SIGN-CHANGING SOLUTIONS TO ELLIPTIC SECOND ORDER EQUATIONS: GLUEING A PEAK TO A DEGENERATE CRITICAL MANIFOLD

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Abstract. We construct blowing-up sign-changing solutions to some nonlinear critical equations by glueing a standard bubble to a degenerate function. We develop a method based on analyticity to perform the glueing when the critical manifold of solutions is degenerate and no Bianchi–Egnell type condition holds.

1. Introduction and statement of the results

Let (M,g) be a smooth compact Riemannian manifold of dimension $n \geq 3$, and let $h \in C^{0,\theta}(M)$ ($\theta \in (0,1)$) be such that $\Delta_g + h$ is coercive where $\Delta_g = -\text{div}_g(\nabla)$ is the Laplace-Beltrami operator. In [24], we addressed the question of the existence of a family $(u_\varepsilon)_{\varepsilon>0} \in C^{2,\theta}(M)$ of blowing-up solutions of type $(u_0 - B)$ to

$$\Delta_g u_\varepsilon + h u_\varepsilon = |u_\varepsilon|^{2^\star - 2 - \varepsilon} u_\varepsilon \text{ in } M, \tag{1}$$

where $2^\star := \frac{2n}{n-2}$. Concerning terminology, we say that $(u_\varepsilon)_\varepsilon$ is of type $(u_0 - B)$ when there exists a function $u_0 \in C^{2,\theta}(M)$ positive that is a solution to

$$\Delta_g u_0 + h u_0 = u_0^{2^\star - 1} \text{ in } M \tag{2}$$

and such that

$$u_\varepsilon = u_0 - B_\varepsilon + o(1),$$

where $(B_\varepsilon)_\varepsilon$ is a bubble as defined in (6) below and $\lim_{\varepsilon \to 0} o(1) = 0$ in $H_1^2(M)$, the completion of $C^\infty(M)$ for the norm $u \mapsto \|u\|_{H_1^2} := \|u\|_2 + \|\nabla u\|_2$. Solutions of type $(u_0 - B)$ are sign-changing. When $h \equiv c_n R_g$, where $c_n := \frac{n-2}{4(n-1)}$ and R_g is the scalar curvature, equation (2) is the Yamabe equation, and $\Delta_g + h$ is coercive if and only if (M,g) has positive Yamabe invariant. There is an extensive literature on the existence of positive blowing-up solutions to equations of type (1): see for instance Rey [23] for a historical reference, Brendle–Marques [4] for the Yamabe equation, Druet–Hebey [13] and Esposito–Pistoia–Vétois [14] for perturbations of the Yamabe equation, Chen–Wei–Yan [6] and Hebey–Wei [15] for equations on the sphere, and the references therein. Sign-changing blowing-up solutions to (1) on the canonical sphere have been constructed by del Pino–Musso–Pacard–Pistoia [10,11] and Pistoia–Vétois [22]. We refer to Robert–Vétois [24] for a discussion and references on the compactness of solutions to (1).

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In [24], we gave sufficient conditions to get blowing-up solutions of type $(u_0 - B)$ to (1) provided that u_0 is a nondegenerate solution to (2), that is $K_0 = \{0\}$ where

$$K_0 := \{\varphi \in C^{2,\theta}(M) / \, \Delta_g \varphi + h\varphi = (2^\star - 1) u_0^{2^\star - 2} \varphi \text{ in } M\}. \tag{3}$$

When u_0 is degenerate, the situation can be different. In [24], we showed that there is no blowing-up solutions of type $(u_0 - B)$ to the constant scalar curvature equation on the canonical sphere: in this case, u_0 is necessarily degenerate.

The present article is devoted to the analysis of the degenerate case, that is when $K_0 \neq \{0\}$. We say that $u_0 \in C^{2,\theta}(M) \setminus \{0\}$ is a strict local minimizer of I_0 if there exists $\nu > 0$ such that

$$I_0(u) > I_0(u_0) \text{ for all } u \in B_\nu(u_0) \setminus \mathbb{R} u_0 \,,$$

where

$$I_0(u) := \frac{\int_M \left(|\nabla u|_g^2 + h u^2\right) dv_g}{\left(\int_M |u|^{2^\star} dv_g\right)^{\frac{2}{2^\star}}}$$

for all $u \in H_1^2(M) \setminus \{0\}$. Our main result is the following:

Theorem 1.1. *We let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ with positive Yamabe invariant and we fix $h \equiv \frac{n-2}{4(n-1)} R_g$. We assume that there exists $u_0 \in C^{2,\theta}(M)$ that is a positive solution to (2) and a strict local minimizer of I_0. We assume either that $\{3 \leq n \leq 9\}$ or that $\{(M, g)$ is locally conformally flat$\}$. Then there exists a solution of type $(u_0 - B)$ to (1).*

It follows from the compactness results of Schoen [26] and Khuri-Marques–Schoen [17] (see also Druet [12]) that blowing-up solutions to (1) must change sign under the assumptions of Theorem 1.1.

As a remark, any nondegenerate local minimizer of I_0 is a strict local minimizer, so we recover the main Theorem of [24]. Moreover no solution of the scalar curvature equation on the sphere is a strict local minimizer. However, as soon as one takes the product of a sphere with another manifold, one gets examples of degenerate strict local minimizers. We refer to Section 7 for such examples, in particular to Corollary 7.1.

We prove Theorem 1.1 by performing a finite-dimensional reduction modeled on $(u - B)$ where B is a bubble and $u \in \mathcal{M}$, and where $\mathcal{M}$ is a suitable finite-dimensional analytic manifold containing u_0. The manifold $\mathcal{M}$ is locally parametrized by K_0, and the tangent space of $\mathcal{M}$ at u_0 is K_0. The general construction in Robert–Vétois [25] reduces the proof of Theorem 1.1 to finding stable critical points to a functional that is the sum of two terms: the first one is an explicit local well involving essentially the bubble and the second one is the restriction to $\mathcal{M}$ of a nontrivial global functional J_0.

In general, the elements of $\mathcal{M}$ are not solutions to (2), that is $\mathcal{M}$ is not the critical manifold of the problem. Following the terminology of Chapter 2 of the monograph Ambrosetti–Malchiodi [1], a critical manifold is a finite-dimensional manifold of solutions to (2). The critical manifold is nondegenerate if and only if there exists $\tilde{u} \in C^1(B_1(0) \subset K_0, M)$ such that $\tilde{u}(z)$ is a solution to (2) for all $z \in B_1(0) \subset K_0$ and that $K_0 = \text{Span}\{\partial_{z_i} \tilde{u}(0) / \, i = 1, ..., d\}$. This condition is standard and reminiscent in the finite-dimensional reduction: it holds for K_0 when $M = \mathbb{R}^n$ and $h \equiv 0$ (see Rey [23], Bianchi–Egnell [3], and the recent example of Musso–Wei

[20] for sign-changing solution), see also Ambrosetti–Malchiodi [1] for an abstract general setting. When this condition holds, the manifold $\mathcal{M}$ is the nondegenerate critical manifold, and minimizing $J_{0|\mathcal{M}}$ exactly amounts to minimizing $I_{0|\mathcal{M}}$. Despite the nondegeneracy of the critical manifold is a natural assumption, it does not necessarily hold, and is even exceptional in general: in Section 7, we exhibit examples of degenerate minimizers u_0 that are isolated among solutions to (2), and therefore, the only possible critical manifold is $\{u_0\}$ and is degenerate (see Propositions 7.1 and 7.3). We refer to Del Pino–Felmer [9], Jeanjean–Tanaka [16], Byeon–Jeanjean [5], and Dancer [8] for an analysis on $\mathbb{R}^n$ without nondegeneracy condition based on topological arguments. Here, we develop a method to deal with the absence of nondegenerate critical manifold by using analyticity. Indeed, we prove that all the terms in the analytic expansion of I_0 and J_0 on $\mathcal{M}$ can be compared, and we prove that the restriction of J_0 to $\mathcal{M}$ has a strict local minimum at u_0 if and only if u_0 is a strict local minimizer of I_0 (Theorem 6.1). This allows us to get a stable critical point for our problem.

This article is organized as follows. In Section 2, we state byproducts of our analysis. In Section 3, we define bubbles, we state the general construction theorem via finite-dimensional reduction and we recall existing results. In Section 4, we perform a first Lyapunov-Schmidt reduction to construct the analytic manifold $\mathcal{M}$ of approximations of u_0. In Section 5, we reduce the proof of Theorem 1.1 to obtaining a stable well for J_0 restricted to $\mathcal{M}$. In Section 6, we use the analyticity to prove the equivalence of strict local minimization for I_0 and J_0 on $\mathcal{M}$. In Section 7, we construct examples of degenerate strict local minimizers.

2. Miscellaneous further results

Theorem 1.1 is a particular case of the Theorem 2.1 below:

Theorem 2.1. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Let $h \in C^{0,\theta}(M)$ be such that $\Delta_g + h$ is coercive. Assume that there exists $u_0 \in C^{2,\theta}(M)$ that is a solution to (2) and a strict local minimizer of I_0. Assume that one of the following situations holds:*

$$(4) \qquad \left\{ \begin{array}{l} 3 \leq n \leq 5, \\ n = 6 \text{ and } c_n R_g - h < 2u_0, \\ 3 \leq n \leq 9 \text{ and } h \equiv c_n R_g, \\ n = 10 \text{ , } h \equiv c_n R_g \text{ and } u_0 > \frac{5}{567} |Weyl_g|_g^2, \\ n \geq 3, \ (M,g) \text{ is locally conformally flat and } h \equiv c_n R_g. \end{array} \right\}$$

Then there exist a solution of type $(u_0 - B)$ to (1).

We are also in position to construct positive solutions in dimension $n = 6$.

Theorem 2.2. *Let (M,g) be a smooth compact Riemannian manifold of dimension $n = 6$ and let $h \in C^{0,\theta}(M)$ be such that $\Delta_g + h$ is coercive. Assume that there exists $u_0 \in C^{2,\theta}(M)$ that is both a solution to (2) and an strict local minimizer of I_0. Assume that*

$$(5) \qquad h - c_6 R_g > 2u_0 > 0 \text{ in } M.$$

Then for $\varepsilon > 0$ small, equation (1) admits a solution $u_\varepsilon > 0$ such that $u_\varepsilon = u_0 + B_\varepsilon + o(1)$, where $(B_\varepsilon)_\varepsilon$ is a bubble and $\lim_{\varepsilon \to 0} o(1) = 0$ in $H_1^2(M)$.

3. Bubbles, general existence theorem and preliminary computations

This section essentially collects existing results from Robert–Vétois [24, 25].

3.1. **Bubbles.** We follow the terminology in [25]. We say that $(B_\varepsilon)_\varepsilon$ is a bubble if there exists $(x_\varepsilon)_\varepsilon \in M$ and $(\mu_\varepsilon)_\varepsilon \in (0,+\infty)$ such that $\lim_{\varepsilon\to 0}\mu_\varepsilon = 0$ and

$$B_\varepsilon(x) := \left(\frac{\sqrt{n(n-2)}\mu_\varepsilon}{\mu_\varepsilon^2 + d_g(x,x_\varepsilon)^2}\right)^{\frac{n-2}{2}} \quad \text{for all } x\in M. \tag{6}$$

There exists $r_0 \in (0, i_g(M))$ and $\Lambda \in C^\infty(M\times M)$ such that $(\xi, x)\mapsto \Lambda_\xi(x) > 0$, $\Lambda_\xi(\xi) = 1$ and :

(i) If (M,g) is locally conformally flat (lcf), then $g_\xi = \Lambda_\xi^{4/(n-2)} g$ is flat in $B_\xi(r_0)$.

(ii) If (M,g) is not locally conformally flat (non lcf) then $g_\xi := \Lambda_\xi^{\frac{4}{n-2}} g$ satisfies $dv_{g_\xi} = (1+O(d_{g_\xi}(\xi,\cdot)^n))\,dx$ in a geodesic normal chart. An immediate consequence is that $R_{g_\xi}(\xi) = |\nabla R_{g_\xi}(\xi)|_{g_\xi} = 0$ and $\Delta_{g_\xi} R_{g_\xi}(\xi) = \frac{1}{6}|\mathrm{Weyl}_g(\xi)|_g^2$. Moreover, $\nabla\Lambda_\xi(\xi) = 0$. This change of metric is due to Lee–Parker [19].

We let χ be a smooth cutoff function such that $0 \le \chi \le 1$ in $\mathbb{R}$, $\chi = 1$ in $[-r_0/2, r_0/2]$, and $\chi = 0$ in $\mathbb{R}\backslash(-r_0, r_0)$. For any $\kappa \in \{-1, 1\}$, any positive real number δ and any point ξ in M, we define the function $W_{\kappa,\delta,\xi}$ on M by

$$W_{\kappa,\delta,\xi}(x) := \kappa\chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)\left(\frac{\sqrt{n(n-2)}\delta}{\delta^2 + d_{g_\xi}(x,\xi)^2}\right)^{\frac{n-2}{2}},$$

where d_{g_ξ} is the geodesic distance on M associated with the metric g_ξ, the exponential map is taken with respect to the same metric g_ξ. As one checks, for any family $(\delta_\varepsilon)_\varepsilon \in (0,+\infty)$ going to 0 as $\varepsilon\to 0$, there exists a bubble $(B_\varepsilon)_\varepsilon$ such that

$$W_{\kappa,\delta_\varepsilon,\xi_\varepsilon} = \kappa B_\varepsilon + o(1) \tag{7}$$

in $H_1^2(M)$ when $\varepsilon \to 0$.

Notations: Here and in the sequel, $(\Delta_g + h)^{-1}$ denotes the inverse of the natural isometric isomorphism

$$\begin{array}{cccc} \Delta_g + h : & H_1^2(M) & \to & (H_1^2(M))' \\ & \phi & \mapsto & \left(\tau \mapsto \int_M ((\nabla\phi,\nabla\tau)_g + h\phi\tau)\,dv_g\right). \end{array}$$

Any function $f \in L^{\frac{2n}{n+2}}(M) = (L^{2^\star}(M))'$ is seen as a linear form on $H_1^2(M)$. In the sequel C will denote a constant independent of $\xi, \delta, \varphi, \varepsilon$. The value of C can change from one line to the other for simplicity.

3.2. **General existence theorem.** For any $\nu_0 > 0$ and $\varepsilon > 0$, we define

$$\mathcal{D}_\varepsilon(\nu_0) := \{(\delta,\xi)\in(0,\nu_0)\times M \,/\, |\delta^\varepsilon - 1| < \nu_0\}.$$

We define for $\epsilon \in [0, 2^\star - 2)$

$$J_\varepsilon(u) := \frac{1}{2}\int_M \left(|\nabla u|_g^2 + hu^2\right) dv_g - \frac{1}{2^\star-\varepsilon}\int_M |u|^{2^\star-\varepsilon}\,dv_g = \frac{1}{2}\|u\|_h^2 - F_\varepsilon(u)$$

for all $u \in H_1^2(M)$, where

$$\|u\|_h^2 = (u,u)_h = \int_M \left(|\nabla u|_g^2 + hu^2\right) dv_g \text{ and } F_\varepsilon(u) := \frac{1}{2^\star-\varepsilon}\int_M H(u)^{2^\star-\varepsilon}\,dv_g.$$

Here, $H(u) := |u|$ if $\kappa = -1$ and $H(u) := u_+$ if $\kappa = 1$. For any closed subspace $L \subset H_1^2(M)$, Π_L will denote the orthogonal projection onto L and $L^\perp$ the orthogonal complement of L with respect to the Hilbert structure $(\cdot,\cdot)_h$.

We let $u \in C^1(B_{\nu_0}(0) \subset K_0, H_1^2(M))$ be such that $u(0) = u_0$ and

$$|\det(\Pi_{K_0}\partial_1 u(\varphi), \cdots, \Pi_{K_0}\partial_d u(\varphi))| \geq c_0 \prod_{i=1}^{d} \|\partial_i u(\varphi)\|_{H_1^2} \tag{8}$$

for some $c_0 > 0$ and all $\varphi \in B_{\nu_0}(0) \subset K_0$. Here, $d := \dim_{\mathbb{R}}(K_0)$ and derivatives refer to a fixed basis of K_0. The following existence theorem is a consequence of Theorem 1.1 in Robert–Vétois [25]:

Theorem 3.1. *There exists $\nu_0 > 0$ and there exists $\phi_\varepsilon \in C^1(B_{\nu_0}(0) \times \mathcal{D}_\varepsilon(\nu_0), K_0^\perp)$ such that for all $\varphi \in B_{\nu_0}(0) \subset K_0$, $(\delta,\xi) \in \mathcal{D}_\varepsilon(\nu_0)$, the function $u_\varepsilon(\varphi,\delta,\xi) := u(\varphi) + W_{\kappa,\delta,\xi} + \phi_\varepsilon(\varphi,\delta,\xi)$ is a critical point for J_ε if and only if (φ,δ,ξ) is a critical point of $(\varphi,\delta,\xi) \mapsto J_\varepsilon(u_\varepsilon(\varphi,\delta,\xi))$. Moreover, $\|\phi_\varepsilon(\varphi,\delta,\xi)\|_h \leq C \cdot R_\varepsilon(\varphi,\delta,\xi)$ where*

$$R_\varepsilon(\varphi,\delta,\xi) := \|\Pi_{K_{\delta,\xi}^\perp}\left(u(\varphi) + W_{\kappa,\delta,\xi} - (\Delta_g + h)^{-1}(F_\varepsilon'(u(\varphi) + W_{\kappa,\delta,\xi}))\right)\|_h\,. \tag{9}$$

The space $K_{\delta,\xi}$ is defined below.

The projection onto $K_{\delta,\xi}^\perp$ in the rest $R_\varepsilon(\varphi,\delta,\xi)$ follows from Subsection 5.3 in [25]. The function ϕ_ε is defined implicitely as follows: given $(\varphi,\delta,\xi) \in B_{\nu_0}(0) \times \mathcal{D}_\varepsilon(\nu_0)$, $\phi_\varepsilon(\varphi,\delta,\xi)$ is the sole element of $K_{\delta,\xi}^\perp$ such that

$$\Pi_{K_{\delta,\xi}^\perp}\left(u_\varepsilon(\varphi,\delta,\xi) - (\Delta_g + h)^{-1}(F_\varepsilon'(u_\varepsilon(\varphi,\delta,\xi)))\right) = 0\,.$$

The linear space $K_{\delta,\xi}$ is defined as

$$K_{\delta,\xi} := \text{Span}\,\{\varphi\,,\, Z_{\delta,\xi}\,,\, Z_{\delta,\xi,X}\,,\, \varphi \in K_0 \text{ and } X \in T_\xi M\}\,,$$

where

$$Z_{\delta,\xi}(x) := \chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)\delta^{\frac{n-2}{2}} \frac{d_{g_\xi}(x,\xi)^2 - \delta^2}{(\delta^2 + d_{g_\xi}(x,\xi)^2)^{\frac{n}{2}}}\,,$$

$$Z_{\delta,\xi,X}(x) := \chi(d_{g_\xi}(x,\xi))\Lambda_\xi(x)\delta^{\frac{n}{2}} \frac{\langle(\exp_\xi^{g_\xi})^{-1}(x), X\rangle_{g_\xi(\xi)}}{(\delta^2 + d_{g_\xi}(x,\xi)^2)^{\frac{n}{2}}}$$

for all $x \in M$.

3.3. **Estimate of the error term.** For simplicity, we will often write $W := W_{\kappa,\delta,\xi}$ and $\phi := \phi_\varepsilon(\varphi,\delta,\xi)$ in this section. It follows from [24, Sections 5 and 7], that

$$\|F_\varepsilon'(u(\varphi) + W) - F_\varepsilon'(u(\varphi)) - F_\varepsilon'(W)\|_{H_1^2(M)'} \leq C \cdot \varepsilon_1(\delta), \tag{10}$$

$$F_\varepsilon(u(\varphi) + W) - F_\varepsilon(u(\varphi)) - F_\varepsilon(W) - F_\varepsilon'(u(\varphi))W - F_\varepsilon'(W)u(\varphi) = O\left(\varepsilon_2(\delta)\right), \tag{11}$$

and

$$\begin{aligned}&\|W - (\Delta_g + h)^{-1}(F_\varepsilon'(W))\|_h \\ &\leq C \cdot \left(\varepsilon \ln\frac{1}{\delta} + \varepsilon_1(\delta) + \mathbf{1}_{\{n\geq 7\}}\|h - c_n R_g\|_\infty \delta^2 + \mathbf{1}_{\{n\geq 15 \text{ and non lcf}\}}\delta^4\right),\end{aligned} \tag{12}$$

where

$$\varepsilon_1(\delta) := \left\{ \begin{array}{ll} \delta^{\frac{n-2}{2}} & \text{if } n < 6 \\ \delta^2 \left(\ln \frac{1}{\delta}\right)^{\frac{2}{3}} & \text{if } n = 6 \\ \delta^{\frac{n+2}{4}} & \text{if } n > 6 \end{array} \right\} \text{ and } \varepsilon_2(\delta) := \left\{ \begin{array}{ll} \delta & \text{if } n = 3 \\ \delta^2 \ln \frac{1}{\delta} & \text{if } n = 4 \\ \delta^{\frac{n}{2}} & \text{if } n \geq 5 \end{array} \right\}. \tag{13}$$

Plugging (10) and (12) in (9) yields

$$\begin{aligned} R_\varepsilon(\varphi, \delta, \xi) \leq C \cdot \|\Pi_{K_{\delta,\xi}^\perp} \left(u(\varphi) - (\Delta_g + h)^{-1}(F_0'(u(\varphi))\right) \|_h \\ + O\left(\varepsilon \ln \frac{1}{\delta} + \varepsilon_1(\delta) + \mathbf{1}_{\{n\geq 7\}} \|h - c_n R_g\|_\infty \delta^2 + \mathbf{1}_{\{n \geq 15 \text{ and non lcf}\}} \delta^4\right). \end{aligned} \tag{14}$$

3.4. **First expansion of the energy J_ε.** The Taylor expansion of J_ε, the control of ϕ_ε in Theorem 3.1 and the definition (9) of $R_\varepsilon(\varphi, \delta, \xi)$ yield

$$\begin{aligned} & J_\varepsilon(u(\varphi) + W + \phi) \\ =& J_\varepsilon(u(\varphi) + W) + (u(\varphi) + W - (\Delta_g + h)^{-1}(F_\varepsilon'(u(\varphi) + W)), \phi)_h + O(\|\phi\|_h^2) \\ =& J_\varepsilon(u(\varphi) + W) + (\Pi_{K_{\delta,\xi}^\perp}(u(\varphi) + W - (\Delta_g + h)^{-1}(F_\varepsilon'(u(\varphi) + W))), \phi)_h + O(\|\phi\|_h^2) \\ =& J_\varepsilon(u(\varphi) + W) + O(R_\varepsilon(\varphi, \delta, \xi)^2). \end{aligned}$$

It then follows from (11) and (13) that

$$\begin{aligned} J_\varepsilon(u(\varphi) + W + \phi) = J_\varepsilon(u(\varphi)) + J_\varepsilon(W_{\kappa,\delta,\xi}) \\ + \left(u(\varphi) - (\Delta_g + h)^{-1}(|u(\varphi)|^{2^*-2-\varepsilon} u(\varphi)), W\right)_h \\ - F_\varepsilon'(W) u(\varphi) + O\left(R_\varepsilon(\varphi, \delta, \xi)^2 + \varepsilon_2(\delta)\right). \end{aligned} \tag{15}$$

Since $\varphi \mapsto u(\varphi) > 0$ is C^1, $u(0) = u_0$ is a solution to (2), we get that

$$\left(u(\varphi) - (\Delta_g + h)^{-1}(u(\varphi)^{2^\star-1-\varepsilon}), W\right)_h = f_1(\varphi)\delta^{\frac{n-2}{2}} + o(\delta^{\frac{n-2}{2}}) \tag{16}$$

when $\delta, \varepsilon \to 0$ and $f_1 \in C^1(B_{\nu_0}(0) \subset K_0, \mathbb{R})$, $f_1(0) = 0$. It follows from [24] that

$$F_\varepsilon'(W) u(\varphi) = \frac{\kappa 2^n \omega_{n-1} K_n^{-n}}{n(n(n-2))^{\frac{n-2}{4}} \omega_n} u(\varphi)[\xi] \delta^{\frac{n-2}{2}} + O(\delta^{\frac{n-2}{2}}(o(1) + |\delta^\varepsilon - 1|)) \tag{17}$$

when $(\delta, \varepsilon) \to 0$. Here, ω_k is the volume of the canonical unit $k-$sphere in $\mathbb{R}^{k+1}$ and K_n is the best constant of the Sobolev inequality $\|u\|_{2^\star} \leq K \|\nabla u\|_2$ in $\mathbb{R}^n$. Finally, expanding $J_\varepsilon(u(\varphi))$ with respect to ε and collecting (15), (16) and (17) yield

$$\begin{aligned} J_\varepsilon(u(\varphi) + W + \phi) = J_0(u(\varphi)) + \varepsilon f_2(\varphi) + J_\varepsilon(W_{\kappa,\delta,\xi}) \\ + \left(f_1(\varphi) - \frac{\kappa 2^n \omega_{n-1} K_n^{-n}}{n(n(n-2))^{\frac{n-2}{4}} \omega_n} u(\varphi)[\xi]\right) \delta^{\frac{n-2}{2}} \\ + O\left(R_\varepsilon(\varphi, \delta, \xi)^2 + \varepsilon_2(\delta) + \delta^{\frac{n-2}{2}}(o(1) + |\delta^\varepsilon - 1|)\right) + o(\varepsilon) \end{aligned} \tag{18}$$

when $\delta, \varepsilon \to 0$. Here, $f_2 \in C^1(B_{\nu_0}(0) \subset K_0, \mathbb{R})$

3.5. **Expansion of $J_\varepsilon(W_{\kappa,\delta,\xi})$.** The following result was obtained in [24]: there exists $\beta_n > 0$ such that

$$J_\varepsilon(W_{\kappa,\delta,\xi}) = \frac{K_n^{-n}}{n}\left(1-\beta_n\varepsilon - \frac{(n-2)^2}{4}(\delta^\varepsilon - 1)\right) + O\left(\varepsilon\delta^2+\varepsilon^2+(\delta^\varepsilon-1)^2\right) + O\left(\mathbf{1}_{\{n\le 5 \text{ or lcf}\}}\delta^{n-2}\right) \tag{19}$$

$$+\frac{K_n^{-n}}{n}\left\{\begin{array}{ll} O\left(\|h-c_3R_g\|_{C^{0,\theta}}\delta\right) & \text{if } n=3\\ 3(h-c_4R_g)(\xi)\delta^2\ln\frac{1}{\delta} + O\left(\|h-c_4R_g\|_{C^{0,\theta}}\delta^2\right) & \text{if } n=4\\ \frac{2(n-1)}{(n-2)(n-4)}(h-c_nR_g)(\xi)\delta^2 + O\left(\|h-c_nR_g\|_{C^{0,\theta}}\delta^{2+\theta}\right) & \text{if } n\ge 5\end{array}\right\}$$
$$+\frac{K_n^{-n}}{n}\left\{\begin{array}{ll} -\frac{1}{64}|\mathrm{Weyl}_g(\xi)|_g^2\delta^4\ln\frac{1}{\delta} + O\left(\delta^4\right) & \text{if } n=6 \text{ and non lcf}\\ -\frac{1}{24(n-4)(n-6)}|\mathrm{Weyl}_g(\xi)|_g^2\delta^4 + O(\delta^5) & \text{if } n\ge 7 \text{ and non lcf}\end{array}\right\}.$$

4. Suitable approximation of u_0 and analyticity

In [24], the blowing-up solutions of type $(u_0 - B)$ are directly modeled on a nondegenerate function u_0. When u_0 is degenerate, the kernel K_0 plays a role in the finite-dimensional reduction and we consider a manifold of functions around u_0 parametrized locally by K_0.

Proposition 4.1. *There exist $\nu_0 > 0$ small and $\phi \in C^1(B_{\nu_0}(0)\subset K_0, K_0^\perp)$ such that for all $\varphi\in K_0$ and $\psi\in K_0^\perp$ satisfying $\|\varphi\|_h, \|\psi\|_h < \nu_0$, we have that*

$$\Pi_{K_0^\perp}(u_0+\varphi+\psi-(\Delta_g+h)^{-1}((u_0+\varphi+\psi)^{2^*-1})) = 0 \Leftrightarrow \psi = \phi(\varphi).$$

In particular, ϕ vanishes up to order 1 at 0. Moreover, taking ν_0 smaller if necessary, $u_0+\varphi+\phi(\varphi)\in C^{2,\theta}(M)$ is positive for all $\varphi\in B_{\nu_0}(0)$ and $\phi : B_{\nu_0}(0)\to C^{2,\theta}(M)$ is analytic with respect to the associated topologies.

The analytic manifold of approximation is $\mathcal{M} := \{u_0+\varphi+\phi(\varphi)/\ \varphi\in B_{\nu_0}(0)\subset K_0\}$.

Proposition 4.1 is a particular case of a more general result. Some definitions and notations are required in order to state the general result. We fix $f\in C^1(\mathbb{R})$ and we assume that there exists $u_0\in C^{2,\theta}(M)$ such that

$$\Delta_g u_0 + hu_0 = f(u_0) \text{ in } M. \tag{20}$$

We define

$$K_0 := \{\varphi\in C^{2,\theta}/\ \Delta_g\varphi + h\varphi = f'(u_0)\varphi\}. \tag{21}$$

In the sequel, K_0 will be regarded as a subset of $H^1(M)$. It follows from Fredholm's theory that K_0 is of finite dimension $d\in\mathbb{N}$. We prove the following result in the spirit of Dancer [7]:

Proposition 4.2. *We let $f\in C^1(\mathbb{R})$ and $u_0\in C^{2,\theta}(M)$ be a solution to (20). We let K_0 be as in (21). Then there exists $\nu>0$ and $\phi\in C^\infty(B_\nu(0)\subset K_0, K_0^\perp\cap C^{2,\theta}(M))$ such that for all $\varphi\in B_\nu(0)\subset K_0$ and $\psi\in B_\nu(0)\subset K_0^\perp$,*

$$\Pi_{K_0^\perp}(u_0+\varphi+\psi-(\Delta_g+h)^{-1}(f(u_0+\varphi+\psi))) = 0 \Leftrightarrow \psi = \phi(\varphi). \tag{22}$$

Moreover, if f is analytic on an open interval I and $u_0(x)\in I$ for all $x\in M$, then ϕ is analytic around 0.

As one checks, the function $x \mapsto |x|^{2^\star-2}x$ is C^1 on $\mathbb{R}$ and analytic on $(0,+\infty)$. Therefore Proposition 4.1 is a direct consequence of Proposition 4.2.

Proof of Proposition 4.2. The first part of the statement is a direct application of the implicit function theorem and regularity theory. Since M is compact and u_0 is continuous, it follows from the analyticity of f that there exists $A, B > 0$ such that

$$|a_k(u_0(x))| \leq A \cdot B^k \text{ for all } k \geq 0 \text{ and } x \in M, \tag{23}$$

where

$$f(u_0(x)+h) = \sum_{k=0}^{\infty} a_k(u_0(x))h^k \text{ for all } x \in M \text{ and } h \in (-B^{-1}, B^{-1}).$$

Since ϕ is C^∞ its differential vanishes at 0, we write for any $L \geq 2$ that

$$\phi(\varphi) = \sum_{l=2}^{L} P_l(\varphi) + o(\|\varphi\|^L) \text{ when } \varphi \to 0\,,$$

where for all $l \geq 2$ and $\varphi \in B_\nu(0) \subset K_0$, $P_l(\varphi) \in K_0^\perp$ is a homogeneous polynomial of degree l. We set $P_1(\varphi) := \varphi \in K_0$. Therefore, for any $L \geq 1$, we have that

$$f(u_0 + \varphi + \phi(\varphi)) = \sum_{k=0}^{L} a_k(u_0)\left(\sum_{l=1}^{L} P_l(\varphi)\right)^k + o(\|\varphi\|^L)$$

when $\varphi \to 0$. We write that

$$\left(\sum_{i=1}^{L} X_i\right)^k = \sum_{j=0}^{\infty} Q_{k,L,j}(X_1, ..., X_L), \tag{24}$$

where

$$Q_{k,L,j}(X_1,...,X_L) := \sum_{\sum_1^L r_l = k\,;\, \sum_1^L l r_l = j} \frac{k!}{\prod_{l=1}^L r_l!} \prod_{l=1}^{L} X_l^{r_l}.$$

Note that $Q_{k,L,j}(X_1,...,X_L) = 0$ when $j \notin [k, Lk]$, so all the sums make sense. Therefore, for any $L \geq 2$, the term of degree L in (22) is

$$\Pi_{K_0^\perp}\left(P_L(\varphi) - (\Delta_g + h)^{-1}\left(\sum_{k=0}^{L} a_k(u_0) Q_{k,L,L}(P_1(\varphi),...,P_L(\varphi))\right)\right) = 0$$

for all $L \geq 2$. In the sum, the term for $k=0$ is 0, and the term for $k=1$ is $a_1(u_0)P_L(\varphi) = f'(u_0)P_L(\varphi)$. Therefore, we have that

$$P_L(\varphi) = L_0^{-1}\Pi_{K_0^\perp}\left((\Delta_g + h)^{-1}\left(\sum_{k=2}^{L} a_k(u_0) Q_{k,L,L}(P_1(\varphi),...,P_L(\varphi))\right)\right) \tag{25}$$

for all $L \geq 2$, where $L_0 : K_0^\perp \to K_0^\perp$ is the isomorphism given by

$$L_0(\psi) = \Pi_{K_0^\perp}\left(\psi - (\Delta_g + h)^{-1}\left(f'(u_0)\psi\right)\right) \text{ for all } \psi \in K_0^\perp.$$

Note that since $k, L \geq 2$, the right-hand side of (25) is independent of $P_L(\varphi)$. We fix $\alpha \in (0,1)$. It follows from elliptic theory that there exists $C > 0$ depending on (M,g), h and $f'(u_0)$ such that

$$\|P_L(\varphi)\|_{C^{1,\alpha}} \leq C\|\sum_{k=2}^{L} a_k(u_0) Q_{k,L,L}(P_1(\varphi),...,P_L(\varphi))\|_\infty \tag{26}$$

for all $L \geq 2$. We fix $K \geq 2$. Summing (26) from $L = 2$ to K, using (23), (24) and the nonnegativity of the coefficients of $Q_{k,L,L}$, we get that

$$\begin{aligned}(27)\qquad \sum_{L=2}^{K} \|P_L(\varphi)\|_{C^{1,\alpha}} &\leq C \cdot A \sum_{k=2}^{K}\sum_{L=k}^{K} B^k Q_{k,L,L}(\|P_1(\varphi)\|_\infty, ..., \|P_L(\varphi)\|_\infty)\\ &\leq C \cdot A \sum_{k=2}^{K}\sum_{L=k}^{K} B^k Q_{k,K,L}(\|P_1(\varphi)\|_\infty, ..., \|P_K(\varphi)\|_\infty)\\ &\leq C \cdot A \sum_{k=2}^{K}\left(B\sum_{l=1}^{K}\|P_l(\varphi)\|_\infty\right)^k .\end{aligned}$$

We define

$$h_K(t) := \sup_{\|\varphi\|_\infty \leq t} \sum_{L=2}^{K} \|P_L(\varphi)\|_\infty .$$

It follows from (27) that

$$t + h_K(t) \leq \frac{1}{2B} \ \Rightarrow\ h_K(t) \leq 2C \cdot A \cdot B^2 \cdot (t + h_K(t))^2 .$$

Therefore, since h_K is continuous and non-decreasing, we get that

$$t < \varepsilon_0 := \min\left(\frac{1}{4B}, \frac{1}{16AB^2C}\right) \ \Rightarrow\ h_K(t) \leq \varepsilon_0 \text{ for all } K \geq 2 .$$

As a consequence, the series $(\sum_{L=2}^{\infty} P_L(\varphi))$ converges uniformly on $B_{\varepsilon_0/2}(0) \subset K_0$ in the $C^{0,\alpha}$–norm. Inequality (27) yields the convergence in $C^{1,\alpha}(M)$. The characterization (22) then yields

$$\phi(\varphi) = \sum_{l=2}^{\infty} P_l(\varphi) \text{ for all } \varphi \in B_{\varepsilon_0}(0) \subset K_0 .$$

Elliptic theory yields convergence in $C^{2,\theta}(M)$. This proves analyticity. □

5. Reduction of the problem to the analysis of $J_0(u_0 + \varphi + \phi(\varphi))$

From now on, we define:

$$u(\varphi) := u_0 + \varphi + \phi(\varphi)$$

for all $\varphi \in B_{\nu_0}(0) \subset K_0$, where $\phi(\varphi)$ is defined in Proposition 4.1. In particular,

$$(28)\qquad \Pi_{K_0^\perp}\left(u(\varphi) - (\Delta_g + h)^{-1}(F_0'(u(\varphi)))\right) = 0$$

for all $\varphi \in B_{\nu_0}(0) \subset K_0$. Since $d\phi_0 \equiv 0$, it then follows from Proposition 4.1 that u satisfies the hypothesis (8). For $0 < a < b$ to be fixed later, we define

$$\delta := t\varepsilon^{\frac{2}{n-2}}$$

for $t \in [a, b]$. We assume that

$$\{3 \leq n \leq 6\} \text{ or } \{h \equiv c_n R_g \text{ and } 3 \leq n \leq 10\} \text{ or } \{h \equiv c_n R_g \text{ and lcf}\}.$$

Taking into account the expressions (13), (14), (18), (19), and (28), we then get that

$$J_\varepsilon(u(\varphi)+W+\phi) = J_0(u(\varphi)) + \varepsilon f_2(\varphi) + \frac{K_n^{-n}}{n}\left(1-\beta_n\varepsilon+\frac{n-2}{2}\varepsilon\ln\frac{1}{\varepsilon}\right) + \varepsilon\cdot\frac{K_n^{-n}}{n}\cdot\left(\frac{(n-2)^2}{4}\ln\frac{1}{t}+F(\varphi,\xi)t^{\frac{n-2}{2}}\right)+o(\varepsilon) \tag{29}$$

when $\varepsilon \to 0$ uniformly with respect to $t \in [a,b]$. Here, $F \in C^1(B_{\nu_0}(0)\times M)$ and we have that

$$F(0,\xi) = -\kappa\frac{2^n\omega_{n-1}u_0(\xi)}{(n(n-2))^{\frac{n-2}{4}}\omega_n} + \begin{cases} \frac{2(n-1)}{(n-2)(n-4)}(h-c_nR_g)(\xi) & \text{if } n=6 \\ -\frac{1}{24(n-4)(n-6)}|\text{Weyl}_g(\xi)|^2 & \text{if } n=10 \text{ and } h\equiv c_nR_g \\ 0 & \text{otherwise.}\end{cases}$$

The assumptions (4) (for $\kappa=-1$) and (5) (for $\kappa=1$) then yield

$$F(0,\xi)>0 \text{ for all } \xi\in M.$$

We define

$$a := \frac{1}{2}\left(\frac{n-2}{2\min_{\xi\in M}F(0,\xi)}\right)^{\frac{2}{n-2}} \text{ and } b := 2\left(\frac{n-2}{2\min_{\xi\in M}F(0,\xi)}\right)^{\frac{2}{n-2}}.$$

Since u_0 is a strict local minimizer of I_0, it follows from Theorem 6.1 of next section that there exists $\nu_1 \in (0,\nu_0/2)$ such that

$$J_0(u(\varphi)) > J_0(u_0) \text{ for all } \varphi\in B_{2\nu_1}(0)\setminus\{0\}. \tag{30}$$

Due to compactness, for any $\varepsilon>0$, there exists $(\varphi_\varepsilon,t_\varepsilon,\xi_\varepsilon)\in\overline{B}_{\nu_1}(0)\times[a,b]\times M$ such that

$$\min_{(\varphi,t,\xi)\in\overline{B}_{\nu_1}(0)\times[a,b]\times M} J_\varepsilon(u(\varphi)+W_{\kappa,t\varepsilon^{\frac{2}{n-2}},\xi}+\phi_\varepsilon(\varphi,t\varepsilon^{\frac{2}{n-2}},\xi)) = J_\varepsilon(u(\varphi_\varepsilon)+W_{\kappa,t_\varepsilon\varepsilon^{\frac{2}{n-2}},\xi_\varepsilon}+\phi_\varepsilon(\varphi_\varepsilon,t_\varepsilon\varepsilon^{\frac{2}{n-2}},\xi_\varepsilon)).$$

It then follows from the Taylor expansion (29), the choice of $0<a<b$ and (30) that $t_\varepsilon\in(a,b)$ and $\varphi_\varepsilon\in B_{\nu_1}(0)$ for small $\varepsilon>0$. Moreover, we have that

$$\lim_{\varepsilon\to0}t_\varepsilon = \left(\frac{n-2}{2\min_{\xi\in M}F(0,\xi)}\right)^{\frac{2}{n-2}} \text{ and } \lim_{\varepsilon\to0}\varphi_\varepsilon = 0\,,$$

and $(\xi_\varepsilon)_{\varepsilon>0}$ approaches the set of minimizers of $F(0,\cdot)$ when $\varepsilon>0$ is small. Therefore, since $(\varphi_\varepsilon,t_\varepsilon,\xi_\varepsilon)$ lies in the interior of the domain, it is a critical point for the minimizing functional, and therefore, $(\varphi_\varepsilon,t_\varepsilon\varepsilon^{\frac{2}{n-2}},\xi_\varepsilon)$ is a critical point for

$$(\varphi,\delta,\xi)\mapsto J_\varepsilon(u(\varphi)+W_{\kappa,\delta,\xi}+\phi_\varepsilon(\varphi,\delta,\xi)).$$

It then follows from Theorem 3.1 that $u_\varepsilon := u(\varphi_\varepsilon)+W_{\kappa,t_\varepsilon\varepsilon^{\frac{2}{n-2}},\xi_\varepsilon}+\phi_\varepsilon(\varphi_\varepsilon,t_\varepsilon\varepsilon^{\frac{2}{n-2}},\xi_\varepsilon)$ is a solution to

$$\Delta_g u_\varepsilon + hu_\varepsilon = |u_\varepsilon|^{2^\star-2-\varepsilon}u_\varepsilon \text{ in } M$$

for $\varepsilon > 0$ small, and in addition, due to (7) and the error control of ϕ_ε in Theorem 3.1, we have that

$$u_\varepsilon = u_0 + \kappa B_\varepsilon + o(1)$$

in $H_1^2(M)$ when $\varepsilon \to 0$, where B_ε is as in (6) with $\mu_\varepsilon := t_\varepsilon \varepsilon^{\frac{2}{n-2}}$. This proves Theorems 2.1 and 2.2, and therefore Theorem 1.1.

We are now left with proving Theorem 6.1.

6. Equivalence of strict local minimizers

This section is devoted to the proof of the following:

Theorem 6.1. *The function u_0 is a strict local minimizer of I_0 iff 0 is a strict local minimizer of $\varphi \mapsto J_0(u_0 + \varphi + \phi(\varphi))$.*

The proof goes through four claims and uses the analyticity of $\varphi \mapsto \phi(\varphi)$.

Claim 6.1. *There exists $\nu_0 > 0$ such that*

$$\|u_0 + \varphi + \phi(\varphi)\|_h^2 - \|u_0 + \varphi + \phi(\varphi)\|_{2^\star}^{2^\star} = \sum_{L=3}^{\infty} A_L(\varphi)$$

and

$$\|u_0 + \varphi + \phi(\varphi)\|_{2^\star}^{2^\star} = \|u_0\|_{2^\star}^{2^\star} - \frac{n}{2} \sum_{L=3}^{\infty} \frac{L-2}{L} A_L(\varphi)$$

for $\varphi \in B_{\nu_0}(0) \subset K_0$, where for any $L \geq 3$, $A_L(\varphi)$ is a homogeneous polynomial of degree L.

Proof of Claim 6.1. We are going to compute the Taylor expansions of the two left-hand-sides and we will use the analyticity of $\varphi \mapsto \phi(\varphi)$ to prove Claim 6.1. We fix $N \geq 2$. It follows from (24) that

$$\|u_0 + \varphi + \phi(\varphi)\|_{2^\star}^{2^\star} = \int_M \left(u_0 + \sum_{l=1}^{N} P_l(\varphi) \right)^{2^\star} dv_g + o\left(\|\varphi\|^N\right) \tag{31}$$

$$= \|u_0\|_{2^\star}^{2^\star} + \sum_{L=1}^{N} \sum_{j=1}^{L} \sum_{\sum_{l=1}^{L} r_l = j \,;\, \sum_{l=1}^{L} l r_l = L} \frac{\prod_{i=0}^{j-1}(2^\star - i)}{\prod_{l=1}^{L} r_l!} \int_M u_0^{2^\star - j} \prod_{l=1}^{L} P_l(\varphi)^{r_l} \, dv_g + o\left(\|\varphi\|^N\right).$$

We claim that

$$u_0 \in K_0^\perp. \tag{32}$$

We prove the claim. We let φ be in K_0. The self-adjointness of the Laplacian yields

$$(u_0, \varphi)_h = \int_M (\Delta_g u_0 + h u_0) \varphi \, dv_g = \int_M (\Delta_g \varphi + h\varphi) u_0 \, dv_g.$$

It then follows from equation (2) and the definition (3) of K_0 that $(u_0, \varphi)_h = 0$. This proves the claim.

It follows from (32) that the term for $L = 1$ in (31) is $2^\star \int_M u_0^{2^\star-1}\varphi\, dv_g = 0$. Separating the cases $j = 1$ and $j \geq 2$, we get that

$$\begin{aligned}(33)\quad & \|u_0+\varphi+\phi(\varphi)\|_{2^\star}^{2^\star} \\ & = \|u_0\|_{2^\star}^{2^\star} + \sum_{L=2}^{N}\sum_{j=2}^{L}\sum_{\sum_l r_l = j\,;\,\sum_l l r_l = L} \frac{\prod_{i=0}^{j-1}(2^\star - i)}{\prod_{l=1}^{L} r_l!}\int_M u_0^{2^\star-j}\prod_{l=1}^{L} P_l(\varphi)^{r_l}\, dv_g \\ & \quad + 2^\star \sum_{L=2}^{N}\int_M u_0^{2^\star-1}P_L(\varphi)\, dv_g + o\left(\|\varphi\|^N\right).\end{aligned}$$

For $L \geq 2$, it follows from the expression (25) of $P_L(\varphi)$ that

$$(34)\quad (L_0 P_L(\varphi), u_0)_h = \sum_{j=2}^{L}\sum_{\sum_l r_l = j\,;\,\sum_l l r_l = L}\frac{\prod_{i=1}^{j}(2^\star - i)}{\prod_{l=1}^{L} r_l!}\int_M u_0^{2^\star-j}\prod_{l=1}^{L}P_l(\varphi)^{r_l}\, dv_g.$$

Since the operator L_0 is symmetric, we have that

$$(35)\qquad (L_0 P_L(\varphi), u_0)_h = (P_L(\varphi), L_0 u_0)_h = -(2^\star - 2)\int_M u_0^{2^\star-1}P_L(\varphi)\, dv_g.$$

Plugging into (33) the expression of $\int_M u_0^{2^\star-1}P_L(\varphi)\, dv_g$ obtained by combining (35) and (34), we get that

$$\begin{aligned}(36)\quad & \|u_0+\varphi+\phi(\varphi)\|_{2^\star}^{2^\star} \\ & = \|u_0\|_{2^\star}^{2^\star} + \frac{n}{2}\sum_{L=2}^{N}\sum_{j=2}^{L}\sum_{\sum_l r_l = j\,;\,\sum_l l r_l = L}\frac{(j-2)\prod_{i=1}^{j-1}(2^\star - i)}{\prod_{l=1}^{L} r_l!}\int_M u_0^{2^\star-j}\prod_{l=1}^{L}P_l(\varphi)^{r_l}\, dv_g \\ & \quad + o\left(\|\varphi\|^N\right).\end{aligned}$$

Note that the term in the above sum vanishes for $j = 2$. As one checks, for any $3 \leq j \leq L$, we have that

$$\begin{aligned}& \sum_{q=1}^{L-1}\frac{L-2q}{L}\sum_{\sum_l s_l = j-1\,;\,\sum_l l s_l = L-q}\frac{1}{\prod_l s_l!}\int_M u_0^{2^\star-j}\Big(\prod_l P_l(\varphi)^{s_l}\Big)P_q(\varphi)\, dv_g \\ & = \sum_{q=1}^{L-1}\frac{L-2q}{L}\sum_{\sum_l r_l = j\,;\,\sum_l l r_l = L}\frac{r_q}{\prod_l r_l!}\int_M u_0^{2^\star-j}\prod_l P_l(\varphi)^{r_l}\, dv_g \\ & = \sum_{\sum_l r_l = j\,;\,\sum_l l r_l = L}\left(\sum_{q=1}^{L-1}\frac{L-2q}{L}r_q\right)\frac{1}{\prod_l r_l!}\int_M u_0^{2^\star-j}\prod_l P_l(\varphi)^{r_l}\, dv_g \\ & = (j-2)\sum_{\sum_l r_l = j\,;\,\sum_l l r_l = L}\frac{1}{\prod_l r_l!}\int_M u_0^{2^\star-j}\prod_l P_l(\varphi)^{r_l}\, dv_g.\end{aligned}$$

Plugging this identity into (36) yields

$$(37)\qquad \|u_0+\varphi+\phi(\varphi)\|_{2^\star}^{2^\star} = \|u_0\|_{2^\star}^{2^\star} + \frac{n}{2}\sum_{L=3}^{N}\sum_{q=1}^{L-2}\frac{L-2q}{L}u_{L-q,q}(\varphi) + o\left(\|\varphi\|^N\right),$$

where

$$u_{k,q}(\varphi) := \sum_{j=2}^{k} \left(\prod_{i=1}^{j} (2^\star - i) \right) \times \sum_{\sum_l s_l = j\,;\, \sum_l l s_l = k} \frac{1}{\prod_l s_l!} \int_M u_0^{2^\star - 1 - j} \Big(\prod_l P_l(\varphi)^{s_l} \Big) P_q(\varphi)\, dv_g \,.$$

For any L, q such that $q \geq 2$ and $L - q \geq 2$, the self-adjointness of L_0 yields $(L_0 P_q(\varphi), P_{L-q}(\varphi))_h = (P_q(\varphi), L_0 P_{L-q}(\varphi))_h$. Taking the explicit expression of (25) then yields

$$u_{L-q,q}(\varphi) = u_{q,L-q}(\varphi) \text{ for } 2 \leq q \leq L-2\,.$$

Therefore, for $L \geq 4$, we get that

$$\sum_{q=2}^{L-2} \frac{L-2q}{L} u_{L-q,q}(\varphi) = 0\,,$$

and then (37) yields

$$\|u_0 + \varphi + \phi(\varphi)\|_{2^\star}^{2^\star} = \|u_0\|_{2^\star}^{2^\star} + \frac{n}{2} \sum_{L=3}^{N} \frac{L-2}{L} u_{L-1,1}(\varphi) + o\left(\|\varphi\|^N\right). \tag{38}$$

We now estimate $\|u_0 + \varphi + \phi(\varphi)\|_h^2 - \|u_0 + \varphi + \phi\|_{2^\star}^{2^\star}$. Using (22) and that $u_0, \phi(\varphi) \in K_0^\perp$ for all $\varphi \in K_0$, we get that (writing $\phi = \phi(\varphi)$ for simplicity)

$$\begin{aligned}
&\|u_0 + \varphi + \phi\|_h^2 - \|u_0 + \varphi + \phi\|_{2^\star}^{2^\star} \\
&= \big(u_0 + \varphi + \phi, u_0 + \varphi + \phi - (\Delta_g + h)^{-1}(u_0 + \varphi + \phi)^{2^\star - 1}\big)_h \\
&= \big(\Pi_{K_0}(u_0 + \varphi + \phi), \Pi_{K_0}(u_0 + \varphi + \phi - (\Delta_g + h)^{-1}(u_0 + \varphi + \phi)^{2^\star - 1})\big)_h \\
&= \big(\varphi, \varphi - (\Delta_g + h)^{-1}(u_0 + \varphi + \phi)^{2^\star - 1})\big)_h \\
&= \|\varphi\|_h^2 - \int_M (u_0 + \varphi + \phi)^{2^\star - 1} \varphi\, dv_g\,.
\end{aligned} \tag{39}$$

We fix $N \geq 3$ and write $\phi(\varphi) = \sum_{L=2}^{N-1} P_l(\varphi) + o\left(\|\varphi\|^{N-1}\right)$ when $\varphi \to 0$. A Taylor expansion and (24) yield

$$\begin{aligned}
&\int_M (u_0 + \varphi + \phi(\varphi))^{2^\star - 1} \varphi\, dv_g \\
&= \int_M u_0^{2^\star - 1} \varphi\, dv_g + (2^\star - 1) \int_M u_0^{2^\star - 2} \varphi^2\, dv_g + \sum_{l=2}^{N-1} (2^\star - 1) \int_M u_0^{2^\star - 2} \varphi P_l(\varphi)\, dv_g \\
&+ \sum_{L=3}^{N} \sum_{j=2}^{L-1} \left(\prod_{i=1}^{j} (2^\star - i) \right) \sum_{\sum_l s_l = j\,;\, \sum_l l s_l = L-1} \frac{1}{\prod_l s_l!} \int_M u_0^{2^\star - 1 - j} \Big(\prod_l P_l(\varphi)^{s_l} \Big) \varphi\, dv_g \\
&\qquad + o\left(\|\varphi\|^N\right)
\end{aligned} \tag{40}$$

when $\varphi \to 0$. The definition (3) of K_0 yields

$$(2^\star - 1) \int_M u_0^{2^\star - 2} \varphi^2\, dv_g = \|\varphi\|_h^2\,. \tag{41}$$

Moreover, since $P_l(\varphi) \in K_0^\perp$ for all $l \geq 2$, we get that

$$\sum_{l=2}^{N-1}(2^\star-1)\int_M u_0^{2^\star-2}\varphi P_l(\varphi)\,dv_g = \left(\sum_{l=2}^{N-1} P_l(\varphi),\varphi\right)_h = 0\,. \tag{42}$$

Plugging together (32) and (39)–(42) yields

$$\|u_0+\varphi+\phi(\varphi)\|_h^2 - \|u_0+\varphi+\phi(\varphi)\|_{2^\star}^{2^\star} = -\sum_{L=3}^{N} u_{L-1,1}(\varphi) + o\left(\|\varphi\|^N\right) \tag{43}$$

when $\varphi \to 0$. We define

$$\begin{aligned} A_L(\varphi) &:= -u_{L-1,L} \\ &= -\sum_{j=2}^{L-1}\left(\prod_{i=1}^{j}(2^\star-i)\right)\sum_{\sum_l s_l=j\,;\,\sum_l l s_l=L-1}\frac{1}{\prod_l s_l!}\int_M u_0^{2^\star-1-j}\Big(\prod_l P_l(\varphi)^{s_l}\Big)\varphi\,dv_g \end{aligned} \tag{44}$$

which is a homogenous polynomial of degree L. Claim 6.1 then follows from (38), (39), (43) and the analyticity of $\varphi \mapsto \phi(\varphi)$ (see Proposition 4.2). ☐

We define

$$\mathcal{S}_{K_0} := \{\varphi \in K_0 / \|\varphi\|_h = 1\}.$$

For any $\varphi \in \mathcal{S}_{K_0}$ and any $t \in (-\nu_0, \nu_0)$, we define

$$f_\varphi(t) := \frac{J_0(u_0+t\varphi+\phi(t\varphi)) - J_0(u_0)}{t^2\cdot\|u_0\|_h^2} \text{ if } t \neq 0 \text{ and } f_\varphi(0) = 0\,.$$

It follows from Claim 6.1 that f_φ is analytic on $(-\nu_0, \nu_0)$ and that

$$\|u_0+t\varphi+\phi(t\varphi)\|_{2^\star}^{2^\star} = \|u_0\|_{2^\star}^{2^\star}\left(1-\frac{n}{2}t^3 f'_\varphi(t)\right)$$

for $|t| < \nu_0$. Therefore, we have that

$$\begin{aligned} &I_0(u_0+t\varphi+\phi(t\varphi)) \\ &\qquad = I_0(u_0)\left(1+2t^2 f_\varphi(t) - \frac{n-2}{2}t^3 f'_\varphi(t)\right)\cdot\left(1-\frac{n}{2}t^3 f'_\varphi(t)\right)^{-\frac{2}{2^\star}}. \end{aligned} \tag{45}$$

Claim 6.2. *We assume that u_0 is a strict local minimizer of I_0. Then there exists $\nu_1 \in (0, \nu_0)$ such that for any $\varphi \in \mathcal{S}_{K_0}$ and $t \in (-\nu_1, \nu_1) \setminus \{0\}$, there holds*

$$\begin{aligned} f_\varphi(t) = 0 &\;\Rightarrow\; f'_\varphi(t) \neq 0\,, \\ f'_\varphi(t) = 0 &\;\Rightarrow\; f_\varphi(t) > 0\,. \end{aligned}$$

Proof of Claim 6.2. If $f_\varphi(t) = f'_\varphi(t) = 0$, it then follows from (45) that $u_0 + t\varphi + \phi(t\varphi)$ is a minimizer for I_0 close to u_0, and therefore there exists $\lambda_t > 0$ such that $u_0 + t\varphi + \phi(t\varphi) = \lambda_t \cdot u_0$ for t small. It then follows from the definition (22) of $\phi(t\varphi)$ that $\lambda_t = 1$ and that $t\varphi = 0$, which is a contradiction since $t \neq 0$ and $\varphi \neq 0$. Therefore $f_\varphi(t)$ and $f'_\varphi(t)$ cannot vanish simultaneously for $t \neq 0$. Moreover, if $f'_\varphi(t) = 0$, (45) yields $f_\varphi(t) \geq 0$. Combining these assertions yields Claim 6.2. ☐

Claim 6.3. *We assume that u_0 is a strict local minimizer of I_0. We claim that for all $\varphi \in \mathcal{S}_{K_0}$, there exists $\tilde{t}_\varphi \in (0, \nu_1)$ such that $f_\varphi(t) > 0$ for all $t \in (0, \tilde{t}_\varphi)$.*

Proof of Claim 6.3. It follows from Claim 6.2 that f_φ does not vanish identically. Since it is analytic, there exists $a \neq 0$ and $k \geq 1$ (both depending on φ) such that $f_\varphi(t) = at^k + o(t^k)$ when $t \to 0$. Obtaining from this the expansion of $f'_\varphi(t)$ and plugging these expressions into (45) yield

$$I_0(u_0 + t\varphi + \phi(t\varphi)) = I_0(u_0)(1 + 2at^{k+2} + o(t^{k+2}))$$

when $t \to 0$. Since u_0 is a local minimizer, we get that $a \geq 0$, and then $a > 0$. This yields the existence of $\tilde{t}_\varphi$. This proves Claim 6.3. □

It follows from Claims 6.2 and 6.3 that for any $\varphi \in \mathcal{S}_{K_0}$, there exists $t_\varphi \in (0, \nu_1]$ such that $f_\varphi(t) > 0$ for all $t \in (0, t_\varphi)$, and in case $t_\varphi < \nu_1$, we have that $f_\varphi(t) < 0$ for all $t \in (t_\varphi, \nu_1)$.

Claim 6.4. *We assume that u_0 is a strict local minimizer of I_0. We claim that there exists $\nu_2 > 0$ such that $t_\varphi > \nu_2$ for all $\varphi \in \mathcal{S}_{K_0}$.*

Proof of Claim 6.4. We prove Claim 6.4 by contradiction. Indeed, otherwise, there exists a sequence $(\varphi_i) \in \mathcal{S}_{K_0}$ such that $t_{\varphi_i} \to 0$ when $i \to +\infty$ and $f_{\varphi_i}(t_{\varphi_i}) = 0$ for all i. Up to a subsequence, we can assume that $\varphi_i \to \varphi \in \mathcal{S}_{K_0}$ when $i \to +\infty$. We fix $t \in (0, \nu_1)$. Then for i large enough, we have $t_{\varphi_i} < t$, and therefore $f_{\varphi_i}(t) < 0$. Passing to the limit when $i \to +\infty$ yields $f_\varphi(t) \leq 0$ for all $t \in (0, \nu_1)$. This is a contradiction with Claim 6.3. This proves Claim 6.4. □

Proof of Theorem 6.1, first implication: We assume that u_0 is a strict local minimizer of I_0. It follows from Claim 6.4 that $J_0(u_0 + \varphi + \phi(\varphi)) > J_0(u_0)$ for all $\varphi \in B_{\nu_2}(0) \setminus \{0\}$. This proves the first implication of Theorem 6.1.

Proof of Theorem 6.1, second implication: We assume that there exists $\nu_1 > 0$ such that $J_0(u_0+\varphi+\phi(\varphi)) > J_0(u_0)$ for all $\varphi \in B_{\nu_1}(0) \setminus \{0\}$. For $\varphi \in B_{\nu_1}(0)$, we define $\delta A(\varphi)$ and $\delta B(\varphi)$ such that

$$\|u_0+\varphi+\phi(\varphi)\|_h^2 = \|u_0\|_h^2 \cdot (1+\delta A(\varphi)) \text{ and } \|u_0+\varphi+\phi(\varphi)\|_{2^\star}^{2^\star} = \|u_0\|_h^2 \cdot (1+\delta B(\varphi)).$$

Therefore, we have that

$$J_0(u_0 + \varphi + \phi(\varphi)) = J_0(u_0) + \|u_0\|_h^2 \cdot \left(\frac{1}{2}\delta A(\varphi) - \frac{1}{2^\star}\delta B(\varphi)\right), \tag{46}$$

$$I_0(u_0 + \varphi + \phi(\varphi)) = I_0(u_0) \cdot (1 + \delta A(\varphi)) \left(1 + \delta B(\varphi)\right)^{-2/2^\star} \tag{47}$$

for all $\varphi \in B_{\nu_1}(0)$. It follows from our assumption and (46) that $\delta A(\varphi) > \frac{2}{2^\star}\delta B(\varphi)$ for all $\varphi \in B_{\nu_1}(0) \setminus \{0\}$. It then follows from (47) that

$$I_0(u_0 + \varphi + \phi(\varphi)) > I_0(u_0) \text{ for all } \varphi \in B_{\nu_1}(0) \setminus \{0\}. \tag{48}$$

We now let $(u_i) \in H_1^2(M)$ be minimizers for I_0 such that $\lim_{i\to+\infty} u_i = u_0$. It follows from regularity theory that $u_i \in C^{2,\theta}(M)$ for all i and that the convergence holds in $C^{2,\theta}(M)$. Without loss of generality, we can assume that u_i is a solution to (2) for all i. It then follows from the definition of ϕ (see Proposition 4.1) that there exists $\varphi_i \in K_0$ such that $u_i = u_0 + \varphi_i + \phi(\varphi_i)$ for all i. Since u_i is a local minimizer, it then follows from (48) that $\varphi_i = 0$ for i large, and thus $u_i = u_0$. Then u_0 is a strict local minimizer of I_0. This proves the second implication of Theorem 6.1. □

7. Examples

In this section, we provide examples of strict local minimizers for the functional I_0, and therefore for J_0 by Theorem 6.1. A preliminary remark is that it follows from the expression (44) of $A_L(\varphi)$ that

$$A_3(\varphi) = -\frac{(2^\star-1)(2^\star-2)}{2}\int_M u_0^{2^\star-3}\varphi^3\,dv_g\,, \tag{49}$$

$$A_4(\varphi) = -(2^\star-1)(2^\star-2)\left(\int_M u_0^{2^\star-3}\varphi^2 P_2(\varphi)\,dv_g + \frac{2^\star-3}{6}\int_M u_0^{2^\star-4}\varphi^4\,dv_g\right) \tag{50}$$

for all $\varphi\in K_0$. Moreover, it follows from Claim 6.1 that

$$I_0(u_0+\varphi+\phi(\varphi)) = I_0(u_0)\cdot\left(1+\frac{2A_3(\varphi)}{3\|u_0\|_{2^\star}^{2^\star}}+\frac{A_4(\varphi)}{2\|u_0\|_{2^\star}^{2^\star}}+o(\|\varphi\|^4)\right) \tag{51}$$

when $\varphi\to 0$. Therefore,

(52) if u_0 is a local minimizer of I_0 then $A_3\equiv 0$ and $A_4(\varphi)\geq 0$ for all $\varphi\in K_0$.

In the case of the Yamabe equation, this condition appeared in Kobayashi [18]. Conversely, we have the following result:

Proposition 7.1. *Assume that $A_3\equiv 0$ and $A_4(\varphi)>0$ for all $\varphi\in K_0\setminus\{0\}$. Then u_0 is a strict local minimizer for I_0. Moreover, there exists $\nu_1>0$ such that u_0 is the only solution to $\Delta_g u+hu=u^{2^\star-1}$ in $B_{\nu_1}(u_0)$.*

Proof of Proposition 7.1. The first part of the proposition is classical. For the second part, for any solution $u\in B_{\nu_1}(u_0)$, we decompose $u:=u_0+\varphi+\psi$ where $\varphi\in K_0$ and $\psi\in K_0^\perp$. We have that $\|\varphi\|<\nu_1$ and $\|\psi\|<\nu_1$. It follows from Proposition 4.2 that if $\nu_1>0$ is small enough, then $\psi=\phi(\varphi)$ and $u=u(\varphi)$. The positivity of A_4 yields the existence of $c>0$ such that $A_4(\varphi)\geq 2c\|\varphi\|^4$ for all $\varphi\in K_0$. It then follows from Claim 6.1 that $\|u\|_h^2-\|u\|_{2^\star}^{2^\star}\geq c\|\varphi\|^4$. Since u is a solution to the equation, we then get that $\varphi=0$ and then $u=u_0$. □

In this section, we exhibit situations in which the hypothesis of Proposition 7.1 hold, which yields strict local minimizers for I_0.

7.1. The expression of A_4 when u_0 is constant.

We assume here that $h,u_0>0$ are positive constants. In particular, we have that $h=u_0^{2^\star-2}$ and that

$$K_0=\{\varphi\in C^2(M)/\,\Delta_g\varphi=\lambda\varphi\},$$

where $\lambda:=(2^\star-2)u_0^{2^\star-2}>0$. In other words, u_0 is degenerate if and only if λ is an eigenvalue of Δ_g. As one checks, the operator

$$\begin{array}{rccc}\Delta_g-\lambda: & K_0^\perp & \to & (K_0^\perp)' \\ & \phi & \mapsto & \left(\tau\mapsto\int_M((\nabla\phi,\nabla\tau)_g-\lambda\phi\tau)\,dv_g\right)\end{array}$$

is a bi-continuous isomorphism and then definition (25) yields

$$P_2(\varphi)=\frac{(2^\star-1)(2^\star-2)}{2}(\Delta_g-\lambda)^{-1}(u_0^{2^\star-3}\varphi^2)$$

for all $\varphi \in K_0$. As a consequence, the expression (50) of A_4 can be rewritten

$$A_4(\varphi) = (2^\star - 1)(2^\star - 2)u_0^{2^\star - 4}\Big(- \frac{(2^\star - 1)\lambda}{2}\int_M \varphi^2(\Delta_g - \lambda)^{-1}(\varphi^2)\, dv_g - \frac{2^\star - 3}{6}\int_M \varphi^4\, dv_g\Big) \tag{53}$$

for all $\varphi \in K_0$.

7.2. **The case of the Yamabe equation on the canonical sphere.** In the case of the Yamabe equation on the sphere, the kernel K_0 parametrizes exactly the noncompact set of minimizers, which makes A_4 vanish. More precisely,

Proposition 7.2. *[Kobayashi [18]] Assume that $(M, g) = (\mathbb{S}^n, \mathrm{can})$ and that $h \equiv c_n R_{\mathrm{can}}$. Then any solution u_0 to (2) is minimizing and $A_4 \equiv 0$ for all u_0.*

Proof of Proposition 7.2. This result is a consequence of Theorem 2.1 in Kobayashi [18]. We give here an independent proof for the sake of self-content. The vanishing of A_4 is a consequence of the direct computation in the proof of (ii) of Proposition 7.3 below. We give here a shorter and less technical proof that stresses on properties of solutions to the scalar curvature equation on the sphere

$$\Delta_{\mathrm{can}} + c_n R_{\mathrm{can}} u = u^{2^\star - 1} \text{ in } \mathbb{S}^n. \tag{54}$$

The proof relies on two facts: first, the elements of the kernel K_0 satisfy a Bianchi–Egnell condition; second, all solutions to (54) minimize I_0 (see Obata [21]).

We fix $\varphi \in K_0$. It follows from properties of the canonical sphere (see below) that there exists $t \in \mathbb{R} \mapsto u(t)$ a smooth function such that $u(t) \in C^\infty(\mathbb{S}^n)$ is a solution to (54) for all t, $u(0) = u_0$ and $u'(0) = \varphi$. This is Bianchi–Egnell condition. Since $u(t)$ is a positive solution to (54), it follows from Proposition 4.1 that for t small, there exists $\varphi(t) \in K_0$ such that $u(t) = u_0 + \varphi(t) + \phi(\varphi(t))$. Moreover, $t \mapsto \varphi(t)$ is smooth, $\varphi(0) = 0$ and $\varphi'(0) = \varphi$. It follows from (52) that $A_3 \equiv 0$ since u_0 minimizes I_0. It then follows from the expansion (51) of A_4 that

$$\frac{A_4(\varphi)}{2\|u_0\|_{2^\star}^2} = \lim_{t\to 0} \frac{I_0(u_0 + \varphi(t) + \phi(\varphi(t))) - I_0(u_0)}{t^4 I_0(u_0)} = \lim_{t\to 0} \frac{I_0(u(t)) - I_0(u_0)}{t^4 I_0(u_0)}\,.$$

Moreover, it follows from Obata [21] that positive solutions to (54) are all minimizing, and then $I_0(u(t)) = I_0(u_0)$ for all small t. Therefore, we get that $A_4(\varphi) = 0$ for all $\varphi \in K_0$.

We are now left with proving the existence of $t \mapsto u(t)$. Up to conformal transformation (see Obata [21]), we assume that u_0 is the sole positive constant solution to (54). In this case, $K_0 = \{\varphi \in C^2(\mathbb{S}^n) \,/\, \Delta_{\mathrm{can}}\varphi = n\varphi\}$ is the space of first spherical harmonics. We fix $\varphi \in K_0$ and we let $Z := \vec{grad}(\varphi)$ be the associated vector field. This is a conformal vector field and, denoting f_t the associated flow, we have that $f_t^\star \mathrm{can} = \omega(t)^{4/(n-2)}\mathrm{can}$ for some positive function $t \mapsto \omega(t) \in C^\infty(\mathbb{S}^n)$ such that $\omega(0) = 1$. It follows from the conformal invariance of the scalar curvature equation that $u(t) := \omega(t)u_0$ is also a solution to (54) for all t. Moreover, since $f_t^\star \mathrm{can} = \omega(t)^{4/(n-2)}\mathrm{can}$, we have that $\omega'(0) = -\frac{n-2}{2n}\Delta_{\mathrm{can}}\varphi = \frac{n-2}{2n}\mathrm{div}_{\mathrm{can}}(Z) = -\frac{n-2}{2}\varphi$, and then $u'(0) = c\varphi$ for some $c \neq 0$. This proves the result after rescaling. $\square$

7.3. **Product of manifolds and examples of degenerate strict local minimizers.** Let (M_1, g_1) and (M_2, g_2) be two compact manifolds of respective dimensions $d \geq 1$ and $n - d \geq 1$ with $n \geq 3$. We consider the Riemannian manifold $M := M_1 \times M_2$ endowed with the product metric $g := g_1 \oplus g_2$. For $i = 1, 2$, we let $\lambda_1(M_i, g_i) > 0$ be the first nonzero eigenvalue of Δ_{g_i} on M_i. We define

$$h := \frac{\lambda_1(M_1, g_1)}{2^\star - 2} \text{ and } u_0 := \left(\frac{\lambda_1(M_1, g_1)}{2^\star - 2}\right)^{\frac{n-2}{4}}, \tag{55}$$

so that u_0 is the only positive constant solution to $\Delta_g u_0 + h u_0 = u_0^{2^\star - 1}$ in M. When $d \geq 3$, we define

$$\tilde{h} := \frac{\lambda_1(M_1, g_1)}{2_d^\star - 2} \text{ and } \tilde{u}_0 := \left(\frac{\lambda_1(M_1, g_1)}{2_d^\star - 2}\right)^{\frac{d-2}{4}}, \text{ where } 2_d^\star := \frac{2d}{d-2}$$

so that $\tilde{u}_0$ is the only positive constant solution to

$$\Delta_{g_1} \tilde{u}_0 + \tilde{h} \tilde{u}_0 = \tilde{u}_0^{2_d^\star - 1} \text{ in } M_1.$$

In particular, $\tilde{u}_0$ is a critical point for the functional

$$\tilde{I}_0(u) := \frac{\int_{M_1} \left(|\nabla u|_{g_1}^2 + \tilde{h} u^2\right) dv_{g_1}}{\left(\int_{M_1} |u|^{2_d^\star} dv_{g_1}\right)^{\frac{2}{2_d^\star}}}$$

for $u \in H_1^2(M_1) \setminus \{0\}$. We prove the following:

Proposition 7.3. *Let (M_1, g_1) and (M_2, g_2) be two compact manifolds of respective dimensions $d \geq 1$ and $n - d \geq 1$ with $n \geq 3$. We consider the Riemannian manifold $M := M_1 \times M_2$ of dimension $n \geq 3$ endowed with the product metric $g := g_1 \oplus g_2$. We let $h, u_0 > 0$ be as in* (55). *We assume that one of the following cases hold:*

(i) *$d \geq 3$, $\lambda_1(M_1, g_1) < \lambda_1(M_2, g_2)$, and $\tilde{u}_0$ is a local minimizer of $\tilde{I}_0$,*
(ii) *$d \geq 1$ and $(M_1, g_1) = (\mathbb{S}^d(r), \mathrm{can})$ with $r > \sqrt{\frac{d}{\lambda_1(M_2, g_2)}}$.*

Then u_0 is a degenerate solution to (2). *We have that $A_3(\varphi) = 0$ and $A_4(\varphi) > 0$ for all $\varphi \in K_0 \setminus \{0\}$. In particular, u_0 is a strict local minimizer of I_0.*

In the case of the Yamabe equation on the product of spheres, this proposition is a consequence of Kobayashi [18].

Proof of Proposition 7.3. We let (M_1, g_1), (M_2, g_2) be as in the proposition. Since $\lambda_1(\mathbb{S}^d(r), \mathrm{can}) = dr^{-2}$ (see Berger–Gauduchon–Mazet [2]), we have that

$$\lambda_1(M_1, g_1) < \lambda_1(M_2, g_2) \tag{56}$$

in both Cases (i) and (ii). As one checks,

$$K_0 = \{\varphi \in C^2(M) / \, \Delta_g \varphi = \lambda_1(M_1, g_1)\varphi\}.$$

It follows from spectral theory for products that K_0 is spanned by the functions $(x, y) \mapsto u_1(x) u_2(y)$ where for $i = 1, 2$, $u_i : M_i \to \mathbb{R}$ is an eigenfunction for the eigenvalue μ_i for Δ_{g_i}, where $\mu_1 + \mu_2 = \lambda_1(M_1, g_1)$. It then follows from (56) that

$$K_0 = \{(x, y) \in M \mapsto \varphi(x) / \, \varphi \in \Lambda_1(M_1, g_1)\},$$

where

$$\Lambda_1(M_1, g_1) := \{\varphi \in C^2(M_1) / \, \Delta_{g_1} \varphi = \lambda_1(M_1, g_1)\varphi\}$$

is the eigenspace associated to the first eigenvalue $\lambda_1(M_1,g_1)$. We claim that

$$\int_{M_1}\varphi^3\,dv_{g_1}=0 \text{ for all } \varphi\in\Lambda_1(M_1,g_1). \tag{57}$$

We prove the claim. In Case (i), since $d\geq 3$ and $\tilde{u}_0$ is a local minimizer, (57) follows from (49) and (52). In Case (ii), since $(M_1,g_1)=(\mathbb{S}^d(r),\mathrm{can})$, $\Lambda_1(M_1,g_1)$ is the restriction to $\mathbb{S}^d(r)$ of linear functions on $\mathbb{R}^{d+1}$, and then (57) follows from symmetry. This proves the claim.

Since the elements of K_0 are independent of the second variable, we get that

$$(\Delta_g-\lambda_1(M_1,g_1))^{-1}((x,y)\mapsto\varphi^2(x))=(x,y)\mapsto(\Delta_{g_1}-\lambda_1(M_1,g_1))^{-1}(\varphi^2(x))$$

for all $\varphi\in\Lambda_1(M_1,g_1)$ where $(\Delta_{g_1}-\lambda_1(M_1,g_1))^{-1}$ is the inverse of the isomorphism

$$\begin{array}{ccc}\Lambda_1(M_1,g_1)^\perp & \to & (\Lambda_1(M_1,g_1)^\perp)'\\ \phi & \mapsto & \left(\tau\mapsto\int_{M_1}((\nabla\phi,\nabla\tau)_{g_1}-\lambda_1(M_1,g_1)\phi\tau)\,dv_{g_1}\right)\end{array}$$

where the orthogonality in $H_1^2(M_1)$ is considered with respect to the standard L^2-product. As a consequence, the expression (53) can be rewritten

$$\begin{aligned}A_4(\varphi)=\frac{c_1\mathrm{Vol}_{g_2}(M_2)}{2}\Bigg(&-(2^\star-1)\lambda_1(M_1,g_1)\int_{M_1}\varphi^2(\Delta_{g_1}-\lambda_1(M_1,g_1))^{-1}(\varphi^2)\,dv_{g_1}\\&-\frac{2^\star-3}{3}\int_{M_1}\varphi^4\,dv_{g_1}\Bigg)\end{aligned} \tag{58}$$

for all $\varphi\in K_0$, where $c_1:=(2^\star-1)(2^\star-2)u_0^{2^\star-4}$ and, for simplicity, we have written K_0 for $\Lambda_1(M_1,g_1)$. We now distinguish Cases (i) and (ii) of Proposition 7.3:

Case (i): $d\geq 3$ and $\tilde{u}_0$ is a local minimizer. As one checks,

$$\tilde{K}_0:=\left\{\varphi\in C^2(M_1)/\,\Delta_{g_1}\varphi+\tilde{h}\varphi=(2_d^\star-1)\tilde{u}_0^{2_d^\star-2}\varphi\right\}=\Lambda_1(M_1,g_1).$$

We define $\tilde{A}_4$ for $\tilde{u}_0$ and therefore (53) yields

$$\begin{aligned}&\frac{2}{(2_d^\star-1)(2_d^\star-2)\tilde{u}_0^{2_d^\star-4}}\tilde{A}_4(\varphi)\\&\qquad=-(2_d^\star-1)\lambda_1(M_1,g_1)\int_{M_1}\varphi^2(\Delta_{g_1}-\lambda_1(M_1,g_1))^{-1}(\varphi^2)\,dv_{g_1}\\&\qquad\qquad-\frac{2_d^\star-3}{3}\int_{M_1}\varphi^4\,dv_{g_1}\end{aligned}$$

for all $\varphi\in\Lambda_1(M_1,g_1)$. Plugging this expression into (58) yields

$$A_4(\varphi)=c_2\cdot\left(\frac{(2^\star-1)\tilde{A}_4(\varphi)}{4(2_d^\star-1)^2(2_d^\star-2)\tilde{u}_0^{2_d^\star-4}}+\frac{(n-d)}{3(n-2)(d+2)}\int_{M_1}\varphi^4\,dv_{g_1}\right)$$

for all $\varphi\in\Lambda_1(M_1,g_1)$, where $c_2:=4(2^\star-1)(2^\star-2)u_0^{2^\star-4}\mathrm{Vol}_{g_2}(M_2)$. In particular, if $\tilde{u}_0$ is a local minimizer for $\tilde{I}_0$, then (52) yields $\tilde{A}_4\geq 0$. Therefore, $A_4(\varphi)>0$ for all $\varphi\in\Lambda_1(M_1,g_1)\setminus\{0\}$ since $n-d>0$. This proves Proposition 7.3 in Case (i).

Case (ii): $(M_1,g_1)=(\mathbb{S}^d(r),\mathrm{can})$. The case $d\geq 3$ is covered by Case (i), and only the cases $d=1,2$ remain to be covered. For simplicity, we assume that $r=1$.

It follows from Berger–Gauduchon–Mazet [2] that the second positive eigenvalue $\lambda_2(\mathbb{S}^d,\text{can})$ is $2(d+1)$ and the eigenfunctions are the restrictions to $\mathbb{S}^d$ of second-order homogeneous harmonic polynomials on $\mathbb{R}^{d+1}$.

We let Eucl be the Euclidean metric on $\mathbb{R}^{d+1}$. We claim that

$$(\Delta_{\text{can}}-\lambda_1)^{-1}(\varphi^2)=\frac{\varphi^2+\frac{\lambda_2\Delta_{\text{Eucl}}(\varphi^2)}{2(d+1)\lambda_1}}{\lambda_2-\lambda_1}\text{ for all }\varphi\in\Lambda_1(\mathbb{S}^d,\text{can}). \tag{59}$$

where $\lambda_1=d$ and $\lambda_2=2(d+1)$. We prove the claim. We fix $\varphi\in\Lambda_1(\mathbb{S}^d,\text{can})$. In particular φ^2 is a second-order homogeneous polynomial on $\mathbb{R}^{d+1}$, and $\varphi^2+\frac{\Delta_{\text{Eucl}}(\varphi^2)}{2(d+1)}|x|^2$ is a harmonic second-order homogenous polynomial, and therefore its restriction to $\mathbb{S}^d$ is an eigenfunction for λ_2. Since $\Delta_{\text{Eucl}}(\varphi^2)$ is constant and $|x|^2$ is constant on $\mathbb{S}^d$, (59) follows from a direct computation. This proves the claim.

We claim that

$$\int_{\mathbb{S}^d}\varphi^4\,dv_{\text{can}}=-\frac{3}{2(d+3)}\Delta_{\text{Eucl}}(\varphi^2)\int_{\mathbb{S}^d}\varphi^2\,dv_{\text{can}}\text{ for all }\varphi\in\Lambda_1(\mathbb{S}^d,\text{can}). \tag{60}$$

We prove the claim. Since, up to homothetic transformation, φ is a coordinate function, proving (60) is equivalent to proving $\int_{\mathbb{S}^d}x^4\,dv_{\text{can}}=(3/(d+3))\int_{\mathbb{S}^d}x^2\,dv_{\text{can}}$ where x is the first coordinate in $\mathbb{R}^{d+1}$. This latest identity follows from the change of variable $(t,\sigma)\mapsto(t,\sqrt{1-t^2}\sigma)$ from $(-1,1)\times\mathbb{S}^{d-1}$ to $\mathbb{S}^d\setminus\{(\pm1,...,0)\}$. This proves the claim.

Plugging (59) and (60) into (58) yields

$$A_4(\varphi)=\frac{4(2^\star-1)(2^\star-2)u_0^{2^\star-4}\text{Vol}_{g_2}(M_2)(n-d)}{3(n-2)(d+2)}\int_{\mathbb{S}^d}\varphi^4\,dv_{\text{can}}$$

for all $\varphi\in\Lambda_1(\mathbb{S}^d,\text{can})$. In particular, since $d<n$, we have that $A_4(\varphi)>0$ for all $\varphi\in\Lambda_1(\mathbb{S}^d,\text{can})\setminus\{0\}$. This proves Case (ii) of Proposition 7.3 when $r=1$. The general case follows by rescaling. This proves Proposition 7.3. □

As a remark, the computations made for Case (ii) are valid when $d=n\geq 3$ (that is $M=\mathbb{S}^d=\mathbb{S}^n$), and we get that $A_4\equiv 0$, which has been obtained by another method in Proposition 7.2.

When $h\equiv c_nR_g$, an immediate consequence of Proposition 7.3 is the following:

Corollary 7.1. *Let (N,g_N) be a compact Riemannian manifold of positive constant scalar curvature. We choose $d\geq 1$ and we assume that*

$$R_{g_N}<\mathit{dim}(N)\lambda_1(N,g_N)\text{ and }n:=d+\mathit{dim}(N)\geq 3\,. \tag{61}$$

We endow the manifold $M:=\mathbb{S}^d\big(\sqrt{\mathit{dim}(N)\cdot d/R_{g_N}}\big)\times N$ with the product metric $g:=\text{can}\oplus g_N$. Then the positive constant solution to the scalar curvature equation $\Delta_g u+c_nR_g u=u^{2^\star-1}$ on M is a degenerate strict local minimizer.

Inequality (61) holds if g_N is a Yamabe metric, that is a minimizer of the Yamabe functional. From the pde point of view, a metric g on M is a Yamabe metric iff R_g is constant and the minimum of I_0 (with $h\equiv c_nR_g$) is achieved by constants.

As a remark, Corollary 7.1 can be generalized by replacing the sphere by a manifold V of dimension $d\geq 3$ with a Yamabe metric g_V of positive scalar curvature satisfying $R_{g_N}=\dim(N)\lambda_1(V,g_V)$ and $\lambda_1(V,g_V)<\lambda_1(N,g_N)$.

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