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Asymptotic Profile for a Fourth Order PDE with Critical Exponential Growth in Dimension Four

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Abstract

Let Ω be a smooth domain of $\mathbb{R}^4$. In this paper, we consider a family $(u_\epsilon)_{\epsilon>0}$ of positive solutions in $C^4(\overline{\Omega})$ to the equation

$$\begin{cases} \Delta^2 u_\epsilon = \lambda u_\epsilon e^{32\pi^2 u_\epsilon^2} & \text{in } \Omega \\ u_\epsilon = \frac{\partial u_\epsilon}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\lambda \in \mathbb{R}$. Assuming that $u_\epsilon \to 0$ weakly in $H^2_{2,0}(\Omega)$ while $\sup_\Omega u_\epsilon \to \infty$, we describe the asymptotics of u_ϵ as $\epsilon \to 0$.

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1 Introduction

Let Ω be a smooth domain of $\mathbb{R}^4$. We denote by $H^2_{2,0}(\Omega)$ the Beppo-Levi space defined as the completion of $C^\infty_c(\Omega)$, the set of smooth compactly supported functions

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in Ω, with respect to the norm

$$\|u\|_{H^2_{2,0}(\Omega)} = \|\Delta u_\epsilon\|_2 = \sqrt{\int_\Omega (\Delta u)^2\, dx},$$

where $\Delta = -\sum \partial_{ii}$ is the Laplacian (with the geometers' sign convention) and where $\|\cdot\|_p$ denotes the L^p-norm. It follows from Sobolev's embedding theorem that $H^2_{2,0}(\Omega)$ is embedded in the Lebesgue spaces $L^q(\Omega)$ for all $q \geq 1$, and that these embeddings are compact. On the other hand, as is well-known, $H^2_{2,0}(\Omega)$ is not embedded in $L^\infty(\Omega)$. However, generalizing work of Trudinger [19] and Moser [14], Adams [1] showed that there exists $C > 0$ such that there holds

$$\int_\Omega e^{32\pi^2 u^2}\, dx \leq C \tag{1}$$

for all $u \in H^2_{2,0}(\Omega)$ with $\|\Delta u\|_2 \leq 1$. Moreover, the constant $32\pi^2$ is sharp in the sense that for any $\alpha > 32\pi^2$, there exists a sequence $(u_k)_{k\in\mathbb{N}}$ such that $\|\Delta u_k\|_2 = 1$ and $\int_\Omega e^{\alpha u_k^2}\, dx \to \infty$ as $k \to \infty$.

We let $\lambda \in \mathbb{R}$. For any $u \in H^2_{2,0}(\Omega)$, we define

$$F(u) = \frac{1}{2}\int_\Omega (\Delta u)^2\, dx - \frac{\lambda}{64\pi^2}\int_\Omega e^{32\pi^2 u^2}\, dx.$$

It follows from the Adams inequality (1) that F is well-defined and smooth. However, F fails to satisfy the Palais-Smale condition: There exist sequences $(u_k)_{k\in\mathbb{N}}$ such that $dF(u_k) \to 0$ strongly in the dual space of $H^2_{2,0}(\Omega)$, $F(u_k) = O(1)$ when $k \to \infty$, but no subsequence of (u_k) converges in $H^2_{2,0}(\Omega)$ when $k \to \infty$.

In order to understand this failure of compactness, we consider families of solutions $(u_\epsilon)_{\epsilon>0} \in C^4(\overline{\Omega})$ of the equation

$$\begin{cases} \Delta^2 u_\epsilon = \lambda u_\epsilon e^{32\pi^2 u_\epsilon^2} & \text{in } \Omega \\ u_\epsilon > 0 & \text{in } \Omega \\ u_\epsilon = \frac{\partial u_\epsilon}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \tag{E_ϵ}$$

where $\partial/\partial n$ denotes the outward normal derivative on the boundary of Ω. As is easily checked, any such u_ϵ is a critical point of F. Then we seek to describe the asymptotic behavior of (u_ϵ) as $\epsilon \to 0$.

From standard elliptic theory, see for instance [4], it follows that whenever $\max_\Omega u_\epsilon$ is bounded as $\epsilon \to 0$, then a subsequence (u_ϵ) converges in $C^4(\overline{\Omega})$ as $\epsilon \to 0$. In the following therefore we may assume that $\max_\Omega u_\epsilon \to \infty$ as $\epsilon \to 0$.

Our main result then is the following:

Theorem 1.1 *Let $\lambda > 0$. Let $(u_\epsilon)_{\epsilon>0}$ be a family of positive solutions of (E_ϵ). Choose $x_\epsilon \in \Omega$ such that* $\max_\Omega u_\epsilon = u_\epsilon(x_\epsilon)$. *Assume that, as $\epsilon \to 0$,*

$$u_\epsilon(x_\epsilon) \to \infty, \quad \|\Delta u_\epsilon\|_2^2 \to \Lambda, \tag{2}$$

where $\Lambda > 0$. *Then, with* $\mu_\epsilon > 0$ *given by*

$$\mu_\epsilon^{-1} = u_\epsilon(x_\epsilon)^{\frac{1}{2}} e^{8\pi^2 u_\epsilon(x_\epsilon)^2} \tag{3}$$

as $\epsilon \to 0$ *we have*

$$\eta_\epsilon(x) := u_\epsilon(x_\epsilon)(u_\epsilon(x_\epsilon + \mu_\epsilon x) - u_\epsilon(x_\epsilon)) \to \eta(x) = -\frac{1}{16\pi^2} \log\left(1 + \frac{\pi\sqrt{\lambda}}{\sqrt{6}}|x|^2\right) \tag{4}$$

in $C^4_{loc}(\mathbb{R}^4)$, *where* η *solves the equation*

$$\Delta^2 \eta = \lambda e^{64\pi^2 \eta} \text{ in } \mathbb{R}^4 \tag{5}$$

with

$$\lambda \int_{\mathbb{R}^4} e^{64\pi^2\eta}\, dx = 1, \tag{6}$$

and for any $R > 0$

$$\int_{B_{R\mu_\epsilon}(x_\epsilon)} u_\epsilon^2 e^{32\pi^2 u_\epsilon^2}\, dx \to \int_{B_R(0)} e^{64\pi^2\eta}\, dx. \tag{7}$$

Moreover, there exist $C > 0$, $I \in \mathbb{N}$, $I \leq \Lambda$, *and families of points* $x_{i,\epsilon} \in \Omega$, *scale factors* $\mu_{i,\epsilon} > 0$, *such that the analogues of (4) - (7) hold when we blow up* u_ϵ *with scale* $\mu_{i,\epsilon}$ *around* $x_{i,\epsilon}$, $i = 1, ..., I$. *In particular,* $I = 1$ *if* $\Lambda < 2$. *In addition,*

$$\frac{|x_{i,\epsilon} - x_{j,\epsilon}|}{\mu_{i,\epsilon}} \to \infty \text{ for all } 1 \leq i \neq j \leq I, \tag{8}$$

and the pointwise estimate

$$\inf_{i=1,\dots,I} |x - x_{i,\epsilon}|^2 u_\epsilon(x) e^{16\pi^2 u_\epsilon(x)^2} \leq C \tag{9}$$

holds for all $x \in \Omega$ *and all* $\epsilon > 0$.

Remark that when Ω is a ball we must have $\lambda \in (0, \lambda_1(\Omega))$; see Lemma 4.1 in the Appendix.

Problem (E_ϵ) is the four-dimensional analogue of the critical $n-$dimensional equation

$$\Delta^2 u = u^{2^\sharp - 1},$$

where $n \geq 5$ and $2^\sharp = \frac{2n}{n-4}$ is the limiting exponent for the embeddings of $H^2_{2,0}(\Omega)$ into Lebesgue's spaces. The asymptotics for this equation were described in Hebey-Robert [11], Robert [15] and Robert-Sandeep [16]. We refer also to Struwe [17] and Druet-Hebey-Robert [8].

Problem (E_ϵ) also is the fourth order extension of the two-dimensional elliptic problem

$$\begin{cases} \Delta \bar{u}_\epsilon = \lambda \bar{u}_\epsilon e^{4\pi \bar{u}_\epsilon^2} & \text{in } \Omega' \\ \bar{u}_\epsilon > 0 & \text{in } \Omega' \\ \bar{u}_\epsilon = 0 & \text{on } \partial\Omega' \end{cases}$$

where Ω' is a smooth domain of $\mathbb{R}^2$ and $(\bar{u}_\epsilon)_{\epsilon>0}$ is a family of smooth functions on Ω'. Such a problem was studied by Struwe [18], Atkinson-Peletier [5], Adimurthi-Struwe [3] and Adimurthi-Druet [2].

The main difficulties when generalizing these results to equations of higher order are due to the lack of the maximum principle and failure of Harnack's inequality for the biharmonic operator. Moreover, in contrast to Liouville's equation on $\mathbb{R}^2$, the conformally invariant limit equation that we encounter in (5) admits a whole family of radially symmetric solutions having arbitrarily small energies. In consequence, the blow-up behavior for this limit equation is more complicated than in the case of two space dimensions, analyzed by Brezis-Merle [7], which makes it more difficult to determine the concentration energy threshold for (E_ϵ).

We thank Adimurthi for having pointed out the problem and for valuable discussions in an early phase of our work.

2 Proof of theorem 1.1

We consider a family of solutions $(u_\epsilon)_{\epsilon>0} \in C^4(\overline{\Omega})$ to the system (E_ϵ) for some $\lambda > 0$, satisfying (2). In particular, we have

$$\int_\Omega (\Delta u_\epsilon)^2 \, dx = \int_\Omega \lambda u_\epsilon^2 e^{32\pi^2 u_\epsilon^2} \, dx \to \Lambda \tag{10}$$

when $\epsilon \to 0$. Then (u_ϵ) is bounded in $H^2_{2,0}(\Omega)$ when $\epsilon \to 0$ and there exists $u_0 \in H^2_{2,0}(\Omega)$ such that a subsequence

$$u_\epsilon \rightharpoonup u_0 \tag{11}$$

weakly in $H^2_{2,0}(\Omega)$ when $\epsilon \to 0$. The macroscopic concentration behavior is captured in the following result, reminiscent of Theorem I.6 in [13].

Proposition 2.1 *Let $(u_\epsilon)_{\epsilon>0}$ a family of solutions to the system (E_ϵ). We assume that (2) holds with $\Lambda \in (0,2)$. Then $\Lambda \geq 1$. Moreover, there exists $x_0 \in \overline{\Omega}$ such that, as $\epsilon \to 0$ suitably,*

$$(\Delta u_\epsilon)^2 \, dx \rightharpoonup \left(\Lambda - \|\Delta u_0\|_2^2\right) \delta_{x_0} + (\Delta u_0)^2 \, dx$$

weakly in the sense of measures and

$$u_\epsilon \to u_0 \text{ in } C^4_{loc}(\overline{\Omega} \backslash \{x_0\}).$$

The proof is similar to the proof of Lemma 3.3 in [3] in the context of second order equations on domains of $\mathbb{R}^2$ and may be omitted.

For any $\epsilon > 0$, we choose $x_\epsilon \in \Omega$ such that $u_\epsilon(x_\epsilon) = \max_\Omega u_\epsilon$. With $\mu_\epsilon > 0$ as defined in (3) we then let

$$\Omega_\epsilon = \{x; \, x_\epsilon + \mu_\epsilon x \in \Omega\} \, .$$

The following lemma shows that the domains Ω_ϵ as $\epsilon \to 0$ will exhaust all of $\mathbb{R}^4$.

Lemma 2.1 *Let $(u_\epsilon)_{\epsilon>0}$ be a family of solutions of (E_ϵ) such that (2) holds. Then, with x_ϵ and μ_ϵ as above, there holds*

$$\frac{d(x_\epsilon, \partial\Omega)}{\mu_\epsilon} \to +\infty$$

when $\epsilon \to 0$.

Proof. We let

$$\bar{u}_\epsilon(x) = \frac{u_\epsilon(x_\epsilon + \mu_\epsilon x)}{u_\epsilon(x_\epsilon)} \tag{12}$$

for all $x \in \Omega_\epsilon$ and $\epsilon > 0$. Clearly, $\bar{u}_\epsilon$ verifies the system

$$\begin{cases} \Delta^2 \bar{u}_\epsilon = \frac{\lambda}{u_\epsilon(x_\epsilon)^2} \bar{u}_\epsilon e^{32\pi^2 u_\epsilon(x_\epsilon)^2(\bar{u}_\epsilon^2 - 1)} & \text{in } \Omega_\epsilon \\ \bar{u}_\epsilon > 0 & \text{in } \Omega_\epsilon \\ \bar{u}_\epsilon = \frac{\partial \bar{u}_\epsilon}{\partial n} = 0 & \text{on } \partial\Omega_\epsilon \end{cases}$$

Moreover, $0 \le \bar{u}_\epsilon \le \bar{u}_\epsilon(0) = 1$. Assume that for a subsequence $\epsilon \to 0$ we have

$$\frac{d(x_\epsilon, \partial\Omega)}{\mu_\epsilon} \to R_0 < \infty.$$

Then, passing to a further subsequence, if necessary, we obtain convergence $\Omega_\epsilon \to \mathcal{P}$, where $\mathcal{P}$ is a half-plane. Standard elliptic theory, as given, for instance, in [4], shows that, up to yet another subsequence, $(\bar{u}_\epsilon)$ converges in C^4 to a function $\bar{u}$ satisfying

$$\begin{cases} \Delta^2 \bar{u} = 0 & \text{in } \mathcal{P} \\ \bar{u} \ge 0 & \text{in } \mathcal{P} \\ \bar{u} = \frac{\partial \bar{u}}{\partial n} = 0 & \text{on } \partial\mathcal{P} \end{cases}$$

and $\bar{u}(0) = 1$. After a change of variable, with error $o(1) \to 0$ as $\epsilon \to 0$, we find that

$$\int_{\Omega_\epsilon} |\nabla^2 \bar{u}_\epsilon|^2 \, dx = \int_\Omega |\nabla^2 u_\epsilon|^2 \, dx / u_\epsilon^2(x_\epsilon) \le \Lambda / u_\epsilon^2(x_\epsilon) + o(1).$$

Passing to the limit $\epsilon \to 0$, we then get that $\nabla^2 \bar{u} = 0$. In view of the boundary condition we conclude that $\bar{u} = 0$ on all of $\mathcal{P}$, in contradiction with $\bar{u}(0) = 1$. □

We now prove a first asymptotic estimate for u_ϵ:

Lemma 2.2 *Let $(u_\epsilon)_{\epsilon>0}$ be a family of solutions of (E_ϵ) such that (2) holds, and let x_ϵ and μ_ϵ be defined as above. Then for any $x \in \mathbb{R}^4$ we have that*

$$u_\epsilon(x_\epsilon + \mu_\epsilon x) - u_\epsilon(x_\epsilon) \to 0$$

when $\epsilon \to 0$. In fact, the convergence holds in $C^3_{loc}(\mathbb{R}^4)$.

Note that it follows from Lemma 2.1 that the statement of Lemma 2.2 is meaningful.

Proof. We let

$$w_\epsilon(x) = u_\epsilon(x_\epsilon + \mu_\epsilon x) - u_\epsilon(x_\epsilon)$$

for all $x \in \Omega_\epsilon$. The function w_ϵ is a solution of the equation

$$\Delta^2 w_\epsilon = \frac{\lambda}{u_\epsilon(x_\epsilon)} \bar{u}_\epsilon e^{32\pi^2 u_\epsilon(x_\epsilon)^2(\bar{u}_\epsilon^2 - 1)},$$

on Ω_ϵ, where $\bar{u}_\epsilon$ is defined in (12). Given $R > 0$, we get that

$$\|\Delta w_\epsilon\|_{L^2(B_R(0))} = \|\Delta u_\epsilon\|_{L^2(B_{R\mu_\epsilon}(x_\epsilon))} \le C,$$

where C is independent of R. By standard estimates for the Laplace operator, as given, for instance, in Theorems 8.17 and 8.18 of [10], we then conclude that there is $w \in C^4(\mathbb{R}^4)$ such that

$$w_\epsilon \to w \text{ in } C^3(\mathbb{R}^4), \quad \Delta w \in L^2(\mathbb{R}^4), \quad w \le w(0) = 0.$$

Moreover, $\Delta^2 w = 0$ in the weak sense. From Lemma 4.3 in the Appendix it then follows that w is affine. Since $w \le w(0) = 0$, we find that $w \equiv 0$, which proves the claim. □

We now define the *maximum rescaling* η_ϵ of u_ϵ by letting

$$\eta_\epsilon(x) = u_\epsilon(x_\epsilon)(u_\epsilon(x_\epsilon + \mu_\epsilon x) - u_\epsilon(x_\epsilon)) = u_\epsilon(x_\epsilon)^2(\bar{u}_\epsilon(x) - 1) \tag{13}$$

for all $x \in \Omega_\epsilon$. Then η_ϵ satisfies the equation

$$\Delta^2 \eta_\epsilon = \lambda \bar{u}_\epsilon e^{64\pi^2 \eta_\epsilon \left(1 + \frac{1}{2}(\bar{u}_\epsilon - 1)\right)} \tag{14}$$

for all $x \in \Omega_\epsilon$ with $\eta_\epsilon(x) \le \eta_\epsilon(0) = 0$. Set

$$V_\epsilon = \lambda \bar{u}_\epsilon, \quad a_\epsilon = 1 + \frac{1}{2}(\bar{u}_\epsilon - 1)$$

for all $\epsilon > 0$. From Lemma 2.2 it follows that

$$V_\epsilon \to \lambda \text{ and } a_\epsilon \to 1$$

in $C^1_{loc}(\mathbb{R}^4)$ as $\epsilon \to 0$. Note that in view of the boundary condition for u_ϵ, letting

$$\begin{cases} v_\epsilon(x) = u_\epsilon(x_\epsilon + \mu_\epsilon x) & \text{when } x \in \Omega_\epsilon, \\ v_\epsilon(x) = 0 & \text{when } x \in \mathbb{R}^4 \backslash \Omega_\epsilon, \end{cases}$$

we obtain a function $v_\epsilon \in H^2_{2,0}(\mathbb{R}^4)$. Finally, for any $y \in \mathbb{R}^4$ and any $r > 0$, we define

$$c_\epsilon^{(y,r)} = \fint_{B_r(y)} v_\epsilon \, dx, \tag{15}$$

where we denote as $\fint_A = \frac{1}{Vol(A)} \int_A$ the mean value on a domain $A \subset \mathbb{R}^4$.

Extending Lemma 4.2 in [3], we can bound the oscillation of $c_\epsilon^{(y,r)}$ as follows.

Proposition 2.2 *Let $(u_\epsilon)_{\epsilon>0}$ be a family of solutions of (E_ϵ) such that (2) holds. Then there exists a constant $C>0$ such that for any $y_1,y_2\in\mathbb{R}^4$ and $r_1,r_2>0$ we have that*

$$|c_\epsilon^{(y_1,r_1)}-c_\epsilon^{(y_2,r_2)}|\le C+4\log\frac{r^2}{r_1r_2}$$

for all $\epsilon>0$, where $2r=|y_1-y_2|+r_1+r_2$, and with $c_\epsilon^{(y,r)}$ as defined in (15).

Proof. We follow the proof of [3]. We first need some notations. Define the affine functions

$$l_\epsilon^{(y,r)}(x)=⨍_{B_r(y)}(x-y,\nabla v_\epsilon(z))\,dz$$

and let $w_\epsilon^{(y,r)}(x)=v_\epsilon(x)-c_\epsilon^{(y,r)}-l_\epsilon^{(y,r)}(x)$ for any $x\in\mathbb{R}$. From Hölder's inequality we have

$$⨍_{B_r(y)}|\nabla v_\epsilon|\,dx\le\left(⨍_{B_{r(y)}}|\nabla v_\epsilon|^4\,dx\right)^{1/4}.$$

On the other hand, the definition of v_ϵ and Sobolev's embedding $H^2_{2,0}(\Omega)\hookrightarrow H^4_1(\Omega)$ together with (2) give the uniform bound

$$\int_{B_r(y)}|\nabla v_\epsilon|^4\,dx=\int_{B_{r\cdot\mu_\epsilon}(x_\epsilon+\mu_\epsilon y)}|\nabla u_\epsilon|^4\,dx\le\int_\Omega|\nabla u_\epsilon|^4\,dx\le C.$$

It then follows that for any $y\in\mathbb{R}^4$, $r>0$, and any $\epsilon>0$ we have

$$|\nabla l_\epsilon^{(y,r)}|\le\frac{C}{r},\tag{16}$$

where $C>0$ is independent of the choice of y,r and ϵ.

We now prove the proposition. Let $y_1,y_2\in\mathbb{R}^4$ and $r_1,r_2>0$. Also let $y=\frac{y_1+y_2}{2}$ and define $2r=|y_1-y_2|+r_1+r_2$. For simplicity, we write $w_\epsilon=w_\epsilon^{(y,r)}$, $l_\epsilon=l_\epsilon^{(y,r)}$, $c_\epsilon=c_\epsilon^{(y,r)}$; moreover, for $j=1,2$ we let $w_\epsilon^j=w_\epsilon^{(y_j,r_j)}$, $l_\epsilon^j=l_\epsilon^{(y_j,r_j)}$, and $c_\epsilon^j=c_\epsilon^{(y_j,r_j)}$. By using (16), then for $j=1,2$ from the pointwise identity $c_\epsilon^j-c_\epsilon=w_\epsilon-w_\epsilon^j+l_\epsilon-l_\epsilon^j$ we obtain

$$\begin{aligned}
&|c_\epsilon^j-c_\epsilon|=\left|⨍_{B_{r_j}(y_j)}\left(w_\epsilon-w_\epsilon^j+l_\epsilon-l_\epsilon^j\right)\,dx\right|=\left|⨍_{B_{r_j}(y_j)}\left(w_\epsilon+l_\epsilon\right)\,dx\right|\\
&\le⨍_{B_{r_j}(y_j)}|w_\epsilon|\,dx+\|l_\epsilon\|_{L^\infty(B_{r_j}(y_j))}\le\log\left(⨍_{B_{r_j}(y_j)}e^{|w_\epsilon|}\,dx\right)+C\\
&\le\log\left(\left(\frac{r}{r_j}\right)^4⨍_{B_r(y)}e^{|w_\epsilon|}\,dx\right)+C\le\log\left(⨍_{B_r(y)}e^{|w_\epsilon|}\,dx\right)+4\log\frac{r}{r_j}+C
\end{aligned}$$

where $C>0$ is independent of r,y and ϵ.

For any $\delta > 0$ we can estimate

$$|w_\epsilon| \le \frac{\delta^2 w_\epsilon^2}{2\|\nabla^2 w_\epsilon\|^2_{L^2(B_r(y))}} + \frac{\|\nabla^2 w_\epsilon\|^2_{L^2(B_r(y))}}{2\delta^2}$$

to obtain the bound

$$|c_\epsilon^j - c_\epsilon| \le \log\left(\fint_{B_r(y)} e^{\frac{\delta^2 w_\epsilon^2}{2\|\nabla^2 w_\epsilon\|^2_{L^2(B_r(y))}}}\, dx\right) + C(\delta) + 4\log\frac{r}{r_j}.$$

Choosing $\delta^2 = k$, where $k > 0$ is the constant in the John-Nirenberg inequality, Lemma 4.2 in the Appendix, we find the estimate

$$|c_\epsilon^j - c_\epsilon| \le C + 4\log\frac{r}{r_j},$$

with a uniform constant C for $j = 1, 2$ and for all $\epsilon > 0$, which implies the claim. □

In particular, Proposition 2.2 and Lemma 2.2 yield the uniform bound

$$\left|c_\epsilon^{(y,1)} - u_\epsilon(x_\epsilon)\right| \le C + 8\log(|y| + 1) \tag{17}$$

for all $y \in \mathbb{R}^4$ and $\epsilon > 0$. The next result is the key step in showing subconvergence of η_ϵ.

Proposition 2.3 *Let $(u_\epsilon)_{\epsilon>0}$ be a family of solutions of (E_ϵ) such that (2) holds and let η_ϵ be defined as in (13). Then for any $R > 0$, there exists a constant $C(R) > 0$ depending only on R such that*

$$\Delta\eta_\epsilon(x) \le C(R)$$

for all $x \in B_R(0)$ and sufficiently small $\epsilon > 0$.

Proof. Since v_ϵ satisfies $\frac{\partial v_\epsilon}{\partial n} = 0$ on $\partial\Omega_\epsilon$, Hopf's maximum principle implies that $\Delta\eta_\epsilon = u_\epsilon(x_\epsilon)\Delta v_\epsilon \le 0$ on $\partial\Omega_\epsilon$. Green's representation formula and the maximum principle then yield the pointwise bound

$$(\Delta\eta_\epsilon)^+(x) \le \frac{1}{4\pi^2}\int_{\Omega_\epsilon} \frac{\Delta^2\eta_\epsilon(y)}{|x-y|^2}\, dy,$$

for any $x \in \Omega_\epsilon$. Hence with Fubini's theorem for any $R \ge 1$ we obtain the estimate

$$\int_{B_{2R}(0)} (\Delta\eta_\epsilon)^+(x)\, dx \le \frac{\lambda}{4\pi^2}\int_{y\in\mathbb{R}^4} u_\epsilon(x_\epsilon)\mu_\epsilon^4 v_\epsilon(y) e^{32\pi^2 v_\epsilon(y)^2}\left(\int_{x\in B_{2R}(0)} \frac{dx}{|x-y|^2}\right) dy.$$

Observe that for any $R \ge 1$ and $y \in \mathbb{R}^4$ we have

$$\int_{x\in B_{2R}(0)} \frac{dx}{|x-y|^2} \le C\min\left\{R^2, \frac{R^4}{1+|y|^2}\right\}$$

with $C > 0$ independent of R and y. Fix $K > 0$. With the preceding inequality we then succeed to bound

$$\begin{aligned}
&\int_{B_{2R}(0)} (\Delta\eta_\epsilon)^+(x)\,dx \le C \cdot R^2 \int_{|y|\le K} \lambda u_\epsilon(x_\epsilon)\mu_\epsilon^4 v_\epsilon(y) e^{32\pi^2 v_\epsilon(y)^2}\,dy \\
&+C \cdot R^4 \int_{|y|\ge K} \frac{\lambda|u_\epsilon(x_\epsilon) - c_\epsilon^{y,1}|\mu_\epsilon^4 v_\epsilon(y) e^{32\pi^2 v_\epsilon(y)^2}}{1+|y|^2}\,dy \\
&+C \cdot R^4 \int_{|y|\ge K} \frac{\lambda c_\epsilon^{y,1}\mu_\epsilon^4 v_\epsilon(y) e^{32\pi^2 v_\epsilon(y)^2}}{1+|y|^2}\,dy = I + II + III,
\end{aligned} \tag{18}$$

where $C > 0$ again is independent of $R \ge 1$ and $\epsilon > 0$. We estimate the terms on the right of (18) separately.

The definition (3) of μ_ϵ immediately yields the bound

$$I = C \cdot R^2 \int_{|y|\le K} \lambda u_\epsilon(x_\epsilon)\mu_\epsilon^4 v_\epsilon(y) e^{32\pi^2 v_\epsilon(y)^2}\,dy \le C(K) \cdot R^2 \tag{19}$$

for all $\epsilon > 0$ with a constant $C(K) > 0$ depending only on K and λ.

Choosing $K \ge 3$, we may use (17) and (10), (11) to estimate

$$\begin{aligned}
II &= C \cdot R^4 \int_{|y|\ge K} \frac{\lambda|u_\epsilon(x_\epsilon) - c_\epsilon^{y,1}|\mu_\epsilon^4 v_\epsilon(y) e^{32\pi^2 v_\epsilon(y)^2}}{1+|y|^2}\,dy \\
&\le C \cdot R^4 \int_{|y|\ge K} \frac{1+\log(|y|+1)}{1+|y|^2}\mu_\epsilon^4 v_\epsilon(y) e^{32\pi^2 v_\epsilon(y)^2}\,dy \\
&\le C \cdot R^4 \frac{1+\log(K+1)}{1+K^2} \cdot \int_{|y|\ge K} \mu_\epsilon^4 v_\epsilon(y) e^{32\pi^2 v_\epsilon(y)^2}\,dy \\
&\le C \cdot R^4 \frac{1+\log K}{1+K^2} \int_\Omega u_\epsilon e^{32\pi^2 u_\epsilon^2} \le C \cdot R^4 \frac{1+\log K}{1+K^2},
\end{aligned} \tag{20}$$

for all $\epsilon > 0$.

Finally, using Jensen's inequality, for any $y \in \mathbb{R}^4$ we can estimate

$$\begin{aligned}
&c_\epsilon^{y,1} v_\epsilon(y) e^{32\pi^2 v_\epsilon(y)^2} \le (c_\epsilon^{y,1})^2 e^{32\pi^2 (c_\epsilon^{y,1})^2} + v_\epsilon(y)^2 e^{32\pi^2 v_\epsilon(y)^2} \\
&\le \left(⨍_{B_1(y)} v_\epsilon(x)^2 e^{32\pi^2 v_\epsilon(x)^2}\,dx \right) + v_\epsilon(y)^2 e^{32\pi^2 v_\epsilon(y)^2}.
\end{aligned}$$

Hence for any $K \ge 3$ from (10) we obtain that

$$\begin{aligned}
&III = C \cdot R^4 \int_{|y|\ge K} \frac{\lambda c_\epsilon^{y,1}\mu_\epsilon^4 v_\epsilon(y) e^{32\pi^2 v_\epsilon(y)^2}}{1+|y|^2}\,dy \\
&\le \frac{C \cdot R^4}{1+K^2} \int_{\mathbb{R}^4} \mu_\epsilon^4 \left(\left(⨍_{B_1(y))} v_\epsilon(x)^2 e^{32\pi^2 v_\epsilon(x)^2}\,dx \right) + v_\epsilon(y)^2 e^{32\pi^2 v_\epsilon(y)^2} \right) dy \\
&\le \frac{C \cdot R^4}{1+K^2}
\end{aligned} \tag{21}$$

for all $\epsilon > 0$, where $C > 0$ is independent of K. Inserting the estimates (19), (20) and (21) into (18), we find the uniform bound

$$\int_{B_{2R}(0)} (\Delta\eta_\epsilon)^+(x)\,dx \leq C\frac{1+\log K}{1+K^2}R^4 + C(K)\cdot R^2 \tag{22}$$

for all $\epsilon > 0$ and any $K \geq 3$.

Now let $\eta_\epsilon^{(1)} \in H_2^2(B_{2R}(0))$ be defined as follows:

$$\begin{cases} \Delta^2\eta_\epsilon^{(1)} = \Delta^2\eta_\epsilon = \lambda\bar{u}_\epsilon e^{64\pi^2\eta_\epsilon\left(1+\frac{1}{2}(\bar{u}_\epsilon-1)\right)} & \text{in } B_{2R}(0), \\ \eta_\epsilon^{(1)} = \Delta\eta_\epsilon^{(1)} = 0 & \text{in } \partial B_{2R}(0). \end{cases}$$

It follows from the definitions (13) and (12) of η_ϵ and $\bar{u}_\epsilon$ that $\Delta^2\eta_\epsilon^{(1)}$ is bounded in $L^\infty(B_{2R}(0))$ when $\epsilon \to 0$. From [4] we obtain that with a uniform constant $C(R) > 0$ there holds

$$\|\eta_\epsilon^{(1)}\|_{C^2(\overline{B}_{2R}(0))} \leq C(R) \tag{23}$$

for all $\epsilon > 0$.

Finally, letting $\eta_\epsilon^{(2)} = \eta_\epsilon - \eta_\epsilon^{(1)}$, we have $\Delta(\Delta\eta_\epsilon^{(2)}) = 0$. Hence with (22) and (23) it follows that

$$\begin{aligned} \Delta\eta_\epsilon^{(2)}(x) &= ⨍_{B_R(x)} \Delta\eta_\epsilon^{(2)}(y)\,dy = ⨍_{B_R(x)} \Delta\eta_\epsilon(y)\,dy - ⨍_{B_R(x)} \Delta\eta_\epsilon^{(1)}(y)\,dy \\ &\leq C(R)\int_{B_{2R}(0)} (\Delta\eta_\epsilon)^+(y)\,dy + \|\eta_\epsilon^{(1)}\|_{C^2(\overline{B}_{2R}(0))} \leq C'(R), \end{aligned}$$

for any $x \in B_R(0)$ and any $\epsilon > 0$, where $C'(R) > 0$ is independent of $\epsilon > 0$ and $x \in B_R(0)$. The proposition is a consequence of this last inequality combined with (23) and $\eta_\epsilon = \eta_\epsilon^{(1)} + \eta_\epsilon^{(2)}$. □

The first part of Theorem 1.1 now is a consequence of the following proposition.

Proposition 2.4 *Let $(u_\epsilon)_{\epsilon>0}$ be a family of solutions to (E_ϵ) such that (2) holds. Let η_ϵ as in (13). Then,*

$$\eta_\epsilon(x) \to -\frac{1}{16\pi^2}\log\left(1+\frac{\pi\sqrt{\lambda}}{\sqrt{6}}|x|^2\right),$$

in $C^4_{loc}(\mathbb{R}^4)$ as $\epsilon \to 0$. Moreover, also (7) holds.

Proof. Recall that η_ϵ satisfies

$$\Delta^2\eta_\epsilon = V_\epsilon e^{64\pi^2\eta_\epsilon\cdot a_\epsilon}$$

for all $x \in \Omega_\epsilon$, where

$$V_\epsilon \to \lambda \text{ and } a_\epsilon \to 1$$

in $C^1_{loc}(\mathbb{R}^4)$ as $\epsilon \to 0$. With Proposition 2.3 and the fact that $\eta_\epsilon \le \eta_\epsilon(0) = 0$, it follows from standard estimates for the Laplace operator as in Theorems 8.17 and 8.18 in [10] that there exists $\eta \in C^4(\mathbb{R}^4)$ such that $\Delta^2\eta = \lambda e^{64\pi^2\eta}$, and, up to a subsequence,

$$\eta_\epsilon \to \eta$$

in $C^4_{loc}(\mathbb{R}^4)$. Note that for any $R > 0$

$$\begin{aligned} &\int_{B_R(0)} \bar{u}_\epsilon^2 e^{64\pi^2\eta_\epsilon\left(1+\frac{1}{2}(\bar{u}_\epsilon - 1)\right)}\,dx \\ &= \int_{B_{R\mu_\epsilon}(x_\epsilon)} u_\epsilon^2 e^{32\pi^2 u_\epsilon^2}\,dx \le \int_\Omega u_\epsilon^2 e^{32\pi^2 u_\epsilon^2}\,dx \le C \end{aligned} \tag{24}$$

when $\epsilon \to 0$. Passing to the limit $\epsilon \to 0$ and then $R \to +\infty$, we get that

$$e^{64\pi^2\eta} \in L^1(\mathbb{R}^4).$$

We claim that $\eta(x) = o(|x|^2)$ when $|x| \to \infty$. Otherwise, it follows from Lin [12] that there exists $a > 0$ such that $\Delta\eta(x) \ge a$ for all $x \in \mathbb{R}^4$. Letting $\epsilon \to 0$ in (22), we get that

$$a \cdot R^4 \cdot Vol(B_1(0)) \le \int_{B_R(0)} (\Delta\eta)^+(x)\,dx \le C\frac{1+\log K}{1+K^2}R^4 + C(K)\cdot R^2$$

for all $R > 1$ and $K > 3$. Dividing by R^4 and then first letting $R \to +\infty$ and afterwards $K \to +\infty$, we get that $a \le 0$. A contradiction. This proves our claim.

But if $\eta(x) = o(|x|^2)$ when $|x| \to \infty$ the result of Lin [12] shows that

$$\eta(x) = -\frac{1}{16\pi^2}\log\left(1 + \frac{\pi\sqrt{\lambda}}{\sqrt{6}}|x|^2\right)$$

for all $x \in \mathbb{R}^4$, which yields the first assertion. The relation (7) now follows from (24). □

3 Pointwise estimate

We now prove the pointwise estimate of Theorem 1.1. When $k \in \mathbb{N}$, we say that (H_k) holds if there exists $C > 0$ and k families of points $(x_{i,\epsilon})_{\epsilon>0}$, $i = 1, ..., k$ such that

$$\inf_{i=1,\dots,k} |x - x_{i,\epsilon}|^2 u_\epsilon(x) e^{16\pi^2 u_\epsilon(x)^2} \le C \tag{H_k}$$

for all $x \in \Omega$ and all $\epsilon > 0$. We say that (E_k) holds if there exist k families of points $(x_{i,\epsilon})_{\epsilon>0}$, $i = 1, ..., k$, such that for any $i = 1, ..., k$, and for any $x \in \mathbb{R}^4$,

$$\left.\begin{aligned} &\lim_{\epsilon\to 0}\frac{d(x_{i,\epsilon},\partial\Omega)}{\mu_{i,\epsilon}} = \infty,\ \lim_{\epsilon\to 0}\inf_{i\neq j}\frac{|x_{i,\epsilon}-x_{j,\epsilon}|}{\mu_{i,\epsilon}} = \infty \text{ for all } 1\leq i\neq j\leq k \\ &\lim_{\epsilon\to 0} u_\epsilon(x_{i,\epsilon})\left(u_\epsilon(x_{i,\epsilon}+\mu_{i,\epsilon}x)-u_\epsilon(x_{i,\epsilon})\right) = -\frac{1}{16\pi^2}\log\left(1+\frac{\pi\sqrt{\lambda}}{\sqrt{6}}|x|^2\right) \\ &\lim_{R\to\infty}\lim_{\epsilon\to 0}\int_{\cup_{i=1}^k B_{R\mu_{i,\epsilon}}(x_{i,\epsilon})}\lambda u_\epsilon^2 e^{32\pi^2 u_\epsilon^2}\,dx = k, \end{aligned}\right\}\quad (E_k)$$

where $\mu_{i,\epsilon} > 0$ is given by

$$\mu_{i,\epsilon}^{-1} = u_\epsilon(x_{i,\epsilon})^{\frac{1}{2}} e^{8\pi^2 u_\epsilon(x_{i,\epsilon})^2}.$$

Taking $x_{1,\epsilon} = x_\epsilon$, it follows from the two first parts of Theorem 1.1 that (E_1) holds.

Inspired by [2] and [8], we now argue by induction via the following proposition.

Proposition 3.1 *If (H_k) does not hold and (E_k) holds, then (E_{k+1}) holds.*

Clearly, (E_k) can only hold for integers $k \leq \Lambda$, where Λ is as in (10). It then follows from a straightforward induction that there exists $I \in \mathbb{N}$ such that (H_I) and (E_I) both hold. Proposition 3.1 therefore yields the remaining part of Theorem 1.1.

Thus, for some $k \in \mathbb{N}$ we now assume that (H_k) fails to be satisfied while (E_k) holds. Since (H_k) does not hold, for any $\epsilon > 0$ there exists $y_\epsilon \in \Omega$ such that

$$\begin{aligned} &\sup_{x\in\Omega}\left(\inf_{i=1,\dots,k}|x-x_{i,\epsilon}|^2\right)u_\epsilon(x)e^{16\pi^2 u_\epsilon(x)^2} \\ &= \left(\inf_{i=1,\dots,k}|y_\epsilon-x_{i,\epsilon}|^2\right)u_\epsilon(y_\epsilon)e^{16\pi^2 u_\epsilon(y_\epsilon)^2} \to \infty \end{aligned}\tag{25}$$

when $\epsilon \to 0$. Similar to (3) we define $\nu_\epsilon > 0$ by letting

$$\nu_\epsilon^{-1} = \sqrt{u_\epsilon(y_\epsilon)}e^{8\pi^2 u_\epsilon(y_\epsilon)^2}\tag{26}$$

for all $\epsilon > 0$ and set

$$\hat{\Omega}_\epsilon = \{y; y_\epsilon + \nu_\epsilon y \in \Omega\}.$$

It easily follows from (25) that for any $i = 1, ..., k$ there holds

$$\lim_{\epsilon\to 0}\frac{|y_\epsilon - x_{i,\epsilon}|}{\nu_\epsilon} = \infty.\tag{27}$$

For $x \in \hat{\Omega}_\epsilon$ we also let

$$\hat{u}_\epsilon(x) = \frac{u_\epsilon(y_\epsilon+\nu_\epsilon x)}{u_\epsilon(y_\epsilon)}.\tag{28}$$

Lemma 3.1 *Let $(u_\epsilon)_{\epsilon>0}$ be a family of solutions of (E_ϵ) such that (2) holds. Let $k \in \mathbb{N}$. We assume that (H_k) does not hold and that (E_k) holds. Let y_ϵ, ν_ϵ, $\hat{\Omega}_\epsilon$ and $\hat{u}_\epsilon$ as in (25), (26) and (28). Then*

$$\lim_{\epsilon\to 0} \frac{d(y_\epsilon, \partial\Omega)}{\nu_\epsilon} = +\infty, \ \hat{\Omega}_\epsilon \to \mathbb{R}^4 \ \text{and} \ \max_{B_R(0)} \hat{u}_\epsilon = 1 + o\left(u_\epsilon(y_\epsilon)^{-2}\right)$$

for all $R > 0$ when $\epsilon \to 0$.

Proof. Let $R > 0$ and $x \in B_R(0) \cap \hat{\Omega}_\epsilon$. It follows from the definition (25) of y_ϵ that, as $\epsilon \to 0$,

$$\begin{aligned}
&\left(\inf_{i=1,\dots,k} |y_\epsilon - x_{i,\epsilon} + \nu_\epsilon x|^2\right) u_\epsilon(y_\epsilon + \nu_\epsilon x) e^{16\pi^2 u_\epsilon(y_\epsilon + \nu_\epsilon x)^2} \\
&\quad \le \left(\inf_{i=1,\dots,k} |y_\epsilon - x_{i,\epsilon}|^2\right) u_\epsilon(y_\epsilon) e^{16\pi^2 u_\epsilon(y_\epsilon)^2} = \inf_{i=1,\dots,k} |\frac{y_\epsilon - x_{i,\epsilon}}{\nu_\epsilon}|^2 \to \infty.
\end{aligned}$$

By (27) and definition of $\hat{u}_\epsilon$, for any $x \in B_R(0) \cap \hat{\Omega}_\epsilon$ we have

$$\hat{u}_\epsilon(x) e^{16\pi^2 u_\epsilon(y_\epsilon)^2(\hat{u}_\epsilon(x)^2 - 1)} \le \frac{\inf_{i=1,\dots,k} |y_\epsilon - x_{i,\epsilon}|^2}{\inf_{i=1,\dots,k} (|y_\epsilon - x_{i,\epsilon}| - \nu_\epsilon R)^2} =: \tau_\epsilon(R),$$

where $\lim_{\epsilon\to 0} \tau_\epsilon(R) = 1$. Since $\hat{u}_\epsilon(0) = 1$, we then get that

$$1 \le \max_{B_R(0)\cap\hat{\Omega}_\epsilon} \hat{u}_\epsilon \le 1 + C\frac{\tau_\epsilon(R) - 1}{u_\epsilon(y_\epsilon)^2} = 1 + o\left(u_\epsilon(y_\epsilon)^{-2}\right)$$

when $\epsilon \to 0$. Therefore $\hat{u}_\epsilon$ is bounded in $L^\infty(B_R(0))$ when $\epsilon \to 0$. Following the proof of Lemma 2.1, we then get that $\hat{\Omega}_\epsilon \to \mathbb{R}^4$ when $\epsilon \to 0$. Note that this is equivalent to asserting that $\lim_{\epsilon\to 0} \frac{d(y_\epsilon, \partial\Omega)}{\nu_\epsilon} = \infty$. In particular, then for any $R > 0$ there holds $B_R(0) \subset \hat{\Omega}_\epsilon$ when $\epsilon > 0$ is sufficiently small. This ends the proof of the Lemma. □

We define

$$\hat{\eta}_\epsilon(x) = u_\epsilon(y_\epsilon)\left(u_\epsilon(y_\epsilon + \nu_\epsilon x) - u_\epsilon(y_\epsilon)\right) = u_\epsilon(y_\epsilon)^2(\hat{u}_\epsilon(x) - 1) \tag{29}$$

for all $\epsilon > 0$ and all $x \in \hat{\Omega}_\epsilon$. Similarly to what was done in Section 2, we prove the convergence of $\hat{\eta}_\epsilon$ when $\epsilon \to 0$.

Lemma 3.2 *Let $(u_\epsilon)_{\epsilon>0}$ be a family of solutions of (E_ϵ) such that (2) holds, and let $k \in \mathbb{N}$. We assume that (H_k) does not hold while (E_k) holds. Letting $\hat{\eta}_\epsilon$ as in (29), then we have*

$$\hat{\eta}_\epsilon(x) \to \ \hat{\eta}(x) := -\frac{1}{16\pi^2} \log\left(1 + \frac{\pi\sqrt{\lambda}}{\sqrt{6}}|x|^2\right),$$

in $C^4_{loc}(\mathbb{R}^4)$ when $\epsilon \to 0$. In particular, (6) also holds for $\hat{\eta}$.

Proof. Similar to (14) we have

$$\Delta^2 \hat{\eta}_\epsilon = \lambda \hat{u}_\epsilon e^{64\pi^2 \hat{\eta}_\epsilon \cdot \left(1+\frac{1}{2}(\hat{u}_\epsilon - 1)\right)}$$

for all $x \in \hat{\Omega}_\epsilon$ and all $\epsilon > 0$. Let $R > 0$. It follows from Lemma 3.1 that

$$\max_{x \in B_R(0)} \hat{\eta}_\epsilon(x) = o(1)$$

when $\epsilon \to 0$. Arguments similar to the ones developped in the proof of Proposition 2.3 give that for any $R > 0$, there exists $C(R) > 0$ such that

$$\Delta \hat{\eta}_\epsilon(x) \leq C(R)$$

for all $x \in B_R(0)$. It then follows from standard elliptic theory that there exists $\hat{\eta} \in C^4(\mathbb{R}^4)$ such that $\hat{\eta}_\epsilon \to \hat{\eta}$ in $C^4_{loc}(\mathbb{R}^4)$. As in Proposition 2.4, we then get that

$$\hat{\eta}(x) = -\frac{1}{16\pi^2} \log \left(1 + \frac{\pi\sqrt{\lambda}}{\sqrt{6}} |x|^2\right),$$

for all $x \in \mathbb{R}^4$. This ends the proof of the Lemma. □

Proof of Proposition 3.1: We now prove that (E_{k+1}) holds with $x_{k+1,\epsilon} = y_\epsilon$.

We claim that for any $i = 1, ..., k$,

$$\lim_{\epsilon \to 0} \frac{|y_\epsilon - x_{i,\epsilon}|}{\mu_{i,\epsilon}} = +\infty. \tag{30}$$

We argue by contradiction and assume that there exists $i_0 \in \{1, ..., k\}$ such that $|y_\epsilon - x_{i_0,\epsilon}| = O(\mu_{i_0,\epsilon})$ when $\epsilon \to 0$. Then for any $\epsilon > 0$, there exists $z_\epsilon \in \mathbb{R}^4$ such that $y_\epsilon = x_{i_0,\epsilon} + \mu_{i_0,\epsilon} z_\epsilon$, and a subsequence $z_\epsilon \to z \in \mathbb{R}^4$ when $\epsilon \to 0$. It follows from the second assertion of (E_k) that there exists a positive constant $C > 0$ such that

$$|y_\epsilon - x_{i_0,\epsilon}|^2 u_\epsilon(y_\epsilon) e^{16\pi^2 u_\epsilon(y_\epsilon)^2} \leq C\mu^2_{i_0,\epsilon} u_\epsilon(x_{i_0,\epsilon}) e^{16\pi^2 u_\epsilon(x_{i_0,\epsilon})^2} \leq C,$$

contradicting the definition (25) of y_ϵ. This proves the claim.

The two first assertions of (E_{k+1}) follow from (27), (30), Lemmas 3.1 and 3.2.

We are left with proving the last assertion about the energy. By (27) and (30) we have

$$B_{R\nu_\epsilon}(y_\epsilon) \cap \cup_{i=1}^{k+1} B_{R\mu_{i,\epsilon}}(x_{i,\epsilon}) = \emptyset$$

for $\epsilon > 0$ small. Therefore

$$\begin{aligned} &\int_{\cup_{i=1}^{k+1} B_{R\mu_{i,\epsilon}}(x_{i,\epsilon})} \lambda u_\epsilon^2 e^{3\pi^2 u_\epsilon^2}\, dx \\ &= \int_{\cup_{i=1}^{k} B_{R\mu_{i,\epsilon}}(x_{i,\epsilon})} \lambda u_\epsilon^2 e^{3\pi^2 u_\epsilon^2}\, dx + \int_{B_{R\nu_\epsilon}(y_\epsilon)} \lambda u_\epsilon^2 e^{32\pi^2 u_\epsilon^2}\, dx. \end{aligned} \tag{31}$$

Similar to (23), moreover, we obtain

$$\int_{B_{R\nu_\epsilon}(y_\epsilon)} \lambda u_\epsilon^2 e^{32\pi^2 u_\epsilon^2}\, dx = \int_{B_R(0)} \lambda \hat{u}_\epsilon^2 e^{64\pi^2 \hat{\eta}_\epsilon \cdot \left(1+\frac{1}{2}(\hat{u}_\epsilon - 1)\right)}\, dx$$
$$\to \int_{B_R(0)} \lambda e^{64\pi^2 \hat{\eta}}\, dx \tag{32}$$

as $\epsilon \to 0$. Combining (31), (32) and Lemma 3.2 with the last assertion of (E_k), we get that

$$\lim_{R\to+\infty} \lim_{\epsilon\to 0} \int_{\cup_{i=1}^{k+1} B_{R\mu_{i,\epsilon}}(x_{i,\epsilon})} \lambda u_\epsilon^2 e^{3\pi^2 u_\epsilon^2}\, dx = k+1,$$

as desired. Thus, assertion (E_{k+1}) holds, and the proof of Proposition 3.1 is complete. □

4 Appendix

In this appendix, we prove some auxiliary results required in the preceding section. We let $\lambda_1(B) > 0$ be the first eigenvalue of Δ^2 on the ball, that is

$$\lambda_1(B) = \min_{u\in H_{2,0}^2(\Omega)\backslash\{0\}} \frac{\int_B (\Delta u)^2\, dx}{\int_B u^2\, dx}. \tag{33}$$

In case Ω is a ball, we have the following

Lemma 4.1 *Assume that Ω is a ball of $\mathbb{R}^4$ and that there exists $\epsilon > 0$ and u_ϵ a solution to (E_ϵ) with $\lambda \in \mathbb{R}$. Then $\lambda \in (0, \lambda_1(\Omega))$, where $\lambda_1(\Omega) > 0$ is defined in (33).*

Proof. It follows from standard variational techniques that there exists a minimizer $\varphi \in C^4(\overline{\Omega})\backslash\{0\}$ for (33), and that

$$\Delta^2\varphi = \lambda_1\varphi \text{ in } \Omega\ ,\ \varphi = \frac{\partial\varphi}{\partial n} = 0 \text{ on } \partial\Omega.$$

We borrow ideas from Van Der Vorst [20]. We let $\varphi_1 \in C^4(\overline{\Omega})$ such that

$$\Delta^2\varphi_1 = |\Delta^2\varphi| \text{ in } \Omega\ ,\ \varphi_1 = \frac{\partial\varphi_1}{\partial n} = 0 \text{ on } \partial\Omega.$$

Since Ω is a ball, the Green's function for Δ^2 with Dirichlet boundary condition is positive (see for instance [6]). We get that

$$\varphi_1 \geq |\varphi| \text{ and } \varphi_1 > 0 \text{ in } \Omega.$$

Since φ is a minimizer for (33), we get that

$$\lambda_1(\Omega) \leq \frac{\int_\Omega (\Delta\varphi_1)^2\, dx}{\int_\Omega \varphi_1^2\, dx} = \frac{\int_\Omega \varphi_1 \Delta^2\varphi_1\, dx}{\int_\Omega \varphi_1^2\, dx} = \frac{\int_\Omega \lambda_1(\Omega)\varphi_1|\varphi|\, dx}{\int_\Omega \varphi_1^2\, dx} \leq \lambda_1(\Omega).$$

Then all these terms are equal, and $|\varphi| = \varphi_1 > 0$ in Ω. It then follows that φ does not change signe. Without loss of generality, we can assume that $\varphi > 0$ in Ω. Multiplying (E_ϵ) by φ and integrating, we get that

$$\begin{aligned}\lambda_1(\Omega)\int u_\epsilon\varphi\,dx &= \int_\Omega u_\epsilon\Delta^2\varphi\,dx = \int_\Omega \Delta^2 u_\epsilon\varphi\,dx = \int_\Omega \lambda u_\epsilon e^{32\pi^2 u_\epsilon^2}\varphi\,dx\\ &> \lambda\int_\Omega u_\epsilon\varphi\,dx\end{aligned}$$

and then $\lambda < \lambda_1(\Omega)$. Multiplying (E_ϵ) by u_ϵ and integrating, we easily get that $\lambda > 0$. □

Lemma 4.2 *There exists $k, C > 0$ such that for any $y \in \mathbb{R}^4$ and $r > 0$*

$$\fint_{B_r(y)} e^{k\frac{w^2}{\|\nabla^2 w\|_2^2}}\,dx \le C$$

for all $w \in H_2^2(B_r(y))$ such that $\fint_{B_r(y)} w\,dx = \fint_{B_r(y)} \partial_i w\,dx = 0$ for all $i = 1...4$.

Proof. Since this inequality is invariant under affine transformation of the domain, we only need to prove the result for B, the unit ball of $\mathbb{R}^4$. It follows from the John-Nirenberg inequality that there exists $C, K > 0$ such that

$$\fint_B e^{K\frac{w^2}{\|w\|_{H_2^2}^2}}\,dx \le C$$

for any $w \in H_2^2(B)$. By a variant of Poincaré's inequality as in [9] there exists $C_1 > 0$ such that

$$\|w\|_{H_2^2(B)}^2 \le C_1\|\nabla^2 w\|_{L^2(B)}^2,$$

for all $w \in H_2^2(B)$ such that $\int_B w\,dx = \int_B \partial_i w\,dx = 0$ for $i = 1...4$. Taking $k = \frac{K}{C_1}$, we obtain the lemma for the unit ball B. As already noticed, this proves the lemma in general. □

We now prove a rigidity result for bi-harmonic functions:

Lemma 4.3 *Let $n \ge 1$ and $u \in H_{2,loc}^2(\mathbb{R}^n)$ such that $\Delta^2 u = 0$ in the distribution sense. Assume that $\nabla^2 u \in L^2(\mathbb{R}^n)$. Then u is affine.*

Proof. It follows from standard elliptic theory that $u \in C^\infty(\mathbb{R}^n)$. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ on $\mathbb{R}^n \backslash B_2(0)$. For any $R > 0$ and $x \in \mathbb{R}^n$, we define $\eta_R(x) = \eta(R^{-1}x)$. For any $r > 0$ and $x \in \mathbb{R}^n$, we define $\varphi_r(x) = \fint_{B_r(0)} u\,dx + x^j \cdot \fint_{B_r(0)} \partial_j u\,dx$. Integrating by parts, we get that

$$0 = \int_{\mathbb{R}^n} (\Delta^2 u)\cdot\eta_R\cdot(u - \varphi_{2R})\,dx = \int_{\mathbb{R}^n} \Delta u\cdot\Delta(\eta_R\cdot(u - \varphi_{2R}))\,dx$$

and then

$$\begin{aligned}
&\int_{\mathbb{R}^n} \eta_R (\Delta u)^2\, dx = -2 \int_{\mathbb{R}^n} (\Delta u) \nabla \eta_R \nabla (u - \varphi_{2R})\, dx - \int_{\mathbb{R}^n} (u - \varphi_{2R}) \Delta u \cdot \Delta \eta_R\, dx \\
&\leq \frac{C}{R} \int_{R \leq |x| \leq 2R} |\Delta u| \cdot |\nabla (u - \varphi_{2R})|\, dx + \frac{C}{R^2} \int_{R \leq |x| \leq 2R} |u - \varphi_{2R}| |\Delta u|\, dx \\
&\leq C \|\Delta u\|_{L^2(\mathbb{R}^n \setminus B_R(0)} \sqrt{\sum_i \frac{1}{R^2} \int_{B_{2R}(0)} \left(\partial_i u - \fint_{B_{2R}(0)} \partial_i u\, dx \right)^2 dx} \\
&+ C \|\Delta u\|_{L^2(\mathbb{R}^n \setminus B_R(0)} \sqrt{\frac{1}{R^4} \int_{B_{2R}(0)} (u - \varphi_{2R})^2\, dx}.
\end{aligned}$$

It now follows from the Poincaré inequality that there exists $C > 0$ such that for any $R > 0$,

$$\frac{1}{R^2} \int_{B_{2R}(0)} \left(v - \fint_{B_{2R}(0)} v\, dx \right)^2 dx \leq C \int_{B_{2R}(0)} |\nabla v|^2\, dx$$

for all $v \in H_1^2(B_{2R}(0))$ and

$$\frac{1}{R^4} \int_{B_{2R}(0)} \left(v - \fint_{B_{2R}(0)} v\, dx - x^j \cdot \fint_{B_{2R}(0)} \partial_j v\, dx \right)^2 dx \leq C \int_{B_{2R}(0)} |\nabla^2 v|^2\, dx$$

for all $v \in H_2^2(B_{2R}(0))$. With these inequalities, we then get that

$$\int_{\mathbb{R}^n} \eta_R (\Delta u)^2\, dx \leq C \|\Delta u\|_{L^2(\mathbb{R}^n \setminus B_R(0)} \cdot \|\nabla^2 u\|_{L^2(\mathbb{R}^n)}$$

for any $R > 0$. Since $\nabla^2 u \in L^2(\mathbb{R}^n)$, letting $R \to +\infty$, we get that $\Delta u = 0$. Using once again the two preceding Poincaré inequalities, we get with similar computations that

$$\int_{\mathbb{R}^n} (\Delta(\eta_R (u - \varphi_{2R})))^2\, dx \leq C \|\nabla^2 u\|_{L^2(\mathbb{R}^n \setminus B_R(0)}.$$

Integrating by parts, using the definition of η_R and that φ_{2R} is affine, we get that

$$\begin{aligned}
\int_{\mathbb{R}^n} (\Delta(\eta_R (u - \varphi_{2R})))^2\, dx &= \int_{\mathbb{R}^n} |\nabla^2 (\eta_R (u - \varphi_{2R}))|^2\, dx \\
&\geq \int_{B_R(0)} |\nabla^2 (u - \varphi_{2R})|^2\, dx = \int_{B_R(0)} |\nabla^2 u|^2\, dx.
\end{aligned}$$

Combining these last two nequalities and letting $R \to +\infty$, we then get that $\nabla^2 u = 0$. Then u is affine. □

Finally, we prove a rigidity result on a half plane:

Lemma 4.4 *Let $n \geq 1$ and $u \in C^3(\mathbb{P})$ such that $u = \frac{\partial u}{\partial n} = 0$ on $\partial\mathbb{P}$, where $\mathbb{P}$ is a half plane of $\mathbb{R}^n$. Assume that $\Delta^2 u = 0$ in the distribution sense in $\mathbb{P}$, that $\nabla^2 u \in L^2(\mathbb{P})$ and that $u \geq 0$. Then $u \equiv 0$.*

Proof. Without loss of generality, we can assume that $0 \in \partial\mathbb{P}$. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ on $B_1(0)$ and $\eta \equiv 0$ on $\mathbb{R}^n \backslash B_2(0)$. For any $R > 0$ and $x \in \mathbb{R}^n$, we define $\eta_R(x) = \eta(R^{-1}x)$. Multiplying $\Delta^2 u$ by $\eta_R u$ and integrating by parts, we get that

$$0 = \int_{\mathbb{P}} \Delta u \cdot \Delta(\eta_R u)\, dx = \int_{\mathbb{P}} \eta_R (\Delta u)^2\, dx + 2\int_{\mathbb{RP}} (\Delta u) \nabla \eta_R \nabla u\, dx + \int_{\mathbb{P}} u \Delta u \cdot \Delta \eta_R\, dx.$$

It now follows from the Poincaré inequality that there exists $C > 0$ such that for any $R > 0$,

$$\frac{1}{R^2} \int_{B_{2R}(0)\cap\mathcal{P}} v^2\, dx \leq C \int_{B_{2R}(0)\mathcal{P}} |\nabla v|^2\, dx$$

for all $v \in H_1^2(B_{2R}(0) \cap \mathcal{P})$ such that $v = 0$ on $B_{2R}(0) \cap \partial\mathcal{P}$, and

$$\frac{1}{R^4} \int_{B_{2R}(0)\cap\mathcal{P}} v^2\, dx \leq C \int_{B_{2R}(0)\mathcal{P}} |\nabla^2 v|^2\, dx$$

for all $v \in H_2^2(B_{2R}(0) \cap \mathcal{P})$ such that $v = \frac{\partial v}{\partial n} = 0$ on $B_{2R}(0) \cap \partial\mathcal{P}$. With these inequalities, we then get that

$$\int_{\mathcal{P}} \eta_R (\Delta u)^2\, dx \leq C \|\Delta u\|_{L^2(\mathcal{P}\backslash B_R(0)} \cdot \|\nabla^2 u\|_{L^2(\mathcal{P})}$$

for any $R > 0$. Since $\nabla^2 u \in L^2(\mathcal{P})$, letting $R \to +\infty$, we get that $\Delta u = 0$. Since $u \geq 0$ and $u = \frac{\partial u}{\partial n} = 0$ on $\partial\mathcal{P}$, Hopf's maximum principle yields that $u \equiv 0$. □

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