Omega_\epsilon = B\backslash B(0, R\mu_\epsilon)$: if $\varphi \in C^2(\Omega_\epsilon) \cap C^0(\overline{\Omega_\epsilon})$ verifies $L_\epsilon\varphi \geq 0$ in Ω_ϵ and $\varphi \geq 0$ on $\partial\Omega_\epsilon$, then $\varphi \geq 0$ in $\overline{\Omega_\epsilon}$. We now prove that L_ϵ is coercive on Ω_ϵ, which will prove the maximum principle we need. Since $\Delta + a$ is coercive on B, we know that there exists $\lambda > 0$ such that

$$\int_B |\nabla\varphi|^2\,dx + \int_B a\varphi^2\,dx \geq \lambda\,||\varphi||^2_{L^{p+1}(B)}, \quad \forall\varphi \in \mathcal{D}(B)\ .$$

Now we take $\varphi \in \mathcal{D}(\Omega_\epsilon)$. We get

$$\begin{aligned}\int_{\Omega_\epsilon} (L_\epsilon\varphi)\varphi\,dx \quad &\geq \quad \int_B |\nabla\varphi|^2\,dx + \int_B a\varphi^2\,dx \\ &\quad -N(N-2)||f||_\infty \int_{B\backslash B(0,R\mu_\epsilon)} u_\epsilon^{p-1-\epsilon}\varphi^2\,dx\ .\end{aligned}$$

But

$$\begin{aligned}&\int_{B\backslash B(0,R\mu_\epsilon)} u_\epsilon^{p-1-\epsilon}\varphi^2\,dx \\ &\leq \left(\int_{B\backslash B(0,R\mu_\epsilon)} u_\epsilon^{p+1-\epsilon}\,dx\right)^{\frac{p-1-\epsilon}{p+1-\epsilon}} \left(\int_B |\varphi|^{p+1-\epsilon}\,dx\right)^{\frac{2}{p+1-\epsilon}}\ .\end{aligned}$$

Since $\int_B u_\epsilon^{p+1-\epsilon}\,dx = 1$, $\frac{\int_{B(0,R\mu_\epsilon)} u_\epsilon^{p+1-\epsilon}\,dx}{\int_{B(0,R)} v_\epsilon^{p+1-\epsilon}\,dx} \to 1$ and $v_\epsilon \to v$ in $C^0_{loc}(\mathbb{R}^N)$, we obtain that there exists $\epsilon_0(R) > 0$ such that $\epsilon_0(R) \to 0$ when R goes to $+\infty$ and

$$\int_{B\backslash B(0,R\mu_\epsilon)} u_\epsilon^{p-1-\epsilon}\varphi^2\,dx \leq \epsilon_0(R)||\varphi||^2_{L^{p+1}(B)}\ .$$

Then

$$\int_{\Omega_\epsilon} (L_\epsilon\varphi)\varphi\,dx \geq (\lambda - \epsilon_0(R))||\varphi||^2_{L^{p+1}(B)}\ .$$

Choosing R large enough, we then obtain that the operator L_ϵ is coercive on Ω_ϵ. Consequently, it verifies the maximum principle stated above, and the lemma is proved. □

3.4 Proof of proposition 1

We now prove proposition 1. As one may easily check, estimate (3) is equivalent to the existence of a constant A such that for all $\epsilon > 0$ and all $x \in B$,

$$|x|^{N-2}u_\epsilon(x_\epsilon)u_\epsilon(x) \leq A\ . \tag{7}$$

(Here, we use the fact that $x_\epsilon = o(k_\epsilon)$). Let $y_\epsilon \in B$ be a point where $x \mapsto |x|^{N-2} u_\epsilon(x)$ achieves its maximum. In order to prove (7), we assume by contradiction that $|x|^{N-2} u_\epsilon(x_\epsilon) u_\epsilon(x)$ is unbounded. Up to a subsequence, we get that

$$|y_\epsilon|^{N-2} u_\epsilon(x_\epsilon) u_\epsilon(y_\epsilon) \to +\infty \quad . \tag{8}$$

Without loss of generality, up to another subsequence, we can assume that $y_\epsilon \to y_0$ in $\overline{B}$. As a first remark, we claim that $|y_0| < 1$. We prove this claim by contradiction, and assume that $|y_0| = 1$. We let

$$z_\epsilon(x) = \frac{u_\epsilon(x)}{u_\epsilon(y_\epsilon)} \quad .$$

The equation satisfied in B by z_ϵ is

$$\Delta z_\epsilon + a(x) z_\epsilon = N(N-2) f(x) u_\epsilon(y_\epsilon)^{p-1-\epsilon} z_\epsilon^{p-\epsilon}$$

and z_ϵ is radially symmetrical. Since $|x|^{N-2} u_\epsilon(x)$ achieves its maximum at $x = y_\epsilon$, we get that

$$z_\epsilon(x) \leq \frac{|y_\epsilon|^{N-2}}{|x|^{N-2}}$$

and z_ϵ is bounded on any compact subset of $\overline{B} \backslash \{0\}$. By point 5. of lemma 2.1, $u_\epsilon(y_\epsilon) \to 0$, since if $|y_0| = 1$, then $y_0 \neq 0$. By standard elliptic theory, see for instance [GT], it follows that (z_ϵ) is actually $C^{1,\alpha}$-bounded in any compact subset of $\overline{B} \backslash \{0\}$. In particular, since $y_0 \in \partial B$ and $z_\epsilon = 0$ on ∂B,

$$|z_\epsilon(y_\epsilon)| = |z_\epsilon(y_\epsilon) - z_\epsilon(y_0)| \leq A |y_\epsilon - y_0|$$

where $A > 0$ does not depend on ϵ. A contradiction, since $z_\epsilon(y_\epsilon) = 1$. This proves the above claim.

Now we set $y_\epsilon = k_\epsilon \hat{x}_\epsilon$. As another remark, we claim that $|\hat{x}_\epsilon| \to +\infty$. If not, then, up to another subsequence,

$$\begin{aligned} |y_\epsilon|^{N-2} u_\epsilon(x_\epsilon) u_\epsilon(y_\epsilon) &= k_\epsilon^{N-2} |\hat{x}_\epsilon|^{N-2} \mu_\epsilon^{-\frac{N-2}{2}} u_\epsilon(k_\epsilon \hat{x}_\epsilon) \\ &\approx |\hat{x}_\epsilon|^{N-2} \mu_\epsilon^{\frac{N-2}{2}} u_\epsilon(k_\epsilon \hat{x}_\epsilon) \\ &= |\hat{x}_\epsilon|^{N-2} v_\epsilon(\hat{x}_\epsilon) \end{aligned}$$

which is bounded since v_ϵ uniformly converges on any compact subset of $\mathbb{R}^N$. This proves the claim.

Now, let G be the Green's function for the operator $\Delta + a$, as defined in the introduction. In addition to be radially symmetrical, one of its classical properties is that for all compact subset $K \subset B$, there exists a constant $A > 0$ such that for all $x \in K$ and all $y \in B$,

$$|y - x|^{N-2} G(x, y) \leq A \quad .$$

Then, we write

$$u_\epsilon(y_\epsilon) = \int_B G(y_\epsilon, \tilde{x}) \left(\Delta u_\epsilon(\tilde{x}) + a(\tilde{x}) u_\epsilon(\tilde{x})\right) d\tilde{x} \quad .$$

From the equation satisfied by u_ϵ, the equivalence of k_ϵ and μ_ϵ, and the change of variable $\tilde{x} = k_\epsilon x$, it follows that

$$u_\epsilon(y_\epsilon) \approx N(N-2)\mu_\epsilon^{\frac{N-2}{2}} \int_{B(0,\frac{1}{k_\epsilon})} f(k_\epsilon x) v_\epsilon^{p-\epsilon}(x) G(y_\epsilon, k_\epsilon x) dx$$

and then that

$$u_\epsilon(x_\epsilon) u_\epsilon(y_\epsilon) \le A \int_{B(0,\frac{1}{k_\epsilon})} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx$$

where A does not depend on ϵ. Let us now define

$$\Omega_\epsilon^1 = \left\{ x \in B\left(0, \frac{1}{k_\epsilon}\right) \ / \ |y_\epsilon - k_\epsilon x| \ge \frac{1}{2}|y_\epsilon| \right\},$$

$$\Omega_\epsilon^2 = \left\{ x \in B\left(0, \frac{1}{k_\epsilon}\right) \ / \ |y_\epsilon - k_\epsilon x| < \frac{1}{2}|y_\epsilon| \right\} \quad .$$

We write

$$\begin{aligned} \int_{B(0,\frac{1}{k_\epsilon})} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx &= \int_{\Omega_\epsilon^1} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx \\ &\quad + \int_{\Omega_\epsilon^2} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx \quad . \end{aligned}$$

According to the above mentioned property of the Green's function, and since $|y_0| < 1$ so that the y_ϵ's are in a compact subset of B,

$$\begin{aligned} \int_{\Omega_\epsilon^1} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx &\le A \int_{\Omega_\epsilon^1} \frac{v_\epsilon^{p-\epsilon}(x)}{|y_\epsilon - k_\epsilon x|^{N-2}} dx \\ &\le \frac{2^{N-2}}{|y_\epsilon|^{N-2}} A \int_{B(0,\frac{1}{k_\epsilon})} v_\epsilon^{p-\epsilon}(x) dx \quad . \end{aligned}$$

Together with the remark we made at the end of lemma 6, we get that

$$\int_{\Omega_\epsilon^1} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx \le \frac{A}{|y_\epsilon|^{N-2}} \quad .$$

Similarly,

$$\int_{\Omega_\epsilon^2} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx \le A \int_{\Omega_\epsilon^2} \frac{v_\epsilon^{p-\epsilon}(x)}{|y_\epsilon - k_\epsilon x|^{N-2}} dx$$

and if $\Omega_\epsilon = \left\{ x \ / \ |x| < \frac{1}{2}|y_\epsilon| \right\}$, then, with the change of variable $y = k_\epsilon x - y_\epsilon$,

$$\int_{\Omega_\epsilon^2} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx \le \frac{A}{k_\epsilon^N} \int_{\Omega_\epsilon} \frac{1}{|y|^{N-2}} v_\epsilon^{p-\epsilon} \left(\frac{y + y_\epsilon}{k_\epsilon} \right) dy \quad .$$

Since $|\frac{y+y_\epsilon}{k_\epsilon}| \ge \frac{1}{2}|\hat{x}_\epsilon|$, and by lemma 6,

$$\begin{aligned} \frac{1}{k_\epsilon^N} \int_{\Omega_\epsilon} \frac{1}{|y|^{N-2}} v_\epsilon^{p-\epsilon} \left(\frac{y+y_\epsilon}{k_\epsilon} \right) dy \quad &\le \quad \frac{A}{|\hat{x}_\epsilon|^{(N-2-\nu)(p-\epsilon)} k_\epsilon^N} \int_{\Omega_\epsilon} \frac{1}{|y|^{N-2}} dy \\ &\le \quad \frac{A}{|\hat{x}_\epsilon|^{(N-2-\nu)(p-\epsilon)} k_\epsilon^N} \int_0^{\frac{1}{2}|y_\epsilon|} t dt \\ &\le \quad \frac{A|y_\epsilon|^2}{|\hat{x}_\epsilon|^{(N-2-\nu)(p-\epsilon)} k_\epsilon^N} \quad . \end{aligned}$$

Since $k_\epsilon \le |y_\epsilon| \le 1$, we get with lemma 2.2 that $|y_\epsilon|^\epsilon \to 1$. It follows that $|\hat{x}_\epsilon|^\epsilon \to 1$, and we can write that

$$\frac{|y_\epsilon|^2}{|\hat{x}_\epsilon|^{(N-2-\nu)(p-\epsilon)} k_\epsilon^N} \le \frac{A}{|\hat{x}_\epsilon|^{2-p\nu} |y_\epsilon|^{N-2}} \quad .$$

Choosing ν such that $\nu < \frac{2}{p}$, we obtain that

$$\int_{\Omega_\epsilon^2} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx \le \frac{o(1)}{|y_\epsilon|^{N-2}} \quad .$$

It follows that

$$\begin{aligned} |y_\epsilon|^{N-2} u_\epsilon(x_\epsilon) u_\epsilon(y_\epsilon) \quad &\le \quad A|y_\epsilon|^{N-2} \int_{B(0,\frac{1}{k_\epsilon})} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx \\ &\le \quad A|y_\epsilon|^{N-2} \int_{\Omega_\epsilon^1} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx \\ &\qquad + A|y_\epsilon|^{N-2} \int_{\Omega_\epsilon^2} G(y_\epsilon, k_\epsilon x) v_\epsilon^{p-\epsilon}(x) dx \\ &\le \quad A + o(1) \end{aligned}$$

which contradicts (8). It follows that (7) is true, and then (3) is also true. The proposition is proved. □

Now that proposition 1 is proved, we go on with the study of the asymptotic behaviour of the u_ϵ's. This is the aim of the following section, where the first assertion in theorem 3 is proved.

4 Convergence to the Green's function

Here again, we assume that the u_ϵ's develop a concentration. First, we recall a result obtained by Brézis and Peletier [BrPe]:

Lemma 4.1 *Let u be a C^2 solution of*

$$\begin{cases} \Delta u = f & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

and let ω be a neighbourhood of ∂B. Then

$$||u||_{W^{1,q}(B)} + ||\nabla u||_{C^{0,\beta}(\omega')} \le A \left(||f||_{L^1(B)} + ||f||_{L^\infty(\omega)} \right)$$

for all $q < \frac{N}{N-1}$, all $0 < \beta < 1$, and all $\omega' \subset\subset \omega$.

Note that it follows from this result that

$$\int_{\partial B} |\nabla u_\epsilon|^2 d\sigma = O({\mu_\epsilon}^{N-2}) \ .$$

By lemma 4.1 we indeed just need to get estimates for the L^1-norm in B and the L^∞-norm in a neighbourhood of ∂B, of the function g_ϵ given by

$$g_\epsilon(x) = N(N-2)f(x)u_\epsilon(x)^{p-\epsilon} - a(x)u_\epsilon(x) \ .$$

As easily seen, these estimates follow from proposition 1.

Now we prove the first assertion in theorem 3. This is the aim of the following lemma where, as in the introduction, G denotes the Green's function of the operator $\Delta + a$.

Lemma 4.2 $\lim_{\epsilon\to 0} u_\epsilon(x_\epsilon)u_\epsilon(x) = (N-2)\omega_{N-1}G(x,0)$ *in* $C^2_{loc}(\overline{B}\backslash\{0\})$.

Proof: We use the same method as in the proof of proposition 1. See [Rob1] for details.

5 Convergence to a solution

In this section, we consider a sequence of functions $(\tilde{u_\epsilon})$ such that

$$\begin{cases} \Delta \tilde{u_\epsilon} + a\tilde{u_\epsilon} = N(N-2)\lambda_\epsilon f(x){\tilde{u_\epsilon}}^{p-\epsilon} \ \text{ in } B \\ \\ \tilde{u_\epsilon} > 0 \ \text{ in } B \ \text{ and } \ \tilde{u_\epsilon} = 0 \ \text{ on } \partial B \\ \\ N(N-2)\int_B f(x){\tilde{u_\epsilon}}^{p+1-\epsilon}dx = 1 \end{cases}$$

where

$$\lambda_\epsilon = \inf_{v\in\mathcal{D}(B)_R\backslash\{0\}} \frac{\int_B \left(|\nabla v|^2 + av^2\right) dx}{\left(N(N-2)\int_B f|v|^{p+1-\epsilon}dx\right)^{\frac{2}{p+1-\epsilon}}} \quad .$$

We set

$$\lambda = \inf_{v\in\mathcal{D}(B)_R\backslash\{0\}} \frac{\int_B \left(|\nabla v|^2 + av^2\right) dx}{\left(N(N-2)\int_B f|v|^{p+1}dx\right)^{\frac{2}{p+1}}} \quad .$$

The following results are by now classical. We therefore restrict ourselves to brief comments on their proofs. For details, see for instance [Heb2].

Lemma 5.1 $\lim_{\epsilon\to 0}\lambda_\epsilon = \lambda$.

Proof: Let $u\in\mathcal{D}(B)_R\backslash\{0\}$. By Hölder's inequality,

$$\left(N(N-2)\int_B f|u|^{p+1-\epsilon}dx\right)^{\frac{2}{p+1-\epsilon}}$$
$$\leq Vol(B)^{\frac{2\epsilon}{(p+1)(p+1-\epsilon)}}\left(N(N-2)\int_B f|u|^{p+1}dx\right)^{\frac{2}{p+1}} \quad .$$

It follows that $\lambda \leq \liminf_{\epsilon\to 0}\lambda_\epsilon$. Conversely, let $\alpha > 0$ be any positive real number, and let $u\in\mathcal{D}(B)_R\backslash\{0\}$ be such that

$$\frac{\int_B \left(|\nabla u|^2 + au^2\right) dx}{\left(N(N-2)\int_B f|u|^{p+1}dx\right)^{\frac{2}{p+1}}} < \lambda + \alpha \quad .$$

Clearly, when $\epsilon\to 0$,

$$\frac{\int_B \left(|\nabla u|^2 + au^2\right) dx}{\left(N(N-2)\int_B f|u|^{p+1-\epsilon}dx\right)^{\frac{2}{p+1-\epsilon}}} \longrightarrow \frac{\int_B \left(|\nabla u|^2 + au^2\right) dx}{\left(N(N-2)\int_B f|u|^{p+1}dx\right)^{\frac{2}{p+1}}} \quad .$$

We then obtain that $\limsup_{\epsilon\to 0}\lambda_\epsilon \leq \lambda+\alpha$. Since $\alpha > 0$ is arbitrary, the result follows. □

We now state the following result.

Lemma 5.2 *: Assume that a subsequence of* $(\tilde{u_\epsilon})$ *converges almost everywhere to a function* $\tilde{u}\neq 0$*. Then:*

1. $\tilde{u}$ *is a MRS solution of the problem*

$$(\star)\quad \begin{cases} \Delta u + a(x)u = N(N-2)\lambda f(x)u^p & \text{in } B \\ u > 0 \ \text{ in } B\ , \quad \text{and } u = 0 & \text{on } \partial B \end{cases}$$

2. $\lim_{\epsilon\to 0}\tilde{u_\epsilon} = \tilde{u}$ *in* $C^2(\overline{B})$.

Proof: Point 1 easily follows from classical arguments of variational theory, like the ones developed, for example, in the study of the Yamabe problem. We first prove that $\tilde{u}$ is a solution of $(\star)$, and then that $\tilde{u}$ is minimizing. Point 2 follows from classical arguments of elliptic theory. $\square$

At last, we state the following result.

Lemma 5.3 *We always have $\lambda \leq \frac{1}{4}\left(N(N-2)\omega_N\right)^{\frac{2}{N}}$, and if this inequality is strict, then, up to a subsequence, $\tilde{u_\epsilon}$ converges almost everywhere to a function $\tilde{u} \neq 0$. Together with lemma 5.2, the convergence is then C^2, and $\tilde{u}$ is a MRS solution of problem $(\star)$.*

Proof: Here again, the result follows from classical variational arguments. We obtain the first assertion with the function z_ϵ given by

$$z_\epsilon(x) = \frac{\phi(|x|)}{\left(\epsilon^2 + |x|^2\right)^{\frac{N-2}{2}}}$$

where ϕ is a cut-off function that equals 1 around 0. As $\epsilon \to 0$, we get indeed that

$$\frac{\int_B \left(|\nabla z_\epsilon|^2 + a z_\epsilon^2\right) dx}{\left(N(N-2)\int_B f|z_\epsilon|^{p+1} dx\right)^{\frac{2}{p+1}}} \longrightarrow \frac{\left(N(N-2)\omega_N\right)^{\frac{2}{N}}}{4} \quad .$$

For the second assertion, the energy associated to the problem goes under the critical energy. The fact that the $\tilde{u_\epsilon}$'s do not develop a concentration under such an assumption is by now classical. $\square$

6 Proof of the theorems

For length reasons, some details are omited in this section. They can be found in [Rob1]. Theorem 2 immediately follows from what we said in section 2, and from lemma 3.3. The first assertion of theorem 3 was proved in section 4. Only theorem 1 and points 1, 2 and 3 of theorem 3 remain to be proved. Everything here comes from the estimate obtained in proposition 1, and from the Pohozaev identity [Poh]. When applied to the functions u_ϵ, this identity gives

$$\underbrace{\frac{N(N-2)^2\epsilon}{2(p+1-\epsilon)}\int_B f(|x|)u_\epsilon^{p+1-\epsilon}dx}_{I_\epsilon} + \underbrace{\frac{N(N-2)}{p+1-\epsilon}\int_B |x|f'(|x|)u_\epsilon^{p+1-\epsilon}dx}_{II_\epsilon}$$
$$\underbrace{-\int_B \left(a(|x|) + \frac{1}{2}|x|a'(|x|)\right) {u_\epsilon}^2 dx}_{III_\epsilon} = \underbrace{\frac{1}{2}\int_{\partial B} |\nabla u_\epsilon|^2 d\sigma}_{IV_\epsilon} \quad .$$

In what follows, we assume that the u_ϵ's develop a concentration. With the notations of section 5, this gives that $\lambda = \frac{1}{4}\left(N(N-2)\omega_N\right)^{\frac{2}{N}}$. In particular, we recover the results of sections 2, 3, and 4. We estimate in what follows the terms I_ϵ, II_ϵ, III_ϵ, and IV_ϵ of the Pohozaev identity.

The terms I_ϵ and IV_ϵ are the easiest to estimate. We straightforwardly obtain that

$$I_\epsilon = \frac{(N-2)^3\omega_N}{2^{N+2}}\left(1+o(1)\right)\epsilon$$

and it follows from lemma 4.2 that

$$IV_\epsilon = \frac{1}{2}(N-2)^2\omega_{N-1}^3 g'(1)^2\mu_\epsilon^{N-2} + o\left(\mu_\epsilon^{N-2}\right)$$

where g is as in the introduction.

Concerning the term II_ϵ, we write that

$$f'(r) = \frac{f^{(k_f)}(0)}{(k_f-1)!}r^{k_f-1} + O(r^{k_f}) \ .$$

Then,

$$\begin{aligned}\int_B |x|f'(|x|)u_\epsilon^{p+1-\epsilon}dx &= \frac{f^{(k_f)}(0)}{(k_f-1)!}\int_B |x|^{k_f}u_\epsilon^{p+1-\epsilon}dx \\ &\quad + O\left(\int_B |x|^{k_f+1}u_\epsilon^{p+1-\epsilon}dx\right) \\ &= \frac{f^{(k_f)}(0)}{(k_f-1)!}\left(1+o(1)\right)\mu_\epsilon^{k_f}\underbrace{\int_{B(0,\frac{1}{k_\epsilon})}|x|^{k_f}v_\epsilon^{p+1-\epsilon}dx}_{II_\epsilon^1} \\ &\quad + O\Bigg(\mu_\epsilon^{k_f+1}\underbrace{\int_{B(0,\frac{1}{k_\epsilon})}|x|^{k_f+1}v_\epsilon^{p+1-\epsilon}dx}_{II_\epsilon^2}\Bigg) \ .\end{aligned}$$

If $k_f < N$, and together with proposition 1, II_ϵ^1 converges by the dominated convergence theorem. Watching closely what occurs, we get that

$$II_\epsilon = \frac{(N-2)^2}{2}\,\frac{f^{(k_f)}(0)}{(k_f-1)!}\,\mu_\epsilon^{k_f}\int_{\mathbb{R}^N}|x|^{k_f}v^{p+1}dx + o\left(\mu_\epsilon^{k_f}\right)$$

if $k_f \le N-2$, while $II_\epsilon = o\left(\mu_\epsilon^{N-2}\right)$ if $k_f > N-2$.

We are finally concerned with the term III_ϵ. The study there is more intricate, and we separate the cases $k_a < N-4$, $k_a > N-4$, and $k_a = N-4$. We first write that

$$\begin{aligned} a(r) &= \frac{a^{(k_a)}(0)}{k_a!} r^{k_a} + O\left(r^{k_a+1}\right) \quad , \\ a'(r) &= \frac{a^{(k_a)}(0)}{(k_a-1)!} r^{k_a-1} + O\left(r^{k_a}\right) \quad . \end{aligned}$$

If $k_a < N-4$, we obtain with the same kind of arguments than the ones used above that

$$III_\epsilon = \frac{a^{(k_a)}(0)}{k_a!}\left(1+\frac{k_a}{2}\right)\mu_\epsilon^{k_a+2}\int_{\mathbb{R}^N} |x|^{k_a} v^2 dx + o\left(\mu_\epsilon^{k_a+2}\right) \quad .$$

We get point 1 of theorem 3 with what has been said before using the Pohozaev identity.

We now assume that $k_a > N-4$. Integrating separately on $B\backslash B(0,\delta)$ and $B(0,\delta)$ with $0<\delta<1$ small, we obtain

$$\begin{aligned} &\frac{1}{\mu_\epsilon^{N-2}}\int_B \left(a(|x|)+\frac{1}{2}|x|a'(|x|)\right) u_\epsilon^2(x) dx \\ &= (N-2)^2\omega_{N-1}^2 \int_B \left(a(|x|)+\frac{1}{2}|x|a'(|x|)\right) G(x,0)^2 dx + o(1) \end{aligned}$$

and then that

$$III_\epsilon = (N-2)^2\omega_{N-1}^3 \Phi(a)\mu_\epsilon^{N-2} + o\left(\mu_\epsilon^{N-2}\right) \quad .$$

Using the Pohozaev identity, we then obtain points $3(a)$ and $3(b)$ and $3(c)$ of theorem 3.

At last, we assume that $k_a = N-4$. By proposition 1, we easily obtain that

$$\begin{aligned} \int_B \left(a(|x|)+\frac{1}{2}|x|a'(|x|)\right) u_\epsilon^2 dx &= \frac{a^{(k_a)}(0)}{k_a!}\left(1+\frac{k_a}{2}\right)\mu_\epsilon^{k_a+2}\int_{\mathbb{R}^N} |x|^{k_a} v_\epsilon^2 dx \\ &\quad + O\left(\mu_\epsilon^{k_a+2}\right) \\ &= \frac{(N-2)a^{(N-4)}(0)}{2(N-4)!}\mu_\epsilon^{N-2}\int_{\mathbb{R}^N} |x|^{N-4} v_\epsilon^2 dx \\ &\quad + O\left(\mu_\epsilon^{N-2}\right) \end{aligned}$$

and we are now left with getting an estimate for the term

$$III_\epsilon^1 = \int_{\mathbb{R}^N} |x|^{N-4} v_\epsilon^2 dx \quad .$$

Let us consider $\delta \in]0,1[$ to be chosen later. By proposition 1,

$$III_\epsilon^1 = \int_{B(0,\frac{\delta}{k_\epsilon})} |x|^{N-4} v_\epsilon^2 dx + O(1) \quad .$$

Let $(\hat{x}_\epsilon)$ be a sequence of points such that $|\hat{x}_\epsilon| \le \frac{\delta}{k_\epsilon}$. We set

$$R_\epsilon = \frac{v_\epsilon(\hat{x}_\epsilon)}{v(\hat{x}_\epsilon)} \quad .$$

If $|\hat{x}_\epsilon|$ is bounded, then $R_\epsilon \to 1$ since $v_\epsilon \to v$ uniformly on every compact subset of $\mathbb{R}^N$. Otherwise, $|\hat{x}_\epsilon| \to +\infty$, and, up to a subsequence, two cases occur: Either there exists $\delta_0 > 0$ such that $k_\epsilon |\hat{x}_\epsilon| \to \delta_0$, or $k_\epsilon |\hat{x}_\epsilon| \to 0$. In the first case, we set $y_\epsilon = k_\epsilon \hat{x}_\epsilon$. Then $|y_\epsilon| \le \delta$ and

$$R_\epsilon \approx |y_\epsilon|^{N-2} u_\epsilon(x_\epsilon) u_\epsilon(y_\epsilon) \quad .$$

It follows from lemma 4.2 that

$$R_\epsilon \to (N-2)\omega_{N-1} \delta_0^{N-2} g(\delta_0) \quad .$$

In the second case, where $|\hat{x}_\epsilon| \to +\infty$ and $k_\epsilon |\hat{x}_\epsilon| \to 0$, integrating on two domains like in section 3, we obtain $R_\epsilon \to 1$. Summarizing: either $k_\epsilon |\hat{x}_\epsilon| \to 0$, and then $R_\epsilon \to 1$, or $k_\epsilon |\hat{x}_\epsilon| \to \delta_0$, where $\delta_0 > 0$, and then $R_\epsilon \to (N-2)\omega_{N-1}\delta_0^{N-2} g(\delta_0)$. Let $\alpha \in]0,1[$ be given. We note that

$$\lim_{\delta_0 \to 0^+} (N-2)\omega_{N-1} \delta_0^{N-2} g(\delta_0) = 1$$

and we choose $\delta > 0$ such that for all $\delta_0 \in]0, \delta[$,

$$1 - \alpha \le (N-2)\omega_{N-1} \delta_0^{N-2} g(\delta_0) \le 1 + \alpha \quad .$$

Then,

$$1 - \alpha \le R_\epsilon \le 1 + \alpha \quad .$$

Therefore, as easily checked,

$$\frac{1}{|\ln k_\epsilon|} III_\epsilon^1 \to \omega_{N-1}$$

and we thus proved that

$$III_\epsilon = \frac{(N-2)\omega_{N-1} a^{(N-4)}(0)}{2(N-4)!} \mu_\epsilon^{N-2} |\ln \mu_\epsilon| + o\left(\mu_\epsilon^{N-2} |\ln k_\epsilon|\right) \quad .$$

Taking the Pohozaev identity again, we obtain the required estimate.

We are now left with the proof of theorem 1. According to the results of section 5, it suffices to show that, under the assumptions of this theorem, at least one subsequence of (u_ϵ) converges almost everywhere to a nonzero function. If not, the u_ϵ's develop a concentration and we are back to one of the situations described in theorem 3. Noting the assumptions of theorem 1 are those that make the limits of the different points of theorem 3 negative, theorem 1 is proved. □

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