Positive solutions for a fourth order equation invariant under isometries

by

Frédéric ROBERT

Université de Cergy-Pontoise
Département de Mathématiques - Site Saint-Martin
2, Avenue Adolphe Chauvin
F 95302 Cergy-Pontoise Cedex, France
e-mail: `Frederic.Robert@math.u-cergy.fr`

Abstract

Let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$. We consider the problem

$$\Delta_g^2 u + \alpha \Delta_g u + au = f u^{\frac{n+4}{n-4}}, \qquad (\star)$$

where $\Delta_g = -div_g(\nabla)$, $\alpha, a \in \mathbb{R}$, $u, f \in C^\infty(M)$. We require u to be positive and invariant under isometries. We prove existence results for $(\star)$ on arbitrary compact manifolds. This includes the case of the geometric Paneitz-Branson operator on the sphere.

In 1983, Paneitz [Pan] introduced a fourth order operator defined on 4-dimensional Riemannian manifolds. Branson [Bra] generalized the definition to n-dimensional Riemannian manifolds. Given (M^n, g), $n \geq 5$, a compact Riemannian manifold, and $u \in C^\infty(M^n)$, we let

$$P_g^n u = \Delta_g^2 u - div_g(a_n S_g g + b_n Ric_g) du + \frac{n-4}{2} Q_g^n u$$

In this expression, $\Delta_g u = -div_g(\nabla u)$, S_g is the scalar curvature of g, Ric_g its Ricci curvature, $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$, $b_n = -\frac{4}{n-2}$, and

$$Q_g^n = \frac{1}{2(n-1)} \Delta_g S_g + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |Ric_g|_g^2$$

If $\tilde{g} = \varphi^{4/(n-4)} g$ is a conformal metric to g, then, see Branson [Bra],

$$P_g^n(u\varphi) = \varphi^{\frac{n+4}{n-4}} P_{\tilde{g}}^n(u) \text{ and } P_g^n \varphi = \frac{n-4}{2} Q_{\tilde{g}}^n \varphi^{\frac{n+4}{n-4}}$$

where the first of these two equations holds for all smooth functions u on M^n. Let $(\mathbb{S}^n, h)$ be the unit n-sphere. Then,

$$P_h^n u = \Delta_h^2 u + c_n \Delta_h u + d_n u,$$

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where $c_n = \frac{n^2-2n-4}{2}$, and $d_n = \frac{(n-4)n(n^2-4)}{16}$. We still refer to P_g^n as the Paneitz operator. Given $\alpha, a \in \mathbb{R}$, let P_g be the constant coefficient Paneitz type operator whose expression is $P_g u = \Delta_g^2 u + \alpha \Delta_g u + au$, where $u \in C^\infty(M^n)$. If G is a group of isometries of (M^n, g) and $f \in C^\infty(M)$ is invariant under the action of G, we are interested in this paper in finding smooth positive G-invariant solutions of the fourth order equation

$$P_g u = f u^{2^\sharp - 1} \tag{1}$$

where $2^\sharp = \frac{2n}{n-4}$ is the critical Sobolev exponent for the embeddings of $H_2^2(M)$ in L^p-spaces. When (M^n, g) is the unit n-sphere $(\mathbb{S}^n, h)$, $\alpha = c_n$, and $a = d_n$, equation (1) reads as

$$\Delta_h^2 u + c_n \Delta_h u + d_n u = f u^{2^\sharp - 1} \tag{2}$$

Then it follows from the above transformation laws that the existence of a smooth positive solution to (2) is equivalent to the existence of a conformal metric g to h such that $Q_g^n = f$. Equation (2) has its exact analogue when passing from the Paneitz operator to the conformal Laplacian on $\mathbb{S}^n$, $n \geq 3$. The equation associated to the conformal Laplacian reads as

$$\Delta_h u + \frac{n(n-2)}{4} u = f u^{2^\star - 1} \tag{3}$$

where $2^\star = \frac{2n}{n-2}$ and $f \in C^\infty(M)$, and we refer to the problem of finding smooth positive solutions to this equation as the Kazdan-Warner or the Nirenberg problem. Extending a result of Moser [Mos] from $\mathbb{S}^2$ to $\mathbb{S}^3$, Escobar and Schoen [EsSc] proved that if f is a smooth positive function on $\mathbb{S}^3$, invariant under the action of a nontrivial group G of isometries of $(\mathbb{S}^3, h)$ acting freely, then (3) possesses a smooth positive G-invariant solution. This result of Escobar and Schoen [EsSc] was then generalized by Hebey [Heb], when he proved that (3) still possesses a smooth positive G-invariant solution if we only require that the action of G is without fixed points. A nontrivial group G of isometries of a manifold (M^n, g) is said to act freely if M^n/G is still a manifold. We say that G acts without fixed points if for any x, the G-orbit $O_G(x)$ of x has at least two elements. A nontrivial group acting freely acts without fixed points. Returning to (2), it was proved in Djadli-Hebey-Ledoux [DHL] that if f is a smooth positive function on $\mathbb{S}^5$, invariant under the action of a nontrivial group G of isometries of $(\mathbb{S}^5, h)$ acting freely, then (2) possesses a smooth positive G-invariant solution. Hebey put to our attention the question of whether or not such a result holds when the condition that G acts freely is replaced by the less restrictive condition that G acts without fixed points. We answer this question by the affirmative, and prove the following theorem:

Theorem 1 *Let G be a compact subgroup of isometries of the standard sphere $(\mathbb{S}^5, h)$, $f \in C^\infty(\mathbb{S}^5)$ positive and $G-$invariant. Assume that G acts without fixed points. Then (2) possesses a smooth positive $G-$invariant solution, and there exists a conformal G-invariant metric g to h such that $Q_g^5 = f$.*

References where (1) and (2) are studied are Djadli-Hebey-Ledoux [DHL], Hebey-Robert [HeRo], and Jourdain [Jou].

1 The case of an arbitrary Riemannian manifold

Let (M^n, g) be a compact Riemannian manifold of dimension $n \geq 5$. Not to carry heavy notations, we note M instead of M^n. If $Isom_g(M)$ is the isometry group of (M, g), we let G be a compact subgroup of $Isom_g(M)$. Given $f \in C^\infty(M)$, positive and $G-$invariant, and given $a, \alpha > 0$, we let

$$\lambda^G(f) = \inf_{u \in \mathcal{H}_f^G} \int_M \left((\Delta_g u)^2 + \alpha|\nabla u|_g^2 + au^2\right) dv_g$$

where dv_g is the Riemannian volume element for g, and $\mathcal{H}_f^G$ is the set consisting of G-invariant functions in $H_2^2(M)$ which are such that $\int_M f|u|^{2^\sharp} dv_g = 1$. It can be checked that whatever (M, g) is, whatever f is, and whatever a and α are,

$$\lambda^G(f) \leq \frac{|O_G(x)|^{\frac{4}{n}}}{K_0 f(x)^{\frac{2}{2^\sharp}}} \tag{4}$$

for all $x \in M$, where $|O_G(x)|$ is the cardinality of the orbit $O_G(x)$ and K_0 is the best constant for the optimal Sobolev Euclidean inequality

$$\left(\int_{\mathbb{R}^n} |u|^{2^\sharp} dv_\xi\right)^{\frac{2}{2^\sharp}} \leq K_0 \int_{\mathbb{R}^n} (\Delta_\xi u)^2 dv_\xi \tag{5}$$

where dv_ξ is the volume element in $\mathbb{R}^n$ and Δ_ξ is the usual Laplacian with the minus sign convention. The first objective of this section is to prove the following theorem:

Theorem 2 *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$, G a compact subgroup of $Isom_g(M)$, $f \in C^\infty(M)$, positive and $G-$invariant, and $a, \alpha > 0$. If $a \leq \frac{\alpha^2}{4}$, and if for all $x \in M$,*

$$\lambda^G(f) < \frac{|O_G(x)|^{\frac{4}{n}}}{K_0 f(x)^{\frac{2}{2^\sharp}}} \tag{6}$$

then (1) possesses a smooth positive $G-$invariant solution.

We prove this theorem in what follows. For $0 < \epsilon < 2^\sharp - 2$, we define

$$\lambda_\epsilon^G(f) = \inf_{u \in \mathcal{H}_{f,\epsilon}^G} \left(\int_M \left((\Delta_g u)^2 + \alpha|\nabla u|_g^2 + au^2\right) dv_g\right)$$

where $\mathcal{H}_{f,\epsilon}^G$ is the set consisting of G-invariant functions in $H_2^2(M)$ which are such that $\int_M f|u|^{2^\sharp-\epsilon} dv_g = 1$. The following lemma easily follows from what has been done in [DHL].

Lemma 1 *Let* (M,g) *be a compact Riemannian manifold of dimension* $n \geq 5$. *Let* G *be a subgroup of* $Isom_g(M)$, $f \in C^\infty(M)$ *a positive* $G-$*invariant function, and* $a, \alpha > 0$ *such that* $a \leq \frac{\alpha^2}{4}$. *Then* $\lambda_\epsilon^G(f)$ *is attained by a smooth positive* $G-$*invariant function* u_ϵ *which satisfies*

$$\Delta_g^2 u_\epsilon + \alpha \Delta_g u_\epsilon + a u_\epsilon = \lambda_\epsilon^G(f) f u_\epsilon^{2^\sharp - 1 - \epsilon} \tag{7}$$

and $\int_M f u_\epsilon^{2^\sharp - \epsilon} \, dv_g = 1$. *Moreover, up to a subsequence,* (u_ϵ) *converges weakly in* $H_2^2(M)$ *to a function* u. *If* $u \not\equiv 0$, *then* u *is a positive smooth* $G-$*invariant function which realizes* $\lambda^G(f)$, *and, up to a positive constant scale factor,* u *is a solution of (1).*

We proceed with the proof of Theorem 2. We assume that (6) is true. We let (u_ϵ) be the sequence of lemma 1. Let also $\lambda = \limsup \lambda_\epsilon^G(f)$. Then $\lambda \leq \lambda^G(f)$, and with Hölder and Sobolev inequalities we get that $\lambda > 0$. Assume now that there is no positive $G-$invariant solution $u \in C^\infty(M)$ to (1). Then $u_\epsilon \to 0$ almost everywhere. Let $x_\epsilon \in M$ be such that $u_\epsilon(x_\epsilon) = \sup_M u_\epsilon$. If $u_\epsilon(x_\epsilon)$ is bounded, it follows from classical regularity theory (see for instance [GT]) that (u_ϵ) is bounded in $C^{4,\beta}(M)$, $0 < \beta < 1$. Then $u_\epsilon \to 0$ in $C^4(M)$, a contradiction since $\int_M f u_\epsilon^{2^\sharp - \epsilon} \, dv_g = 1$. Hence, $u_\epsilon(x_\epsilon) \to +\infty$. Let $x_1 \in M$ be such that $x_\epsilon \to x_1$. We define $\mu_\epsilon = u_\epsilon(x_\epsilon)^{-\frac{2}{n-4}}$ and $k_\epsilon = \mu_\epsilon^{1 - \epsilon \frac{n-4}{8}}$. For $|x| < \frac{i_g(M)}{k_\epsilon}$, where $i_g(M) > 0$ is the injectivity radius of M, we let

$$v_\epsilon(x) = \mu_\epsilon^{\frac{n-4}{2}} u_\epsilon(exp_{x_\epsilon}(k_\epsilon x)) \text{ and } g_\epsilon = (exp_{x_\epsilon}^\star g)(k_\epsilon x),$$

where exp_{x_ϵ} denotes the exponential map at x_ϵ. Then v_ϵ verifies

$$\Delta_{g_\epsilon}^2 v_\epsilon + \alpha k_\epsilon^2 \Delta_{g_\epsilon} v_\epsilon + a k_\epsilon^4 v_\epsilon = \lambda_\epsilon^G(f) f(exp_{x_\epsilon}(k_\epsilon x)) v_\epsilon^{2^\sharp - 1 - \epsilon}$$

an equation which we can read also as

$$\left(\Delta_{g_\epsilon} + \frac{\alpha k_\epsilon^2}{2}\right)^2 v_\epsilon = \lambda_\epsilon^G(f) f(exp_{x_\epsilon}(k_\epsilon x)) v_\epsilon^{2^\sharp - 1 - \epsilon} + \left(\frac{\alpha^2}{4} - a\right) v_\epsilon$$

We have $0 \leq v_\epsilon \leq 1$ and $k_\epsilon \to 0$. By classical regularity theorems (see for instance [GT]), (v_ϵ) is bounded in $C^{4,\beta}(K)$ for $0 < \beta < 1$ and all compact subsets $K \subset \mathbb{R}^n$. Then, up to a subsequence, there exists $v \in C^4(\mathbb{R}^n)$ such that v_ϵ goes to v in $C^4_{loc}(\mathbb{R}^n)$. In particular $v \geq 0$, $v(0) = 1$, and

$$\Delta_\xi^2 v = \lambda f(x_1) v^{2^\sharp - 1}$$

Then, see [HeRo], we know precisely what v is. Given $x \in M$ and $r > 0$, we let $B_g(x,r)$ be the geodesic ball of center x and radius r in M, and for $p \in \mathbb{R}^n$, we let $B_\xi(p,r)$ be the Euclidean ball in $\mathbb{R}^n$ of center p and radius r. For $R > 0$, we have

$$\begin{aligned}\int_{B_g(x_\epsilon, Rk_\epsilon)} f u_\epsilon^{2^\sharp - \epsilon} \, dv_g &= \left(\mu_\epsilon^{-1}\right)^{\epsilon \frac{(n-4)^2}{8}} \int_{B_\xi(0,R)} f(exp_{x_\epsilon}(k_\epsilon x)) v_\epsilon^{2^\sharp - \epsilon} \, dv_{g_\epsilon} \\ &\geq f(x_1) \int_{B_\xi(0,R)} v^{2^\sharp} \, dv_\xi + o(1)\end{aligned}$$

since $\mu_\epsilon \to 0$ and $v_\epsilon \to v$ in $C^4(B_\xi(0,R))$. Now, since we also have that $x_\epsilon \to x_1$, $k_\epsilon \to 0$ and $f \geq 0$, we obtain that for any $\delta > 0$,

$$\int_{B_g(x_1,\delta)} f u_\epsilon^{2^\sharp - \epsilon} \, dv_g \geq f(x_1) \int_{\mathbb{R}^n} v^{2^\sharp} \, dv_\xi + o(1) \tag{8}$$

Let $O_G(x_1) = \{x_1, ..., x_m\}$. Since f is $G-$invariant and G is a group of isometries,

$$\int_{B_g(x_i,\delta)} f u_\epsilon^{2^\sharp - \epsilon} \, dv_g = \int_{B_g(x_1,\delta)} f u_\epsilon^{2^\sharp - \epsilon} \, dv_g \geq f(x_1) \int_{\mathbb{R}^n} v^{2^\sharp} \, dv_\xi + o(1)$$

for all $i = 1, ..., m$. Taking $\delta > 0$ sufficiently small, we obtain

$$1 = \int_M f u_\epsilon^{2^\sharp - \epsilon} \, dv_g \geq m f(x_1) \int_{\mathbb{R}^n} v^{2^\sharp} \, dv_\xi + o(1)$$

Multiplying by v the equation satisfied by v, and integrating, it follows with (5), (4), and the inequality $\lambda \leq \lambda^G(f)$, that v is minimizing for (5) and that

$$\lambda^G(f) = \lambda = \frac{|O_G(x_1)|^{\frac{4}{n}}}{f(x_1)^{\frac{2}{2^\sharp}} K_0} \tag{9}$$

A contradiction with (6). This proves Theorem 2.

We proceed in what follows with the study of the behaviour of the u_ϵ's. We assume as in the proof of Theorem 2 that $u_\epsilon \to 0$ almost everywhere. It follows from the proof of Theorem 2 that equality holds in (8). Then, for any δ small,

$$\int_{B_g(x_1,\delta)} f u_\epsilon^{2^\sharp - \epsilon} \, dv_g = \frac{1}{|O_G(x_1)|} + o(1) \tag{10}$$

We also get that $\mu_\epsilon^\epsilon \to 1$ and that for any $\Omega \subset\subset M \backslash O_G(x_1)$,

$$\int_\Omega u_\epsilon^{2^\sharp - \epsilon} \, dv_g = o(1) \tag{11}$$

We now give a more precise description of the convergence of (u_ϵ) outside the orbit $O_G(x_1)$. Let $\sigma_1 = Id_M, \sigma_2, ..., \sigma_m \in G$ be such that $x_i = \sigma_i(x_1)$ where $O_G(x_1) = \{x_1, ..., x_m\}$. Define $x_{\epsilon,i} = \sigma_i(x_\epsilon)$. First, we want to prove that there exists $C > 0$ such that for any $x \in M$,

$$\inf_{i=1,\ldots,p} d_g(x, x_{\epsilon,i})^{\frac{4(n-4)}{8-\epsilon(n-4)}} u_\epsilon(x) \leq C \tag{12}$$

We follow an idea of Druet [Dru]. Assume that there exists $y_\epsilon \in M$ such that

$$\sup_{x \in M} \inf_{i=1,\ldots,p} d_g(x, x_{\epsilon,i})^{s_\epsilon} u_\epsilon(x) = \inf_{i=1,\ldots,p} d_g(y_\epsilon, x_{\epsilon,i})^{s_\epsilon} u_\epsilon(y_\epsilon) \to +\infty \tag{13}$$

where $s_\epsilon = \frac{4(n-4)}{8-\epsilon(n-4)}$. Define $\hat{\mu}_\epsilon = u_\epsilon(y_\epsilon)^{-\frac{2}{n-4}}$, $\hat{k}_\epsilon = \hat{\mu}_\epsilon^{1-\epsilon\frac{n-4}{8}}$, and set

$$\hat{v}_\epsilon(x) = \hat{\mu}_\epsilon^{\frac{n-4}{2}} u_\epsilon\left(exp_{y_\epsilon}(\hat{k}_\epsilon x)\right)$$

For $|x| < \frac{i_g(M)}{\hat{k}_\epsilon}$ and $\hat{g}_\epsilon(x) = exp^\star_{y_\epsilon} g(\hat{k}_\epsilon x)$, we have

$$\Delta^2_{\hat{g}_\epsilon} \hat{v}_\epsilon + \alpha k_\epsilon^2 \Delta_{\hat{g}_\epsilon} \hat{v}_\epsilon + a k_\epsilon^4 \hat{v}_\epsilon = f(exp_{y_\epsilon}(\hat{k}_\epsilon x)) \hat{v}_\epsilon^{2^\sharp - 1 - \epsilon} \tag{14}$$

Let $R > 0$. With (13), and $|x| \leq R$, we obtain

$$\hat{v}_\epsilon(x) = \frac{u_\epsilon\left(exp_{y_\epsilon}(\hat{k}_\epsilon x)\right)}{u_\epsilon(y_\epsilon)} \leq \left(\frac{\inf\limits_{i=1,\ldots,p} d_g(y_\epsilon, x_{\epsilon,i})}{\inf\limits_{i=1,\ldots,p} d_g(exp_{y_\epsilon}(\hat{k}_\epsilon x), x_{\epsilon,i})}\right)^{\frac{4(n-4)}{8-\epsilon(n-4)}}$$

Since $\inf\limits_{i=1,\ldots,p} d_g(exp_{y_\epsilon}(\hat{k}_\epsilon x), x_{\epsilon,i}) \geq \inf\limits_{i=1,\ldots,p} d_g(y_\epsilon, x_{\epsilon,i}) - \hat{k}_\epsilon R$,

$$\hat{v}_\epsilon(x) \leq \left(1 - R \frac{\hat{k}_\epsilon}{\inf\limits_{i=1,\ldots,p} d_g(y_\epsilon, x_{\epsilon,i})}\right)^{-\frac{4(n-4)}{8-\epsilon(n-4)}}$$

for all $|x| \leq R$. Now, with (13), we obtain that

$$\frac{\inf\limits_{i=1,\ldots,p} d_g(y_\epsilon, x_{\epsilon,i})}{\hat{k}_\epsilon} \to +\infty \tag{15}$$

Then $\hat{v}_\epsilon$ is uniformly bounded on every compact set. Writing that

$$\left(\Delta_{\hat{g}_\epsilon} + \frac{\alpha \hat{k}_\epsilon^2}{2}\right)^2 \hat{v}_\epsilon = f(exp_{y_\epsilon}(\hat{k}_\epsilon x)) \hat{v}_\epsilon^{2^\sharp - 1 - \epsilon} + \left(\frac{\alpha^2}{4} - a\right) \hat{k}_\epsilon^4 \hat{v}_\epsilon$$

and using classical regularity results (see for instance [GT]), there exists $\hat{v} \in C^4(\mathbb{R}^n)$ such that, up to a subsequence, $\hat{v}_\epsilon \to \hat{v}$ in $C^4_{loc}(\mathbb{R}^n)$, and $\hat{v}(0) = 1$. Now, as easily checked,

$$\int_{B_g(y_\epsilon, \hat{k}_\epsilon)} u_\epsilon^{2^\sharp - \epsilon} \, dv_g = \hat{\mu}_\epsilon^{-\epsilon \frac{(n-4)^2}{8}} \int_{B_\xi(0,1)} \hat{v}_\epsilon^{2^\sharp - \epsilon} \, dv_{\hat{g}_\epsilon}$$

Since $\hat{\mu}_\epsilon \leq 1$, it comes when $\epsilon \to 0$ that

$$\int_{B_g(y_\epsilon, \hat{k}_\epsilon)} u_\epsilon^{2^\sharp - \epsilon} \, dv_g \geq \int_{B_\xi(0,1)} \hat{v}^{2^\sharp} \, dv_\xi + o(1)$$

Now, up to a subsequence, we can assume that $y_\epsilon \to y_0 \in M$. If $y_0 \notin O_G(x_1)$, then, with (11), we get that $\int_{B_g(y_\epsilon, \hat{k}_\epsilon)} u_\epsilon^{2^\sharp - \epsilon} \, dv_g \to 0$. Then $\int_{B_\xi(0,1)} \hat{v}^{2^\sharp} \, dv_\xi = 0$, a contradiction. Hence, up to an isometry of G, we can assume that $y_0 = x_1$. Taking $\delta > 0$ small enough,

$$\int_{B_g(y_\epsilon,\hat{k}_\epsilon)} u_\epsilon^{2^\sharp-\epsilon}\, dv_g \quad = \quad \int_{B_g(y_\epsilon,\hat{k}_\epsilon)\cap B_g(x_1,\delta)} u_\epsilon^{2^\sharp-\epsilon}\, dv_g$$

For any $R' > 0$, we have

$$\int_{B_g(x_1,\delta)\backslash B_g(x_\epsilon,R'k_\epsilon)} u_\epsilon^{2^\sharp-\epsilon}\, dv_g \leq \epsilon(R') + o(1)$$

where $\lim_{R'\to+\infty} \epsilon(R') = 0$. It follows that

$$\int_{B_g(y_\epsilon,\hat{k}_\epsilon)} u_\epsilon^{2^\sharp-\epsilon}\, dv_g \leq \int_{B_g(y_\epsilon,\hat{k}_\epsilon)\cap B_g(x_\epsilon,R'k_\epsilon)} u_\epsilon^{2^\sharp-\epsilon}\, dv_g + \epsilon(R') + o(1)$$

If $B_g(y_\epsilon, \hat{k}_\epsilon) \cap B_g(x_\epsilon, R'k_\epsilon) \neq \emptyset$, then

$$\inf_{i=1,\dots,p} d_g(y_\epsilon, x_{\epsilon,i}) \leq \hat{k}_\epsilon + R'k_\epsilon \tag{16}$$

With (15) and (16), we then obtain that $\hat{k}_\epsilon = o(k_\epsilon)$ and $\frac{d_g(y_\epsilon,x_\epsilon)}{k_\epsilon}$ is bounded. Now we write $y_\epsilon = exp_{x_\epsilon}(k_\epsilon \hat{y}_\epsilon)$ where $\hat{y}_\epsilon$ is bounded. There exists $C_0 > 0$ such that

$$\frac{1}{k_\epsilon} exp_{x_\epsilon}^{-1}\left(B_g(exp_{x_\epsilon}(k_\epsilon \hat{y}_\epsilon), \hat{k}_\epsilon)\right) \subset B_\xi\left(\hat{y}_\epsilon, C_0 \frac{\hat{k}_\epsilon}{k_\epsilon}\right)$$

We thus obtain that

$$\int_{B_g(y_\epsilon,\hat{k}_\epsilon)\cap B_g(x_\epsilon,R'k_\epsilon,)} u_\epsilon^{2^\sharp-\epsilon}\, dv_g \leq \mu_\epsilon^{-\epsilon\frac{(n-4)^2}{8}} \int_{B_\xi\left(\hat{y}_\epsilon, C_0\frac{\hat{k}_\epsilon}{k_\epsilon}\right)} v_\epsilon^{2^\sharp-\epsilon}\, dv_{g_\epsilon} = o(1)$$

since $\hat{k}_\epsilon = o(k_\epsilon)$ and (v_ϵ) is bounded. As a consequence,

$$\int_{B_g(y_\epsilon,\hat{k}_\epsilon)} u_\epsilon^{2^\sharp-\epsilon}\, dv_g \leq \epsilon(R') + o(1)$$

for all $R' > 0$. We then get that $\int_{B_\xi(0,1)} \hat{v}^{2^\sharp}\, dv_\xi = 0$. A contradiction since $\hat{v}(0) = 1$. This proves (12). Given an open subset $\Omega \subset\subset M\backslash O_G(x_1)$, we now get by classical regularity theorems (see for instance [GT]) that (u_ϵ) is bounded in $C^{4,\beta}(\Omega)$. Since u_ϵ goes to 0 almost everywhere, it follows that

$$u_\epsilon \to 0 \text{ in } C^4(\Omega) \tag{17}$$

as $\epsilon \to 0$, a relation we use in the following section.

2 The case of the sphere

Let $x_0 \in \mathbb{S}^n$. For $\beta > 1$, define

$$u_{x_0,\beta}(x) = (\beta - \cos r)^{-\frac{n-4}{2}} \quad \text{and} \quad \tilde{u}_{x_0,\beta} = (\beta^2 - 1)^{\frac{n-4}{4}} u_{x_0,\beta}$$

where $r = d_h(x_0, x)$. Then,

$$P_h^n(\tilde{u}_{x_0,\beta}) = d_n \tilde{u}_{x_0,\beta}^{2^\sharp - 1} \quad \text{and} \quad \int_{\mathbb{S}^n} \tilde{u}_{x_0,\beta}^{2^\sharp} \, dv_h = \omega_n$$

where ω_n is the volume of the unit $n-$sphere. We now make these functions $G-$invariant. Let $x_1 \in M$ be a point of finite orbite $O_G(x_1) = \{x_1, ..., x_m\}$. We define $u_{i\beta} = u_{x_i,\beta}$, $\tilde{u}_{i\beta} = \tilde{u}_{x_i,\beta}$ and $\tilde{u}_\beta = \sum_{i=1}^m \tilde{u}_{i\beta}$ (this function is $G-$invariant). Computing $\int_{\mathbb{S}^n} P_h^n \tilde{u}_\beta \tilde{u}_\beta \, dv_h$ and $\int_{\mathbb{S}^n} f |\tilde{u}_\beta|^{2^\sharp} \, dv_h$ we find that

$$\int_{\mathbb{S}^n} P_h^n \tilde{u}_\beta \tilde{u}_\beta \, dv_h = m d_n \omega_n + d_n \alpha (\beta - 1)^{\frac{n-4}{2}} + o\left((\beta - 1)^{\frac{n-4}{2}}\right)$$

where

$$\alpha = \sum_{i \neq j} (1 - \cos d_h(x_i, x_j))^{-\frac{n-4}{2}} \omega_{n-1} \int_0^{+\infty} \frac{2^n r^{n-1}}{(1 + r^2)^{\frac{n+4}{2}}} \, dr > 0$$

since $|O_G(x_1)| \geq 2$, and

$$\left(\int_{\mathbb{S}^n} f(x) \tilde{u}_\beta^{2^\sharp} \, dv_h\right)^{\frac{2}{2^\sharp}} \geq f(x_1)^{\frac{2}{2^\sharp}} (m\omega_n)^{\frac{2}{2^\sharp}} \left(1 + \frac{2\alpha}{m\omega_n} (\beta - 1)^{\frac{n-4}{2}} + o\left((\beta - 1)^{\frac{n-4}{2}}\right)\right)$$

provided that $\nabla^k f(x_1) = 0$, for all $k = 1, ..., n-4$. We write now that

$$\frac{\int_{\mathbb{S}^n} P_h^n \tilde{u}_\beta \tilde{u}_\beta \, dv_h}{\left(\int_{\mathbb{S}^n} f(x) \tilde{u}_\beta^{2^\sharp} \, dv_h\right)^{\frac{2}{2^\sharp}}} \leq \frac{m^{\frac{4}{n}} d_n \omega_n^{\frac{4}{n}}}{f(x_1)^{\frac{2}{2^\sharp}}} \left(1 - \frac{\alpha}{m\omega_n} (\beta - 1)^{\frac{n-4}{2}} + o\left((\beta - 1)^{\frac{n-4}{2}}\right)\right)$$

Since $d_n \omega_n^{\frac{4}{n}} = 1/K_0$ (see [EFJ]), it follows that

$$\frac{\int_{\mathbb{S}^n} P_h^n \tilde{u}_\beta \tilde{u}_\beta \, dv_h}{\left(\int_{\mathbb{S}^n} f(x) \tilde{u}_\beta^{2^\sharp} \, dv_h\right)^{\frac{2}{2^\sharp}}} \leq \frac{|O_G(x_1)|^{\frac{4}{n}}}{f(x_1)^{\frac{2}{2^\sharp}} K_0} \left(1 - \frac{\alpha}{m\omega_n} (\beta - 1)^{\frac{n-4}{2}} + o\left((\beta - 1)^{\frac{n-4}{2}}\right)\right)$$

Noting that $\alpha > 0$, we get that

$$\lambda^G(f) < \frac{|O_G(x_1)|^{\frac{4}{n}}}{f(x_1)^{\frac{2}{2^\sharp}} K_0} \tag{18}$$

for all $x_1 \in \mathbb{S}^n$ such that $\nabla^k f(x_1) = 0$ for all $k = 1, ..., n-4$. It then follows from Theorem 2 that the following theorem holds:

Theorem 3 *Let G be a compact subgroup of $Isom_g(\mathbb{S}^n)$, $n \geq 5$, acting without fixed point. Let $f \in C^\infty(\mathbb{S}^n)$ be a positive $G-$invariant function, and let $x_0 \in \mathbb{S}^n$ be such that for any $x \in \mathbb{S}^n$,*

$$\frac{f(x_0)}{|O_G(x_0)|^{\frac{4}{n-4}}} \geq \frac{f(x)}{|O_G(x)|^{\frac{4}{n-4}}}$$

Assume that $\nabla^q f(x_0) = 0$ for all $q = 1, ..., n-4$. Then there exists $u \in C^\infty(\mathbb{S}^n)$, positive and $G-$invariant, such that

$$P_h^n u = f u^{2^\sharp - 1}$$

and there exists a G-invariant conformal metric g to h such that $Q_g^n = f$.

We now prove Theorem 1. If there is no solution for (2), then we have (9) with a point $x_1 \in \mathbb{S}^n$ such that (10) and (17) are true. Assume that we have proved that x_1 is a critical point for f. Since $n = 5$, then (18) is true for x_1. A contradiction, and this proves the theorem. Then the proof of Theorem 1 reduces to the proof that x_1 is a critical point for f. We adapt an argument from Aubin. Given (M, g) a compact manifold of dimension n, let (u_ϵ) be as in lemma 1. We suppose that (u_ϵ) converges weakly to 0 and let $x_1 \in M$ be such that (10) and (17) are true. With (7) we have

$$\Delta_g^2 u_\epsilon + \alpha \Delta_g u_\epsilon + a u_\epsilon = \lambda_\epsilon^G(f) f u_\epsilon^{2^\sharp - 1 - \epsilon}$$

Let $0 < \delta < \min\limits_{\substack{x,y \in O_G(x_1) \\ x \neq y}} d_g(x, y)$. We get with (10) that for all $z \in C^0(M)$,

$$\int_{B_g(x_1,\delta)} z u_\epsilon^{2^\sharp - \epsilon} \, dv_g = \frac{z(x_1)}{f(x_1)|O_G(x_1)|} + o(1) \tag{19}$$

Now we choose $\psi \in C^\infty(M)$ such that $Supp\, \psi \subset B_g(x_1, \delta)$, $\nabla\psi(x_1) = \nabla f(x_1)$ and $\nabla_g^2 \psi(x_1) = 0$. We then have

$$\int_M (\nabla f, \nabla \psi)_g u_\epsilon^{2^\sharp - \epsilon} \, dv_g = \frac{|\nabla f|_g^2(x_1)}{f(x_1)|O_G(x_1)|} + o(1)$$

On the other hand, since $\Delta_g \psi(x_1) = 0$, $u_\epsilon \to 0$ strongly in $H_1^2(M)$ and is bounded in $H_2^2(M)$, we have

$$
\begin{aligned}
&\int_M (\nabla f, \nabla\psi)_g u_\epsilon^{2^\sharp-\epsilon}\, dv_g \\
&= \int_M (\nabla(f u_\epsilon^{2^\sharp-\epsilon}), \nabla\psi)\, dv_g \\
&\quad -(2^\sharp-\epsilon)\int_M f u_\epsilon^{2^\sharp-1-\epsilon}(\nabla u_\epsilon, \psi)_g\, dv_g \\
&= \int_M f u_\epsilon^{2^\sharp-\epsilon}\Delta_g\psi\, dv_g \\
&\quad -\frac{2^\sharp-\epsilon}{\lambda_\epsilon^G(f)}\int_M \left(\Delta_g^2 u_\epsilon + \alpha\Delta_g u_\epsilon + a u_\epsilon\right)(\nabla u_\epsilon, \nabla\psi)_g\, dv_g \\
&= -\frac{2^\sharp-\epsilon}{\lambda_\epsilon^G(f)}\int_M \Delta_g^2 u_\epsilon(\nabla u_\epsilon, \nabla\psi)_g\, dv_g + o(1) \\
&= -\frac{2^\sharp-\epsilon}{\lambda_\epsilon^G(f)}\int_M \Delta_g u_\epsilon \Delta_g(\nabla u_\epsilon, \nabla\psi)_g\, dv_g + o(1)
\end{aligned}
$$

where we have used (19). We have

$$
\begin{aligned}
&\Delta_g(\nabla u_\epsilon, \nabla\psi)_g = (\nabla\Delta_g u_\epsilon, \nabla\psi)_g \\
&\quad + O(|\nabla u_\epsilon|_g) + O(|x||\nabla_g^2 u_\epsilon|_g|\nabla\psi|_g) + O(|\nabla_g^2 u_\epsilon|_g|\nabla_g^2\psi|_g)
\end{aligned}
$$

Then, with (17), (19) and since (u_ϵ) is bounded in $H_2^2(M)$, we get that

$$
\begin{aligned}
\int_M (\nabla f, \nabla\psi)_g u_\epsilon^{2^\sharp-\epsilon}\, dv_g &= -\frac{2^\sharp-\epsilon}{\lambda_\epsilon^G(f)}\int_M \Delta_g u_\epsilon(\nabla\Delta_g u_\epsilon, \nabla\psi)_g\, dv_g + o(1) \\
&= -\frac{2^\sharp-\epsilon}{2\lambda_\epsilon^G(f)}\int_M (\nabla(\Delta_g u_\epsilon)^2, \nabla\psi)_g\, dv_g + o(1) \\
&= -\frac{2^\sharp-\epsilon}{2\lambda_\epsilon^G(f)}\int_M (\Delta_g u_\epsilon)^2\Delta_g\psi\, dv_g + o(1) = o(1)
\end{aligned}
$$

since $\Delta_g\psi(x_1) = 0$. Hence $\nabla f(x_1) = 0$. Taking $M = \mathbb{S}^n$, this ends the proof of Theorem 1.

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