CONSTRUCTION AND ASYMPTOTICS FOR THE GREEN'S FUNCTION WITH NEUMANN BOUNDARY CONDITION

INFORMAL NOTES

FRÉDÉRIC ROBERT

Abstract. This notes are devoted to a construction and to pointwise properties of the Green's functions of the Laplacian with Neumann boundary condition on a smooth bounded domain of $\mathbb{R}^n$. These informal notes are essentially self-contained and require only standard elliptic theory.

Let Ω be a smooth bounded domain of $\mathbb{R}^n$ (the definition is in Section 1). We consider the following problem:

$$\left\{\begin{array}{ll}\Delta u = f & \text{in } \Omega\\ \partial_\nu u = 0 & \text{in } \partial\Omega\end{array}\right. \tag{1}$$

where $u \in C^2(\overline{\Omega})$ and $f \in C^0(\overline{\Omega})$. Here and in the sequel, $\Delta := -\sum_i \partial_{ii}$ and for any $x \in \partial\Omega$, $\partial_\nu u(x)$ denotes the normal derivative of u at the boundary point x, that is $\partial_\nu u(x) := du_x(\nu(x))$ where du_x is the differential of u at x and $\nu(x)$ is the outward normal derivative of the oriented hypersurface $\partial\Omega$ (see Section 1). Note that the solution u is defined up to the addition of a constant and that it is necessary that $\int_M f\, dx = 0$ (this is a simple integration by parts). Our objective here is to construct and give properties of the Green kernel associated to (1). In the sequel, for any function $u \in L^1(\Omega)$, we define $\bar{u} := \frac{1}{|\Omega|}\int_\Omega u\, dy$ where $|\Omega|$ is the volume of Ω.

Definition 1. *We say that a function $G : \Omega \times \Omega \setminus \{(x,x)/\, x \in \Omega\} \to \mathbb{R}$ is a Green's function for (1) if for any $x \in \Omega$, noting $G_x := G(x,\cdot)$, we have that*

(i) $G_x \in L^1(\Omega)$,

(ii) $\int_\Omega G_x\, dy = 0$,

(iii) for all $\varphi \in C^2(\overline{\Omega})$ such that $\partial_\nu \varphi = 0$ on $\partial\Omega$, we have that

$$\varphi(x) - \bar{\varphi} = \int_\Omega G_x \Delta\varphi\, dy.$$

Condition (ii) here is required for convenience in order to get uniqueness, symmetry and regularity for the Green's function. Indeed, if G is a Green's function and if $c : \Omega \to \mathbb{R}$ is any function, the function $(x,y) \mapsto G(x,y) + c(x)$ satisfies (i) and (iii). The first result concerns the existence of the Green's function:

Theorem 1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$. Then there exists a unique Green's function G for (1). Moreover, G is symmetric and extends continuously to*

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$\overline{\Omega}\times\overline{\Omega}\setminus\{(x,x)/\,x\in\overline{\Omega}\}$ and for any $x\in\overline{\Omega}$, we have that $G_x\in C^{2,\alpha}(\overline{\Omega}\setminus\{x\})$ and satisfies

$$\begin{cases} \Delta G_x = -\frac{1}{|\Omega|} & \text{in } \Omega\setminus\{x\}\\ \partial_\nu G_x = 0 & \text{on } \partial\Omega\setminus\{x\}.\end{cases}$$

In addition, for all $x\in\overline{\Omega}$ and for all $\varphi\in C^2(\overline{\Omega})$ we have that

$$\varphi(x)-\bar{\varphi}=\int_\Omega G_x\Delta\varphi\,dy+\int_{\partial\Omega}G_x\partial_\nu\varphi\,dy.$$

A standard and useful estimate for Green's function is the following uniform pointwise upper bound:

Proposition 1. *Let G be the Green's function for (1). Then there exist $C(\Omega)>0$ and $m(\Omega)>0$ depending only on Ω such that*

$$\frac{1}{C(\Omega)}|x-y|^{2-n}-m(\Omega)\le G(x,y)\le C(\Omega)|x-y|^{2-n} \text{ for all } x,y\in\Omega,\ x\neq y. \quad (2)$$

Concerning the derivatives, we get that

$$|\nabla_y G_x(y)|\le C|x-y|^{1-n} \text{ for all } x,y\in\Omega,\ x\neq y. \quad (3)$$

Estimate (2) was proved by Rey-Wei [4] with a different method. We also refer to Faddeev [2] for very nice estimates in the two-dimensional case.

Notations: in the sequel, $C(a,b,...)$ denotes a constant that depends only on Ω, a, b... We will often keep the same notation for different constants in a formula, and even in the same line. For U an open subset of $\mathbb{R}^n$, $k\in\mathbb{N}$, $k\ge 1$, and $p\ge 1$, we define $H_k^p(U)$ as the completion of $C^\infty(\bar{U})$ for the norm $\sum_{i=1}^k\|\nabla^i\|_p$

1. Preliminary: Existence and estimates of solutions to Neumann problems via extensions

The main goal here is to prove the regularity and existence (Proposition 2 and Theorem 2 below) for solutions to Neumann-type problems. These results are classical (see Agmon-Douglis-Nirenberg [1]); we give here a self-contained proof that uses only the interior estimates for solutions to elliptic equations. We first define smooth domains:

Definition 2. *Let Ω be an open subset of $\mathbb{R}^n$, $n\ge 2$. We say that Ω is smooth if for all $x\in\partial\Omega$, there exists $\delta_x>0$, there exists U_x an open neighborhood of x in $\mathbb{R}^n$, there exists $\varphi: B_{\delta_x}(0)\to U_x$ such that*

(i) φ *is a* $C^\infty-$ *diffeomorphism*
(ii) $\varphi(0)=x$
(iii) $\varphi(B_{\delta_x}(0)\cap\{x_1<0\})=\varphi(B_{\delta_x}(0))\cap\Omega$
(iv) $\varphi(B_{\delta_x}(0)\cap\{x_1=0\})=\varphi(B_{\delta_x}(0))\cap\partial\Omega$

The outward normal vector is then defined as follows:

Definition 3. *Let Ω be a smooth domain of $\mathbb{R}^n$. For any $x\in\partial\Omega$, there exists a unique $\nu(x)\in\mathbb{R}^n$ such that $\nu(x)\in(T_x\partial\Omega)^\perp$, $\|\nu(x)\|=1$ and $(\partial_1\varphi(0),\nu(x))>0$ for φ as in Definition 2. This definition is independant of the choice of such a φ and the map $x\mapsto\nu(x)$ is in $C^\infty(\partial\Omega,\mathbb{R}^n)$.*

It is useful to extend solutions to (1) to a neighborhood of $\overline{\Omega}$. For this, a variational formulation of (1) is required: multiplying (1) by $\psi \in C^\infty(\overline{\Omega})$ and integrating by parts leads us to the following definition:

Definition 4. *We say that $u \in H_1^1(\Omega)$ is a weak solution to* (1) *with $f \in L^1(\Omega)$ if*

$$\int_\Omega (\nabla u, \nabla \psi)\, dx = \int_\Omega f\psi\, dx \text{ for all } \psi \in C^\infty(\overline{\Omega}).$$

In case $u \in C^2(\overline{\Omega})$, as easily checked, u is a weak solution to (1) iff it is a classical solution to (1).

We let ξ be the standard Euclidean metric on $\mathbb{R}^n$ and we set

$$\begin{cases} \tilde{\pi}: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & (x_1, x') & \mapsto & (-|x_1|, x') \end{cases}$$

We prove an extension lemma:

Lemma 1. *Let $x_0 \in \partial\Omega$. There exist $\delta_{x_0} > 0$, U_{x_0} and a chart φ as in Definition 2 such that the metric $\tilde{g} := (\varphi \circ \tilde{\pi} \circ \varphi^{-1})^\star \xi$ is in $C^{0,1}(U_{x_0})$ (that is Lipschitz continuous), $\tilde{g}_{|\Omega} = \xi$, the Christoffel symbols of the metric $\tilde{g}$ are in $L^\infty(U_{x_0})$ and $d\varphi_0$ is an orthogonal transformation. In addition, consider $u \in H_1^1(\Omega \cap U_{x_0})$ and $f \in L^1(\Omega \cap U_{x_0})$ such that*

$$\int_\Omega (\nabla u, \nabla \psi)\, dx = \int_\Omega f\psi\, dx \text{ for all } \psi \in C_c^\infty(\overline{\Omega} \cap U_{x_0}). \tag{4}$$

For all $v : \Omega \cap U_{x_0} \to \mathbb{R}$, we define

$$\tilde{v} := v \circ \varphi \circ \tilde{\pi} \circ \varphi^{-1} \text{ in } U_{x_0}.$$

Then, we have that $\tilde{u} \in H_1^1(U_{x_0})$, $\tilde{u}_{|\Omega} = u$, $f \in L^1(U_{x_0})$ and

$$\Delta_{\tilde{g}} \tilde{u} = \tilde{f} \text{ in the distribution sense,}$$

where $\Delta_{\tilde{g}} := -div_{\tilde{g}}(\nabla)$.

Here, by "distribution sense", we mean that

$$\int_{U_{x_0}} (\nabla \tilde{u}, \nabla \psi)_{\tilde{g}}\, dv_{\tilde{g}} = \int_{U_{x_0}} \tilde{f}\psi\, dv_{\tilde{g}} \text{ for all } \psi \in C_c^\infty(U_{x_0}).$$

Remark 1: the notation $\tilde{g} = (\varphi \circ \tilde{\pi} \circ \varphi^{-1})^\star \xi$ is a slight abuse of notation. Indeed, the map $\tilde{\pi}$ is not a diffeomorphism, and it is not even C^1. However, $\tilde{g}$ is well-defined and smooth outside $\partial\Omega$, and one proves that it can be extended to a Lipschitz continuous function.

Remark 2: It is natural to wonder whether it is possible to gain regularity for the metric $\tilde{g}$. Indeed, the metric $\tilde{g}$ we construct is C^1 iff $\partial\Omega$ is flat in a neighborhood of x_0.

Proof of Lemma 1: Given a chart $\hat{\varphi}$ at x_0 defined on $B_{\tilde{\delta}_{x_0}}(0)$ as in Definition 2, we define the map

$$\begin{cases} \varphi: & B_{\tilde{\delta}_{x_0}}(0) & \to & \mathbb{R}^n \\ & (x_1, x') & \mapsto & x_1 \nu(\hat{\varphi}(0, x')) + \hat{\varphi}(0, x') \end{cases}$$

The inverse function theorem yields the existence of $\delta_{x_0} > 0$ and $U_{x_0} \subset \mathbb{R}^n$ open such that $\varphi : B_{\delta_{x_0}}(0) \to U_{x_0}$ is a smooth diffeomorphism satisfying (2). Moreover, the pull-back metric satisfies the following properties:

$$(\varphi^\star \xi)_{11} = 1,\ (\varphi^\star \xi)_{1i} = 0\ \forall i \neq 1.$$

In particular, up to a linear transformation on the $\{x_1 = 0\}$ hyperplane, we can assume that $d\varphi_0$ is an orthogonal transformation. It is easily checked that $((\varphi \circ \tilde{\pi})^\star \xi)_{ij} = (\varphi^\star \xi)_{ij} \circ \tilde{\pi}$ outside $\{x_1 = 0\}$ for all i, j, and then we prologate $(\varphi \circ \tilde{\pi})^\star \xi$ as a Lipschitz continuous function in U_{x_0}, and so is $\tilde{g} := (\varphi \circ \tilde{\pi} \circ \varphi^{-1})^\star \xi$. In addition, as easily checked, if $\tilde{\Gamma}_{ij}^k$'s denote the Christoffel symbols for the metric $\tilde{g}$, we have that $\tilde{\Gamma}_{ij}^k \in L^\infty$. Therefore, the coefficients of $\Delta_{\tilde{g}}$ are in L^∞ and the principal part is Lipschitz continuous.

We fix $\psi \in C_c^\infty(U_{x_0})$. For convenience, in the sequel, we define

$$\left\{ \begin{array}{llll} \pi : & \mathbb{R}^n_+ & \to & \mathbb{R}^n_- \\ & (x_1, x') & \mapsto & (-x_1, x'). \end{array} \right.$$

Clearly, π is a smooth diffeomorphism. With changes of variable, we get that

$$\int_{U_{x_0}} (\nabla \tilde{u}, \nabla \psi)_{\tilde{g}}\, dv_{\tilde{g}} = \int_{\Omega \cap U_{x_0}} (\nabla u, \nabla(\psi + \psi \circ \varphi \circ \pi^{-1} \circ \varphi^{-1}))\, dx$$

and

$$\int_{U_{x_0}} \tilde{f} \psi\, dv_{\tilde{g}} = \int_{\Omega \cap U_{x_0}} f(\psi + \psi \circ \varphi \circ \pi^{-1} \circ \varphi^{-1})\, dx.$$

It then follows from (4) that $\Delta_{\tilde{g}} \tilde{u} = \tilde{f}$ in U_{x_0} in the distribution sense. This ends the proof of Lemma 1. □

We prove elliptic estimates close to the boundary. These results are classical (here again, we refer to Agmon-Douglis-Nirenberg [1]): our objective here is to derive these boundary estimates from interior estimates.

Proposition 2 (Regularity). *Let $x_0 \in \partial\Omega$ and let $\delta > 0$ be a real number. Let $u \in H_1^q(\Omega \cap B_\delta(x_0))$ and $f \in L^p(\Omega \cap B_\delta(x_0))$, $p, q > 1$ be such that*

$$\int_\Omega (\nabla u, \nabla \psi)\, dx = \int_\Omega f \psi\, dx \text{ for all } \psi \in C_c^\infty(\overline{\Omega} \cap B_\delta(x_0)). \tag{5}$$

Then $u \in H_2^p(\Omega \cap B_{\delta'}(x_0))$ for all $\delta' \in (0, \delta)$, and for all $r \in (1, p]$, there exists $C = C(\Omega, p, q, r, \delta, \delta') > 0$ such that

$$\|u\|_{H_2^p(\Omega \cap B_{\delta'}(x_0))} \leq C \left(\|f\|_{L^p(\Omega \cap B_\delta(x_0))} + \|u\|_{L^r(\Omega \cap B_\delta(x_0))} \right). \tag{6}$$

Moreover, if $u \in C^1(\Omega \cap B_\delta(x_0))$, then $\partial_\nu u = 0$ in $\partial\Omega \cap B_\delta(x_0)$.
We assume that $f \in C^{0,\alpha}(\overline{\Omega} \cap B_\delta(x_0))$ for some $\alpha \in (0,1)$. Then $u \in C^{2,\alpha}(\overline{\Omega} \cap B_{\delta'}(x_0))$ for all $\delta' \in (0, \delta)$ and for all $r > 1$, there exists $C = C(\Omega, r, \alpha, \delta, \delta') > 0$ such that

$$\|u\|_{C^{2,\alpha}(\Omega \cap B_{\delta'}(x_0))} \leq C \left(\|f\|_{C^{0,\alpha}(\Omega \cap B_\delta(x_0))} + \|u\|_{L^r(\Omega \cap B_\delta(x_0))} \right). \tag{7}$$

Proof. It follows from Lemma 1 that for $\epsilon > 0$ small enough, the function $u, f : \Omega \cap B_{2\epsilon}(x_0) \to \mathbb{R}$ can be extended to $\tilde{u}, \tilde{f} : B_{2\epsilon}(x_0) \to \mathbb{R}$ such that $\|\tilde{u}\|_{H_1^q(B_{2\epsilon}(x_0))} \leq \|u\|_{H_1^q(\Omega \cap B_{2\epsilon}(x_0))}$ and $\|\tilde{u}\|_{L^p(B_{2\epsilon}(x_0))} \leq \|u\|_{L^p(\Omega \cap B_{2\epsilon}(x_0))}$. Moreover, they are solutions to $\Delta_{\tilde{g}} \tilde{u} = \tilde{f}$ weakly in $B_{2\epsilon}(x_0)$. Therefore, it follows from the interior estimates for elliptic pdes (see Theorem 9.11 of [3]) that $\tilde{u} \in H_2^p(B_{2\epsilon}(x_0))$ and that there exists $C(\Omega, \epsilon, x_0, p, r) > 0$ such that

$$\|\tilde{u}\|_{H_2^p(B_\epsilon(x_0))} \leq C(\Omega, \epsilon, x_0, p, r)\left(\|\tilde{f}\|_{L^p(B_{2\epsilon}(x_0))} + \|\tilde{u}\|_{L^r(B_{2\epsilon}(x_0))}\right).$$

Using the control above of the norms of $\tilde{u}, \tilde{f}$ by the norms of u, f and that $\tilde{u}_{|\Omega} = u$, we get that (6) holds with $\delta' = \epsilon$. Applying this estimate for all points of $\partial\Omega \cap B_\delta(x_0)$, using the interior estimates of Theorem 9.11 of [3] and a finite covering, we get that (6) holds for all $\delta' \in (0, \delta)$. This ends the proof of (6).

We now assume that $u \in C^1(\Omega \cap B_\delta(x_0))$. Since $u \in H_2^p(\Omega \cap B_\delta'(x_0))$ for all $\delta' \in (0, \delta)$, integrating (5) by parts yields:

$$\int_\Omega (f - \Delta u)\psi \, dx = \int_{\partial\Omega} \partial_\nu u \psi \, d\sigma$$

for all $\psi \in C_c^\infty(\Omega \cap B_\delta(x_0))$. Taking all function ψ with compact support in $\Omega \cap B_\delta(x_0)$ yields $\Delta u = f$ a.e. in $\Omega \cap B_\delta(x_0)$. Therefore, $\int_{\partial\Omega} \partial_\nu u \psi \, d\sigma = 0$ for all $\psi \in C_c^\infty(\Omega \cap B_\delta(x_0))$, and then $\partial_\nu u = 0$ on $\partial\Omega \cap B_\delta(x_0)$.

We now concentrate on the Hölder case and we assume that there exists $\alpha \in (0, 1)$ such that $f \in C^{0,\alpha}(\Omega \cap B_\delta(x_0))$. In particular, $f \in L^p(\Omega \cap B_{\delta'}(x_0))$ for all $p > 1$, and therefore $u \in H_2^p(\Omega \cap B_{\delta'}(x_0))$ for all $\delta' \in (0, \delta)$. It then follows from Sobolev's embedding theorem that $u \in C^{1,\theta}(\Omega \cap B_\delta(x_0))$ for all $\theta \in (0, 1)$ and all $\delta' \in (0, \delta)$: in particular, $\|u\|_{C^{1,\theta}(\Omega \cap B_{\delta'}(x_0))}$ is controled by $\|u\|_r$ and $\|f\|_{C^{0,\alpha}}$. Another important fact is that we have that $\partial_\nu u = 0$ on the boundary $\partial\Omega$. We take $\epsilon > 0$, $\tilde{u}$ and $\tilde{f}$ as above. Via the chart φ that straightens the boundary, we can assume that $\Omega = \mathbb{R}^n_-$ and that $\partial_\nu \tilde{u} = \partial_1 \tilde{u} = 0$ on the boundary $\partial\mathbb{R}^n_-$. We rewrite the equation $\Delta_{\tilde{g}} \tilde{u} = \tilde{f}$ as

$$-\tilde{g}^{ij}\partial_{ij}\tilde{u} = \tilde{f} - \tilde{g}^{ij}\tilde{\Gamma}_{ij}^k \partial_k \tilde{u} =: \hat{f} \text{ in } B_{2\epsilon}(x_0).$$

Outside the boundary $\{x_1 = 0\}$, the function $\tilde{g}^{ij}\tilde{\Gamma}_{ij}^k \partial_k \tilde{u}$ is $\theta-$Hölder continuous with the $C^{0,\theta}-$norm controled by $\|u\|_r$ and $\|f\|_{C^{0,\alpha}}$. Therefore, $\tilde{g}^{ij}\tilde{\Gamma}_{ij}^k \partial_k \tilde{u}$ is $\theta-$Hölder continuous iff it is continous on $\{x_1 = 0\}$. As easily checked, since we work in the specific chart φ, we have that for all $x' \in \{0\} \times \mathbb{R}^{n-1}$:

$$\tilde{g}^{ij}\tilde{\Gamma}_{ij}^k \partial_k u(0^+, x') - \tilde{g}^{ij}\tilde{\Gamma}_{ij}^k \partial_k \tilde{u}(0^-, x') = 2H(0, x')\partial_1 \tilde{u}(0, x') = 0$$

where H denotes the mean curvature and $v(0^+, x') := \lim_{x_1 \to 0;\, x_1 > 0} v(x_1, x')$. Therefore, $-\tilde{g}^{ij}\partial_{ij} u = \hat{f} \in C^{0,\alpha}$, where $\tilde{g}$ has Lipschitz regularity: it then follows from standard elliptic theory (see Theorems 9.19 and 6.2 of [3]) that u is in $C^{2,\alpha}$, and its norm is controled as in (7) on $B_\epsilon(x_0)$. As for (6), a covering argument yields the control on $B_{\delta'}(x_0)$. $\square$

From these estimates, we obtain the existence and the regularity for solutions to the Neumann problem (here again, this is in Agmon-Douglis-Nirenberg [1]):

Theorem 2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ and let $f \in L^p(\Omega)$, $p > 1$ be such that $\int_\Omega f \, dx = 0$. Then there exists $u \in H_2^p(\Omega)$ which is a weak solution to*

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ \partial_\nu u = 0 & \text{in } \partial\Omega \end{cases}$$

The function u is unique up to the addition of a constant. Moreover, there exists $C(p)>0$ such that

$$\|u-\bar{u}\|_{H_2^p(\Omega)}\leq C(p)\|f\|_p.$$

If $f\in C^{0,\alpha}(\overline{\Omega})$, $\alpha\in(0,1)$, then $u\in C^{2,\alpha}(\overline{\Omega})$ is a strong solution and there exists $C(\alpha)>0$ such that

$$\|u-\bar{u}\|_{C^{2,\alpha}(\Omega)}\leq C(\alpha)\|f\|_{C^{0,\alpha}(\overline{\Omega})}.$$

Proof. Assume that $f\in L^2(\Omega)$. For any $u\in H_1^2(\Omega)$, we define

$$J(u):=\frac{1}{2}\int_\Omega|\nabla u|^2\,dx-\int_\Omega fu\,dx,$$

and $\mathcal{F}:=\{u\in H_1^2(\Omega)/\int_\Omega u\,dx=0\}$. It follows from Poincaré's inequality that there exists $C>0$ such that $\|u\|_2\leq C\|\nabla u\|_2$ for all $u\in\mathcal{F}$. Therefore, there exists $C(\|f\|_2)>0$ such that

$$J(u)\geq\frac{1}{2}\|\nabla u\|_2^2-\|f\|_2\|u\|_2\geq\|\nabla u\|_2\left(\frac{1}{2}\|\nabla u\|_2-C\|f\|_2\right)\geq -C(\|f\|_2)$$

for all $u\in\mathcal{F}$. Therefore, $m:=\inf\{J(u)/\,u\in\mathcal{F}\}$ exists.

Step 1: We claim that m is achieved. Indeed, we let $(u_i)_i\in\mathcal{F}$ be a minimizing sequence, that is $\lim_{i\to+\infty}J(u_i)=m$. The inequalities above yield $\|u_i\|_{H_1^2}=O(1)$ when $i\to+\infty$, and therefore, there exists $u\in H_1^2(\Omega)$ such that $u_i\rightharpoonup u$ weakly in H_1^2 and strongly in L^2 when $i\to+\infty$ (up to a subsequence). We then get that

$$m+o(1)=J(u_i)=J(u)+\frac{1}{2}\|\nabla(u_i-u)\|_2^2+o(1)\geq m+\frac{1}{2}\|\nabla(u_i-u)\|_2^2+o(1)$$

when $i\to+\infty$, and therefore, $u_i\to u$ strongly in H_1^2 and $m=J(u)$ is achieved. This proves the claim.

Step 2: We claim that u is a weak solution to (1). Indeed, given $\psi\in H_1^2(\Omega)$, we have that $\psi-\bar{\psi}\in\mathcal{F}$, and the Euler equation for J at u writes $\int_\Omega(\nabla u,\nabla\psi)\,dx=\int_\Omega f(\psi-\bar{\psi})\,dx$. Since $\int_\Omega f\,dx=0$, we get that u is a weak solution to (1). This proves the claim.

Step 3: We choose $f\in L^p(\Omega)$, $p>1$. We claim that there exists $C(p)>0$ such that

$$\|u-\bar{u}\|_{H_2^p}\leq C(p)\|f\|_p \tag{8}$$

for all $u\in H_2^p(\Omega)$ which is a weak solution to (1).

We prove the claim by contradiction and we assume that there exists sequences $(u_i)_i\in H_2^p(\Omega)$ and $(f_i)_i\in L^p(\Omega)$ such that $\|f_i\|_p=o(\|u_i-\bar{u}_i\|_{H_2^p})$ when $i\to+\infty$. With no loss of generality, we can assume that $\bar{u}_i=0$ and $\|u_i\|_{H_2^p}=1$ for all i. Therefore there exists $u\in H_2^p(\Omega)$ such that $u_i\to u$ weakly in H_2^p and strongly in L^p when $i\to+\infty$. In particular, $\int_\Omega u\,dx=0$ and, passing to the limit in the definition of the weak solution, we get that $\int_\Omega(\nabla u,\nabla\psi)\,dx=0$ for all $\psi\in H_1^2(\Omega)$, and then, taking $\psi=u$, it follows from Poincaré's inequality that $u\equiv0$. Using standard interior estimates and the boundary estimate (6), it follows from a covering argument that

$$1=\|u_i\|_{H_2^p}\leq C(p)\left(\|f_i\|_p+\|u_i\|_p\right)$$

for all i. Since $f_i\to0$ and $u_i\to u\equiv0$ in L^p when $i\to+\infty$, we get a contradiction. This proves the claim.

Step 4: We are now in position to prove Theorem 2. We fix $f \in L^p(\Omega)$ and we let $(f_i)_i \in C_c^\infty(\Omega)$ be such that $\lim_{i\to+\infty} f_i = f$ in $L^p(\Omega)$. Substracting $\bar{f}_i$, we can assume that $f_i \in C^\infty(\overline{\Omega})$ and $\int_\Omega f_i\, dx = 0$. Since $f_i \in L^2(\Omega)$ for all i, it follows from Steps 1 and 2 that there exists $u_i \in H_1^2(\Omega)$ such that $\bar{u}_i = 0$ and u_i is a weak solution to (1) with f_i. Since $f_i \in L^p$, it follows from Proposition 2 that $u_i \in H_2^p(\Omega)$. We fix i, j: we have that $u_i - u_j$ is a weak solution to (1) with $f_i - f_j$. It then follows from inequality (8) that

$$\|u_i - u_j\|_{H_2^p} \le C(p)\|f_i - f_j\|_p$$

for all i, j. Since $f_i \to f$ in L^p, we get that (u_i) is a Cauchy sequence in H_2^p and therefore, it converges in H_2^p to a limit $u \in H_2^p$. Passing to the limit, we get that u is a solution to (1). This proves the claim.

The uniqueness is a direct consequence of (8). The proof of the $C^{2,\alpha}$ regularity goes similarly and we leave it to the reader. This ends the proof of Theorem 2. □

2. Construction of the Green's function and proof of the upper bound (2)

This section is devoted to the proof of Theorem 1.

2.1. **Construction of G_x.**

We define $c_n := \frac{1}{(n-2)\omega_{n-1}}$. We fix $x \in \Omega$ and we take $u_x \in C^2(\overline{\Omega})$ that will be chosen later, and we define

$$H_x := c_n|\cdot - x|^{2-n} + u_x.$$

In particular, $H_x \in L^p(\Omega)$ for all $p \in (1, \frac{n}{n-2})$. We let $u \in C^2(\overline{\Omega})$ be a function. Standard computations (see [3] or [5]) yield

$$\int_\Omega H_x \Delta u\, dy = u(x) + \int_\Omega u \Delta u_x\, dy + \int_{\partial\Omega} (-\partial_\nu u H_x + u \partial_\nu H_x)\, d\sigma. \tag{9}$$

We let $\eta \in C^\infty(\mathbb{R})$ be such that $\eta(t) = 0$ if $t \le 1/3$ and $\eta(t) = 1$ if $t \ge 2/3$. We define

$$v_x(y) := \eta\left(\frac{|x-y|}{d(x, \partial\Omega)}\right) c_n|x-y|^{2-n}$$

for all $y \in \overline{\Omega}$. Clearly, $v_x \in C^\infty(\overline{\Omega})$ and $v_x(y) = c_n|x-y|^{2-n}$ for all $y \in \Omega$ close to $\partial\Omega$. It follows from Theorem 2 that there exists $u'_x \in C^{2,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$ unique such that

$$\begin{cases} \Delta u'_x = \Delta v_x - \overline{\Delta v_x} & \text{in } \Omega \\ \partial_\nu u'_x = 0 & \text{in } \partial\Omega \\ \overline{u'_x} = 0 \end{cases}$$

We define $u_x := u'_x - v_x \in C^{2,\alpha}(\overline{\Omega})$ and $c_x := \overline{\Delta v_x} \in \mathbb{R}$ so that

$$\begin{cases} \Delta u_x = -c_x & \text{in } \Omega \\ \partial_\nu u_x = -\partial_\nu(c_n|\cdot - x|^{2-n}) & \text{in } \partial\Omega \end{cases}$$

Therefore, $\partial_\nu H_x = 0$ on $\partial\Omega$ and (9) rewrites

$$\int_\Omega H_x \Delta u\, dy = u(x) - c_x \int_\Omega u\, dy - \int_{\partial\Omega} \partial_\nu u H_x\, d\sigma$$

for all $u \in C^2(\overline{\Omega})$. Taking $u \equiv 1$ yields $c_x = \frac{1}{|\Omega|}$, and then, we have that

$$\int_\Omega H_x \Delta u\, dy = u(x) - \bar{u} - \int_{\partial\Omega} \partial_\nu u H_x\, d\sigma$$

for all $u \in C^2(\overline{\Omega})$. Finally, we define $G_x := H_x - \overline{H_x}$ and we have that:

$$\int_\Omega G_x \Delta u\, dy = u(x) - \bar{u} - \int_{\partial\Omega} \partial_\nu u G_x\, d\sigma$$

for all $u \in C^2(\overline{\Omega})$. Therefore G is a Green's function for (1). In addition,

$$G_x \in C^{2,\alpha}(\overline{\Omega} \setminus \{x\}) \cap L^p(\Omega) \text{ for all } \alpha \in (0,1) \text{ and } p \in \left(1, \frac{n}{n-2}\right).$$

Taking $u \in C_c^\infty(\overline{\Omega} \setminus \{x\})$ above, and the definition of G_x, we get that

$$\left\{ \begin{array}{ll} \Delta G_x = -\frac{1}{|\Omega|} & \text{in } \Omega \setminus \{x\} \\ \partial_\nu G_x = 0 & \text{in } \partial\Omega. \end{array} \right. \tag{10}$$

2.2. **Uniform L^p-bound.**

Lemma 2. *Fix $x \in \overline{\Omega}$ and assume that there exist $H \in L^1(\Omega)$ such that*

$$\int_\Omega H \Delta u\, dy = u(x) - \bar{u}$$

for all $u \in C^2(\overline{\Omega})$ such that $\partial_\nu u = 0$ on $\partial\Omega$. Then $H \in L^p(\Omega)$ for all $p \in \left(1, \frac{n}{n-2}\right)$ and there exists $C(p) > 0$ independant of x such that

$$\|H - \bar{H}\|_p \leq C(p) \tag{11}$$

for all $x \in \Omega$.

Proof. For p as above, we define $q := \frac{p}{p-1} > \frac{n}{2}$. We fix $\psi \in C^\infty(\overline{\Omega})$ and we let $u \in C^2(\overline{\Omega})$ be such that

$$\left\{ \begin{array}{ll} \Delta u = \psi - \bar{\psi} & \text{in } \Omega \\ \partial_\nu u = 0 & \text{in } \partial\Omega \\ \bar{u} = 0 & \end{array} \right.$$

It follows from the properties of H that

$$\int_\Omega (H - \bar{H})\psi\, dy = \int_\Omega H(\psi - \bar{\psi})\, dy = u(x).$$

It follows from Sobolev's embedding that $H_2^q(\Omega)$ is continously embedded in $L^\infty(\Omega)$: therefore, using the control of the H_2^q-norm of Theorem 2 yields

$$\left| \int_\Omega (H - \bar{H})\psi\, dy \right| \leq \|u\|_\infty \leq C(q)\|u\|_{H_2^q} \leq C'(q)\|\psi - \bar{\psi}\|_q \leq C''(q)\|\psi\|_q$$

for all $\psi \in C_c^\infty(\overline{\Omega})$. It then follows from duality that $H - \bar{H} \in L^p(\Omega)$ and that (11) holds. ☐

2.3. **Uniqueness.** We prove the following uniqueness result:

Lemma 3. *Fix $x \in \overline{\Omega}$ and assume that there exist $G_1, G_2 \in L^1(\Omega)$ such that*

$$\int_\Omega G_i \Delta u\, dy = u(x) - \bar{u}$$

for all $i \in \{1, 2\}$ and for all $u \in C^2(\overline{\Omega})$ such that $\partial_\nu u = 0$ on $\partial\Omega$. Then there exists $c \in \mathbb{R}$ such that $G_1 - G_2 = c$ a.e on Ω.

Proof. We define $g := G_1 - G_2$. We have that

$$\int_\Omega g\Delta u\, dy = 0$$

for all $u \in C^2(\overline{\Omega})$ such that $\partial_\nu u = 0$ on $\partial\Omega$. We fix $\psi \in C_c^\infty(\Omega)$. It follows from Theorem 2 that there exists $u \in C^2(\overline{\Omega})$ such that $\Delta u = \psi - \bar{\psi}$ in Ω, $\partial_\nu u = 0$ on $\partial\Omega$ and $\bar{u} = 0$. Therefore, we get that

$$\int_\Omega (g - \bar{g})\psi\, dy = \int_\Omega g(\psi - \bar{\psi})\, dy = \int_\Omega g\Delta u\, dy = 0.$$

for all $\psi \in C_c^\infty(\Omega)$. Moreover, it follows from Lemma 2 that $g \in L^p(\Omega)$ for some $p > 1$, and then we get that $g - \bar{g} = 0$ a.e, and then $G_1 = G_2 + \bar{g}$. □

As an immediate corollary, we get that the function G constructed above is the unique Green's function for (1).

2.4. **Pointwise control.** We let G be the Green's function for (1). The objective here is to prove that there exists $C(\Omega) > 0$ such that

$$|G_x(y)| \leq C(\Omega)|x - y|^{2-n} \tag{12}$$

for all $x, y \in \Omega$, $x \neq y$.

Proof. The proof of (12) goes through six steps.

Step 1: We fix $K \subset \Omega$ a compact set. We claim that there exists $C(K) > 0$ such that

$$|G_x(y)| \leq C(K)|x - y|^{2-n}$$

for all $x \in K$ and all $y \in \Omega$, $y \neq x$.

We prove the claim. We use the notations u_x, u'_x, v_x above. As easily checked, $v_x \in C^2(\overline{\Omega})$ and $\|v_x\|_{C^2} \leq Cd(x, \partial\Omega)^{-n} \leq Cd(K, \partial\Omega)^{-n} \leq C(K)$. Therefore, it follows from Theorem 2 that $\|u'_x\|_\infty \leq C(K)$, and then $|H_x(y)| \leq C(K)|x - y|^{2-n}$ for all $y \in \Omega$, $y \neq x$. Since $G_x = H_x - \overline{H_x}$ holds, the claim follows.

Step 2: We fix $\delta > 0$. We claim that there exists $C(\delta) > 0$ such that

$$\|G_x\|_{C^2(\Omega \setminus \bar{B}_x(\delta))} \leq C(\delta) \tag{13}$$

for all $x, y \in \Omega$ such that $|x - y| \geq \delta$.

We prove the claim. It follows from (10) and (6) of Proposition 2 that for any $p > 1$, there exists $C(\delta, p) > 0$ such that $\|G_x\|_{C^2(\Omega \setminus \bar{B}_x(\delta))} \leq C(\delta) + C(\delta)\|G_x\|_{L^p(\Omega)}$. Step 2 is then a consequence of (11).

We are now interested in the neighborhood of $\partial\Omega$. We fix $x_0 \in \partial\Omega$ and we choose a chart φ as in Lemma 1. For simplicity, we assume that $\varphi : B_\delta(0) \to \mathbb{R}^n$ and that $\varphi(0) = x_0$ and we define $V := \varphi(B_\delta(0))$. We fix $x \in V \cap \Omega$ and we let $\tilde{G}_x$ be the extension $\tilde{G}_x := G_x \circ \varphi \circ \tilde{\pi} \circ \varphi^{-1}$: we have that

$$\tilde{G}_x : V \setminus \{x, x^\star\} \to \mathbb{R} \text{ with } x^\star := \varphi \circ \pi^{-1} \circ \varphi^{-1}(x) \in \overline{\Omega}^c.$$

Moreover, since G_x is $C^{2,\alpha}$ outside x and $\tilde{\pi}$ is Lipschitz continuous, we have that $\tilde{G}_x \in H^q_{1,loc}(V \setminus \{x, x^\star\})$ for all $q > 1$; in addition, it follows from (11) that $\tilde{G}_x \in L^p(V)$ for all $p \in \left(1, \frac{n}{n-2}\right)$ and that there exists $C(p) > 0$ independant of x such that

$$\|\tilde{G}_x\|_p \leq C(p).$$

Step 3: We claim that

$$\Delta_{\tilde{g}} \tilde{G}_x = \delta_x + \delta_{x^\star} - \frac{1}{|\Omega|} \text{ in } \mathcal{D}'(V). \tag{14}$$

We prove the claim. We let $\psi \in C_c^\infty(V)$ be a smooth function. Separating $V \cap \Omega$ and $V \cap \Omega^c$ and using a change of variable, we get that

$$\int_V \tilde{G}_x \Delta_{\tilde{g}} \psi \, dv_{\tilde{g}} = \int_{V\cap\Omega} G_x \Delta \left(\psi + \psi \circ \varphi \circ \pi^{-1} \circ \varphi^{-1} \right) dy.$$

Noting that $\partial_\nu \left(\psi + \psi \circ \varphi \circ \pi^{-1} \circ \varphi^{-1} \right) = 0$ on $\partial\Omega$ (we have used that $\nu(\varphi(0,x')) = d\varphi_{(0,x')}(\vec{e}_1)$) and using the definition of the Green's function G_x, we get that

$$\begin{aligned}\int_V \tilde{G}_x \Delta_{\tilde{g}} \psi \, dv_{\tilde{g}} &= \psi(x) + \psi(\varphi \circ \pi^{-1} \circ \varphi^{-1}(x)) - \frac{1}{|\Omega|} \int_{V\cap\Omega} \left(\psi + \psi \circ \varphi \circ \pi^{-1} \circ \varphi^{-1} \right) dy \\ &= \psi(x) + \psi(x^\star) - \frac{1}{|\Omega|} \int_V \psi \, dv_{\tilde{g}}.\end{aligned}$$

This proves (14) and ends the claim.

Step 4: We fix $z \in V$. We claim that there exists $\Gamma_z : V \setminus \{z\} \to \mathbb{R}$ such that the following properties hold:

$$\left\{ \begin{array}{ll} \Delta_{\tilde{g}} \Gamma_z = \delta_z & \text{in } \mathcal{D}'(V), \\ & \\ |\Gamma_z(y)| \leq C|z-y|^{2-n} & \text{for all } y \in V \setminus \{z\}, \\ & \\ \Gamma_z \in C^1(V \setminus \{z\}) & \end{array} \right\} \tag{15}$$

We prove the claim. We define $r(y) := \sqrt{\tilde{g}_{ij}(z)(y-z)^i(y-z)^j}$ for all $y \in V$. As easily checked, $r^{2-n} \in C^\infty(V \setminus \{z\})$: we define $f := \Delta_{\tilde{g}} r^{2-n}$ on $V \setminus \{z\}$. It follows from the properties of $\tilde{g}$ that $f \in L^\infty_{loc}(V \setminus \{z\})$. Moreover, straightforward computations yield the existence of $C > 0$ such that

$$|f(y)| \leq C|z-y|^{1-n} \text{ for all } y \in V \setminus \{z\}. \tag{16}$$

Computing $\Delta_{\tilde{g}} r^{2-n}$ in the distribution sense yields

$$\Delta_{\tilde{g}} r^{2-n} = f + K_z \delta_z \text{ in } \mathcal{D}'(V),$$

where $K_z := (n-2) \int_{\partial B_1(0)} (\nu(y), y)_{\tilde{g}(z)} r(y)^{2-n} \, dv_{\tilde{g}(z)} > 0$. Moreover, $\lim_{z \to x_0} K_z = K_{x_0} > 0$.

We define h such that

$$\left\{ \begin{array}{ll} \Delta_{\tilde{g}} h = f & \text{in } V \\ h = 0 & \text{on } \partial V \end{array} \right\}$$

It follows from (16) and elliptic theory that h is well defined and that $h \in H_2^p(V) \cap H_{1,0}^p(V)$ for all $p \in \left(1, \frac{n}{n-1}\right)$ and $h \in C^{1,\theta}_{loc}(V \setminus \{z\})$. Moreover, there exists $C > 0$ such that

$$\|h\|_{H_2^p} \leq C(p) \text{ for all } p \in \left(1, \frac{n}{n-1}\right). \tag{17}$$

We claim that for any $\alpha \in (n-3, n-2)$, there exists $C(\alpha) > 0$ such that

$$|h(y)| \leq C(\alpha) |y-z|^{-\alpha}$$

for all $y \in V \setminus \{z\}$.

We prove the claim. We let $\epsilon > 0$ be a small parameter and we define

$$h_\epsilon(y) := \epsilon^\alpha h(z+\epsilon y) \text{ and } f_\epsilon(y) := \epsilon^{2+\alpha} f(z+\epsilon y)$$

for all $y \in B_2(0) \setminus \bar{B}_{1/2}(0)$. We then have that

$$\Delta_{\tilde{g}_\epsilon} h_\epsilon = f_\epsilon \text{ in } B_2(0) \setminus \bar{B}_{1/2}(0), \tag{18}$$

where $\tilde{g}_\epsilon = \tilde{g}(\epsilon \cdot)$. Since $\alpha > n-3$, we have with (16) that

$$|f_\epsilon(y)| \leq C\epsilon^{\alpha-(n-3)}|y|^{1-n} \leq 2^{n-1}C \tag{19}$$

for all $y \in B_2(0) \setminus \bar{B}_{1/2}(0)$. We fix $p := \frac{n}{\alpha+2} \in \left(1, \frac{n}{n-1}\right)$ and $q := \frac{n}{\alpha}$. A change of variable, Sobolev's embedding theorem and (17) yield

$$\|h_\epsilon\|_{L^q(B_2(0)\setminus \bar{B}_{1/2}(0))} \leq C\|h\|_q \leq C\|h\|_{H_2^p} \leq C \tag{20}$$

for all $\epsilon > 0$ small. It then follows from (18), (19), (20) and Theorem 8.17 of [3] that there exists $C > 0$ such that

$$|h_\epsilon(y)| \leq C \text{ for all } y \in \mathbb{R}^n \text{ such that } |y| = 1.$$

Therefore, coming back to h, we get that $|h(y)| \leq C|y-z|^{-\alpha}$ for all $|y-z| = \epsilon$. Since ϵ can be chosen arbitrary small and h is bounded outside y, the claim is proved.

We now set $\Gamma_z := \frac{1}{K_z}\left(r^{2-n} - h\right)$. It follows from the above estimates that Γ satisfies (15). This ends Step 4.

We define $\mu_x := \tilde{G}_x - \Gamma_x - \Gamma_{x^\star}$. It follows from Steps 2 and 3 above that

$$\Delta_{\tilde{g}} \mu_x = -\frac{1}{|\Omega|} \text{ in } \mathcal{D}'(V). \tag{21}$$

Moreover, we have that $\mu_x \in H_1^q(V \setminus \{x, x^\star\})$ for all $q > 1$ and that

$$\|\mu_x\|_p \leq C(p) \text{ for all } p \in \left(1, \frac{n}{n-2}\right). \tag{22}$$

Step 5: We claim that for all $V' \subset\subset V$, there exists $C(V') > 0$ such that

$$\|\mu_x\|_{L^\infty(V')} \leq C(V'), \tag{23}$$

where $C(V')$ is independent of x.

We prove the claim. Since $x \in \Omega \cap V$, we have that $\tilde{g} = \xi$ in a neighborhood of x, and then $\tilde{g}$ is hypoelliptic around x: therefore, it follows from (21) that μ_x is C^∞ around x. Similarly, around $x^\star \in V \cap \overline{\Omega}^c$, $\tilde{g} = (\varphi \circ \tilde{\pi} \circ \varphi^{-1})^\star \xi$ is also hypoelliptic, and therefore, μ_x is C^∞ around $x^\star$. It then follows that $\mu_x \in H_1^q(V)$ for $q > 1$ and (21) rewrites

$$\int_V (\nabla \mu_x, \nabla \psi)_{\tilde{g}}\, dv_{\tilde{g}} = -\frac{1}{|\Omega|}\int_V \psi\, dv_g \text{ for all } \psi \in C_c^\infty(V).$$

Therefore, it follows from Theorem 8.17 of [3] that $\mu_x \in L^\infty_{loc}(V)$ and that there exists $C(V, V', p) > 0$ such that

$$\|\mu_x\|_{L^\infty(V')} \leq C(V, V', p)\left(1 + \|\mu_x\|_{L^p(V)}\right)$$

for all $p > 1$. Taking $p \in \left(1, \frac{n}{n-2}\right)$ and using (22), we get (23) and the claim is proved.

Step 6: We are now in position to conclude. It follows from the definition of μ_x from (23) and from (15) that there exists $C(V')>0$ such that

$$|\tilde{G}_x(y)|\leq C+C|x-y|^{2-n}+|x^\star-y|^{2-n}$$

for all $x,y\in V'$ such that $x\neq y$. As easily checked, one has that $|x^\star-y|\geq c|x-y|$ for all $x,y\in V'\cap\Omega$, and therefore

$$|G_x(y)|\leq C|x-y|^{2-n} \tag{24}$$

for all $x,y\in V'\cap\Omega$ such that $x\neq y$. Recall that V' is a small neighborhood of $x_0\in\partial\Omega$. Combining (24) with Step 1, we get that there exists $\delta(\Omega)>0$ such that (24) holds for all $x,y\in\Omega$ distinct such that $|x-y|\leq\delta(\Omega)$. For points x,y such that $|x-y|\geq\delta(\Omega)$, this is Step 2. This ends the proof of the pointwise estimate (12). □

2.5. **Extension to the boundary and regularity with respect to the two variables.** We are now in position to extend the Green's function to the boundary.

Proposition 3. *The Green's function extends continuously to* $\overline{\Omega}\times\overline{\Omega}\setminus\{(x,x)/\,x\in\overline{\Omega}\}\to\mathbb{R}$.

Proof. As above, we denote G the Green's function for (1). We fix $x\in\partial\Omega$ and $y\in\overline{\Omega}\setminus\{x\}$ and we define

$$G_x(y):=\lim_{i\to+\infty}G(x_i,y)\text{ for all }y\in\overline{\Omega}\setminus\{x\},$$

where $(x_i)_i\in\Omega$ is any sequence such that $\lim_{i\to+\infty}x_i=x$.

We claim that this definition makes sense. It follows from (13) that for all $\delta>0$, we have that

$$\|G_{x_i}\|_{C^2(\Omega\setminus\bar{B}_\delta(x))}\leq C(\delta)$$

for all i. Let (i') be a subsequence of i: it then follows from Ascoli's theorem that there exists $G'\in C^1(\overline{\Omega}\setminus\{x\})$ and a subsequence i" of i' such that

$$\lim_{i\to+\infty}G_{x_i}=G'\text{ in }C^1_{loc}(\overline{\Omega}\setminus\{x\}).$$

Moreover, It follows from (12) that $|G'(y)|\leq C|x-y|^{2-n}$ for all $y\neq x$. We choose $u\in C^2(\overline{\Omega})$ such that $\partial_\nu u=0$ on $\partial\Omega$. We then have that $\int_\Omega G_{x_i}\Delta u\,dy=u(x_i)-\bar{u}$ for all i. Letting $i\to+\infty$ yields

$$\int_\Omega G'\Delta u\,dy=u(x)-\bar{u},$$

and then it follows from Lemma 3 that G' does not depend of the choice of the sequence (x_i) converging to x. We then let $G_x:=G'$ and the definition above makes sense.

We claim that $G\in C^0(\overline{\Omega}\times\overline{\Omega}\setminus\{(x,x)/\,x\in\overline{\Omega}\})$. We only sketch the proof since it is similar to the proof of the extension to the boundary. We fix $x\in\overline{\Omega}$ and we let $(x_i)_i$ be such that $\lim_{i\to+\infty}x_i=x$. Arguing as above, we get that any subsequence of (G_{x_i}) admits another subsequence that converges to some function G" in $C^1_{loc}(\overline{\Omega}\setminus\{x\})$. We choose $u\in C^2(\overline{\Omega})$ such that $\partial_\nu u$ vanishes on $\partial\Omega$ and we get that $\int_\Omega G_{x_i}\Delta u\,dy=u(x_i)-\bar{u}$ for all i. With the pointwise bound (12), we pass to the limit and get that $\int_\Omega G"\Delta u\,dy=u(x)-\bar{u}$: it then follows from Lemma 3 that $G"=G_x$, and then (G_{x_i}) converges uniformly to G_x outside x. The continuity of G outside the diagonal follows immediately. □

Remark: It is essential to assume that G satisfies point (ii) of the definition of the Green's function: indeed, for any $c:\Omega\to\mathbb{R}$, the function $(x,y)\mapsto G(x,y)+c(x)$ satisfies (i) and (iii), but it is not continous outside the diagonal if c is not continuous.

2.6. **Symmetry.**

Proposition 4. *Let G be the Green's function for* (1)*. Then $G(x,y)=G(y,x)$ for all $x,y\in\Omega\times\Omega$, $x\neq y$.*

Proof. Let $f\in C_c^\infty(\Omega)$ be a smooth compactly supported function. We define

$$F(x):=\int_\Omega G(y,x)(f-\bar f)(y)\,dy \text{ for all } x\in\overline{\Omega}.$$

It follows from (12) and Proposition 3 above that $F\in C^0(\overline{\Omega})$ is well defined. We fix $g\in C_c^\infty(\Omega)$ and we let $\varphi,\psi\in C^2(\overline{\Omega})$ be such that

$$\left\{\begin{array}{ll}\Delta\varphi=f-\bar f & \text{in }\Omega\\ \partial_\nu\varphi=0 & \text{in }\partial\Omega\\ \bar\varphi=0\end{array}\right. \text{ and } \left\{\begin{array}{ll}\Delta\psi=g-\bar g & \text{in }\Omega\\ \partial_\nu\psi=0 & \text{in }\partial\Omega\\ \bar\psi=0\end{array}\right.$$

It follows from Fubini's theorem (which is valid here since $G\in L^1(\Omega\times\Omega)$ due to (12) and Proposition 3) that

$$\begin{aligned}\int_\Omega(F-\bar F)g\,dx &= \int_\Omega F(g-\bar g)\,dx=\int_\Omega F\Delta\psi\,dx\\ &= \int_\Omega(f-\bar f)(y)\left(\int_\Omega G(y,x)\Delta\psi(x)\,dx\right)dy=\int_\Omega(\Delta\varphi)\psi\,dy\\ &= \int_\Omega\varphi\Delta\psi\,dy=\int_\Omega\varphi(g-\bar g)\,dy=\int_\Omega g\varphi\,dy,\end{aligned}$$

and therefore $\int_\Omega(F-\bar F-\varphi)g\,dx=0$ for all $g\in C_c^\infty(\Omega)$. Since $F,\varphi\in C^0(\overline{\Omega})$, we then get that $F(x)=\varphi(x)+\bar F$ for all $x\in\Omega$. We now fix $x\in\Omega$. Using the definition of the Green's function and the definition of F, we then get that

$$\int_\Omega G(y,x)(f-\bar f)(y)\,dy=\int_\Omega G(x,y)(f-\bar f)(y)\,dy+\frac{1}{|\Omega|}\int_\Omega\left(\int_\Omega G(y,z)\,dz\right)(f-\bar f)(y)\,dy,$$

and then, setting

$$H_x(y):=G(y,x)-G(x,y)-\frac{1}{|\Omega|}\int_\Omega G(y,z)\,dz$$

for all $y\in\Omega\setminus\{x\}$, we get that

$$0=\int_\Omega H_x(f-\bar f)\,dy=\int_\Omega(H_x-\bar H_x)f\,dy$$

for all $f\in C_c^\infty(\Omega)$. Therefore, $H_x\equiv\bar H_x$, which rewrites

$$G(y,x)-G(x,y)=\frac{1}{|\Omega|}\int_\Omega(G(y,z)-G(x,z))\,dz+h(x),$$

for all $x\neq y$, where $h(x):=\frac{1}{|\Omega|}\int_\Omega G(z,x)\,dz-\frac{1}{|\Omega^2|}\int_{\Omega\times\Omega}G(s,t)\,ds\,dt$ for all $x\in\overline{\Omega}$. Exchanging x,y yields $h(x)+h(y)=0$ for all $x\neq y$, and then $h\equiv 0$ since h is continuous. Therefore, we get that

$$G(y,x)-G(x,y)=\frac{1}{|\Omega|}\int_\Omega (G(y,z)-G(x,z))\,dz=\bar{G}_y-\bar{G}_y \tag{25}$$

for all $x\neq y$. The normalization (ii) in the definition of the Green's function then yields Proposition 4. □

Remark: If one does not impose the normalization (ii), we have already remarked that we just get $G':(x,y)\mapsto G(x,y)+c(x)$ where G is the unique Green's function and c is a function. We then get that $G'(x,y)-G'(y,x)=c(x)-c(y)$ for all $x\neq y$, which is not vanishing when c is nonconstant.

These different lemmae and estimates prove Theorem 1.

3. Asymptotic analysis

This section is devoted to the proof of general asymptotic estimates for the Green's function. As a byproduct, we will get the control (3) of the derivatives of Proposition 1. The following proposition is the main result of this section:

Proposition 5. *Let G be the Green's function for (1). Let $(x_\alpha)_\alpha\in\Omega$ and let $(r_\alpha)_\alpha\in(0,+\infty)$ be such that $\lim_{\alpha\to+\infty}r_\alpha=0$.*
Assume that

$$\lim_{\alpha\to+\infty}\frac{d(x_\alpha,\partial\Omega)}{r_\alpha}=+\infty.$$

Then for all $x,y\in\mathbb{R}^n$, $x\neq y$, we have that

$$\lim_{\alpha\to+\infty}r_\alpha^{n-2}G(x_\alpha+r_\alpha x,x_\alpha+r_\alpha y)=c_n|x-y|^{2-n}.$$

Moreover, for fixed $x\in\mathbb{R}^n$, this convergence holds uniformly in $C^2_{loc}(\mathbb{R}^n\setminus\{x\})$.
Assume that

$$\lim_{\alpha\to+\infty}\frac{d(x_\alpha,\partial\Omega)}{r_\alpha}=\rho\geq 0.$$

Then $\lim_{\alpha\to+\infty}x_\alpha=x_0\in\partial\Omega$. We choose a chart φ at x_0 as in Lemma 1 and we let $(x_{\alpha,1},x'_\alpha)=\varphi^{-1}(x_\alpha)$. Then for all $x,y\in\mathbb{R}^n\cap\{x_1\leq 0\}$, $x\neq y$, we have that

$$\lim_{\alpha\to+\infty}r_\alpha^{n-2}G(\varphi((0,x'_\alpha)+r_\alpha x),\varphi((0,x'_\alpha)+r_\alpha y)=c_n\left(|x-y|^{2-n}+|\pi^{-1}(x)-y|^{2-n}\right),$$

where $\pi^{-1}(x_1,x')=(-x_1,x')$. Moreover, for fixed $x\in\overline{\mathbb{R}^n_-}$, this convergence holds uniformly in $C^2_{loc}(\overline{\mathbb{R}^n_-}\setminus\{x\})$.

Proof of Proposition 5:
Step 1: We first assume that

$$\lim_{\alpha\to+\infty}\frac{d(x_\alpha,\partial\Omega)}{r_\alpha}=+\infty. \tag{26}$$

We define

$$\tilde{G}_\alpha(x,y):=r_\alpha^{n-2}G(x_\alpha+r_\alpha x,x_\alpha+r_\alpha y)$$

for all $\alpha\in\mathbb{N}$ and all $x,y\in\Omega_\alpha:=r_\alpha^{-1}(\Omega-x_\alpha)$, $x\neq y$. We fix $x\in\mathbb{R}^n$. It follows from Theorem 1 that $\tilde{G}_\alpha\in C^2(\Omega_\alpha\times\Omega_\alpha\setminus\{(x,x)/\,x\in\Omega_\alpha\})$ and that

$$\Delta(\tilde{G}_\alpha)_x=-\frac{r_\alpha^n}{|\Omega|}\text{ in }\Omega_\alpha\setminus\{x\} \tag{27}$$

for $\alpha \in \mathbb{N}$ large enough. Moreover, it follows from (12) that there exists $C > 0$ such that

$$|(\tilde{G}_\alpha)_x(y)| \leq C|y - x|^{2-n} \tag{28}$$

for all $\alpha \in \mathbb{N}$ and all $y \in \Omega_\alpha \setminus \{x\}$. It then follows from (26), (27), (28) and standard elliptic theory that, up to a subsequence, there exists $\tilde{G}_x \in C^2(\mathbb{R}^n \setminus \{x\})$ such that

$$\lim_{\alpha \to +\infty} (\tilde{G}_\alpha)_x = \tilde{G}_x \text{ in } C^2_{loc}(\mathbb{R}^n \setminus \{x\}). \tag{29}$$

with

$$|\tilde{G}_x(y)| \leq C|y - x|^{2-n} \tag{30}$$

for all $y \in \mathbb{R}^n \setminus \{x\}$. We consider $f \in C^\infty_c(\mathbb{R}^n)$ and we define $f_\alpha(y) := f(r_\alpha^{-1}(y - x_\alpha))$: it follows from (26) that $f_\alpha \in C^\infty_c(\Omega)$ for $\alpha \in \mathbb{N}$ large enough. Applying Green's representation formula yields

$$f_\alpha(x_\alpha + r_\alpha x) - \overline{f_\alpha} = \int_\Omega G(x_\alpha + r_\alpha x, z)\Delta f_\alpha(z)\, dz.$$

With a change of variable, this equality rewrites

$$f(x) = \int_{\mathbb{R}^n} \tilde{G}_\alpha(x, y)\Delta f(y)\, dy + \overline{f_\alpha}$$

for $\alpha \in \mathbb{N}$ large enough. With (28), (29) and the definition of f_α, we get that

$$f(x) = \int_{\mathbb{R}^n} \tilde{G}_x \Delta f\, dy,$$

and then

$$\Delta(\tilde{G}_x - c_n|\cdot - x|^{2-n}) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

The hypoellipticity of the Laplacian, (30) and Liouville's theorem yield

$$\tilde{G}_x(y) = c_n|y - x|^{2-n} \text{ for all } y \neq x.$$

This ends Step 1.

Step 2:

$$\lim_{\alpha \to +\infty} \frac{d(x_\alpha, \partial\Omega)}{r_\alpha} = \rho \geq 0.$$

We take φ as in the statement of the Proposition and we define

$$\tilde{G}_\alpha(x, y) := r_\alpha^{n-2} G(\varphi((0, x'_\alpha) + r_\alpha x), \varphi((0, x'_\alpha) + r_\alpha y)$$

for all $x, y \in \mathbb{R}^n_-$, $x \neq y$ with $\alpha \in \mathbb{N}$ large enough. We fix $x \in \mathbb{R}^n_-$ and we symmetrize $\tilde{G}$ as usual:

$$\hat{G}_\alpha(x, y) := \tilde{G}_\alpha(x, \tilde{\pi}(y))$$

for all $y \in \mathbb{R}^n$ close enough to 0 and where, as above, $\tilde{\pi} : \mathbb{R}^n \to \overline{\mathbb{R}^n_-}$. As in the first case, we get that there exists $C > 0$ such that

$$|\hat{G}_\alpha(x, y)| \leq C\left(|y - x|^{2-n} + |y - \pi^{-1}(x)|^{2-n}\right)$$

for all $y \neq x, \pi^{-1}(x)$ and there exists $\hat{G}_x \in C^2(\mathbb{R}^n \setminus \{x, \pi^{-1}(x)\})$ such that

$$\lim_{\alpha \to +\infty} (\hat{G}_\alpha)_x = \hat{G}_x \text{ in } C^2_{loc}(\mathbb{R}^n \setminus \{x, \pi^{-1}(x)\}).$$

Moreover, letting $L = d\varphi_0$ be the differential of φ at 0, arguing again as in the first case, we have that

$$\Delta_{L^\star\xi}\hat{G}_x = \delta_x + \delta_{\pi^{-1}(x)} \text{ in } \mathcal{D}'(\mathbb{R}^n_-).$$

Therefore, with a change of variable, we get that

$$\Delta_\xi(\hat{G}_x \circ L^{-1}) = \delta_{L(x)} + \delta_{L\circ\pi^{-1}(x)} \text{ in } \mathcal{D}'(\mathbb{R}^n_-),$$

and then

$$\Delta_\xi\left(\hat{G}_x \circ L^{-1} - c_n\left(|\cdot - L(x)|^{2-n} + |\cdot - L\circ\pi^{-1}(x)|^{2-n}\right)\right) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n_-),$$

Arguing as above, we get that $\hat{G}_x \circ L^{-1} = c_n\left(|\cdot - L(x)|^{2-n} + |\cdot - L\circ\pi^{-1}(x)|^{2-n}\right)$, and then

$$\hat{G}_x = c_n\left(|\cdot - x|^{2-n} + |\cdot - \pi^{-1}(x)|^{2-n}\right)$$

since L is an orthogonal transformation. This ends Step 2.

Proposition 5 is a direct consequence of Steps 1 and 2. □

We now prove Proposition 1:

Corollary 1. *Let G be the Green's function for (1). Then there $C, M > 0$ such that*

$$\frac{1}{C|x-y|^{n-2}} - M \le G(x,y) \le \frac{C}{|x-y|^{n-2}}$$

and

$$|\nabla_y G(x,y)| \le \frac{C}{|x-y|^{n-1}}$$

for all $x, y \in \Omega$, $x \neq y$.

Proof of the corollary: We claim that there exists $m \in \mathbb{R}$ such that

$$G(x,y) \ge -m \text{ for all } x, y \in \Omega,\ x \neq y. \tag{31}$$

We argue by contradiction and we assume that there exists $(x_\alpha)_\alpha, (y_\alpha)_\alpha \in \Omega$ such that

$$\lim_{\alpha\to+\infty} G(x_\alpha, y_\alpha) = -\infty. \tag{32}$$

Assume that $\lim_{\alpha\to+\infty} |y_\alpha - x_\alpha| = 0$. We then define $r_\alpha := |y_\alpha - x_\alpha|$ and we apply Proposition 5:

If $\lim_{\alpha\to+\infty} \frac{d(x_\alpha, \partial\Omega)}{r_\alpha} = +\infty$, we have that

$$|y_\alpha - x_\alpha|^{n-2} G(x_\alpha, y_\alpha) = r_\alpha^{n-2} G\left(x_\alpha, x_\alpha + r_\alpha \frac{y_\alpha - x_\alpha}{|y_\alpha - x_\alpha|}\right) = c_n + o(1)$$

when $\alpha \to +\infty$. This contradicts (32).

If $d(x_\alpha, \partial\Omega) = O(r_\alpha)$ when $\alpha \to +\infty$, we get also a contradiction.

This proves that $\lim_{\alpha\to+\infty} |x_\alpha - y_\alpha| \neq 0$. Therefore, with (2), we get that $G(x_\alpha, y_\alpha) = O(1)$ when $\alpha \to +\infty$: this contradicts (32). Therefore, there exists m such that (31) holds.

We define $M := m + 1$. With (2), there exists also $C > 0$ such that $|G(x,y)| \le C|x-y|^{2-n}$ for all $x \neq y$. We claim that there exists $c > 0$ such that

$$G(x,y) + M \ge c|x-y|^{2-n} \tag{33}$$

for all $x \neq y$. Here again, we argue by contradiction and we assume that there exists $(x_\alpha)_\alpha, (y_\alpha)_\alpha \in \Omega$ such that

$$\lim_{\alpha\to+\infty} |x_\alpha - y_\alpha|^{n-2}(G(x_\alpha, y_\alpha) + M) = 0. \tag{34}$$

Since $G + M \geq 1$, it follows from (34) that $\lim_{\alpha\to+\infty} |x_\alpha - y_\alpha| = 0$. Therefore, as above, we get that the limit of the left-hand-side in (34) is positive: a contradiction. This proves that (33) holds. In particular, this proves the first part of the corollary.

Concerning the estimate of the gradient, we argue by contradiction and we use again Proposition 5. We just sketch the proof. Assume by contradiction that there exists $(x_\alpha)_\alpha, (y_\alpha)_\alpha \in \Omega$ such that

$$\lim_{\alpha\to+\infty} |y_\alpha - x_\alpha|^{n-1} |\nabla_y G(x_\alpha, y_\alpha)| = +\infty.$$

It follows from (13) that $\lim_{\alpha\to+\infty} |y_\alpha - x_\alpha| = 0$. We set $r_\alpha := |y_\alpha - x_\alpha|$. Assume that $r_\alpha = o(d(x_\alpha, \partial\Omega))$ when $\alpha \to +\infty$. It then follows from Proposition 5 that

$$\lim_{\alpha\to+\infty} |y_\alpha - x_\alpha|^{n-1} |\nabla_y G(x_\alpha, y_\alpha)| = \frac{1}{\omega_{n-1}},$$

which contradicts the hypothesis. The proof goes the same way when $d(x_\alpha, \partial\Omega) = O(r_\alpha)$ when $\alpha \to +\infty$. This ends the proof of the gradient estimate. □

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Institut Élie Cartan, Université Henri Poincaré Nancy 1, B.P. 239, F-54506 Vandoeuvre-lès-Nancy Cedex, France

E-mail address: frobert@iecn.u-nancy.fr