FOURTH ORDER EQUATIONS WITH CRITICAL GROWTH IN RIEMANNIAN GEOMETRY.

NOTES FROM A COURSE GIVEN AT THE UNIVERSITY OF WISCONSIN AT MADISON AND AT THE TECHNISCHE UNIVERSITÄT IN BERLIN

PERSONAL NOTES

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Abstract. These notes are devoted to the study of solutions to the equation $\Delta_g^2 u - \mathrm{div}_g(A(\nabla u)^\sharp) + au = f u^{(n+4)/(n-4)}$. They focus on existence and compactness issues related to this equation.

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Date: April 25th, 2009.

1. Background material

1.1. **Riemannian geometry.** The reader is supposed to be familiar with basic concepts in Riemannian geometry. We list below a few definitions and properties that are needed in the sequel. This preliminary part is not supposed to be a course in Riemannian geometry. The interested reader can consult some of the various references available, for instance, we refer to [Cha], [DoC], [Heb1], [GHL], [Sak], [Spi]. In the sequel, we consider M a smooth manifold of dimension $n \geq 1$ (in particular, this is a topological space).

1.1.1. *Tangent and cotangent spaces.* Let $x \in M$, and let $\varphi : U \to \Omega$ a local chart of M where $x \in U$, U being an open subset of M and Ω is an open subset of $\mathbb{R}^n$. We let $\mathcal{C}(M)_x$ denote the set of smooth functions defined in a neigborhood of x. The tangent vectors of M at x are the linear functions $X : \mathcal{C}(M)_x \to \mathbb{R}$ such that $X(f) = 0$ in case the differential of $f \circ \varphi^{-1}$ vanishes at $\varphi(x)$. This notion is independant of th choice of the chart φ. We denote as T_xM the set of tangent vectors at x: it is a linear space. For any $i \in \{1, ..., n\}$, we define $\left(\frac{\partial}{\partial x_i}\right)_x \left(\frac{\partial}{\partial x_i}\right)_x \in TxM$ by

$$\left(\frac{\partial}{\partial x_i}\right)_x (f) := \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(\varphi(x))$$

for all $f \in \mathcal{C}(M)_x$. The space T_xM is $n-$dimensional, and a basis is give, by $\left(\left(\frac{\partial}{\partial x_i}\right)_x\right)_{i\in\{1,\dots,n\}}$. In other words, any vector $X \in T_xM$ can be written as

$$X = \sum_{i=1}^n X^i \left(\frac{\partial}{\partial x_i}\right)_x ,$$

where the X^i's are uniquely determined real numbers. We let $(T_xM)^\star$ be the dual space of T_xM, that is the space of linear forms on T_xM. A possible basis of $(T_xM)^\star$ is $(dx_x^1, ..., dx_x^n)$ the dual basis of $(\left(\frac{\partial}{\partial x_i}\right)_x)_{i\in\{1,\dots,n\}}$. Therefore, any $\eta \in (T_xM)^\star$ is uniquely written as

$$\eta = \sum_{i=1}^{n} \eta_i dx_x^i,$$

where $\eta_i \in \mathbb{R}$ for all $i \in \{1, ..., n\}$.

1.1.2. *Tensors on manifolds.* Given $v = (p, q)$, with $p, q \in \mathbb{N}$, we define the bundle of $(p, q)-$tensors on M as follows

$$\bigotimes_{(p,q)} M := \{(x, \mathcal{L})/\, x \in M,\ \mathcal{L} : (T_xM)^p \times ((T_xM)^\star)^q \to \mathbb{R} \text{ is } (p+q) - \text{linear}\}\,.$$

The bundle of $(p, q)-$tensors has a natural structure of $C^\infty-$manifold of dimension $n + n^{p+q}$. We define $\Pi_{(p,q)} : \bigotimes_{(p,q)} M \to M$ by $\Pi_{(p,q)}(x, \mathcal{L}) = x$ for all $(x, \mathcal{L}) \in \bigotimes_{(p,q)} M$. A smooth field of $(p, q)-$tensors on M is a smooth function $T : M \to \bigotimes_{(p,q)} M$ such that $\Pi_{(p,q)} \circ T = Id_M$. Since T_xM is canonically isomorphic to $(T_xM)^{\star\star}$, a vector field on M is interpretated as a $(0, 1)-$tensor field on M. If T is a smooth $(p, q)-$tensor field (which includes vector fields, as discussed above), we can write $T(x) = (x, \mathcal{L}_x)$, where $\mathcal{L}_x$ is a $(p+q)-$linear form: with a standard abuse of notation, we will often refer to $x \mapsto \mathcal{L}_x$ as the tensor field T. Given a chart φ around x_0 as above, for $i_1, ..., i_p \in \{1, ..., n\}$ and $j_1, ..., j_q \in \{1, ..., n\}$, we let the coordinates of the tensor T as follows

$$T(x)_{i_1...i_p}^{j_1...j_q} = T(x)\left(\left(\frac{\partial}{\partial x_{i_1}}\right)_x, ..., \left(\frac{\partial}{\partial x_{i_p}}\right)_x, dx_x^{j_1}, ..., dx_x^{j_q}\right).$$

In particular, if $X_1, ..., X_p \in T_xM$ and $\eta^1, ..., \eta^q \in (T_xM)^\star$ are written in a chart

$$X_k = \sum_{i=1}^{n} (X_k)^i \left(\frac{\partial}{\partial x_i}\right)_x \text{ and } \eta_k = \sum_{1=1}^{n} (\eta^k)_i dx_x^i,$$

then

$$T(x)(X_1, ..., X_p, \eta^1, ..., \eta^q) = T(x)_{i_1...i_p}^{j_1...j_q} (X_1)^{i_1} \cdots (X_p)^{i_p} \cdot (\eta^1)_{j_1} \cdots (\eta^q)_{j_q}.$$

Here and in the sequel, we use Einstein's summation convention: in the right-hand-side term, we omitted the sum for $i_1 = 1..n$,..., $j_q = 1...n$. This summation is independant of the choice of the chart. We will then often define a tensor through its coordinates. For instance, let $u \in C^1(M)$, let $x \in M$ and $X \in T_xM$ and define $(\nabla u)(x)(X) = du_x(X)$, where du_x denotes the differential of u at x. Then ∇u is a $(1, 0)-$tensor, and we denote by $\partial_i u(x) := (\nabla u)(x)_i = \partial_i(u \circ \varphi^{-1})_{\varphi(x)}$ its coordinate in the chart φ.

1.1.3. *Riemannian manifolds.* Let (M, g) be a smooth Riemannian manifold of dimension $n \geq 1$. Here, g denotes the metric, that is a smooth field of positive symmetric bilinear forms on the tangent bundle (that is a $(2, 0)-$tensor field). The linear space $\mathbb{R}^n$ is systematically endowed with its canonical Euclidean metric that we will denote ξ. We endow M with the Levi-Civita connection (denoted as ∇) associated to the metric g, that is the only torsion-free connection M such that $\nabla g = 0$. The Christoffel symbols of this connection in a chart are

$$\Gamma_{ij}^k = \frac{1}{2} g^{km} \left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right),$$

where (g_{ij}) denote the coordinates of the metric tensor in the chart, and (g^{ij}) denote the coordinates of g^{-1}, the inverse of the metric tensor, in the same chart. Note that g^{-1} is a $(0,2)-$tensor field. Let $x \in M$ and $X, Y, Z \in T_xM$. We define

$$R(X,Y)(x).Z = \nabla_{\tilde{X}(x)}(\nabla_{\tilde{Y}}\tilde{Z}) - \nabla_{\tilde{Y}(x)}(\nabla_{\tilde{X}}\tilde{Z}) - \nabla_{[\tilde{X},\tilde{Y}](x)}\tilde{Z}$$

where $\tilde{X}, \tilde{Y}, \tilde{Z}$ are vector fields on M such that $\tilde{X}(x) = X$, $\tilde{Y}(x) = Y$ and $\tilde{Z}(x) = Z$. This definition is independant of the choice of the extensions $\tilde{X}, \tilde{Y}, \tilde{Z}$. Given $x \in M$, $X, Y, Z \in T_xM$ and $\eta \in (T_xM)^\star$, we define the curvature tensor as follows:

$$R(x)(X,Y,Z,\eta) = \eta\left(R(Y,Z)(x).X\right).$$

The function R is a smooth $(3,1)-$tensor fields. Note that the definition varies from one book to the other, however, two definitions differ only by the multiplication by ± 1 of the curvature tensor. The coordinates of R in a chart are given by

$$R(x)_{ijk}^l = \left(\frac{\partial \Gamma_{ki}^l}{\partial x_j}\right)_x - \left(\frac{\partial \Gamma_{ji}^l}{\partial x_k}\right)_x + \Gamma_{j\alpha}^l(x)\Gamma_{ki}^\alpha(x) - \Gamma_{k\alpha}^l(x)\Gamma_{ji}^\alpha$$

where the Γ_{ij}^k are given above.

The Riemann tensor is the $(4,0)-$tensor field Rm_g whose coordinates in a chart are $R_{ijkl} := g_{\alpha l}R_{ijk}^\alpha$. The Ricci tensor Ric_g is the symmetric $(2,0)$-tensor fields with coordinates $R_{ij} := R_{\alpha i\beta j}g^{\alpha\beta}$. The scalar curvature R_g is the trace of the Ricci tensor $R_g := g^{ij}R_{ij}$.

1.1.4. *Riemannian distance and geodesics.* The distance between two points $x, y \in M$ is, by definition

$$d_g(x,y) = \inf\left\{\int_0^1 |\dot{c}(t)|_{g(c(t))}\,dt/\, c \in C^1([0,1],M) \text{ such that } c(0) = x \text{ and } c(1) = y\right\},$$

where for any $t \in [0,1]$, $\dot{c}(t) \in T_{c(t)}M$ is such that $\dot{c}(t)(f) = \frac{d(f\circ c)}{dt}_t$ for all $f \in \mathcal{C}(M)_{c(t)}$, and where $|\dot{c}(t)|_{g(c(t))}$ is the norm of $\dot{c}(t)$ for the scalar product $g(c(t))$. The function d_g is well-defined as a distance, and the topology induced by d_g is the topology induced by the structure of manifold of M. Given $x \in M$ and $d > 0$, we define the geodesic ball

$$B_d(x) = \{y \in M/\, d_g(x,y) < d\}.$$

Let us consider the exponential map. Let (M,g) be a complete Riemannian manifold, and let $x \in M$. The exponential map at x is defined on the tangent space T_xM by $\exp_x(X) = \gamma(1)$, where $\gamma : [0,2] \to M$ is the unique geodesic such that $\gamma(0) = x$ and $\dot{\gamma}(0) = X \in T_xM$. The definition on the whole space T_xM is not trivial, and is a consequence of Hopf-Rinow's theorem. A particularly important property is the following: given a complete Riemannian manifold, the Riemannian distance between two points is always achieved, and the path between these two points that realizes the distance is a geodesic. One then defines the injectivity radius as $i_g(x) = \inf\{\rho_x(u)/\, u \in T_xM,\ |u|_{g(x)} = 1\}$, where $\rho_x(u) = \inf\{T > 0 \text{ such that } t \mapsto \exp_x(tu) \text{ is minimizing on } [0,T]\}$. The injectivity radius is $i_g(M) = \inf\{i_g(x)/\, x \in M\}$. When M is compact, then $i_g(M) > 0$. Note that when $i_g(M) > 0$, for any $x \in M$, the restriction of $\exp_x$ to $\{X \in T_xM/\, |X|_{g(x)} < i_g(M)\}$ induces a diffeomorphism onto $B_{i_g(M)}(x)$. Assimilating $(T_xM, g(x)$ to $(\mathbb{R}^n, \xi)$ isometrically, one can then consider $\exp_x$ as a local chart around the point x. We will refer to this chart as the exponential chart, and, with a standard abuse of notations, we will sometimes consider $\exp_x$ as defined on an open subset of $\mathbb{R}^n$. We will often use the following useful result: let $x_0 \in M$ and consider the exponential map at x_0. Then, in the chart $\exp_{x_0}$, we have that

$$g_{ij}(x_0) = \delta_{ij} \text{ and } \Gamma_{ij}^k(x_0) = 0 \text{ for all } i, j, k \in \{1, ..., n\}, \tag{1}$$

where δ_{ij} is the Kronecker symbol: that is $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Let us conclude this part with the following important relation between the Riemannian distance and the exponential map:

$$d_g(x, \exp_x(X)) = |X|_{g(x)} \tag{2}$$

for all $x \in M$ and all $X \in T_xM$ such that $|X|_{g(x)} < i_g(M)$.

1.1.5. *Miscellaneous tools in Riemannian geometry.* •**Scalar product for tensors.** Given T, T' two $(p, q)-$tensor, we define their scalar product as follows:

$$(T, T')_g = T^{i_1 \dots i_p}_{j_1 \dots j_q} g_{i_1 k_1} \cdots g_{i_p k_p} \cdot g^{j_1 l_1} \cdots g^{j_q l_q} T^{k_1 \dots k_p}_{l_1 \dots l_q}.$$

Indeed, defining $|T|_g = \sqrt{(T, T)_g}$, we get a smooth family of norms on $(p, q)-$tensors.

•**The musical isomorphism.** Let $x \in M$. We let $\#$ be the musical isomorphism between T_xM and $(T_xM)^\star$ defined as follows:

$$\begin{array}{cccc} \#: & T_xM & \to & (T_xM)^\star \\ & X & \mapsto & \left\{ \begin{array}{ccc} T_xM & \to & \mathbb{R} \\ Y & \mapsto & (X, Y)_{g(x)} \end{array} \right\}. \end{array}$$

This isomorphism is nothing but the canonical identification of a Euclidean space with its dual space. We let $X^\#$ the image of X via $\#$, and $\eta^\#$ the image of $\eta \in (T_xM)^\star$ via the inverse of $\#$. This definition extends naturally to vector field (that is $(0, 1)-$tensor fields) and to $(1, 0)-$tensor fields. If X is a vector field and η is a $(1, 0)-$tensor fields, the coordinates of their images in a chart are $X_i := (X^\#)_i = g_{ij} X^j$ and $\eta^i := (\eta^\#)^i = g^{ij} \eta_j$. Clearly $\left(X^\#\right)^\# = X$ and $\left(\eta^\#\right)^\# = \eta$.

•**Riemannian element of volume.** Given (M, g) a Riemannian manifold of dimension $n \geq 1$, we let dv_g be its Riemannian element of volume. Given $\varphi : U \to \Omega \subset \mathbb{R}^n$ a local chart of M in $U \subset M$, we have that

$$((\varphi^{-1})^\star dv_g)(x) = \sqrt{|g|(x)} dx_1 \wedge \cdots \wedge dx_n,$$

where $|g|(x) = \det(g_{ij}(\varphi^{-1}(x)))$, and the g_{ij}'s are the coordinates of the metric tensor in the chart φ, and $dx_1 \wedge \cdots \wedge dx_n$ is the determinant in the canonical basis of $\mathbb{R}^n$. This element of volume induces a Riemannian measure and a Riemannian integral on M. The volume of the manifold is defined as

$$\mathrm{Vol}_g(M) := \int_M dv_g.$$

Note that this volume can be infinite. However, $\mathrm{Vol}_g(M) < \infty$ as soon as M is compact.

•**Divergence.** Let η be a smooth $(1, 0)-$tensor on M. The divergence of η is defined as $\mathrm{div}_g(\eta) := g^{ij} (\nabla \eta)_{ij} = g^{ij} (\partial_i \eta_j - \Gamma_{ij}^k \eta_k)$, which is an expression independant of the chart.

•Laplace-Beltrami operator. The Laplace Beltrami operator of a function $u \in C^2(M)$ is given in a chart by

$$(\Delta_g u)(x) := -\mathrm{div}_g(\nabla u) = -g^{ij}(x)\left(\partial_{ij}(u\circ\varphi^{-1})_{\varphi(x)} - \Gamma_{ij}^k(x)\partial_k(u\circ\varphi^{-1})_{\varphi(x)}\right).$$

Note that we use here the minus sign convention. When $g = \xi$ the Euclidean metric, one has that $\Delta_\xi = \sum_i \partial_{ii}$.

•The divergence theorem. Let us conclude this part with the following statement that we will use intensively:

Theorem 1.1. *Let (M,g) be a compact Riemannian manifold without boundary. Let η be a smooth $(1,0)-$tensor. Then we have that*

$$\int_M div_g(\eta)\, dv_g = 0.$$

In particular, given $u,v \in C^\infty(M)$, we have that

$$\int_M u\Delta_g v\, dv_g = \int_M (\nabla u, \nabla v)_g\, dv_g$$

1.2. **Sobolev spaces.** Here, we refer systematically to Hebey [Heb1, Heb2].

1.2.1. *Definition.* Let (M,g) be a compact Riemannian manifold with the Riemannian element of volume dv_g. For any $p \geq 1$, we define $L^p(M)$ as the L^p-space of M with the measure dv_g, endowed with the L^p-norm:

$$\|u\|_{L^p(M)} := \left(\int_M |u|^p\, dv_g\right)^{\frac{1}{p}},\ \|u\|_{L^\infty(M)} := \mathrm{supess}_M |u| \text{ when } p = \infty$$

for $u \in L^p(M)$. Then $(L^p(M), \|\cdot\|_{L^p(M)})$ is a Banach space. When there is no ambiguity, we let $\|\cdot\|_p = \|\cdot\|_{L^p(M)}$. For $k \in \mathbb{N}$ and $p \geq 1$, we define the Sobolev space $H_k^p(M)$ as the completion of $C^\infty(M)$ in $L^p(M)$ for the norm $\|\cdot\|_{H_k^p(M)}$ defined as follows:

$$\|u\|_{H_k^p(M)} = \sum_{i=1}^k \|\nabla^k u\|_p$$

for $u \in C^\infty(M)$, where

$$\|\nabla^k u\|_p = \left(\int_M |\nabla^k u|_g^p\, dv_g\right)^{\frac{1}{p}}$$

for all $u \in C^\infty(M)$. This definition naturally extends to $u \in H_k^p(M)$. When there is no ambiguity, we will write $\|\cdot\|_{H_k^p} := \|\cdot\|_{H_k^p(M)}$. Recall that we have the Hölder inequality for the L^p-space: Let $p,q \geq 1$ such that $1/p + 1/q = 1$. Let $u \in L^p(M)$ and $v \in L^q(M)$. Then $uv \in L^1(M)$ and we have that

$$\|uv\|_1 \leq \|u\|_p \|v\|_q. \tag{3}$$

1.2.2. *Weak compactness.* Let $(E, \|\cdot\|)$ be a Banach space. Let $(x_i)_{i\in\mathbb{N}} \in E$ and $x \in E$. We say that (x_i) converges weakly to x if $\lim_{i\to+\infty} \phi(x_i) = \phi(x)$ for all $\phi \in E'$, where E' denotes the continuous linear forms of E. In this case, we write $x_i \rightharpoonup x$ weakly in E' when $i \to +\infty$. In the case of the space $H_2^2(M)$, we can rewrite this definition as follows: if $(u_i)_{i\in\mathbb{N}} \in H_2^2(M)$ converges weakly to $u \in H_2^2(M)$, then we have that

$$\lim_{i\to+\infty} \int_M (\Delta_g u_i \Delta_g \varphi + (\nabla u_i, \nabla\varphi)_g + u_i\varphi)\, dv_g = \int_M (\Delta_g u \Delta_g \varphi + (\nabla u, \nabla\varphi)_g + u\varphi)\, dv_g$$

for all $\varphi \in H_2^2(M)$. Actually, since $H_2^2(M)$ is a Hilbert space when endowed with a suitable scalar product (see Subsection 2.4), this last statement is equivalent to the weak convergence.

Theorem 1.2. *Let (M,g) be a compact Riemannian manifold. Let $k \in \mathbb{N}$ and let $p > 1$. Then the unit ball of $H_k^p(M)$ is weakly compact. In other words, for any sequence $(u_i)_{i\in\mathbb{N}} \in H_k^p(M)$ such that $\|u_i\|_{H_k^p} \le C$ for all $i \in \mathbb{N}$, there exists a subsequence $(u_{i'})_{i\in\mathbb{N}} \in H_k^p(M)$ and there exists $u \in H_k^p(M)$ such that $u_{i'} \rightharpoonup u$ weakly in $H_k^p(M)$ when $i \to +\infty$.*

1.2.3. *Sobolev embeddings and inequalities.*

Theorem 1.3. *Let (M,g) be a compact Riemannian manifold of dimension $n \ge 1$. Let $p \ge 1$ and let $0 \le m < k$ two integers such that $n > p(k-m)$. Then $H_k^p(M)$ is embedded in $H_m^q(M)$, where $\frac{1}{q} = \frac{1}{p} - \frac{k-m}{n}$. Moreover, this embedding is continuous. In other words, there exists $C > 0$ such that for all $u \in H_k^p(M)$, then $u \in H_m^q(M)$ and*

$$\|u\|_{H_m^q} \le C\|u\|_{H_k^p}.$$

In this notes, we will intensively use the following Sobolev inequality: Given (M,g) a manifold of dimension $n \ge 5$, then $H_2^2(M) \rightharpoonup L^{\frac{2n}{n-4}}(M)$ continuously. In other words, there exists $A > 0$ such that

$$\|u\|_{L^{\frac{2n}{n-4}}(M)} \le A\|u\|_{H_2^2(M)} \tag{4}$$

for all $u \in H_2^2(M)$.

Theorem 1.4. *Let (M,g) be a compact Riemannian manifold of dimension $n \ge 1$. Let $p \ge 1$ and let $0 \le m < k$ two integers such that $n = p(k-m)$. Then $H_k^p(M)$ is embedded in $H_m^q(M)$ for all $q \ge 1$. Moreover, this embedding is continuous. In other words, for any $q \ge 1$, there exists $C(q) > 0$ such that for all $u \in H_k^p(M)$, then $u \in H_m^q(M)$ and*

$$\|u\|_{H_m^q} \le C(q)\|u\|_{H_k^p}.$$

Given $\alpha \in (0,1]$, we say that $u \in C^{0,\alpha}(M)$ if there exists $C > 0$ such that $|u(x) - u(y)| \le C\, d_g(x,y)^\alpha$ for all $x, y \in M$. The space $C^{0,\alpha}(M)$ is a Banach space when equiped with the norm

$$\|u\|_{C^{0,\alpha}(M)} := \|u\|_\infty + \sup_{x\neq y\in M} \frac{|u(x)-u(y)|}{d_g(x,y)^\alpha}.$$

Theorem 1.5. *Let (M,g) be a compact Riemannian manifold of dimension $n \ge 1$. Let $p \ge 1$ and let $k \ge 1$ an integer such that $kp > n$. Then $H_k^p(M)$ is embedded in $C^{0,\alpha}(M)$ for all $\alpha \in (0,1)$ such that $\alpha < k - \frac{n}{p}$. Moreover, this embedding is continuous and there exists $C(\alpha) > 0$ such that*

$$\|u\|_{C^{0,\alpha}(M)} \le C(\alpha)\|u\|_{H_k^p(M)}$$

for all $u \in H_k^p(M)$.

Note that there is a slight (but standard) abuse of notation in the above statement. Indeed, for all $u \in H_k^p(M)$, $kp > n$, there exists a continuous representative for the class of u in $L^p(M)$. Since it is unique, we identify the class u to this representative. There are other embedding results for $H_k^p(M)$: for the sake of simplicity, we do not write them here, and we refer to Gilbarg-Trudinger [GiTr] or Adams [Ada].

1.2.4. *Compact embeddings.* In the sequel, we say that an application $T : E \to F$ between two Banach spaces is compact is for any sequence $(x_i)_{i\in\mathbb{N}} \in E$ uniformly bounded for the norm of E, then there exists $y \in F$, there exists a subsequence $(x_{i'})$ such that $\lim_{i\to+\infty} T(x_{i'}) = y$ strongly in F. For the sake of simplicity, we do not state the compact embeddings in Sobolev space in all their generality, but we restrict to the case of $H_2^2(M)$ that will be of interest in the sequel. The following theorem is esentially due to Rellich and Kondrakov. Here again, we refer to [Ada] and to [GiTr].

Theorem 1.6. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 1$. Then the embedding $H_2^2(M) \hookrightarrow H_1^2(M)$ is compact. In case $n \geq 5$, then the embedding $H_2^2(M) \hookrightarrow L^q(M)$ is compact for all $q \in [1, \frac{2n}{n-4})$.*

Indeed, the recurent problem we will have to face will be that the embedding $H_2^2(M) \hookrightarrow L^{2^\sharp}(M)$ is not compact. And these notes are mainly concerned with this issue.

1.3. **Regularity theorems.** The main references here are Agmon-Nirenberg [DoNi], Agmon-Douglis-Nirenberg [ADN] and the celebrated [GiTr] by Gilbarg and Trudinger and the distribution theory [Sch] by L.Schwartz. We present here the statement of two regularity results in the context of Riemannian manifolds. Here, the references are [Aub2], [Heb1] and [EsRo]. Let (M,g) be a smooth compact Riemannian manifold. We let A be a smooth symmetric $(2,0)-$tensor on M and $a \in C^\infty(M)$. We let the operator $P_g := \Delta_g^2 - \mathrm{div}_g(A(\nabla\cdot)^\#) + a$ defined as

$$P_g u = \Delta_g^2 u - \mathrm{div}_g(A(\nabla u)^\#) + au \tag{5}$$

for all $u \in C^\infty(M)$. Here, $A(\nabla u)^\#$ is the $(1,0)-$tensor whose coordinates in a chart are $(A(\nabla u)^\#)_i = A_{ij}((\nabla u)^\#)^j = A_{ij}g^{jk}(\nabla u)_k$. Concerning terminology, we say that $u \in H_2^2(M)$ is a weak solution of $P_g u = f$, where $f \in L^1(M)$, if

$$\int_M \left(\Delta_g u \Delta_g \varphi + A(\nabla u^\#, \nabla\varphi^\#) + au\varphi\right) dv_g = \int_M f\varphi \, dv_g$$

for all $\varphi \in C^\infty(M)$. In the sequel, given $k \in \mathbb{N}$, we define the norm

$$\|u\|_{C^k(M)} := \sum_{i=1}^k \|\nabla^k u\|_\infty,$$

for all $u \in C^k(M)$. In particular $\|u\|_{C^0(M)} = \|u\|_{L^\infty(M)}$ for all $u \in C^0(M)$. Note that this definition of the C^k-norm extends to tensors.

1.3.1. *L^p theory.*

Theorem 1.7. *Let (M,g) be a compact Riemannian manifold. Let $a \in C^\infty(M)$ and let A be a smooth symmetric $(2,0)-$tensor on M. Let $f \in H_k^p(M)$. Let $u \in H_2^2(\Omega)$ be a weak solution of $P_g u = f$.Then $u \in H_{4+k}^p(M)$. Moreover, we have that*

$$\|u\|_{H_{4+k}^p(M)} \leq C\left(\|f\|_{H_k^p(M)} + \|u\|_{L^p(M)}\right)$$

where $C = C(M,g,K)$ and

$$\|a\|_{C^{k+1}(M)} + \|A\|_{C^{k+2}(M)} \leq K.$$

1.3.2. *Schauder theory.*

Theorem 1.8. *Let (M,g) be a compact Riemannian manifold. Let $a \in C^\infty(M)$ and let A be a smooth symmetric $(2,0)-$tensor on M. Let $\alpha \in (0,1)$ and let $f \in C^{0,\alpha}(M)$. Let $u \in H_2^2(\Omega)$ be a weak solution of $P_g u = f$.Then $u \in C^{4,\alpha}(M)$. Moreover, we have that*

$$\|u\|_{C^4(M)} \leq C\left(\|f\|_{C^{0,\alpha}(M)} + \|u\|_{C^0(M)}\right)$$

where $C = C(M,g,K)$ and

$$\|a\|_{C^{k+1}(M)} + \|A\|_{C^{k+2}(M)} \leq K.$$

Convention: in these notes, C will denote a positive constant independant of the various indices and variables, unless the dependance is precised. The constant C may vary from one line to the other, and even in the same line. The notation $C(a,b,...)$ means that the constant C depends only on (M,g), $a,b,....$

2. Motivations

2.1. The geometric operator and its conformal invariance properties.

2.1.1. *The case of dimension four.* Let (M^4,g) be a Riemannian manifold. In 1983, Paneitz [Pan] introduced the fourth order operator $P_g^4 : C^4(M) \to C^0(M)$ defined as follows: given $u \in C^4(\Omega)$, we have that

$$P_g^4 u := \Delta_g^2 u - \mathrm{div}_g\left(\left(\frac{2}{3}R_g g - 2Ric_g\right)(\nabla u)^\#\right).$$

Actually, this operator enjoys some nice conformal invariance properties. Namely, let $\varphi \in C^\infty(M)$ and let $\tilde{g} = e^{2\varphi}g$ be a metric conformal to g. We have that

$$P_{\tilde{g}}^4 = e^{-4\varphi}P_g^4. \tag{6}$$

Associated to this operator is a notion of $Q-$curvature, a curvature that also enjoys some nice conformal properties: namely, let

$$Q_g^4 = \frac{1}{6}\left(\Delta_g R_g - 3|Ric_g|_g^2 + R_g^2\right).$$

Passing from Q_g^4 to $Q_{\tilde{g}}^4$ is easy through the following formula:

$$P_g^4\varphi + Q_g^4 = Q_{\tilde{g}}^4 e^{4\varphi}. \tag{7}$$

A possible survey on the questions raised by the Paneitz operator in dimension four is [ChYa].

2.1.2. *The case of dimension $n \geq 5$.* In these notes, we will not be concerned with the four-dimensional case, but with the generalization of this operator to the dimensions $n \geq 5$. This generalization is due to Branson [Bra]. Let (M,g) be a Riemannian manifold of dimension $n \geq 5$, and define the operator $P_g^n : C^4(M) \to C^0(M)$ by

$$P_g^n u := \Delta_g^2 u - \mathrm{div}_g\left((a_n R_g g + b_n Ric_g)(\nabla u)^\#\right) + \frac{n-4}{2}Q_g^n u, \tag{8}$$

where

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)} \text{ and } b_n = -\frac{4}{n-2},$$

and

$$Q_g^n = \frac{1}{2(n-1)}\Delta_g R_g + \frac{n^3-4n^2+16n-16}{8(n-2)^2(n-2)^2}R_g^2 - \frac{2}{(n-2)^2}|Ric_g|_g^2$$

is the $Q-$curvature in dimension $n \geq 5$. Note that one recovers the Paneitz operator when $n=4$. This operator also enjoys nice conformal invariance properties: indeed, let $u \in C^\infty(M)$, $u>0$ and consider the metric $\tilde{g} = u^{\frac{4}{n-4}}g$ which is conformal to g. Given $\varphi \in C^\infty(M)$, we have that

$$P_{\tilde{g}}^n \varphi = u^{-\frac{n+4}{n-4}} P_g^n(u\varphi). \tag{9}$$

In particular, taking $\varphi \equiv 1$, one gets the equation

$$P_g^n u = \frac{n-4}{2} Q_{\tilde{g}}^n u^{\frac{n+4}{n-4}},\ u>0. \tag{10}$$

2.1.3. *The conformal Laplacian and the Yamabe problem.* Actually, all this framework is very similar to the framework involved with the conformal Laplacian. More precisely, let (M^2, g) a 2-dimensional Riemannian manifold. Let $\varphi \in C^\infty(M^2)$ and consider the metric $\tilde{g} = e^{2\varphi}g$ conformal to g. One gets that

$$\Delta_{\tilde{g}} = e^{-2\varphi}\Delta_g.$$

In addition, the scalar curvature is a natural invariant associated to this operator. Indeed, the scalar curvature of $\tilde{g}$ and the scalar curvature for g are related as follows:

$$\Delta_g \varphi + \frac{1}{2}R_g = \frac{1}{2}R_{\tilde{g}}e^{2\varphi}.$$

These relations are very similar to the relations (6) and (7) enjoyed by the Paneitz operator. This analogy extends to the higher dimensional case. Let (M,g) be a Riemannian manifold of dimension $n \geq 3$, and define the conformal Laplacian $L_g^n : C^2(M) \to C^0(M)$ by

$$L_g^n u := \Delta_g u + \frac{n-2}{4(n-1)}R_g u$$

for all $y \in C^2(M)$. Let $v \in C^\infty(M)$ such that $v>0$ and consider the metric $\tilde{g} = v^{\frac{4}{n-2}}g$ conformal to g, then we have that

$$L_{\tilde{g}}^n \varphi = v^{-\frac{n+2}{n-2}} L_g^n(v\varphi)$$

for all $\varphi \in C^\infty(M)$, and then, taking $\varphi \equiv 1$ yields

$$L_g^n v = \frac{n-2}{4(n-1)} R_{\tilde{g}} v^{\frac{n+2}{n-2}}. \tag{11}$$

In particular, these properties are similar to the properties (9) and (10) enjoyed by the Paneitz-Branson operator. The Paneitz operator can then be seen as an extension of the conformal Laplacian. Note that in [GJMS], Graham and al. constructed operators of order $2k$ on manifold of dimension, with the restriction $n \geq 2k$ in case n is even. The principal part of these operators is Δ_g^k: when $k=1$, they recover the conformal Laplacian, and when $k=2$, they recover the Paneitz operator.

In the conformal class of a metric, is there a metric that is nicer than the other ones? Indeed, it happens that the good idea is to find a metric with constant scalar curvature. For justifications of this assertion, we refer to [Bes] or to the survey [LePa]. What is now refered to the Yamabe problem is the following:

The Yamabe Problem: given (M,g) a compact Riemannian manifold of dimension $n \geq 2$ without boundary, is there a metric $\tilde{g}$ conformal to g such that $R_{\tilde{g}} = C^{st}$?
In dimension $n \geq 3$, the problem can be reformulated as follows:

The Yamabe Problem, pde aspect for $n \geq 3$: given (M,g) a compact Riemannian manifold of dimension $n \geq 2$ without boundary, is there a function $v \in C^\infty(M)$ such that $v > 0$ and $L_g^n v = \epsilon v^{\frac{n+2}{n-2}}$, where $\epsilon \in \{-1, 0, +1\}$?

The resolution of this problem was quite a long history. Let us just mention that Yamabe's initial proof [Yam] was not complete and that the final resolution of the problem is due to Aubin [Aub1] and Schoen [Sch1]. The classical reference for this problem is the very nice survey of Lee and Parker [LePa]. The two-dimensional problem was also answered positively, but the resolution is completely different: in particular, it is related to the topology of the manifold and the uniformization theorem.

The answer to the Yamabe problem in dimension $n \geq 3$ is also positive. However, the answer hides the specificity of the sphere for which the Yamabe invariant achieves its maximum possible value (here again, we refer to [LePa]). Schoen raised the question of the compactness of metrics with constant scalar curvature and conjectured the following:

Conjecture (Schoen): let (M,g) be a compact Riemannian manifold without boundary of dimension $n \geq 3$ with positive Yamabe invariant. Then the set of metrics $\tilde{g}$ conformal to g such that $R_{\tilde{g}} \equiv 1$ is compact in the C^2-topology if (M,g) is not conformally equivalent to the sphere equiped with its round metric.

Note that in the case of the standard sphere, the set of conformal metrics with constant scalar curvature is not compact for the C^2-topology. Schoen proved this conjecture when (M,g) is locally conformally flat [Sch2] and when $n = 3$ [Sch3]. O.Druet [Dru] proved the conjecture in dimension $n = 4, 5$, F.Marques [Mar] proved it in dimensions $n = 6, 7$ and Y.-Y.Li-M.Zhu [LiZh] in dimension $n = 8, 9$. Recently, the final (and positive) answer to the conjecture was given by Khuri and Schoen [KhSc].

The question of compactness happens to be very rich. Indeed, the Yamabe problem consists in saying that a certain set is nonempty, and the compactness issue amounts to consider its structure, to know whether it is compact or not. When dealing with multiplicity questions and degree theory for the Yamabe equation, the compactness is a crucial point, see for instance Schoen and Zhang [ScZh] or Li and Zhu [LiZh].

2.2. **The model equation.** These questions of existence and compactness naturally extend to the fourth order setting. Let (M,g) be a compact Riemannian manifold of dimension $n \geq 5$. Let A be a smooth symmetric $(2,0)-$tensor field and $a, f \in C^\infty(M)$. We consider here function $u \in C^4(M)$ solutions to the following model equation:

$$\Delta_g^2 u - \mathrm{div}_g(A(\nabla u)^\#) + au = fu^{2^\sharp - 1},\ u > 0, \qquad (E)$$

where $2^\sharp = \frac{2n}{n-4}$. Note that when the operator $P_g := \Delta_g^2 - \mathrm{div}_g(A(\nabla \cdot)^\#) + a$ is P_g^n (the Paneitz-Branson operator), equation (E) means that the $Q-$curvature of the metric $\tilde{g} = u^{\frac{4}{n-4}} g$ verifies that $\frac{n-4}{2} Q_{\tilde{g}} = f$. Following the preceding discussion about the Yamabe equation, we address in these notes the following questions: **(Q1):** is there a solution $u \in C^4(M)$ to (E)?

(Q2): is the set of functions u solutions to (E) compact in the C^4-topology?

2.3. **A possible strategy for existence.** From now one, we let $f \in C^\infty(M)$ such that $f > 0$. We let A be a smooth symmetric $(2,0)-$tensor field and $a \in C^\infty(M)$, and we let the operator

$$P_g := \Delta_g^2 - \mathrm{div}_g(A(\nabla\cdot)^\#) + a.$$

Given $q \in [2, 2^\sharp]$, we consider the functional $I_{q,f} : H_2^2(M) \setminus \{0\} \to \mathbb{R}$ such that

$$I_{q,f}(u) := \frac{\int_M \left((\Delta_g u)^2 + A((\nabla u)^\sharp, (\nabla u)^\sharp) + a u^2\right)\, dv_g}{\left(\int_M f|u|^q\, dv_g\right)^{\frac{2}{q}}} \tag{12}$$

for all $u \in H_2^2(M) \setminus \{0\}$. This functional is well defined since the Sobolev embedding $H_2^2(M) \hookrightarrow L^q(M)$ is continuous (see Theorem 1.3). With a standard abuse of notation, we define

$$\int_M uP_g v\, dv_g := \int_M vP_g u\, dv_g \int_M \left(\Delta_g u \Delta_g v + A((\nabla u)^\sharp, (\nabla v)^\sharp) + auv\right)\, dv_g \tag{13}$$

when $u, v \in H_2^2(M)$. This definition is relevant since A is symmetrical and Theorem 1.1 holds. We then say that $\int_M uP_g v\, dv_g$ is defined in the distributional sense. Note that it follows from Theorem 1.1 that when $u \in C^4(M)$, $\int_M uP_g u\, dv_g$ and coincides with (13).

2.3.1. *Differentiability of the functional $I_{q,f}$.*

Lemma 2.1. *The functional $I_{q,f}$ is differentiable in $H_2^2(M) \setminus \{0\}$ and for any $u \in H_2^2(M)$, we have that*

$$I'_{q,f}(u).\varphi = 2\int_M \left(\Delta_g u \Delta_g \varphi + A((\nabla u)^\sharp, (\nabla \varphi)^\sharp) + au\varphi - \lambda_q(u) f|u|^{q-2}u\varphi\right)\, dv_g, \tag{14}$$

for all $\varphi \in H_2^2(M)$, where $\lambda_q(u) = \frac{\int_M uP_g u\, dv_g}{\int_M f|u|^q\, dv_g}$.

Before proving this lemma, let us see why it is useful in our framework: take $u \in C^4(M)$ a solution to (E), and assume that $\lambda_{2^\sharp}(u) > 0$ and let $\tilde{u} := (\lambda_{2^\sharp}(u))^{-\frac{1}{2^\sharp - 2}}$. We then get with Theorem 1.1 that

$$\begin{aligned}
I'_{2^\sharp,f}(\tilde{u}).\varphi &= 2\int_M \left(\Delta_g \tilde{u} \Delta_g \varphi + A((d\tilde{u})^\sharp, (\nabla\varphi)^\sharp) + a\tilde{u}\varphi - \lambda_{2^\sharp}(\tilde{u}) f|\tilde{u}|^{2^\sharp - 2}\tilde{u}\varphi\right)\, dv_g \\
&= 2\left(\lambda_{2^\sharp}(u)\right)^{-\frac{1}{2^\sharp-2}} \int_M \left(\Delta_g u \Delta_g \varphi + A((\nabla u)^\sharp, (\nabla\varphi)^\sharp) + au\varphi - fu^{2^\sharp-1}\varphi\right)\, dv_g \\
&= 2\left(\lambda_{2^\sharp}(u)\right)^{-\frac{1}{2^\sharp-2}} \int_M \left(\varphi\Delta_g^2 u - \mathrm{div}_g\left(A((\nabla u)^\sharp\right)\varphi + au\varphi - fu^{2^\sharp-1}\varphi\right)\, dv_g \\
&= 2\left(\lambda_{2^\sharp}(u)\right)^{-\frac{1}{2^\sharp-2}} \int_M \left(\varphi P_g u - fu^{2^\sharp-1}\varphi\right)\, dv_g = 0
\end{aligned}$$

for all $\varphi \in H_2^2(M)$. Then $I'_{2^\sharp,f}(u) = 0$ and u is a critical point for $I_{2^\sharp,f}$.

Proof of Lemma 2.1: Indeed, let $u \in H_2^2(M)$ and $\varphi \in H_2^2(M)$ such that $\|\varphi\|_{H_2^2(M)} \leq \frac{1}{2}\|u\|_{H_2^2(M)}$. Clearly, we have that $u + \varphi \in H_2^2(M) \setminus \{0\}$, and $I_{q,f}(u+\varphi)$ is well-defined. We have that

$$\begin{aligned}
&\int_M (u+\varphi)P_g(u+\varphi)\,dv_g\\
&=\int_M uPu\,dv_g + 2\int_M \left(\Delta_g u\Delta_g\varphi + A((\nabla u)^\sharp,(\nabla\varphi)^\sharp)+au\varphi\right)\,dv_g + \int_M \varphi P\varphi\,dv_g\\
&=\int_M uPu\,dv_g + 2\int_M uP_g\varphi\,dv_g + O(1)\|\varphi\|_{H_2^2}^2. \qquad (15)
\end{aligned}$$

where $|O(1)|\leq C$ for all $\varphi\in H_2^2(M)$ such that $\|\varphi\|_{H_2^2(M)}\leq\frac{1}{2}\|u\|_{H_2^2(M)}$. Note here that we have used that A is symmetric. Concerning the denominator of $I_{q,f}$, we need the following estimate: for all $q\geq 2\geq\theta\geq 0$, there exists $C(q,\theta)>0$ that depends only on q and θ such that

$$\left||x+y|^q-|x|^q-q|x|^{q-2}xy\right|\leq C(q,\theta)(|x|^{q-\theta}|y|^\theta+|y|^q) \text{ for all } x,y\in\mathbb{R}. \qquad (16)$$

This inequality is straightforward. Using (16) and Hölder's inequality, we get that

$$\begin{aligned}
&\left|\int_M f|u+\varphi|^q\,dv_g-\int_M f|u|^q\,dv_g - q\int_M f|u|^{q-2}u\varphi\,dv_g\right|\\
&\leq C(q,2)\|f\|_\infty\left(\int_M |u|^{q-2}|\varphi|^2\,dv_g+\int_M|\varphi|^q\,dv_g\right)\\
&\leq C(q,2)\|f\|_\infty \mathrm{Vol}_g(M)^{1-\frac{q}{2^\sharp}}\left(\|u\|_{2^\sharp}^{q-2}\|\varphi\|_{2^\sharp}^2+\|\varphi\|_{2^\sharp}^q\right).
\end{aligned}$$

Since $u,\varphi\in H_2^2(M)$, it follows from the Sobolev inequality (4) that

$$\begin{aligned}
&\left|\int_M f|u+\varphi|^q\,dv_g-\int_M f|u|^q\,dv_g - q\int_M f|u|^{q-2}u\varphi\,dv_g\right|\\
&\leq C(q,\|u\|_{H_2^2},\|f\|_\infty\mathrm{Vol}_g(M))\cdot\|\varphi\|_{H_2^2}^2.
\end{aligned}$$

as soon as $\|\varphi\|_{H_2^2(M)}\leq\frac{1}{2}\|u\|_{H_2^2(M)}$. Since $\int_M f|u|^q\,dv_g\neq 0$, we then get that

$$\begin{aligned}
&\left(\int_M f|u+\varphi|^q\,dv_g\right)^{-\frac{2}{q}}\\
&=\left(\int_M f|u|^q\,dv_g\right)^{-\frac{2}{q}}\left(1+\frac{q\int_M f|u|^{q-2}u\varphi\,dv_g}{\int_M f|u|^q\,dv_g}+O(1)\|\varphi\|_{H_2^2}^2\right)^{-\frac{2}{q}}\\
&=\left(\int_M f|u|^q\,dv_g\right)^{\frac{2}{q}}\left(1-\frac{2\int_M f|u|^{q-2}u\varphi\,dv_g}{\int_M f|u|^q\,dv_g}+O(1)\|\varphi\|_{H_2^2}^2\right) \qquad (17)
\end{aligned}$$

where $|O(1)|\leq C$ for all $\varphi\in H_2^2(M)$ such that $\|\varphi\|_{H_2^2(M)}\leq\frac{1}{2}\|u\|_{H_2^2(M)}$. Plugging (15) and (17) in (12), one gets that

$$\begin{aligned}
I_{q,f}(u+\varphi)=I_{q,f}(u)\cdot\Bigg(1+2\int_M&\left(\Delta_g u\Delta_g\varphi+A((\nabla u)^\sharp,(\nabla\varphi)^\sharp)+au\varphi\right)\,dv_g\\
-\frac{2\int_M uP_gu\,dv_g}{\int_M f|u|^q\,dv_g}\int_M& f|u|^{q-2}u\varphi\,dv_g+O(1)\|\varphi\|_{H_2^2}^2\Bigg)
\end{aligned}$$

In particular, $I_{q,f}$ is differentiable at u and we get (14). This ends the proof of the lemma. □.

Exercise: prove that $I_{q,f}\in C^2(H_2^2(M),\mathbb{R})$.

Definition 2.1. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 5$. Let A be a smooth symmetric $(2,0)-$tensor field and let $a, f \in C^\infty(M)$ such that $f > 0$. Let $q \in [2, 2^\sharp]$. Let $u \in H_2^2(M)$. We say that u is a weak solution of*

$$\Delta_g^2 u - div_g(A(\nabla u)^\sharp) + au = f|u|^{q-2}u \tag{E_q}$$

if we have that

$$\int_M \left(\Delta_g u \Delta_g \varphi + A((\nabla u)^\sharp, (\nabla \varphi)^\sharp) + au\varphi\right) \, dv_g = \int_M f|u|^{q-2}u\varphi \, dv_g$$

for all $\varphi \in H_2^2(M)$.

In particular, u is a critical point for $I_{q,f}$ iff u is a weak solution to $P_g u = \lambda_q(u) f|u|^{q-2}u$. We are now in position to suggest a strategy to obtain solutions to (E):

2.3.2. *Step 1: minimization of $I_{2^\sharp,f}$.* Find (if possible) $u \in H_2^2(M) \setminus \{0\}$ such that $I_{2^\sharp,f}(u) = \inf\left\{I_{q,f}(v) / \, v \in H_2^2(M) \setminus \{0\}\right\}$. Since $I_{2^\sharp,f}(\lambda u) = I_{2^\sharp(u)}$ for all $\lambda \neq 0$, up to multiplicating by a positive constant, one can assume that $\int_M f|u|^{2^\sharp} \, dv_g = 1$. In this case, $\lambda_{2^\sharp}(u) = I_{2^\sharp,f}(u)$. Such a function u then verifies $I'_{2^\sharp,f}(u) = 0$, that is

$$\int_M \left(\Delta_g u \Delta_g \varphi + A((\nabla u)^\sharp, (\nabla \varphi)^\sharp) + au\varphi) \, dv_g = I_{2^\sharp,f}(u) \int_M f|u|^{2^\sharp-2}u\varphi\right) \, dv_g$$

for all $\varphi \in H_2^2(M)$. It is then a weak solution of $P_g u = I_{2^\sharp,f}(u) f|u|^{2^\sharp-2}u$.

2.3.3. *Step 2: find a nonnegative minimizer.* In other words, can we choose the function $u \in H_2^2(M)$ above such that $u \geq 0$ a.e.?

2.3.4. *Step 3: regularity of weak solutions.* Prove that when $u \in H_2^2(M)$ is a weak solution to $P_g u = I_{2^\sharp,f}(u) f|u|^{2^\sharp-2}u$, then $u \in C^4(M)$. In this situation, we have that

$$\int_M \left(P_g u - I_{2^\sharp,f}(u) f|u|^{2^\sharp-2}u\right) \varphi \, dv_g = 0$$

for all $\varphi \in H_2^2(M)$, and then $P_g u = I_{2^\sharp,f}(u) f|u|^{2^\sharp-2}u$ in the usual sense.

2.3.5. *Step 4: find a positive solution.* With Steps 1 to 3, we have a function $u \in C^4(M) \setminus \{0\}$ such that $u \geq 0$ and $P_g u = I_{2^\sharp,f}(u) f u^{2^\sharp-1}$: prove that $u > 0$ indeed. At this stage, if $\lambda_{2^\sharp}(u) > 0$, we have that $I_{2^\sharp,f}(u)^{-\frac{1}{2^\sharp-2}} u$ is a solution to (E).

Unfortunately (or fortunately...), each of these steps involves some particular difficulties, either due to the exponent $2^\sharp$ or due to the bi-harmonic operator Δ_g^2.

2.4. **A suitable norm for $H_2^2(M)$.** It is standard here to use an equivalent norm for $H_2^2(M)$ more suitable to functional $I_{q,f}$. On $H_2^2(M)$, we define the following norm

$$\|u\|'_{H_2^2(M)} = \|\Delta_g u\|_2 + \|\nabla u\|_2 + \|u\|_2$$

for all $u \in H_2^2(M)$. This norm will be very convenient in this notes, and it is relevant thanks to the following proposition:

Proposition 2.1. *The norms $\|\cdot\|_{H_2^2}$ and $\|\cdot\|'_{H_2^2}$ are equivalent.*

Proof. The main tool here is the Bochner-Lichnerowitz-Weitzenbock formula. Indeed, we have that

$$\int_M (\Delta_g u)^2\, dv_g = \int_M |\nabla^2 u|_g^2\, dv_g + \int_M Ric_g((\nabla u)^\#, (\nabla u)^\#)\, dv_g$$

for all $u \in H_2^2(M)$. In particular, we have that

$$\begin{aligned}
&\|\nabla^2 u\|_2^2 + \|\nabla u\|_2^2 + \|u\|_g^2 \\
&= \|\Delta_g u\|_2^2 - \int_M Ric_g((\nabla u)^\#, (\nabla u)^\#)\, dv_g + \|\nabla u\|_2^2 + \|u\|_2^2 \\
&\leq \|\Delta_g u\|_2^2 + C\|\nabla u\|_2^2 + \|u\|_2^2
\end{aligned}$$

for all $u \in H_2^2(M)$, and then there exists $C > 0$ such that $\|\cdot\|_{H_2^2} \leq \|\cdot\|'_{H-2^2}$. The reverse inequality goes the same way, and we get that the two norms are equivalent. ☐

In particular, from now on, we will use this new norm, and we will write in the sequel

$$\|u\|_{H_2^2(M)} = \|\Delta_g u\|_2 + \|\nabla u\|_2 + \|u\|_2 \tag{18}$$

for all $u \in H_2^2(M)$.

2.5. **The main difficulties one encounters.**

2.5.1. *The critical exponent (1): minimization.* Let $q \in [2, 2^\sharp]$ and define

$$\mu_q(f) := \inf \left\{ I_{q,f}(u) / \, u \in H_2^2(M) \setminus \{0\} \right\}.$$

Proposition 2.2. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Let A be a smooth symmetric $(2,0)-$tensor field and let $a, f \in C^\infty(M)$ such that $f > 0$. Let $q \in [2, 2^\sharp)$. Then $\mu_q(f)$ is finite and achieved. In other words, $\mu_q(f) \in \mathbb{R}$ and there existe $u \in H_2^2(M) \setminus \{0\}$ such that $I_{q,f}(u) = \mu_q(f)$.*

Proof. We first prove that $\mu_q(f) > -\infty$. Let $u \in H_2^2(M) \setminus \{0\}$. Since A is smooth, there exists $C > 0$ such that

$$\left| \int_M A((\nabla u)^\#, (\nabla u)^\#)\, dv_g \right| \leq C \int_M |\nabla u|_g^2\, dv_g$$

for all $u \in H_2^2(M)$. It is there convenient to control the L^2-norm of the gradient by the L^2-norm of the laplacian:

Lemma 2.2. *Let (M, g) be a compact Riemanian manifold. Then for any $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that*

$$\|\nabla u\|_2 \leq \epsilon \|\Delta_g u\|_2 + C(\epsilon)\|u\|_2$$

for all $u \in H_2^2(M)$.

Proof of Lemma 2.2: The proof goes by contradiction. Let $\epsilon > 0$. We assume that for all $i \in \mathbb{N}^\star$, there exists $u_i \in H_2^2(M)$ such that

$$\|\nabla u_i\|_2 > \epsilon \|\Delta_g u_i\|_2 + i\|u_i\|_2 \text{ and } \|\nabla u_i\|_2 = 1. \tag{19}$$

It then follows from (19) that

$$\|\Delta_g u_i\|_2 + \|\nabla u_i\|_2 + \|u\|_2 \leq \epsilon^{-1} + 1 + i^{-1}$$

for all $i \in \mathbb{N}^\star$. Then there exists $C > 0$ such that $\|u_i\|_{H_2^2(M)} \leq C$ for all $i \in \mathbb{N}^\star$ (we used the norm defined in (18)). It follows from the compactness of the embedding

$H_2^2(M) \hookrightarrow H_1^2(M)$ (see Proposition 1.6) that there exists a subsequence $(u_{i'})$ and there exist $u \in H_2^2(M)$ such that $\lim_{i\to+\infty} u_{i'} = u$ strongly in $H_1^2(M)$. With (19), we get that $\|\nabla u\|_2 = 1$ and that $\|u\|_2 = 0$: a contradiction, and Lemma 2.2 is proved. □

With Lemma 2.2, we have that there exists $C' > 0$ such that

$$\left|\int_M A((\nabla u)^\#, (\nabla u)^\#)\, dv_g\right| \leq \frac{1}{2}\int_M |\nabla u|_g^2\, dv_g + C'\|u\|_2^2$$

for all $u \in H_2^2(M)$. Using this inequality, the fact that $f > 0$ and Hölder's inequality, we get that

$$\begin{aligned} I_{q,f}(u) &= \frac{\int_M uP_g u\, dv_g}{\left(\int_M f|u|^q\, dv_g\right)^{\frac{2}{q}}} \\ &= \frac{\int_M \left((\Delta_g u)^2 + A((\nabla u)^\#, (\nabla u)^\#)\, dv_g + au^2\right)\, dv_g}{\left(\int_M f|u|^q\, dv_g\right)^{\frac{2}{q}}} \\ &\geq \frac{\frac{1}{2}\int_M (\Delta_g u)^2\, dv_g - (C' + \|a\|_\infty)\|u\|_2^2}{\left(\int_M f|u|^q\, dv_g\right)^{\frac{2}{q}}} \\ &\geq \frac{\int_M (\Delta_g u)^2\, dv_g}{\left(\int_M f|u|^q\, dv_g\right)^{\frac{2}{q}}} - (C' + \|a\|_\infty)\frac{\|u\|_2^2}{(\inf_M f)^{\frac{2}{q}}\|u\|_q^2} \\ &\geq \frac{\int_M (\Delta_g u)^2\, dv_g}{\left(\int_M f|u|^q\, dv_g\right)^{\frac{2}{q}}} - \frac{(C' + \|a\|_\infty)\mathrm{Vol}_g(M)^{2-\frac{4}{q}}}{(\inf_M f)^{\frac{2}{q}}} \end{aligned} \tag{20}$$

for all $u \in H_2^2(M) \setminus \{0\}$. This proves that $\mu_q(f) > -\infty$ and then $\mu_q(f) \in \mathbb{R}$.

Let $(u_i)_{i\in\mathbb{N}} \in H_2^2(M) \setminus \{0\}$ be a minimizing sequence for $I_{q,f}$, that is

$$\lim_{i\to+\infty} I_{q,f}(u_i) = \mu_q(f). \tag{21}$$

Without loss of generality, we can assume that

$$\int_M f|u_i|^q\, dv_g = 1 \tag{22}$$

for all $i \in \mathbb{N}$. With (20) and (21), we get that there exists $C > 0$ such that

$$\int_M (\Delta_g u_i)^2\, dv_g \leq C$$

for all $i \in \mathbb{N}$. Since $f > 0$, we get with Hölder's inequality and (22) that

$$\|u_i\|_2 \leq \mathrm{Vol}_g(M)^{\frac{1}{2}-\frac{1}{q}}\|u_i\|_q \leq \frac{\mathrm{Vol}_g(M)^{\frac{1}{2}-\frac{1}{q}}}{(\inf_M f)^{\frac{1}{q}}}$$

for all $i \in \mathbb{N}$. With Lemma 2.2 and the definition (18), we then get that there exists $C > 0$ such that

$$\|u_i\|_{H_2^2} \leq C$$

for all $i \in \mathbb{N}$. It then follows from the weak compactness of the unit ball (see Theorem 1.2) that there exists $u \in H_2^2(M)$ such that there exists a subsequence $(u_{i'})$ of (u_i) such that

$$u_{i'} \rightharpoonup u$$

weakly in $(H_2^2(M))'$ when $i \to +\infty$. Without loss of generality, we can assume that the convergence actually holds for the initial sequence $(u_i)_{i\in\mathbb{N}}$. Since the embedding $H_2^2(M) \rightharpoonup H_1^2(M)$ is compact (see Theorem 1.6), we can assume that $\lim_{i\to+\infty} u_i = u$ in $H_1^2(M)$. Since $2 \le q < 2^\sharp$ and the embedding $H_2^2(M) \hookrightarrow L^q(M)$ is compact (see Theorem 1.6), we can assume that $\lim_{i\to+\infty} u_i = u$ in $L^q(M)$. Consequently, we get that

$$\lim_{i\to+\infty} \int_M f|u_i|^q \, dv_g = \int_M f|u|^q \, dv_g = 1, \tag{23}$$

and then $u \not\equiv 0$. We let $\theta_i = u_i - u \in H_2^2(M)$ for all $i \in \mathbb{N}$. We have that

$$\begin{aligned}\int_M & u_i P_g u_i \, dv_g \\ &= \int_M u P_g u \, dv_g + 2\int_M \left(\Delta_g u \Delta_g \theta_i + A((\nabla u)^\sharp, (d\theta_i)^\sharp) + a u\theta_i\right) dv_g \\ &+ \int_M \left((\Delta_g\theta_i)^2 + A((d\theta_i)^\sharp, (d\theta_i)^\sharp) + a\theta_i^2\right) dv_g\end{aligned}$$

Since $\theta_i \rightharpoonup 0$ in $(H_2^2(M))'$ and $u_i \to u$ in $H_1^2(M)$ when $i \to +\infty$, we get that

$$\int_M u_i P_g u_i \, dv_g = \int_M u P_g u \, dv_g + \int_M (\Delta_g\theta_i)^2 \, dv_g + o(1)$$

where $\lim_{i\to+\infty} o(1) = 0$. Plugging this equality in $I_{q,f}(u_i)$, one gets that

$$\mu_q(f) = \int_M u P_g u \, dv_g + \int_M (\Delta_g\theta_i)^2 \, dv_g + o(1) \tag{24}$$

when $i \to +\infty$. Since $u \not\equiv 0$, we have that $I_{q,f}(u) \ge \mu_q(f)$. With (23), we get that

$$\mu_q(f) \le \int_M u P_g u \, dv_g. \tag{25}$$

Plugging (24) and (25) together, we get that

$$\mu_q(f) = \int_M u P_g u \, dv_g = I_{q,f}(u) \text{ and } \lim_{i\to+\infty} \int_M (\Delta_g\theta_i)^2 \, dv_g = 0.$$

In particular, the infimum $\mu_q(f)$ is achieved at $u \in H_2^2(M)$. □

One crucial point in the preceding proof is the compactness of the embedding $H_2^2(M) \rightharpoonup L^q(M)$ for $2 \le q < 2^\sharp$. For $q = 2^\sharp$, the embedding is continuous, but not compact, and the above variational method does not work. This lack of compactness is actually fundamental: there are obstructions to the existence of critical points for $I_{2^\sharp,f}$, as proved in [DHL]. In the following statement, the first spherical harmonics are the eigenfunctions of Δ_h for the eigenvalue n, the first nonzero eigenvalue of Δ_h (see Subsection 4.4). The following was proved in [DHL]:

Theorem 2.1. *Let $(\mathbb{S}^n, h)$ be the standard $n-$dimensional sphere equiped with its round metric h. Let $u, f \in C^\infty(\mathbb{S}^n)$ such that*

$$P_h^n u = f u^{2^\sharp - 1}, \ u > 0 \text{ in } \mathbb{S}^n.$$

Here, P_h^n denotes the Paneitz-Branson operator on the sphere. Then for first spherical harmonic $\varphi \in C^\infty(M)$, we have that

$$\int_{\mathbb{S}^n} (\nabla f, \nabla\varphi)_g u^{2^\sharp} \, dv_h = 0.$$

In particular, there is no solution to $P_h u = (1+\epsilon\varphi)u^{2^\sharp-1}$ *for all* $\epsilon > 0$ *and all* φ *first spherical harmonic.*

2.5.2. *The critical exponent (2): regularity and compactness.* Here again, a regularity result holds for the subcritical case, but fails in the critical case:

Proposition 2.3. *Let* (M,g) *be a compact Riemannian manifold of dimension* $n \geq 5$. *Let* A *be a smooth symmetric* $(2,0)-$*tensor field and let* $a, f \in C^\infty(M)$ *such that* $f > 0$. *Let* $u \in H_2^2(M)$ *be a weak solution to* (E_q). *Then* $u \in C^4(M)$ *and is a solution to* (E_q) *in the usual sense.*

Proof. The proof is quite standard and is now refered to a bootstrap argument. Let

$$p_0 = \max\{p \geq 1/\, u \in L^p(M)\}.$$

It follows from the Sobolev embedding (see also Theorem 1.3) that $p_0 \geq 2^\sharp$. Assume that $p_0 < +\infty$ and let $p \in (2, p_0)$. Then $u \in L^p(M)$, and then

$$f|u|^{q-2}u \in L^{\frac{p}{q-1}}(M).$$

Since $P_g u = f|u|^{q-2}u$, it then follows from regularity theorems (see Theorem 1.7) that $u \in H_4^{\frac{p}{q-1}}(M)$. It then follows from Sobolev's embedding theorem (see Theorems 1.3, 1.4 and 1.5) that we are in one of the following cases:

(i) If $\frac{q-1}{p} - \frac{4}{n} < 0$, then $u \in C^0(M)$, and then $u \in L^r(M)$ for all $r \geq 1$ and then $p_0 = +\infty$, a contradiction.

(ii) If $\frac{q-1}{p} - \frac{4}{n} = 0$, then $u \in L^r(M)$ for all $r \geq 1$ and then $p_0 = +\infty$, a contradiction.

(iii) If $\frac{q-1}{p} - \frac{4}{n} > 0$ then $u \in L^r(M)$, where $\frac{1}{r} = \frac{q-1}{p} - \frac{4}{n}$, and then $p_0 \geq r > p$, so that we have improved the order integrability of u. Since this is valid for all $p \in (2, p_0)$, letting p go to p_0, we get that

$$\frac{1}{p_0} \leq \frac{q-1}{p_0} - \frac{4}{n} \Rightarrow p_0 \leq \frac{n(q-2)}{4} < 2^\sharp \tag{26}$$

since $q < 2^\sharp$, a contradiction since $p_0 \geq 2^\sharp$.

This proves that $p_0 = +\infty$, and then that $u \in L^p(M)$ for all $p \geq 1$. Then, $P_g u \in L^p(M)$ for all $p \geq 1$. It then follows from regularity theory (see Theorem 1.7) that $u \in H_4^p(M)$ for all $p \geq 1$, and then from Sobolev's embedding theorem (see Theorem 1.5), that $u \in C^{0,\alpha}(M)$ for all $\alpha \in (0,1)$. We then get that $f|u|^{q-2}u \in C^{0,\alpha}(M)$, and by regularity theory (see Theorem 1.8), one gets that $u \in C^4(M)$. □

Actually, the preceding bootstrap can be applied to obtained C^4-bounds from $L^{2^\sharp}-$bounds for u. Of course, still in the subcritical case:

Proposition 2.4. *Let* (M,g) *be a compact Riemannian manifold of dimension* $n \geq 5$. *Let* A *be a smooth symmetric* $(2,0)-$*tensor field and let* $a, f \in C^\infty(M)$ *such that* $f > 0$. *Let* $q \in [2, 2^\sharp)]$. *Then for any* $\Lambda > 0$, *there exists* $C_0(\Lambda, q) > 0$ *such that for any* $u \in C^4(M)$ *solution to* (E_q), *we have that*

$$\|u\|_{2^\sharp} \leq \Lambda \Rightarrow \|u\|_{C^4(M)} \leq C_0(\Lambda, q).$$

Exercise: Prove this proposition using the proof of the regularity of solutions to the subcritical problem and the regularity theorems.

The reader will have pointed out that the subcritical exponent $q < 2^\sharp$ in (26) is crucial in this proof. Actually, in the critical case, one could start with $p = 2^\sharp$, and one would obtain $r = 2^\sharp$! So that there would be no improvement of the integrability. On the other hand, still in the case $q = 2^\sharp$, assume that there exists $p > 2^\sharp$ such that $u \in L^p(M)$: in this situation, the bootstrap works and one recovers that $u \in C^4(M)$.

Exercise [EXO]: Let $u \in H_2^2(M)$ be a weak solution to (E). Prove that if there exists $p > 2^\sharp$ such that $u \in L^p(M)$, then $u \in C^4(M)$.

Trudinger [Tru] was able to prove that a weak solution $u \in H_1^2(M)$ of $\Delta_g u + au = fu^{\frac{n+2}{n-2}}$ (a weak solution to this equation is a function u such that

$$\int_M ((\nabla u, \nabla\varphi)_g + au\varphi)\, dv_g = \int_M fu^{\frac{n+2}{n-2}}\varphi\, dv_g$$

for all $\varphi \in H_2^2(M)$) is actually in $C^2(M)$. But this proof does not easily extends to the bi-harmonic operator.

2.5.3. *The bi-harmonic operator (1): positivity.* Another problem is to recover positive solutions to our equation. A natural approach would be to consider a minimizer u of $I_{q,f}$ and then to see whether $|u|$ has a chance to be another minimizer. This approach is very fruitful for second-order problems, here is how:

Let $r \in [2, \frac{2n}{n-2})$, and let

$$\tilde{I}_r(u) := \frac{\int_M \left(|\nabla u|_g^2 + au^2\right)\, dv_g}{\left(\int_M f|u|^r\, dv_g\right)^{\frac{2}{r}}}$$

for all $u \in H_1^2(M) \setminus \{0\}$ (note that this is well defined thanks to the Sobolev embedding of Theorem 1.3). Assume that there exists a minimizer $u \in H_1^2(M)\setminus\{0\}$ for $\tilde{I}_r$ such that $\tilde{I}_r(u) > 0$, and assume that $\int_M f|u|^r\, dv_g = 1$. We have that $|u| \in H_1^2(M)$ and $|\nabla|u||_g = |\nabla u|_g$ (see for instance [GiTr], Theorem 7.8, or [Heb1]). Therefore, one gets that $\tilde{I}_r(|u|) = \tilde{I}_r(u)$ also minimizes $\tilde{I}_r$. As in Proposition 2.3, we get that $|u|$ is a weak solution to $\Delta_g|u| + a|u| = \tilde{I}_r(|u|)f|u|^{r-1}$. With a bootstrap argument, one gets that $|u| \in C^2(M)$. With $K > 0$ large enough such that $a + K - \tilde{I}_r(|u|)f|u|^{r-2} > 0$, one gets that $\Delta_g|u| + (a + K - \tilde{I}_r(|u|)f|u|^{r-2})|u| = K|u| \geq 0$, and then $|u| > 0$ by the strong comparison principle (see for instance [GiTr], Theorem 3.5 or [Heb1]). In particular, the initial function u is either positive or negative and, up to multiplying by a nonzero constant we have recovered a positive solution to the equation $\Delta_g\tilde{u} + a\tilde{u} = f\tilde{u}^{r-1}$.

This strategy doe not apply to the fourth-order setting for (at least!) one good reason: there exists $u \in H_2^2(M)$ such that $|u| \notin H_2^2(M)$. A very simple illustration of this fact is on $\mathbb{R}^n$, actually. In the distributional sense, one has

$$\begin{aligned}\langle \Delta |x_i|, \varphi\rangle_{\mathcal{D}'(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} |x_i| \Delta\varphi\, dx\\ &= \int_{\{x_i>0\}} x_i \Delta\varphi\, dx - \int_{\{x_i<0\}} x_i \Delta\varphi\, dx\\ &= \int_{\{x_i>0\}} (\Delta x_i)\varphi\, dx + \int_{\partial\{x_i>0\}} (-x_i\partial_\nu\varphi + (\partial_\nu x_i)\varphi)\, d\sigma\\ &\quad - \int_{\{x_i<0\}} x_i \Delta\varphi\, dx\\ &= 2\int_{\{x_i=0\}} \varphi\, d\sigma\end{aligned}$$

here, ∂_ν denotes the normal outer derivative. In particular, $\Delta|x_i| \not\in L^p_{loc}(\mathbb{R}^n)$ for all $p \geq 1$. As we will see, there are situations in which the minimizers of $I_{q,f}$ (when they exist) change sign, contrary to the second-order case

3. Concerning regularity

As already mentioned, the strategy of Trudinger for second-order operators does not adapt nicely to the fourth-order case. Note that Sandeep [San] could perform a De Giorgi-Nash-Moser scheme for fourth-order equations, a technique that is very close to Trudinger's technique.

In these notes, we adapt the techniques developed by Van der Vorst [VdV] for fourth-order problems (see also Djadli-Hebey-Ledoux [DHL] and Esposito-Robert [EsRo] for the context of Riemannian manifolds). We prove the following:

Proposition 3.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Let A be a smooth symmetric $(2,0)-$tensor field and let $a, f \in C^\infty(M)$ such that $f > 0$. Let $u \in H_2^2(M)$ be a weak solution to (E). Then $u \in C^4(M)$ and is a solution to (E) in the usual sense.*

In particular, the question of the regularity of weak solutions is completely solved for our problem. The proof of Proposition 3.1 uses the notion of coercivity:

3.1. **Coercivity.**

Definition 3.1. *We say that P_g as above is coercive if there exists $\lambda > 0$ such that*

$$\int_M \left((\Delta_g u)^2 + A((\nabla u)^\#, (\nabla u)^\#) + au^2\right)\, dv_g \geq \lambda \int_M u^2\, dv_g$$

for all $u \in H_2^2(M)$.

Exercise [COER]: Prove that the following assertions are equivalent:
(i) P_g is coercive
(ii) there exists $\lambda > 0$ such that

$$\int_M \left((\Delta_g u)^2 + A((\nabla u)^\#, (\nabla u)^\#) + au^2\right)\, dv_g \geq \lambda \|u\|_{2^\sharp}^2$$

for all $u \in H_2^2(M)$.

(iii) there exists $\lambda > 0$ such that

$$\int_M \left((\Delta_g u)^2 + A((\nabla u)^\#, (\nabla u)^\#) + a u^2\right)\, dv_g \geq \lambda \|u\|_{H_2^2}^2$$

for all $u \in H_2^2(M)$.

An important Corollary of the coercivity is the following existence result. The proof is postponed to the Appendix.

Proposition 3.2. *Let (M, g) be a compact Riemannian manifold. Let $a \in C^\infty(M)$ and let A be a smooth symmetric $(2,0)-$tensor on M. Assume that the operator $P_g = \Delta_g^2 - div_g(A(\nabla \cdot)^\#) + a$ is coerciv. Then for any $f \in H_k^p(M)$, there exists a unique $u \in H_{4+k}^p(M)$ such that $P_g u = f$. Moreover, we have that*

$$\|u\|_{H_{4+k}^p(M)} \leq C \cdot \|f\|_{H_k^p(M)}.$$

where $C = C(M, g, K)$ and

$$\|a\|_{C^{k+1}(M)} + \|A\|_{C^{k+2}(M)} \leq K.$$

3.2. **Proof of Proposition 3.1:** We prove the proposition in the case the operator P_g is coercive. Let $p \geq 1$. Let $R > 0$ to be chosen later. Let $v \in L^p(M)$. It follows from Hölder's inequality that $f|u|^{2^\sharp - 2}\mathbf{1}_{|u|\geq R} v \in L^r(M)$ with $\frac{1}{r} = \frac{1}{p} + \frac{4}{n}$ and that

$$\|f|u|^{2^\sharp-2}\mathbf{1}_{|u|\geq R} v\|_r \leq \|f\|_\infty \||u|^{2^\sharp-2}\mathbf{1}_{|u|\geq R}\|_{\frac{n}{4}} \|v\|_p.$$

It follows from regularity theory that there exists a unique $w \in H_4^r(M)$ such that $P_g v = f|u|^{2^\sharp-2}\mathbf{1}_{|u|\geq R} v$. Moreover, there exists $C = C(p, r, n) > 0$ such that

$$\|w\|_{H_4^r(M)} \leq C \cdot \|f|u|^{2^\sharp-2}\mathbf{1}_{|u|\geq R} v\|_r.$$

It follows from Sobolev's embedding in Theorem 1.3 that $H_4^r(M)$ is embedded continuously in $L^q(M)$, where $\frac{1}{q} = \frac{1}{r} - \frac{4}{n} = \frac{1}{p}$. Then $w \in L^p(M)$ and there exists $C = C((M, g), p, r, n) > 0$ such that

$$\|w\|_{L^p(M)} \leq C \cdot \|f\|_\infty \||u|^{2^\sharp-2}\mathbf{1}_{|u|\geq R}\|_{\frac{n}{4}} \|v\|_p.$$

We define the operator $T_{p,R} : L^p(M) \to L^p(M)$ such that for any $v \in L^p(M)$, $T_{p,R}(u) = w$ where w is as above. It follows from the above discussion that $T_{p,R}$ is a continuous linear map and that its norm satisfies

$$\|T_{p,R}\|_{L^p \to L^p} \leq C(p, r, n) \cdot \|f\|_\infty \left(\int_{\{|u|\geq R\}} |u|^{2^\sharp}\, dv_g\right)^{\frac{4}{n}}.$$

Therefore, since $u \in L^{2^\sharp(M)}$, there exists $R_0 = R((M, g), p, r, n) > 0$ such that $\|T_{p,R}\|_{L^p \to L^p} \leq \frac{1}{2}$, and then, we get that $Id_{L^p} - T_{p,R} : L^p(M) \to L^p(M)$ is linear continuous with linear continuous inverse.

Since $f|u|^{2^\sharp-2} u \mathbf{1}_{|u|\leq R} \in L^\infty(M)$, we have by Proposition 3.2 that for all $p \geq 2^\sharp$, there exists $\tilde{u} \in H_4^p(M)$ such that $P_g \tilde{u} = f|u|^{2^\sharp-2} u \mathbf{1}_{|u|\leq R}$. We let $\bar{u} = (Id_{L^p} - T_{p,R})^{-1}(\tilde{u}) \in L^p(M)$. We have that

$$P_g u = f|u|^{2^\sharp-2} u \mathbf{1}_{|u|\geq R} + f|u|^{2^\sharp-2} u \mathbf{1}_{|u|\leq R}$$
$$P_g(u - \tilde{u}) = f|u|^{2^\sharp-2}\mathbf{1}_{|u|\geq R} u$$

and then $u-\tilde{u}=T_{2^\sharp,R}(u)$, which yields $(Id_{L^{2^\sharp}}-T_{2^\sharp,R})u=\tilde{u}=(Id_{L^p}-T_{p,R})(\bar{u})=(Id_{L^{2^\sharp}}-T_{2^\sharp,R})(\bar{u})$ since $p\geq 2^\sharp$ and $u,\bar{u}\in L^{2^\sharp}(M)$. Since the operator $(Id_{L^{2^\sharp}}-T_{2^\sharp,R}$ is invertible, we get that $u=\bar{u}\in L^p(M)$ for all $p\geq 2^\sharp$. The bootstrap argument and Exercise [EXO] then yields that $u\in C^4(M)$.

We consider now the case when P_g is not coercive. We let $K>0$ such that P_g+K is coercive, and therefore invertible. We define the map $T_{p,R}:L^p(M)\to L^p(M)$ by $T_{p,R}(v)=(P_g+K)^{-1}(f|u|^{2^\sharp-2}\mathbf{1}_{|u|\geq R}v)$ for all $v\in L^p(M)$. This maps is well-defined. We let $p\geq 2^\sharp$ such that $u\in L^p(M)$ and as above, we get that $u\in L^q(M)$ for some $q>2^\sharp$. The conclusion of the proposition then follows. $\square$

Exercice: Complete the last part of the preceding proof. That is prove Proposition 3.1 in case P_g is not coercive.

4. Concerning positive solutions

4.1. **The main result.** As mentioned, finding positive solutions minimizing $I_{q,f}$ is not so easy... and can sometimes be impossible! We present here a technique that permits in some situations to recover positive minimizers.

Proposition 4.1. *Let (M,g) be a compact Riemannian manifold of dimension $n\geq 5$. Let A be a smooth symmetric $(2,0)-$tensor field and let $a,f\in C^\infty(M)$ such that $f>0$. We assume that P_g verifies the two following properties:*

(i) P_g is coercive

(ii) for all $u\in C^4(M)$ such that $P_gu\geq 0$, then $u>0$ or $u\equiv 0$.

Let $q\in[2,2^\sharp]$ and assume that $u\in H_2^2(M)$ is a minimizer for $I_{q,f}$. Then $u\in C^4(M)$ and either $u>0$ or $u<0$.

Proof. We keep the same notations as in section 3. The regularity $u\in C^4(M)$ is just Proposition 3.1 above. Therefore, there exists $\mu\in\mathbb{R}$ such that

$$P_gu=\mu f|u|^{q-2}u \text{ in } M.$$

With the definition of $I_{q,f}(u)$, we get that

$$\mu=\mu_q(f)\times\left(\int_M f|u|^q\,dv_g\right)^{\frac{2}{q}-1}. \tag{27}$$

We claim that $\mu_q(f)>0$. Indeed, since P_g is coercive, we get that there exists $\lambda>0$ such that

$$\int_M uP_gu\,dv_g\geq\lambda\|u\|_q^2$$

for all $u\in H_2^2(M)$. We then get that

$$I_{q,f}(u)=\frac{\int_M uP_gu\,dv_g}{\left(\int_M f|u|^q\,dv_g\right)^{\frac{2}{q}}}\geq\frac{\lambda}{(\sup_M f)^{\frac{2}{q}}} \tag{28}$$

for all $u\in H_2^2(M)\setminus\{0\}$, and therefore $\mu_q(f)\geq\lambda(\sup_M f)^{-\frac{2}{q}}>0$. This proves the claim.

We let $v\in H_2^2(M)$ such that $P_gv=|P_gu|$ in M. The existence is a consequence of Proposition 3.2. Since $|P_gu|\in C^{0,1}(M)$, one gets with Theorem 1.8 that $v\in C^4(M)$. Then, we have that $P_g(v\pm u)\geq 0$, and then $v\pm u\geq 0$ with point (ii). Then $v\geq|u|$ and $v\neq 0$ (otherwise $u\equiv 0$). Since $P_gv\geq 0$, we then get with point (ii) that $v>0$. Let us compute $I_{q,f}(v)$:

$$\begin{aligned}
I_{q,f}(v) &= \frac{\int_M vP_g v\,dv_g}{\left(\int_M f|v|^q\,dv_g\right)^{\frac{2}{q}}} = \mu\frac{\int_M vf|u|^{q-1}\,dv_g}{\left(\int_M f|v|^q\,dv_g\right)^{\frac{2}{q}}}\\
&\leq \mu\frac{\int_M (f^{\frac{1}{q}}v)\cdot(f|u|^q)^{\frac{q-1}{q}}\,dv_g}{\left(\int_M f|v|^q\,dv_g\right)^{\frac{2}{q}}}\\
&\leq \mu\frac{\left(\int_M fv^q\,dv_g\right)^{\frac{1}{q}}\cdot\left(\int_M f|u|^q\,dv_g\right)^{\frac{q-1}{q}}}{\left(\int_M f|v|^q\,dv_g\right)^{\frac{2}{q}}} \text{ with Hölder's inequality}\\
&\leq \mu\frac{\left(\int_M f|u|^q\,dv_g\right)^{\frac{q-1}{q}}}{\left(\int_M fv^q\,dv_g\right)^{\frac{1}{q}}} \leq \mu\left(\int_M f|u|^q\,dv_g\right)^{\frac{q-2}{q}} \quad \text{since } v\geq |u|\\
&\leq \mu_q(f) \text{ with (27)}
\end{aligned}$$

Since the minimum is $\mu_q(f)$, one gets that $I_{q,f}(v)\geq\mu_q(f)$, and then $I_{q,f}(v)=\mu_q(f)$, and the minimum is achieved at v. There are then equalities everywhere above, and in particular one gets $|u|=v>0$. Since u is continuous, we get therefore that either $u>0$ everywhere or $u<0$ everywhere. □

Point (ii) of Proposition 4.1 is the crucial point. Concerning terminology, we define the pointwise maximum principle as follows:

Definition 4.1. *We say that the operator P_g verifies the pointwise comparison principle if for any $u\in C^4(M)$ such that $P_g u\geq 0$, then either $u>0$ or $u\equiv 0$.*

4.2. **The Rayleigh quotient and the first eigenfunction.** Given an operator P_g as in Proposition 3.1, we define the first eigenvalue of P_g as follows:

$$\lambda_1(P_g)=\inf_{u\in H_2^2(M)\setminus\{0\}}\frac{\int_M uP_g u\,dv_g}{\int_M u^2\,dv_g}. \tag{29}$$

It follows from Propositions 2.2 and 3.1 that $\lambda_1(P_g)$ is achieved by functions in $C^4(M)$. We let

$$E_1(P_g)=\left\{u\in C^4(M)/\,P_g u=\lambda_1(P_g)u\right\}.$$

Clearly, $u\in H_2^2(M)$ is a minimizer for (29) iff $u\not\equiv 0$ and $u\in E_1(P_g)$. Under the hypothesis of Proposition 4.1, we have more informations:

Proposition 4.2. *Let (M,g) be a compact Riemannian manifold of dimension $n\geq 5$. Let A be a smooth symmetric $(2,0)-$tensor field and let $a,f\in C^\infty(M)$ such that $f>0$. We assume that P_g verifies the two following properties:*

(i) P_g is coercive

(ii) P_g satisifies the pointwise comparison principle.

Then every nonzero eigenfunction for $\lambda_1(P_g)$ does not change sign and the eigenspace $E_1(P_g)$ is one-dimensional.

Proof. Let $u\in H_2^2(M)\setminus\{0\}$ be a minimizer for the Rayleigh quotient. With Propositon 3.1, $u\in C^4(M)$ and, up to multipliying by (-1), one can assume that u is positive somewhere. It follows from Proposition 4.1 that actually $u>0$. Let $v\in E_1(P_g)$. Let $x_0\in M$ and let $t=v(x_0)u(x_0)^{-1}$. Then $v-tu\in E_1(P_g)$. In case $v-tu\not\equiv 0$, then $v-tu$ is a minimizer for the Rayleigh quotient and then either $v - tu > 0$ everywhere or $v - tu < 0$ everywhere; a contradiction since $(v - tu)(x_0) = 0$. Then $v = tu \in \mathbb{R}u$, and $E_1(P_g)$ is one-dimensional. $\square$

4.3. **A short detour to second-order equations.** While considering second-order operators, coercivity is equivalent to the pointwise comparison principle. Namely we have the following

Proposition 4.3. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Let $a \in C^\infty(M)$. We say that $\Delta_g + a$ is coercive on $H_1^2(M)$ if there exists $\lambda > 0$ such that*

$$\int_M \left(|\nabla u|_g^2 + au^2\right)\, dv_g \geq \lambda \int_M u^2\, dv_g$$

for all $u \in H_1^2(M)$. Then the two following assertions are equivalent:
(i) $\Delta_g + a$ is coercive
(ii) for all $u \in C^2(M)$ such that $\Delta_g u + au \geq 0$, then $u > 0$ or $u \equiv 0$.

Proof. (i)⇒(ii): Let $u \in C^2(M)$ such that $L_g u \geq 0$. Let $u_- := \max\{-u, 0\}$. Then (see for instance [GiTr], Theorem 7.8, or [Heb1] for the Riemannian setting), $u_- \in H_1^2(M)$ and $\nabla u_- = -\mathbf{1}_{\{u \leq 0\}} \nabla u$. We have that

$$\begin{aligned} 0 &\leq \int_M u_- L_g u\, dv_g = \int_M \left((\nabla u, \nabla u_-)_g + auu_-\right)\, dv_g \\ &\leq -\int_M \left(|\nabla u_-|_g^2 + au_-^2\right)\, dv_g \leq -\lambda \|u_-\|_2 \end{aligned}$$

and then $u_- \equiv 0$, which implies $u \geq 0$. Then we have that $\Delta_g u + (a + \|a\|_\infty)u \geq \|a\|_\infty u \geq 0$. It then follows from the strong comparison principle (see for instance [GiTr], Theorem 3.5 or [Heb1] for the Riemannian setting) that either $u > 0$ or $u \equiv 0$.

(ii)⇒(i) Let $u \in H_1^2(M) \setminus \{0\}$ be a minimizer for the Rayleigh quotient

$$\frac{\int_M \left(|\nabla u|_g^2 + au^2\right)\, dv_g}{\int_M u^2\, dv_g}$$

We let $\mu \in \mathbb{R}$ the value achievd by this minimizer. Following discussion of the proof of Proposition 4.2 above, we can assume that $u \in C^2(M)$ and that $u > 0$ in M and verifies $\Delta_g u + au = \mu u$. Assume that $\mu \leq 0$. Then $\Delta_g(-u) + a(-u) \geq 0$, and it follows from (ii) that either $-u > 0$ or $-u \equiv 0$. A contradiction since $u > 0$. Then $\mu > 0$ and $\Delta_g + a$ is coercive. $\square$

Such a result does not extend to the fourth-order setting:

4.4. **A situation where the minimizer changes sign.**

Proposition 4.4. *Let us consider the unit sphere $(\mathbb{S}^n, h)$, $n \geq 1$, where h is the round metric. Let $a, \alpha \in \mathbb{R}$ such that $\alpha \in (n, 2n)$ and $a > \frac{\alpha^2}{4}$. Then $P_h = \Delta_h^2 - \alpha\Delta_h + a$ is coercive and $E_1(P_g)$ is the $(n+1)-$dimensional space of first spherical harmonics, that is*

$$E_1(P_g) = \{u \in C^2(\mathbb{S}^n) /\, \Delta_h u = nu\} = \{l \circ i /\, l : \mathbb{R}^{n+1} \to \mathbb{R} \text{ is linear}\},$$

wher $i : \mathbb{S}^n \to \mathbb{R}^{n+1}$ is the canonical embedding of the sphere into $\mathbb{R}^{n+1}$. In particular, the minimizers for the Rayleigh quotient change sign.

Proof. Let $u \in E_1(P_g)$. In particular $u \in C^4(\mathbb{S}^n)$. Since $u \in L^2(\mathbb{S}^n)$, we decompose it along the eigenvalues of the Laplacian. This requires some notations: the eigenvalues of Δ_h on $\mathbb{S}^n$ are of the form $\lambda_i = i(i+n-1)$ for all $i \in \mathbb{N}$, and we denote as $E_i(\Delta_h)$ the eigenspace associated to λ_i for all $i \in \mathbb{N}$, each of the $E_i(\Delta_h)$'s being finite-dimensional. Note that here, the first eigenvalue of the Laplacian is 0 and is denoted by λ_0. We refer to [BGM] for a proof of these results. It follows from spectral theory that we can write

$$u = \sum_{i=0}^{\infty} \alpha_i \varphi_i, \tag{30}$$

where $(\alpha_i)_{i\in\mathbb{N}^\star} \in \mathbb{R}$ and $(\varphi_i)_{i\in\mathbb{N}^\star}$ are such that

(i) $\varphi \in E_i(\Delta_h)$ for all $i \in \mathbb{N}$
(ii) $\int_M \varphi_i\varphi_j \, dv_g = \delta_{ij}$, the Kronecker symbol, for all $i, j \in \mathbb{N}$.
(iii) $\sum_{i=0}^{\infty} \alpha_i^2 < \infty$.

Here, the sum (30) must be understand in the L^2-sense, that is $\lim_{N\to+\infty} \|u - \sum_{i=0}^{N} \alpha_i\varphi_i\|_2 = 0$. Let $i \in \mathbb{N}$. We have that $\varphi_i \in C^\infty(M)$ and $P_h\varphi_i = p_0(\lambda_i)\varphi_i$, where $p_0(X) = X^2 - \alpha X + a$. In particular, integrating by parts, one gets

$$\lambda_1(P_h) \int_{\mathbb{S}^n} u\varphi_i \, dv_h = \int_{\mathbb{S}^n} \varphi_i P_h u \, dv_h = \int_{\mathbb{S}^n} u P_h \varphi_i \, dv_h = p_0(\lambda_i) \int_{\mathbb{S}^n} u\varphi_i \, dv_h. \tag{31}$$

With (30) and (31), one gets that

$$(\lambda_1(P_h) - p_0(\lambda_i))\alpha_i = 0. \tag{32}$$

Since $u \not\equiv 0$, there exists $j_0 \in \mathbb{N}$ such that $\alpha_{j_0} \neq 0$, and therefore $p_0(\lambda_{j_0}) = \lambda_1(P_h)$. In particular, since $p_0(\lambda_j)$ is an eigenfunction for P_h for all $j \in \mathbb{N}$, one gets that

$$\lambda_1(P_h) = \inf\{p_0(\lambda_j) / \, j \in \mathbb{N}\}.$$

Since p_0 is a quartic, there are two possibilities:

(i) either $p_0(\lambda_j) \neq \lambda_1(P_h)$ for all $j \neq j_0$
(ii) or there exists $j_1 \neq j_0$ such that $p_0(\lambda_j) = \lambda_1(P_h)$ iff $j \in \{j_0, j_1\}$.

In case (i), one gets with (32) that $\alpha_j = 0$ for all $j \neq j_0$ and then $E_1(P_h) = E_{j_0}(\Delta_h)$.

In case (ii), one gets with (32) that $\alpha_j = 0$ for all $j \notin \{j_0, j_1\}$ and then $E_1(P_h) = E_{j_0}(\Delta_h) \bigoplus E_{j_1}(\Delta_h)$.

Since $p_0(X) = X^2 - \alpha X + a$, one gets that p_0 is increasing on $(\frac{\alpha}{2}, +\infty)$, and since $\alpha \in (n, 2n)$, p_0 is increasing on $[n, +\infty)$. Since $\lambda_i = i(i+n-1)$ for all $i \in \mathbb{N}$, one gets that $p_0(\lambda_i) > p_0(\lambda_1) = p_0(n)$ for all $i \geq 2$. Moreover, $p_0(\lambda_0) = p_0(0) < p_0(\lambda_1)$ since $\alpha > n$. Then $\lambda_1(P_h) = p_0(\lambda_1) = p_0(n) = n^2 - \alpha n + a$ and

$$E_1(P_h) = E_1(\Delta_h) = \{u \in C^2(\mathbb{S}^n) / \, \Delta_h u = nu\},$$

which is exactly the linear space of restrictions of linear forms of $\mathbb{R}^{n+1}$ to the sphere (here again, we refer to [BGM]). Note that the condition $a > \alpha^2/4$ implies that $\lambda_1(P_h) = p_0(n) > 0$, and then P_h is coercive. □

A consequence of this result is that the operator P_g does not satisfy the pointwise comparison principle, despite it is coercive: an important difference with second-order operators. Actually, there are similar situations situations in the Euclidean case: there are simply connected, and even convex domains for which there exists smooth functions such that $\Delta^2 u \geq 0$, $u = \partial_\nu u = 0$ on the boundary, but u is not positive on the domain. These remarks seem to go back to Hadamard [Had].

Note that the situation is particularly surprising for annuli in dimension two: if $D_\epsilon = \{x \in \mathbb{R}^2 / \epsilon < |x| < 1\}$, then Δ^2 can verify or not the above comparison principle depending on the value of $\epsilon \in (0,1)$. We refer to [CDS] for discussions and results about this fact. These propertie are deeply related to the Green's function: indeed, the operator P_g verifies the pointwise comparison principle if and only if its Green's function is positive. We do not intend to discuss on the Green's function here and we refer to Grunau-Sweers [GrSw], for instance, for considerations about it. It is now important to know in which situations the operator P_g satisfies the pointwise comparison principle.

4.5. **When does P_g satisfy the pointwise maximum principle?** We begin with the following simple, but crucial remark:

Proposition 4.5. *Let $a, a' \in C^\infty(M)$ such that $\Delta + a$ and $\Delta_g + a'$ are coercive on $H_1^2(M)$ (as defined in Proposition 4.3). Then the operator $P_g = (\Delta_g + a) \circ (\Delta_g + a')$ satisfies the pointwise maximum principle.*

Proof. Let $u \in C^4(M)$ such that $(\Delta_g + a) \circ (\Delta_g + a')u \geq 0$. It follows from Proposition 4.3 that $(\Delta_g + a')u \geq 0$, and then applying Proposition 4.3 again yields $u > 0$ or $u \equiv 0$ □

We are now interested in knowing which of the operators $P_g = \Delta^2 - \text{div}_g(A(\nabla)^\#) + a$ are product of two second-order operators.

Proposition 4.6. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 1$. Let A be a smooth symmetric $(2,0)-$tensor and $a \in C^\infty(M)$. Then P_g is the product of two second order operators as in Proposition 4.5 if and only if there exists $f \in C^\infty(M)$ such that*

$$A = fg \text{ and } f^2 - 2\Delta_g f - 4a \text{ is a nonnegative constant.}$$

Proof. We first proove the "only if" part of the proposition. Let $a_1, a_2 \in C^\infty(M)$ such that $P_g = (\Delta_g + a) \circ (\Delta_g + a')$. Writing P_g in two different ways, we get that

$$\begin{aligned} P_g u &= \Delta_g^2 u - A^{ij}\nabla_{ij}u - \nabla_i A^{ij}\nabla_j u + au \\ &= \Delta_g^2 u - (a_1 + a_2)g^{ij}\nabla_{ij}u - 2\nabla^i a_2 \nabla_i u + (\Delta_g a_2 + a_1 a_2)u. \end{aligned}$$

Identifying these terms, and using that A is symmetric, we get that

$$A^{ij} = (a_1 + a_2)g^{ij} \text{ for all } i, j \tag{33}$$

$$\nabla_i A^{ij} = 2\nabla^j a_2 \text{ for all } j \tag{34}$$

$$a = \Delta_g a_2 + a_1 a_2 \tag{35}$$

Letting $f := a_1 + a_2$, we then get with (33) that $A = fg$. Since $A = (a_1 + a_2)g$, (34) yields $(\nabla_i(a_1 + a_2))g^{ij} = 2\nabla^j a_2 = 2g^{ij}\nabla_i a_2$ for all j, and then $\nabla_i a_1 = \nabla_i a_2$ for all i. In other words, there exists $K \in \mathbb{R}$ such that $a_2 = a_1 + K$, and then, since $a_1 + a_2 = f$, we get with (34) that

$$f^2 + 2\Delta_g f - 4\alpha = K^2 \in \mathbb{R}_{\geq 0},$$

and the "only if" part of the proposition is proved. The "if" part follows from the preceding proof. □

There are two interesting corollaries to this result:

Corollary 4.1. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 1$. Let $\alpha, a \in \mathbb{R}$. Then the operator $\Delta_g + \alpha\Delta_g + a$ is a product of two elliptic operators as in Proposition 4.5 if and only if $4a \leq \alpha^2$.*

Corollary 4.2. *The Paneitz-Branson operator is the product of two second-order operators if and only if the metric is Einstein. In this situation, the Paneitz-Branson operator has constant coefficients.*

Proof. Let $n \geq 4$ and assume that P_g^n is a product of two second-order operators. It follows from Proposition 4.6 and the definition (8) of P_g^n that there exists $f \in C^\infty(M)$ such that

$$\frac{(n-2)^2+4}{2(n-1)(n-2)}R_g g - \frac{4}{n-2}Ric_g = fg.$$

We then get that there exists $\tilde{f} \in C^\infty(M)$ such that $Ric_g = \tilde{f}g$. Since $n \geq 4$, we then get that there exists $\lambda \in \mathbb{R}$ such that $Ric_g = \lambda g$, and then g is Einstein. □

5. The minimization techniques

We have now enough material to perform some of the steps we mentioned in the strategy we would like to apply. Indeed, with Theorem 3.1, we are left with proving the existence of a minimizer. Actually the minimizers do not necessarily exist, and even solutions to (E). Let us recall that this is due to the lack of compactness of the embedding $H_2^2(M) \hookrightarrow L^{2^\sharp}(M)$. A possiblity to recover compactness is to use the best constants in Sobolev inequalities.

5.1. **The optimal Sobolev inequality.** Recall that it follows from (4) (see also Theorem 1.3) that there exists $A, B > 0$ such that

$$\|u\|_{2^\sharp}^2 \leq A\|\Delta_g u\|_2^2 + B\|u\|_{H_1^2(M)}^2 \tag{36}$$

for all $u \in H_2^2(M)$. We address here the question of the optimality of the different constants A and B. More precisely, we will be interested in taking A as small as possible.

5.1.1. *Preliminary discussion: the Euclidean setting.* In $\mathbb{R}^n$, there exists $A > 0$ such that

$$\|u|_{L^{2^\sharp}(\mathbb{R}^n)}^2 \leq A\int_{\mathbb{R}^n}(\Delta_\xi u)^2\,dx \tag{37}$$

for all $u \in C_c^\infty(\mathbb{R}^n)$, the set of smooth compactly supported functions in $\mathbb{R}^n$ and ξ is the Euclidean metric on $\mathbb{R}^n$. Define

$$\frac{1}{K_n} = \inf_{u \in D_2^2(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n}(\Delta_\xi u)^2\,dx}{\left(\int_{\mathbb{R}^n}|u|^{2^\sharp}\,dx\right)^{\frac{2}{2^\sharp}}} \tag{38}$$

Here,

$$D_2^2(\mathbb{R}^n) = \left\{ \text{Completion of } C_c^\infty(\mathbb{R}^n) \text{ for the norm } \|u\|_{D_2^2(\mathbb{R}^n)} := \|\Delta_\xi u\|_2 \right\} \tag{39}$$

It follows from Sobolev's theorem that the constant $K_n > 0$ is well-defined. It has been computed by Lieb [Lie], Lions [Lio], Edmunds-Fortunato-Jannelli [EFJ], Swanson [Swa] and we have that

$$\frac{1}{K_n} = \frac{n(n^2-4)(n-4)\omega_n^{\frac{4}{n}}}{16},$$

where ω_n is the volume of $(\mathbb{S}^n, h)$, the standard unit sphere of $\mathbb{R}^{n+1}$ endowed with its round metric. Moreover, the extremals for the optimal inequality (that is functions in $D_2^2(\mathbb{R}^n)$ that achieve the infimum in (38)) are known and are of the form

$$u_{\lambda,\mu,x_0}(x) = \mu \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n-4}{2}} \text{ for all } x \in \mathbb{R}^n, \tag{40}$$

where $\mu \neq 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ are arbitrary.

5.1.2. *Best first constant in the Riemannian setting.* Let us consider the Riemannian setting. The following optimal result concerning the best constant A is due to Djadli-Hebey-Ledoux [DHL] for the first part, and to [Heb3] for the second:

Theorem 5.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Then*

$$K_n = \inf\{A/\exists B \in \mathbb{R} \text{ such that (36) holds } \forall u \in H_2^2(M)\}. \tag{41}$$

Moreover, the infimum is achieved, that is there exists $B_0 > 0$ such that

$$\left(\int_M |u|^{2^\sharp}\, dv_g \right)^{\frac{2}{2^\sharp}} \leq K_n \int_M (\Delta_g u)^2\, dv_g + B_0 \int_M \left(|\nabla u|_g^2 + u^2 \right) dv_g \tag{42}$$

for all $u \in H_2^2(M)$.

Proof. We let A_0 be the right-hand-side of (41). The proof proceeds in three steps:

Step 1: We claim that $A_0 \geq K_n$. Let $A > A_0$. By the definition of A_0, we get that there exists $A < K_n$ and $B > 0$ such that (36) holds for all $u \in H_2^2(M)$. The idea is then to prove that in this situations, the optimal Sobolev inequality (37) holds with the constant A, which implies that $A \geq K_n$. let us prove this claim. let $x_0 \in M$ and consider a local chart of M around x_0, namely let $U \subset M$ an open subset such that $x_0 \in U$, let Ω be an open subset of $\mathbb{R}^n$ and $\varphi : U \to \Omega$ be a chart around x_0. Without loss of generality, we can assume that there exists $\delta \in (0, 1)$ such that

$$\varphi(x_0) = 0,\ \Omega = B_\delta(0) \text{ and } g_{ij}(x_0) = \delta_{ij} \text{for all } i, j \in \{1, ..., n\}$$

here, δ_{ij} denotes the Kronecker symbol. For instance, see (1)), one can take φ as the exponential map. From now on, with a standard abuse of notation, when $x \in B_\delta(0) \subset \mathbb{R}^n$, we define $g_{ij}(x)$ as $g_{ij}(\varphi^{-1}(x))$ (this last notion was defined in 1.1.2). In particular, with this convention, one has that

$$g_{ij}(0) = \delta_{ij} \text{ and } \Gamma_{ij}^k(0) = 0 \text{ for all } i, j, k \in \{1, ..., n\}. \tag{43}$$

Let $u \in C_c^\infty(\mathbb{R}^n)$. Let $R > 0$ such that $\operatorname{supp} u \subset B_{R/2}(0)$. Let $\epsilon \in (0, R^{-1}\delta)$, and consider

$$u_\epsilon(x) = \epsilon^{2-\frac{n}{2}} u \left(\frac{\varphi(x)}{\epsilon} \right) \text{ if } x \in U \text{ and } u_\epsilon(x) = 0 \text{ elsewhere.} \tag{44}$$

Clearly $u \in C^\infty(M)$ is well defined. With the Sobolev inequality (36), we get that

$$\left(\int_M |u_\epsilon|^{2^\sharp}\, dv_g \right)^{\frac{2}{2^\sharp}} \leq A \int_M (\Delta_g u_\epsilon)^2\, dv_g + B \int_M \left(|\nabla u_\epsilon|_g^2 + u_\epsilon^2 \right) dv_g \tag{45}$$

for all $\epsilon > 0$ small enough. We get that

$$\begin{aligned}
\int_M u_\epsilon^2\, dv_g &= \int_{B_\delta(0)} \epsilon^{4-n} u(\epsilon^{-1}x)^2 \sqrt{|g|}(x)\, dx = \epsilon^4 \int_{\mathbb{R}^n} u(x)^2 \sqrt{|g|}(\epsilon x)\, dx, \\
\int_M |\nabla u_\epsilon|_g^2\, dv_g &= \int_{B_\delta(0)} \epsilon^{4-n} g^{ij}(x)\partial_i u(\epsilon^{-1}x)\partial_j u(\epsilon^{-1}x) \sqrt{|g|}(x)\, dx \\
&= \epsilon^2 \int_{\mathbb{R}^n} g^{ij}(\epsilon x)\partial_i u(x)\partial_j u(x)\sqrt{|g|}(\epsilon x)\, dx \\
\int_M |u_\epsilon|^{2^\sharp}\, dv_g &= \int_{B_\delta(0)} \epsilon^{-n} u(\epsilon^{-1}x)^{2^\sharp} \sqrt{|g|}(x)\, dx = \int_{\mathbb{R}^n} u(x)^{2^\sharp}\sqrt{|g|}(\epsilon x)\, dx, \\
\int_M (\Delta_g u_\epsilon)^2\, dv_g &= \int_{B_\delta(0)} \epsilon^{4-n} \left(g^{ij}(x)\left(\partial_{ij}(u(\epsilon^{-1}x)) - \Gamma_{ij}^k(x)\partial_k(u(\epsilon^{-1}x))\right)\right)^2 \sqrt{|g|}(x)\, dx \\
&= \int_{\mathbb{R}^n} \left(g^{ij}(\epsilon x)\left(\partial_{ij}u(x) - \epsilon\Gamma_{ij}^k(\epsilon x)\partial_k u(x)\right)\right)^2 \sqrt{|g|}(\epsilon x)\, dx.
\end{aligned}$$

Letting $\epsilon \to 0$ and using (43), and plugging these terms in inequality (45), we get that

$$\left(\int_{\mathbb{R}^n} |u|^{2^\sharp}\, dx\right)^{\frac{2}{2^\sharp}} \leq A \int_{\mathbb{R}^n} (\Delta_\xi u)^2\, dx$$

for all $u \in C_c^\infty(\mathbb{R}^n)$. With the definition (39) of $D_2^2(\mathbb{R}^n)$ and the definition (38) of K_n, we get that $A \geq K_n$. Since this is valid for all $A > A_0$, one gets that $A \geq K_n$, and the claim is proved.

Step 2: We claim that $A_0 \leq K_n$. The proof uses the following idea: in the neighborhood of any point of M, we can choose a chart such that this neighborhood is isometric to an open subset of $\mathbb{R}^n$ with a metric "close" to the Euclidean metric. Using the optimal Sobolev inequality (38), we get that for any $\epsilon > 0$, there exists $B_\epsilon > 0$ such that (36) holds with $A = K_n + \epsilon$ for all smooth function with compact support in the considered neighborhood. With a finite covering of the manifold, one finally finds that there exists B'_ϵ such that

$$\|u\|_{2^\sharp}^2 \leq (K_n + \epsilon)\|\Delta_g u\|_2^2 + B'_\epsilon \|u\|_{H_1^2(M)}^2 \tag{46}$$

for all $u \in H_2^2(M)$, and then $A_0 \leq K_n$. We omit the proof and refer to [DHL]. The proof if detailed in Appendix 2.

Step 3: We claim that $A_0 = K_n$ is achieved. Actually this is the difficult part. The argument goes by contradiction, and we assume that the infimum $A_0 = K_n$ is not achieved. This is equivalent to say that

$$\inf_{u \in H_2^2(M)\setminus\{0\}} \frac{\int_M \left((\Delta_g u)^2 + \alpha|\nabla u|_g^2 + \frac{\alpha^2}{4}u^2\right) dv_g}{\left(\int_M |u|^{2^\sharp}\, dv_g\right)^{\frac{2}{2^\sharp}}} < \frac{1}{K_n}$$

for all $\alpha > 0$. Then, see Theorem 5.3 below, for any $\alpha > 0$, there exists $u_\alpha \in C^4(M)$, $u_\alpha > 0$ such that

$$\Delta_g^2 u_\alpha + \alpha\Delta_g u_\alpha + \frac{\alpha^2}{4}u_\alpha = \lambda_\alpha u_\alpha^{2^\sharp - 1} \text{ with } \int_M u_\alpha^{2^\sharp}\, dv_g = 1 \text{ and } \lambda_\alpha \in \left(0, \frac{1}{K_n}\right).$$

The proof is then a delicate description of the asymptotic behavior of u_α when $\alpha \to +\infty$. We refer to [Heb3] for the proof of this result. □

5.2. **The main result.**

Theorem 5.2. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 5$. Let A be a smooth symmetric $(2,0)-$tensor field and let $a, f \in C^\infty(M)$ such that $f > 0$. Let $P_g = \Delta_g^2 - div_g((\nabla \cdot)^\#) + a$ and assume that P_g is coercive. Assume that*

$$\inf_{u \in H_2^2(M)\setminus\{0\}} \frac{\int_M \left((\Delta_g u)^2 + A((\nabla u)^\sharp, (\nabla u)^\sharp) + au^2\right) dv_g}{\left(\int_M f|u|^{2^\sharp} dv_g\right)^{\frac{2}{2^\sharp}}} < \frac{1}{(\sup_M f)^{\frac{2}{2^\sharp}} K_n}. \quad (47)$$

Then there exists $u \in C^4(M)$ such that $u \neq 0$ and $P_g u = f|u|^{2^\sharp-2}u$. Moreover, u can be chosen as a minimizer in (47).

When requiring positive solutions, one has the following result:

Theorem 5.3. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 5$. Let $a, \alpha \in \mathbb{R}$ such that $a, \alpha > 0$ and $a \leq \frac{\alpha^2}{4}$. Let $f \in C^\infty(M)$ such that $f > 0$. Let $P_g = \Delta_g^2 + \alpha\Delta_g + a$. Assume that*

$$\inf_{u \in H_2^2(M)\setminus\{0\}} \frac{\int_M \left((\Delta_g u)^2 + \alpha|\nabla u|_g^2 + au^2\right) dv_g}{\left(\int_M f|u|^{2^\sharp} dv_g\right)^{\frac{2}{2^\sharp}}} < \frac{1}{(\sup_M f)^{\frac{2}{2^\sharp}} K_n}. \quad (48)$$

Then there exists $u \in C^4(M)$ such that $u > 0$ and $P_g u = fu^{2^\sharp-1}$. Moreover, u can be chosen as a minimizer in (48).

Proof of Theorem 5.3: As a preliminary remark, note that for any $u \in H_2^2(M)$, one has that

$$\int_M \left((\Delta_g u)^2 + \alpha|\nabla u|_g^2 + au^2\right) dv_g \geq a\|u\|_2,$$

and therefore P_g is coercive. Since (48) holds, we apply Theorem 5.2 and we get a function $u \in C^4(M)$ such that $P_g u = f|u|^{2^\sharp-2}u$ and u is a minimizer for (48). It follows from Proposition 4.6 that P_g verifies the hypothesis of Proposition 3.1, and then, with Proposition 3.1, $u > 0$ or $u < 0$. Up to multiplying by (-1), one gets that $u > 0$ and $P_g u = fu^{2^\sharp-1}$. □

Proof of Theorem 5.3: Since P_g is coercive, one gets that $\mu_{2^\sharp}(f) > 0$ (see (28) in the proof of Proposition 4.1). We let $(u_i)_{i\in\mathbb{N}} \in H_2^2(M) \setminus \{0\}$ be a minimizing sequence for $I_{2^\sharp,f}$, that is

$$\lim_{i\to+\infty} I_{q,f}(u_i) = \mu_q(f). \quad (49)$$

Without loss of generality, we can assume that

$$\int_M f|u_i|^{2^\sharp} dv_g = 1 \quad (50)$$

for all $i \in \mathbb{N}$. Mimicking what was done in the proof of Proposition 2.2, we get that there exists $u \in H_2^2(M)$ such that there exists a subsequence $(u_{i'})$ of (u_i) such that

$$u_{i'} \rightharpoonup u$$

weakly in $(H_2^2(M))'$ and strongly in $H_1^2(M)$. Letting $\theta_i = u_i - u \in H_2^2(M)$ for all $i \in \mathbb{N}$. We have that

$$\mu_{2^\sharp}(f) = \int_M uP_g u\, dv_g + \int_M (\Delta_g\theta_i)^2\, dv_g + o(1) \quad (51)$$

where $\lim_{i\to+\infty} o(1) = 0$. Without loss of generality, we can assume that $\lim_{i\to+\infty} \theta_i(x) = 0$ for a.e. $x \in M$. We claim that

$$\lim_{i\to+\infty} \int_M f|\theta_i|^{2^\sharp}\, dv_g = 1 - \int_M f|u|^{2^\sharp}\, dv_g. \tag{52}$$

We prove the claim. Indeed, it follows from the equality (16) taken with $q = 2^\sharp$ and $\theta = 2$ that there exists $C > 0$ such that

$$\begin{aligned}
&\left|\int_M f|u+\theta_i|^{2^\sharp}\, dv_g - \int_M f|u|^{2^\sharp}\, dv_g - \int_M f|\theta_i|^{2^\sharp}\, dv_g\right| \\
&\leq \|f\|_\infty \int_M \left||u+\theta_i|^{2^\sharp} - |u|^{2^\sharp} - |\theta_i|^{2^\sharp}\right|\, dv_g \\
&\leq C\|f\|_\infty \int_M \left(|u|^{2^\sharp-2}|\theta_i|^2 + |u|^2|\theta_i|^{2^\sharp-2}\right)\, dv_g.
\end{aligned} \tag{53}$$

We need the following useful lemma (a proof can be found in [Heb1]):

Lemma 5.1. *Let $(u_i)_{\in\mathbb{N}} \in L^p(M)$ such that $\|u_i\|_p \leq C$ for all $i \in \mathbb{N}$ and such that $\lim_{i\to+\infty} u_i(x) = 0$ a.e in M. Then for any $v \in L^{p'}(M)$, we have that $\lim_{i\to+\infty} u_i v\, dv_g = 0$, where $\frac{1}{p'} + \frac{1}{p} = 1$.*

Proof. With Hölder's inequality, we get that

$$\begin{aligned}
\left|\int_M u_i v\, dv_g\right| &\leq \int_M \mathbf{1}_{|u_i|\leq 1}|u_i v|\, dv_g + \int_M |u_i|\mathbf{1}_{|u_i|\geq 1}|v|\, dv_g \\
&\leq \int_M \mathbf{1}_{|u_i|\leq 1}|u_i v|\, dv_g + \|u_i\|_p \left(\int_M \mathbf{1}_{|u_i|\geq 1}|v|^{p'}\, dv_g\right)^{\frac{1}{p'}}.
\end{aligned}$$

We deal with the first integral of the right-hand-side. Since $\mathbf{1}_{|u_i|\leq 1}|u_i v| \leq |v| \in L^{p'}(M)$ and since $\lim_{i\to+\infty} u_i v = 0$ for a.e. $x \in M$, it follows from Lebesgue's theorem that the first integral of the right-hand-side goes to 0 when $i \to +\infty$. We deal with the second integral of the right-hand-side. Since $\mathbf{1}_{|u_i|\leq 1}|v|^{p'} \leq |v|^{p'} \in L^1(M)$ and since $\lim_{i\to+\infty} \mathbf{1}_{|u_i|\leq 1}|v|^{p'} = 0$ for a.e. $x \in M$, it follows from Lebesgue's theorem that the second integral of the right-hand-side goes to 0 when $i \to +\infty$. These two results prove the lemma. $\square$

It follows from (53) and Lemma 5.1 that

$$\lim_{i\to+\infty} \int_M f|u+\theta_i|^{2^\sharp}\, dv_g - \int_M f|u|^{2^\sharp}\, dv_g - \int_M f|\theta_i|^{2^\sharp}\, dv_g = 0, \tag{54}$$

and the claim is proved.

Let $\epsilon > 0$. With the Sobolev inequality (42) and the strong convergence of θ_i in $H_1^2(M)$, we get that

$$\begin{aligned}
\left(\int_M f|\theta_i|^{2^\sharp}\, dv_g\right)^{\frac{2}{2^\sharp}} &\leq \|f\|_\infty^{\frac{2}{2^\sharp}} \left((K_n+\epsilon)\int_M (\Delta_g\theta_i)^2\, dv_g + B_\epsilon\|\theta_i\|_{H_1^2}^2\right) \\
&\leq \|f\|_\infty^{\frac{2}{2^\sharp}} (K_n+\epsilon)\int_M (\Delta_g\theta_i)^2\, dv_g + o(1).
\end{aligned} \tag{55}$$

With the definition of $I_{2^\sharp,f}$, we get that

$$\int_M uP_g u\, dv_g \geq \mu_{2^\sharp}(f)\left(\int_M f|u|^{2^\sharp}\, dv_g\right)^{\frac{2}{2^\sharp}}. \tag{56}$$

Plugging (55) and (56) into (51) and using (50) and (54), we get that

$$\begin{aligned}\mu_{2^\sharp}(f) &\geq \mu_{2^\sharp}(f)\left(\int_M f|u|^{2^\sharp}\,dv_g\right)^{\frac{2}{2^\sharp}}+\left((K_n+\epsilon)\|f\|_\infty^{\frac{2}{2^\sharp}}\right)^{-1}\left(\int_M f|\theta_i|^{2^\sharp}\,dv_g\right)^{\frac{2}{2^\sharp}}\\ &\geq \mu_{2^\sharp}(f)\left(\int_M f|u|^{2^\sharp}\,dv_g\right)^{\frac{2}{2^\sharp}}\\ &\quad+\left((K_n+\epsilon)\|f\|_\infty^{\frac{2}{2^\sharp}}\right)^{-1}\left(1-\int_M f|u|^{2^\sharp}\,dv_g\right)^{\frac{2}{2^\sharp}}+o(1).\end{aligned}$$

letting $i\to+\infty$, this last inequality then yields

$$\mu_{2^\sharp}(f)(K_n+\epsilon)\|f\|_\infty^{\frac{2}{2^\sharp}}\left(1-\left(\int_M f|u|^{2^\sharp}\,dv_g\right)^{\frac{2}{2^\sharp}}\right)\geq\left(1-\int_M f|u|^{2^\sharp}\,dv_g\right)^{\frac{2}{2^\sharp}}.$$

Since $1-X^p\geq(1-X)^p$ for all $X\in[0,1]$ and all $p\geq 1$, we get that

$$\left(\mu_{2^\sharp}(f)(K_n+\epsilon)\|f\|_\infty^{\frac{2}{2^\sharp}}-1\right)\left(1-\left(\int_M f|u|^{2^\sharp}\,dv_g\right)^{\frac{2}{2^\sharp}}\right)\geq 0$$

Now, hypothesis (47) implies that for $\epsilon>0$ small enough, we have that $\int_M f|u|^{2^\sharp}\,dv_g\geq 1$, and then with (50) and (54), we get that

$$\int_M f|u|^{2^\sharp}\,dv_g=1.$$

As in the proof of Proposition (2.2), this last equality yields that $\lim_{i\to+\infty}\theta_i=0$ in $H_2^2(M)$ and that $u\in H_2^2(M)\setminus\{0\}$ is a minimizer for $I_{2^\sharp,f}$, and $I'_{2^\sharp,f}(u)=0$. With Proposition 3.1, we get that $u\in C^4(M)$ and that there exists $\lambda\in\mathbb{R}$ such that $P_g u=\lambda f|u|^{2^\sharp-2}u$. Multiplying by u and integrating, we get that $\lambda=I_{2^\sharp,f}(u)=\lambda_{2^\sharp}(f)$. Letting $\tilde{u}=\lambda_{2^\sharp}(f)^{\frac{n-4}{8}}$, we get that $\tilde{u}\not\equiv 0$ is a solution to $P_g\tilde{u}=f|\tilde{u}|^{2^\sharp-2}\tilde{u}$. $\square$

Exercise (Alternative proof): Assume here again that P_g is coercive. It follows from (2.2) that for any $q\in[2,2^\sharp)$, there is a minimizer $u_q\in C^4(M)$ for $I_{q,f}$ such that $\int_M f|u_q|^q\,dv_g=1$. Prove that under the assumptions of Theorem 5.2, we have that, up to a subsequence, $\lim_{q\to+\infty}u_q=u$ in $C^4(M)$, where $u\in C^4(M)\setminus\{0\}$ is a minimizer for $I_{2^\sharp,f}$.

5.3. **An important remark.** One is naturally interested in the validity of inequality (47). The following proposition actually says that it is "not far" from being true:

Proposition 5.1. *Let (M,g) be a compact Riemannian manifold of dimension $n\geq 5$ and A,a,f,P_g as in Theorem 5.2. Then*

$$\mu_{2^\sharp}(f)\leq\frac{1}{(\sup_M f)^{\frac{2}{2^\sharp}}K_n}. \tag{57}$$

Proof. Let $u\in C_c^\infty(\mathbb{R}^n)\setminus\{0\}$. Let $\epsilon>0$ small such that $u_\epsilon\in C^4(M)$ in (44) is well defined. With the computations provided in Step 1 of the proof of Theorem 41, we get that

$$\lim_{\epsilon\to 0}I_{2^\sharp,f}(u_\epsilon)=\frac{\int_{\mathbb{R}^n}(\Delta_\xi u)^2\,dx}{f(x_0)^{\frac{2}{2^\sharp}}\left(\int_{\mathbb{R}^n}|u|^{2^\sharp}\,dx\right)^{\frac{2}{2^\sharp}}}.$$

Since $\mu_{2^\sharp}(f) \leq I_{2^\sharp,f}(u_\epsilon)$, we get with the definition (38) of K_n that (57) holds. $\square$

5.4. **Some applications.** We are then left with finding situations in which the strict inequality (47) or the strict inequality (48) holds. The natural strategy is to evaluate the functional $I_{2^\sharp,f}$ at some good test-functions. Regarding to the role of the best constant in the Euclidean Sobolev, the good test-functions will be pull-back of the extremals for the Euclidean optimal inequality given by (40) via the exponential map. Let $x_0 \in M$ and define

$$u_\epsilon(x) := \eta(x)\left(\frac{\epsilon}{\epsilon^2 + d_g(x,x_0)^2}\right)^{\frac{n-4}{2}} = \eta(x)\epsilon^{-\frac{n-4}{2}} u_{1,1,0}\left(\frac{\exp_{x_0}^{-1}(x)}{\epsilon}\right) \tag{58}$$

for all $x \in M$. Here, $\eta \in C^\infty(M)$ is such that $\eta \equiv 1$ in $B_\delta(x_0)$ and $\eta \equiv 0$ in $M \setminus B_{2\delta}(x_0)$, where $\delta < \frac{i_g(M)}{2}$. With computations similar to Step 1 in the proof of Theorem 41, one gets that

$$\lim_{\epsilon\to 0} I_{2^\sharp,f}(u_\epsilon) = \frac{1}{f(x_0)^{\frac{2}{2^\sharp}} K_n},$$

which does not give the strict inequality we want. We then take x_0 such that $f(x_0) = \sup_M f$, calculate a Taylor expansion of $I_{2^\sharp,f}(u_\epsilon)$ to go below the critical level. These computations were done in [EsRo]. Letting

$$F(x_0) = 8(n-1)Tr_g A(x_0) + (n-6)(n+2)(n-4)\frac{\Delta_g f(x_0)}{f(x_0)} - 4(n^2-2n-4)R_g(x_0),$$

where $Tr_g A = g^{ij}A_{ij}$, we get that

$$I_{2^\sharp,f}(u_\epsilon) = \frac{1}{K_n f(x_0)^{\frac{2}{2^\sharp}}}\left(1 + \frac{1}{2n(n-6)(n^2-4)}F(x_0)\epsilon^2 + o(\epsilon^2)\right)$$

when $n \geq 7$ and

$$I_{2^\sharp,f}(u_\epsilon) = \frac{1}{K_n f(x_0)^{\frac{2}{2^\sharp}}}\left(1 + \frac{\omega_5}{360\omega_6}F(x_0)\epsilon^2|\ln\epsilon| + o(\epsilon^2|\ln\epsilon|)\right)$$

when $n = 6$. In particular, using Theorem 5.2, we get the following existence theorem:

Theorem 5.4. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 6$. Let A be a smooth symmetric $(2,0)-$tensor field and let $a, f \in C^\infty(M)$ such that $f > 0$. Assume that the operator $\Delta_g^2 - div_g(A(\nabla\cdot)^\#) + a$ is coercive and that there exists $x_0 \in M$ such that $f(x_0) = \sup_M f$ and*

$$8(n-1)Tr_g A(x_0) + (n-6)(n+2)(n-4)\frac{\Delta_g f(x_0)}{f(x_0)} - 4(n^2-2n-4)R_g(x_0) < 0.$$

Then there exists $u \in C^4(M)$ such that $u \not\equiv 0$ and $\Delta_g^2 u - div_g(A(\nabla u)^\#) + au = f|u|^{2^\sharp-2}u$.

Concerning positive solutions, we get the following corollary:

Theorem 5.5. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 6$. Let $\alpha, a \in \mathbb{R}$ such that $\alpha, a > 0$ and $a \leq \frac{\alpha^2}{4}$. Let $f \in C^\infty(M)$ such that $f > 0$. Assume that there exists $x_0 \in M$ such that $f(x_0) = \sup_M f$ and*

$$8n(n-1)\alpha + (n-6)(n+2)(n-4)\frac{\Delta_g f(x_0)}{f(x_0)} - 4(n^2-2n-4)R_g(x_0) < 0.$$

Then there exists $u \in C^4(M)$ *such that* $u > 0$ *and* $\Delta_g^2 u + \alpha\Delta_g u + au = fu^{2^\sharp-1}$.

Naturly, in case $F(x_0) = 0$ (which is the case when $P_g = P_g^n$, the geometric Paneitz-Branson operator), we must push the development further to obtain informations on $I_{2^\sharp,f}(u_\epsilon)$. We refer to [EsRo] for the calculations to the next order.

The test-functions u_ϵ constructed above concentrate at x_0: indeed, we have that $\lim_{\epsilon\to 0} u_\epsilon(x_0) = +\infty$ and $\lim_{\epsilon\to 0} u_\epsilon(x) = 0$ for all $x \in M \setminus \{x_0\}$. These test-functions are efficient in dimension $n \geq 6$, but not in dimension $n = 5$. Indeed, we say that the L^2-norm of the gradient concentrates at x_0 if

$$\lim_{\epsilon\to 0} \frac{\int_{B_\delta(x_0)} |\nabla u_\epsilon|_g^2\, dv_g}{\int_M u_\epsilon^2\, dv_g} = 1$$

for all $\delta > 0$. As easily checked, The L^2-norm of the gradient concentrates iff $n \geq 6$: this is why the choice of the cut-off function η was not very important. However, in dimension $n = 5$, the gradient does not concentrate, and we have to consider the behavior of u_ϵ on the whole manifold, and it is not possible to use any test-function η.

5.5. **Invariance under isometries.** We present here a result in dimension $n = 5$. It involves test-functions in its proof, and, as discussed above, they must be defined on the whole manifold. Since our initial test-functions above are the pull-back of functions on $\mathbb{R}^n$ via a chart, they are only defined locally on the manifold, and this is why we had to multiply by a cut-off function to define them everywhere. But on the standard sphere, there is a chart that covers all of the sphere but one point: this is how one can construct global test-functions on $\mathbb{S}^5$. In the sequel, we say that a function φ is an isometry of $\mathbb{S}^n$ is $\varphi^\star h = h$, where h is the round metric of $\mathbb{S}^n$. We say that a function f is $G-$invariant if $f \circ \sigma = f$ for all $\sigma \in G$. We have the following result, proved in [Rob]:

Theorem 5.6. *Let G be a compact subgroup of isometries of the $(\mathbb{S}^5, h)$, the standard $5-$sphere endowed with its round metric. Let $f \in C^\infty(\mathbb{S}^5)$ be a positive $G-$invariant function. Assume that G acts without fixed point. Then there exists $u \in C^\infty(M)$ such that $u > 0$ and $P_h^5 u = \frac{n-4}{2} fu^{2^\sharp-1}$. In particular, the metric $g := u^4 h$ verifies $Q_g^5 = f$.*

In the latest sections, we were mainly concerned with finding a way to make converge sequences in $H_2^2(M)$ that were not bound to converge due to the lack of compactness of the embedding $H_2^2(M) \hookrightarrow L^{2^\sharp}(M)$. In a sense, this is satisfactory because we could finally find solutions to an equation that did not necessary had one. In another sense, it is not satisfactory because we have avoided the generic situation, that is the lack of convergence. In the following two sections, we will tackle this question, which is a much more intricate problem.

6. General H_2^2-theory

6.1. **Palais-Smale sequences.** Let (M, g) be a smooth Riemannian manifold of dimension $n \geq 5$, and let A be a smooth symmetric $(2,0)-$tensor and $a \in C6\infty(M)$. Let $P_g = \Delta_g^2 - \mathrm{div}_g(A(\nabla\cdot)^\sharp) + a$. We consider the functional

$$J(u) = \frac{1}{2}\int_M uP_g u\, dv_g - \frac{1}{2^\sharp}\int_M |u|^{2^\sharp}\, dv_g$$

for all $u \in H_2^2(M)$. Here, $\int_M uP_g u\, dv_g$ is again defined in the distribution sense as in (13). The functional J is well defined thanks to the Sobolev embedding (4) (see also Theorem 1.3). As in Subsection 2.3, $J \in C^1(H_2^2(M))$ and

$$J'(u).\varphi = \int_M \left(\Delta_g u \Delta_g \varphi + A((\nabla u)^\sharp, (\nabla\varphi)^\sharp) + au\varphi\right)\, dv_g - \int_M |u|^{2^\sharp-2}u\varphi\, dv_g$$

for all $u, \varphi \in H_2^2(M)$. With the regularity result of Proposition 3.1, one gets that

$$\left(u \in H_2^2(M) \text{ and } J'(u) = 0\right) \Leftrightarrow \left(u \in C^4(M) \text{ and } P_g u = |u|^{2^\sharp-2}u\right).$$

A notion more general than the notion of critical point (that is $J'(u) = 0$) is the notion of Palais-Smale sequence:

Definition 6.1. *Let $(u_k)_{k\in\mathbb{N}} \in H_2^2(M)$. The sequence (u_k) is a Palais-Smale sequence for the functional J if*

(i) $J(u_k) = O(1)$ when $k \to +\infty$,

(ii) $\lim_{k\to+\infty} J'(u_k) = 0$ in $(H_2^2(M))'$.

In other words, there exists $C > 0$ and there exists $(\epsilon_k)_{k\in\mathbb{N}} \in \mathbb{R}$ such that $|J(u_k)| \leq C$ for all $k \in \mathbb{N}$ and $|J'(u_k), \varphi| \leq \epsilon_k \|\varphi\|_{H_2^2}$ for all $\varphi \in H_2^2(M)$ and $\lim_{k\to+\infty} \epsilon_k = 0$.

Palais-Smale sequences arise quite often in elliptic critical problems: the Mountain Pass Lemma of Ambrosetti and Rabinowitz [AmRa] naturally produces these sequences (see [EsRo] for an application to fourth order problems), and sequences of solutions to equation $P_g u = u^{2^\sharp-1}$ with uniformly bounded H_2^2-norm are Palais-Smale sequences for J. A simple behavior for a Palais-Smale sequence would be convergence in $H_2^2(M)$. Namely, does one have

$$(u_k)_{k\in\mathbb{N}} \text{ Palais-Smale sequence for } J \Rightarrow \lim_{k\to+\infty} u_k = u \text{ in } H_2^2(M)? \tag{59}$$

(at least up to a subsequence). Actually this is not true in general, and here again, this is due to the critical exponent and the lack of compactness of the embedding $H_2^2(M) \hookrightarrow L^{2^\sharp}(M)$.

6.2. **A non-converging Palais-Smale sequence.** Let us consider the following example. Let $(x_k)_{k\in\mathbb{N}} \in M$ be a converging sequence and let $(\mu_k)_{k\in\mathbb{N}} \in \mathbb{R}$ such that $\mu_k > 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to+\infty} \mu_k = 0$. Let $\delta \in (0, i_g(M)/2)$ and let $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta \equiv 1$ in $B_\delta(0)$ and $\eta \equiv 0$ in $\mathbb{R}^n \setminus B_{2\delta}(0)$. Define

$$u_{x_k,\mu_k}(x) := \alpha_n \eta_{x_k}(x) \left(\frac{\mu_k}{\mu_k^2 + d_g(x, x_k)^2}\right)^{\frac{n-4}{2}} \tag{60}$$

for all $x \in M$. Here, $\eta_{x_k} \in C^\infty(M)$ is defined by $\eta_{x_k} = \eta \circ \exp_{x_k}^{-1}$ and verifies then that $\eta_k \equiv 1$ in $B_\delta(x_k)$, $\eta_k \equiv 0$ in $M \setminus B_{2\delta}(x_k)$. The constant α_n is $\alpha_n = \left(n(n-4)(n^2-4)\right)^{\frac{n-4}{8}}$. Note that we have that

$$u_{x_k,\mu_k}(x) = \eta_{x_k}(x)\mu_k^{-\frac{n-4}{2}} u\left(\frac{\exp_{x_k}^{-1}(x)}{\mu_k}\right) \tag{61}$$

for all $x \in M$, where we have considered $\exp_x : B_{2\delta}(0) \to B_{2\delta}(x_k)$ as a chart defined on $B_{2\delta}(0)$ (the Euclidean ball of $\mathbb{R}^n$) and

$$u(x) = \alpha_n \left(\frac{1}{1+|x|^2}\right)^{\frac{n-4}{2}}$$

for all $x \in \mathbb{R}^n$. Note that $u \in D_2^2(\mathbb{R}^n)$ is an extremal for the Sobolev inequality (38) (see (40)) and that

$$\Delta_\xi^2 u = u^{2^\sharp - 1} \text{ in } \mathbb{R}^n. \tag{62}$$

Then we have that

Proposition 6.1. *Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$. Let $(x_k)_{k\in\mathbb{N}} \in M$ and $(\mu_k)_{k\in\mathbb{N}} \in \mathbb{R}_{>0}$ two converging sequences such that $\lim_{k\to+\infty} \mu_k = 0$. Then $(u_{x_k,\mu_k})_{k\in\mathbb{N}} \in C^\infty(M)$ is a Palais-Smale sequence for J. More precisely, we have that*

$$\begin{cases} (i) & \lim_{k\to+\infty} J(u_{x_k,\mu_k}) = \frac{2}{n} K_n^{-\frac{n}{4}}, \\ (ii) & \lim_{k\to+\infty} J'(u_{x_k,\mu_k}) = 0 \textit{ in } H_2^2(M)', \\ (iii) & u_{x_k,\mu_k} \rightharpoonup 0 \textit{ weakly in } H_2^2(M)' \textit{ when } k \to +\infty \end{cases}$$

In particular, $(u_k)_{k\in\mathbb{N}}$ does not converge strongly in $H_2^2(M)$, since otherwise, it would converge to zero (with (iii)), a contradiction with (i). We prove the Proposition. We let (x_k) and (μ_k) as in Proposition 6.1. For the sake of simplicity, we let $u_k := u_{x_k,\mu_k}$. In the sequel, we define the metric $g_k := \exp_{x_k}^\star g$, which is defined on $B_{2\delta}(0)$, via the usual assimilation of $T_{x_k}M$ to $\mathbb{R}^n$. This metric satisfies that

$$(g_k)_{ij}(0) = \delta_{ij} \text{ and } (\Gamma_k)_{ij}^p(0) = 0 \text{ for all } i, j, \in \{1, ..., n\}, \tag{63}$$

where we denote as $(\Gamma_k)_{ij}^p(x)$ the Christoffel symbols with index i, j, p associated to the metric g_k. In the sequel, we will often use that

$$d_g(x_k, \exp_{x_k}(x)) = |x|$$

for all $k \in \mathbb{N}$ and all $x \in B_{2\delta(0)}$. This assertion is a consequence of (2) and the isometric assimilation of $T_{x_k}M$ to $\mathbb{R}^n$ discussed in 1.1.4 before formula (1).

6.2.1. *Estimates of zeroth and first order.* We claim here that

$$\lim_{k\to+\infty} \|\nabla u_k\|_2 + \|u_k\|_2 = 0. \tag{64}$$

Indeed, We get that

$$\begin{aligned} \int_M u_k^2 \, dv_g &= \int_M \eta_{x_k}(x)^2 \left(\frac{\mu_k}{\mu_k^2 + d_g(x, x_k)^2} \right)^{n-4} dv_g \\ &\leq C \int_{B_{2\delta}(x_k)} \left(\frac{\mu_k}{\mu_k^2 + d_g(x, x_k)^2} \right)^{n-4} dv_g \\ &\leq C \int_{B_{2\delta}(0)} \left(\frac{\mu_k}{\mu_k^2 + |x|^2} \right)^{n-4} \sqrt{|g_k|(x)} \, dx \\ &\leq C\mu_k^4 \int_{B_{2\delta\mu_k^{-1}}(0)} \left(\frac{1}{1 + |x|^2} \right)^{n-4} \sqrt{|g_k|(\mu_k x)} \, dx = o(1) \end{aligned}$$

when $k \to +\infty$ (note that one must distinguish the case $n > 8$, the case $n = 8$ and the case $n < 8$). Similarly,

$$
\begin{aligned}
&\int_M |\nabla u_k|_g^2 \, dv_g \\
&= \int_{B_\delta(x_k)} |\nabla u_k|_g^2 \, dv_g + \int_{M \setminus B_\delta(x_k)} |\nabla u_k|_g^2 \, dv_g \\
&\leq \int_{B_\delta(0)} (g_k)^{ij}(x) \partial_i \left(\left(\frac{\mu_k}{\mu_k^2 + |x|^2} \right)^{\frac{n-4}{2}} \right) \partial_j \left(\left(\frac{\mu_k}{\mu_k^2 + |x|^2} \right)^{\frac{n-4}{2}} \right) \sqrt{|g_k|(x)} \, dx + C\mu_k^{n-4} \\
&\leq \mu_k^2 \int_{B_{\delta\mu_k^{-1}}(0)} (g_k)^{ij}(\mu_k x) \partial_i \left(\left(\frac{1}{1 + |x|^2} \right)^{\frac{n-4}{2}} \right) \partial_j \left(\left(\frac{1}{1 + |x|^2} \right)^{\frac{n-4}{2}} \right) \sqrt{|g_k|(\mu_k x)} \, dx + C\mu_k^{n-4} \\
&\leq C\mu_k^2 \int_{B_{\delta\mu_k^{-1}}(0)} \frac{dx}{1 + |x|^{2n-6}} + C\mu_k^{n-4} = o(1)
\end{aligned}
$$

when $k \to +\infty$. Here again, one must differentiate the case $n > 6$, the case $n = 6$ and the case $n < 6$. Then (64) is proved.

6.2.2. *Estimates of the $L^{2^\sharp}$−norm.* We claim that

$$
\lim_{R \to +\infty} \lim_{k \to +\infty} \int_{M \setminus B_{R\mu_k}(x_k)} u_k^{2^\sharp} \, dv_g = 0. \tag{65}
$$

Indeed, we have that

$$
\begin{aligned}
&\int_{M \setminus B_{R\mu_k}(x_k)} u_k^{2^\sharp} \, dv_g \\
&= \int_{B_\delta(x_k) \setminus B_{R\mu_k}(x_k)} \left(\frac{\mu_k}{\mu_k^2 + d_g(x, x_k)^2} \right)^n dv_g \\
&+ \int_{M \setminus B_\delta(x_k)} \eta_{x_k}(x)^{2^\sharp} \left(\frac{\mu_k}{\mu_k^2 + d_g(x, x_k)^2} \right)^n dv_g \\
&= \int_{B_\delta(0) \setminus B_{R\mu_k}(0)} \left(\frac{\mu_k}{\mu_k^2 + |x|^2} \right)^n \sqrt{|g_k|(x)} \, dx + O(\mu_k^n) \\
&= \int_{B_{\delta\mu_k^{-1}}(0) \setminus B_R(0)} \left(\frac{1}{1 + |x|^2} \right)^n \sqrt{|g_k|(\mu_k x)} \, dx + O(\mu_k^n) \\
&= \int_{\mathbb{R}^n \setminus B_R(0)} \left(\frac{1}{1 + |x|^2} \right)^n \sqrt{|g_k|(0)} \, dx + o(1) = \int_{\mathbb{R}^n \setminus B_R(0)} u^{2^\sharp} \, dx + o(1)
\end{aligned}
$$

when $k \to +\infty$. We have used (63). Then (65) is proved.

Concerning the behavior on $B_{R\mu_k}(x_k)$, we claim that

$$
\int_{B_{R\mu_k}(x_k)} u_k^{2^\sharp - 1} \psi \, dv_g = \int_{B_R(0)} u(x)^{2^\sharp - 1} \mu_k^{\frac{n-4}{2}} \psi(\exp_{x_k}(\mu_k x)) \, dx + o(1) \|u_k\|_{2^\sharp}^{2^\sharp - 1} \|\psi\|_{2^\sharp}, \tag{66}
$$

where $\lim_{k\to+\infty} o(1) = 0$ uniformly in $\psi \in C^\infty(M)$. We prove the claim:

$$\int_{B_{R\mu_k}(x_k)} u_k^{2^\sharp-1}\psi\, dv_g = \int_{B_{R\mu_k}(0)} u_k(\exp_{x_k}(x))^{2^\sharp-1}\psi(\exp_{x_k}(x))\sqrt{|g_k|(x)}\, dx$$
$$= \int_{B_{R\mu_k}(0)} u_k(\exp_{x_k}(x))^{2^\sharp-1}\psi(\exp_{x_k}(x))\, dx + o(1)\int_{B_{R\mu_k}(x_k)} u_k^{2^\sharp-1}\psi\, dv_g$$
$$= \int_{B_R(0)} u(x)^{2^\sharp-1}\mu_k^{\frac{n-4}{2}}\psi(\exp_{x_k}(\mu_k x))\, dx + o(1)\|u_k\|_{2^\sharp}^{2^\sharp-1}\|\psi\|_{2^\sharp}$$

when $k \to +\infty$ (we have used (63)), and then (66) is proved.

6.2.3. *Estimates of the second-order term.* We claim that

$$\lim_{R\to+\infty}\lim_{k\to+\infty}\|\Delta_g u_k\|_{L^2(M\setminus B_{R\mu_k}(x_k))} = 0. \tag{67}$$

$$\int_{M\setminus B_{R\mu_k}(x_k)} (\Delta_g u_k)^2\, dv_g = \int_{B_\delta(x_k)\setminus B_{R\mu_k}(x_k)} + \int_{M\setminus B_\delta(x_k)}$$
$$= \int_{B_\delta(0)\setminus B_{R\mu_k}(0)} \left((g_k)^{ij}(x)\left(\partial_{ij}\left(\frac{\mu_k}{\mu_k^2+|x|^2}\right)^{\frac{n-4}{2}} - (\Gamma_k)_{ij}^p\partial_p\left(\frac{\mu_k}{\mu_k^2+|x|^2}\right)^{\frac{n-4}{2}}\right)\right)^2\sqrt{|g_k|(x)}\, dx$$
$$+O(\mu_k^{n-4})$$
$$= \int_{B_{\delta\mu_k^{-1}}(0)\setminus B_R(0)} ((g_k)^{ij}(\mu_k x)\left(\partial_{ij}\left(\frac{1}{1+|x|^2}\right)^{\frac{n-4}{2}}\right.$$
$$\left.-(\Gamma_k)_{ij}^p(\mu_k x)\partial_p\left(\frac{1}{1+|x|^2}\right)^{\frac{n-4}{2}}\right))^2\sqrt{|g_k|(\mu_k x)}\, dx + o(1)$$
$$= \int_{\mathbb{R}^n\setminus B_R(0)} ((g_k)^{ij}(0)\partial_{ij}\left(\frac{1}{1+|x|^2}\right)^{\frac{n-4}{2}})^2\sqrt{|g_k|(0)}\, dx + o(1)$$
$$= \int_{\mathbb{R}^n\setminus B_R(0)} (\Delta_\xi u)^2\, dx + o(1)$$

when $k \to +\infty$ (here again, we have used (63) and the computations are different depending on the dimension greater or smaller to 6), and then (67) is proved.

We claim that for all $R > 0$ and all function $(G_k)_{k\in\mathbb{N}} \in C^\infty(M)$ such that $\lim_{k\to+\infty}\sup_{B_{R\mu_k}(0)}|G_k - 1| = 0$, there exists $(\epsilon_k(R))_{k\in\mathbb{N}} > 0$ such that we have that

$$\|(\Delta_g\psi)\circ\exp_{x_k}G_k - \Delta_\xi(\varphi\circ\exp_{x_k})\|_{L^2(B_{R\mu_k}(0))} \le \epsilon_k(R)\|\psi\|_{H_2^2} \tag{68}$$

for all $k \in \mathbb{N}$ and for all $\psi \in C^\infty(M)$, with $\lim_{k\to+\infty} \epsilon_k(R) = 0$. We prove the claim

$$\begin{aligned}
&\int_{B_{R\mu_k}} \left((\Delta_g\psi) \circ \exp_{x_k} G_k - \Delta_\xi(\varphi \circ \exp_{x_k})\right)^2 dx\\
&= \int_{B_{R\mu_k}(0)} \left(\left((g_k)^{ij} G_k - \delta^{ij}\right) \partial_{ij}(\psi \circ \exp_{x_k}) + G_k (\Gamma_k)_{ij}^p \partial_p(\varphi \circ \exp_{x_k})\right)^2 dx\\
&\leq C\mu_k \int_{B_{R\mu_k}(0)} |\nabla_\xi^2(\varphi \circ \exp_{x_k})|^2 + |\nabla_\xi(\varphi \circ \exp_{x_k})|^2 \, dx\\
&\leq C\mu_k \int_{B_{R\mu_k}(x_k)} |\nabla_g^2\varphi|_g^2 + |\nabla\varphi|_g^2 \, dv_g
\end{aligned}$$

here, we have used (63). This proves (68).

Exercise: prove the last inequality above..

We claim that

$$\int_{B_{R\mu_k}(x_k)} \Delta_g u_k \Delta_g \psi \, dv_g = \int_{B_R(0)} \Delta_\xi u \Delta_\xi (\mu_k^{\frac{n-4}{2}} \psi \circ \exp_{x_k}(\mu_k x)) \, dx + o(1) \|\psi\|_{H_2^2} \|u_k\|_{H_2^2} \tag{69}$$

where $\lim_{k\to+\infty} o(1) = 0$ uniformly for all $\psi \in C^\infty(M)$. We prove the claim. Using (68) alternatively with $G_k \equiv \sqrt{|g_k|}$ or $G_k \equiv 1$, we have that

$$\begin{aligned}
&\int_{B_{R\mu_k}(x_k)} \Delta_g u_k \Delta_g \psi \, dv_g = \int_{B_{R\mu_k}(0)} (\Delta_g u_k) \circ \exp_{x_k} (\Delta_g \psi) \circ \exp_{x_k} \sqrt{|g_k|} \, dx\\
&= \int_{B_{R\mu_k}(0)} \Delta_\xi (u_k \circ \exp_{x_k}) (\Delta_g \psi) \circ \exp_{x_k} \, dx + o(\|u_k\|_{H_2^2} \|\psi\|_{H_2^2})\\
&= \int_{B_{R\mu_k}(0)} \Delta_\xi (u_k \circ \exp_{x_k}) \Delta_\xi (\psi \circ \exp_{x_k}) \, dx + o(\|u_k\|_{H_2^2} \|\psi\|_{H_2^2})\\
&= \int_{B_R(0)} \Delta_\xi u \Delta_\xi (\mu_k^{\frac{n-4}{2}} \psi \circ \exp_{x_k}(\mu_k x)) \, dx + o(\|\psi\|_{H_2^2} \|u_k\|_{H_2^2})
\end{aligned}$$

and the claim is proved.

6.2.4. *Proof that $J(u_k)$ is bounded.* Taking $\psi \equiv u_k$ in (69) and (66) and using (67), (65) and (64), we get that

$$\lim_{k\to+\infty} \int_M (\Delta_g u_k)^2 \, dv_g = \int_{\mathbb{R}^n} (\Delta_\xi u)^2 \, dx \text{ and } \lim_{k\to+\infty} \int_M u_k^{2^\sharp} \, dv_g = \int_{\mathbb{R}^n} u^{2^\sharp} \, dx,$$

and $\|u_k\|_{H_2^2} = O(1)$ when $k \to +\infty$. Then, we get that

$$J(u_k) = \frac{1}{2} \int_{\mathbb{R}^n} (\Delta_\xi u)^2 \, dx - \frac{1}{2^\sharp} \int_{\mathbb{R}^n} |u|^{2^\sharp} \, dx + o(1)$$

where $\lim_{k\to+\infty} o(1) = 0$. We compute explicitely the right-hand-side. Let $R > 0$. Integrating by parts, we get that

$$\begin{aligned}
\int_{B_R(0)} (\Delta_\xi u)^2 \, dx &= \int_{B_R(0)} u \Delta_\xi^2 u \, dx + \int_{\partial B_R(0)} (u \partial_\nu \Delta_\xi u - \partial_\nu \Delta_\xi u) \, d\sigma\\
&= \int_{B_R(0)} u^{2^\sharp} \, dx + \int_{\partial B_R(0)} (u \partial_\nu \Delta_\xi u - \partial_\nu u \Delta_\xi u) \, d\sigma
\end{aligned}$$

Letting $R \to +\infty$ and using the explicit expression of u, on gets that $\int_{\mathbb{R}^n} (\Delta_\xi u)^2\, dx = \int_{\mathbb{R}^n} u^{2^\sharp}\, dx$. Since u is an extremal for the Sobolev inequality (38), one gets that $\int_{\mathbb{R}^n} u^{2^\sharp}\, dx = K_n^{-n/4}$, and then

$$\lim_{k\to+\infty} J(u_k) = \frac{2}{n} K_n^{-\frac{n}{4}}.$$

6.2.5. *Proof that $J'(u_k)$ goes to zero.* Let $\varphi \in C^\infty(M)$. Recall that we have that

$$J'(u_k).\varphi = \int_M \left(\Delta_g u_k \Delta_g \varphi + A((\nabla u_k)^\sharp, (\nabla\varphi)^\sharp) + a u_k \varphi\right)\, dv_g - \int_M u_k^{2^\sharp-1}\varphi\, dv_g.$$

With (64), we have that

$$J'(u_k).\varphi = \int_M \Delta_g u_k \Delta_g \varphi\, dv_g - \int_M u_k^{2^\sharp-1}\varphi\, dv_g + o(1)\|u_k\|_{H_1^2}\|\varphi\|_{H_2^2},$$

where $\lim_{k\to+\infty} o(1) = 0$ independantly of φ. Let $R > 0$. With (65), (66), (67) and (69), we get that

$$J'(u_k).\varphi = \int_{B_R(0)} \Delta_\xi u \Delta_\xi \varphi_k\, dx - \int_{B_R(0)} u^{2^\sharp-1}\varphi_k + \epsilon_k(R,\varphi)\|\psi\|_{H_2^2} + o(1)\|\psi\|_{H_2^2} \tag{70}$$

where $\lim_{k\to+\infty} \epsilon_k(R,\varphi) = 0$ uniformly in φ and

$$\varphi_k(x) = \eta(\mu_k x)\mu_k^{\frac{n-4}{2}}\psi \circ \exp_{x_k}(\mu_k x)$$

for all $x \in \mathbb{R}^n$ and k large enough. Note that $\varphi_k \in C_c^\infty(\mathbb{R}^n)$. We have that

$$\begin{aligned}\left|\int_{\mathbb{R}^n\setminus B_R(0)} \Delta_\xi u \Delta_\xi \varphi_k\, dx\right| &\leq \left(\int_{\mathbb{R}^n\setminus B_R(0)} (\Delta_\xi u)^2\, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} (\Delta_\xi \varphi_k)^2\, dx\right)^{\frac{1}{2}} \\ &\leq \epsilon(R)\|\varphi_k\|_{D_2^2(\mathbb{R}^n)} \end{aligned} \tag{71}$$

where $\lim_{R\to+\infty}\epsilon(R) = 0$ (we have used here that $\Delta_\xi u \in L^2(\mathbb{R}^n)$). Again with Hölder's inequality, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^n\setminus B_R(0)} u^{2^\sharp-1}\varphi_k\, dx\right| &\leq \left(\int_{\mathbb{R}^n\setminus B_R(0)} u^{2^\sharp}\, dx\right)^{\frac{2^\sharp-1}{2^\sharp}} \left(\int_{\mathbb{R}^n} \varphi_k^{2^\sharp}\, dx\right)^{\frac{1}{2^\sharp}} \\ &\leq \epsilon'(R)\|\varphi_k\|_{D_2^2(\mathbb{R}^n)} \end{aligned} \tag{72}$$

where $\lim_{R\to+\infty}\epsilon'(R) = 0$ (we have used here that $u \in L^{2^\sharp}(\mathbb{R}^n)$ and the Sobolev inequality (38)). Now, one easily gets that there exists $C > 0$ such that

$$\|\varphi_k\|_{D_2^2(\mathbb{R}^n)} \leq C\|\varphi\|_{H_2^2} \tag{73}$$

for all $k \in \mathbb{N}$ and all $\varphi \in C^\infty(M)$.

Exercise: prove (73).

Plugging (71) and (72) in (70), we get that

$$J'(u_k).\varphi = \int_{\mathbb{R}^n} \Delta_\xi u \Delta_\xi \varphi_k\, dx - \int_{\mathbb{R}^n} u^{2^\sharp-1}\varphi_k\, dx + \epsilon_k(R,\varphi)\|\varphi\|_{H_2^2} + o(1)\|\varphi\|_{H_2^2} \tag{74}$$

where $\lim_{R\to+\infty}\lim_{k\to+\infty}\epsilon_k(R,\varphi)=0$ uniformly in φ. Integrating by parts and using equation (62), one gets

$$\int_{\mathbb{R}^n}\Delta_\xi u\Delta_\xi\varphi_k\,dx=\int_{\mathbb{R}^n}\Delta_\xi^2 u\varphi_k\,dx=\int_{\mathbb{R}^n}u^{2^\sharp-1}\varphi_k\,dx,$$

for all $k\in\mathbb{N}$. Plugging this inequality in (74) and letting $R\to+\infty$, one gets that there exists $(\epsilon_k)_{k\in\mathbb{N}}>0$ such that

$$|J'(u_k).\varphi|\le\epsilon_k\|\varphi\|_{H_2^2}$$

for all $k\in\mathbb{N}$ and all $\varphi\in C^\infty(M)$, with $\lim_{k\to+\infty}\epsilon_k=0$. Then $J'(u_k)\to0$ in $H_2^2(M)'$ when $k\to+\infty$ (we have used here that $H_2^2(M)$ is the completion of $C^\infty(M)$ for $\|\cdot\|_{H_2^2}$).

6.2.6. *Conclusion and remark.* Let $\varphi\in C^\infty(M)$. We have that

$$\begin{aligned}\left|\int_M\Delta_g u_k\Delta_g\varphi\,dv_g\right|&\le\left|\int_{B_{R\mu_k}(x_k)}+\int_{M\setminus B_{R\mu_k}(x_k)}\right|\\&\le\|\Delta_g u_k\|_2\|\Delta_g\varphi\|_{L^2(B_{R\mu_k}(x_k))}+\|\Delta_g u_k\|_{L^2(M\setminus B_{R\mu_k}(x_k))}\|\Delta_g\varphi\|_2\\&\le C\|\Delta_g\varphi\|_{L^2(B_{R\mu_k}(x_k))}+C\|\Delta_g u_k\|_{L^2(M\setminus B_{R\mu_k}(x_k))}.\end{aligned}$$

Since $\varphi\in C^\infty(M)$, $\lim_{k\to+\infty}\mu_k=0$ and (67) holds, we get that

$$\lim_{k\to+\infty}\int_M\Delta_g u_k\Delta_g\varphi\,dv_g=0$$

for all $\varphi\in C^\infty(M)$. In particular, with (64), we get that $u_k\rightharpoonup0$ weakly in $H_2^2(M)$ when $k\to+\infty$. To conclude, we have constructed a sequence $(u_k)_{k\in\mathbb{N}}\in H_2^2(M)$ such that

$$\begin{cases}(i) & \lim_{k\to+\infty}J(u_k)=\frac{2}{n}K_n^{-\frac{n}{4}},\\(ii) & \lim_{k\to+\infty}J'(u_k)=0\text{ in }H_2^2(M)',\\(iii) & u_k\rightharpoonup0\text{ weakly in }H_2^2(M)'\text{ when }k\to+\infty\end{cases}$$

This proves Proposition 6.1.

This example shows that situations more subtle than (59) can happen. This is due to the critical exponent $2^\sharp$.

Exercice: Let $q\in[2,2^\sharp)$ and let the functional $J_q:u\mapsto\frac12\int_M uP_gu\,dv_g-\frac1q\int_M|u|^q\,dv_g$ for $u\in H_2^2(M)$. Show that every Palais-Smale sequence for J_q converges strongly in $H_2^2(M)$ up to the extraction of a subsequence. It is recommended to take inspiration from the proof ot Theorem 2.2.

Indeed, the lack of strong convergence of the Palais-Smale sequences for J can be described by the functions u_{x_k,μ_k} above. Following standard terminology, we denote the functions $(u_{x_k,\mu_k})_{k\in\mathbb{N}}$ as bubbles. The following theorem shows how fundamental they are for the description of Palais-Smale sequences.

6.3. **The main result.** The description of Palais-Smale sequences for critical functionals goes back to Sacks-Uhlenbeck [Sac] and to Wente [Wen]. A very beautiful and optimal description is due to Struwe [Str], where the Palais-Smale sequences for a critical functional associated to a second order elliptic operator on an open subset of $\mathbb{R}^n$ was provided. The result, due to Hebey and Robert [HeRo1] we present here is the extension of Struwe's result to the functional J, that is a functional associated to a fourth order operator P_g on a Riemannian manifold.

Theorem 6.1. *Let $(u_k)_{k\in\mathbb{N}} \in H_2^2(M)$ be a Palais-Smale sequence for J. We assume that $u_k \geq 0$ for all $k\in\mathbb{N}$. Then there exists $u_\infty \in H_2^2(M)$, there exists $N\in\mathbb{N}$, there exists N sequences of converging points $(x_{k,1})_{k\in\mathbb{N}}\in M$, ..., $(x_{k,N})_{k\in\mathbb{N}}\in M$, there exists N sequences of positive numbers $(\mu_{k,1})_{k\in\mathbb{N}}\in\mathbb{R}_{>0}$, ..., $(\mu_{k,N})_{k\in\mathbb{N}}\in\mathbb{R}_{>0}$ such that $\lim_{k\to+\infty}\mu_{k,i}=0$ for all $i\in\{1,...,N\}$ and such that*

$$u_k = u_\infty + \sum_{i=1}^N B_{k,i} + R_k,$$

where $\lim_{k\to+\infty} R_k = 0$ in $H_2^2(M)$ and $B_{k,i} := u_{x_{k,i},\mu_{k,i}}$ for all $i\in\{1,...,N\}$ and all $k\in\mathbb{N}$ are bubbles. Moreover, the energy splits, that is

$$J(u_k) = J(u_\infty) + \sum_{i=1}^N J(B_{k,i}) + o(1) = J(u_\infty) + \left(\frac{2}{n}K_n^{-\frac{n}{4}}\right)N + o(1)$$

where $\lim_{k\to+\infty} o(1) = 0$.

In other words, the lack of strong convergence of the sequence (u_k) to its weak limit u_∞ is entirely contained in the functions $(B_{k,i})_{k\in\mathbb{N}}$, and this lack of convergence is quantified: the difference between the energy of u_k and the energy of u_∞ is a multiple of a fixed threshold.

For the clarity of theses notes, we have taken $u_k \geq 0$. Actually, a similar decomposition holds with bubbles defined as in (61), where $u\in D_2^2(\mathbb{R}^n)\cap C^\infty(\mathbb{R}^n)$ is a solution of $\Delta_\xi^2 u = |u|^{2^\sharp-2}u$ on $\mathbb{R}^n$. We refer to [HeRo1] for this point.

In the following, we proove Theorem 6.1. Actually, we will not prove Step 4, which the most complicated step, and we refer to [HeRo1] for the details. From now on, we let $(u_k)_{k\in\mathbb{N}}\in H_2^2(M)$ be a Palais-Smale sequence for J. The idea of the proof is as follows: we first prove that the Palais-Smale sequence converges weakly, and we substract the weak limit to the sequence. We then obtain a new Palais-Smale sequence (for a new functional). If this new sequence carry enough energy, we substract a bubble and get another Palais-Smale sequence whose energy is lowered by a quantum. We do this process again, and it must finish since we substract a quantum at each step. Then the ultimate sequence has got enough small energy to converge strongly.

6.4. **Proof of Theorem 6.1: Step 1.** We claim that there exists $C>0$ such that

$$\|u_k\|_{H_2^2(M)} \leq C \tag{75}$$

for all $k\in\mathbb{N}$. We prove the claim. Coimputing $J'(u_k).u_k$ and using that (u_k) is a Palais-Smale sequence, we get that

$$\int_M u_k P_g u_k\, dv_g = \int_M |u_k|^{2^\sharp}\, dv_g + o(1)\|u_k\|_{H_2^2},$$

where $\lim_{k\to+\infty} o(1)=0$. Since $J(u_k)$ is bounded, we get that

$$\left\{\begin{array}{l} \int_M |u_k|^{2^\sharp}\, dv_g = O(1) + o(1)\|u_k\|_{H_2^2} \\ \int_M u_k P_g u_k\, dv_g = \int_M |u_k|^{2^\sharp}\, dv_g = O(1) + o(1)\|u_k\|_{H_2^2} \end{array}\right\} \tag{76}$$

With Hölder's inequality and the first equality on (76), we get that

$$\int_M u_k^2\, dv_g = O(1) + o(1)\|u_k\|_{H_2^2}. \tag{77}$$

The second equality of (76) yields that there exists $C>0$ such that

$$\int_M (\Delta_g u_k)^2\, dv_g \le C\int_M |\nabla u_k|_g^2\, dv_g + C\int_M u_k^2\, dv_g + O(1) + o(1)\|u_k\|_{H_2^2}.$$

Using inequality (2.2) with $\epsilon = (2C)^{-1}$, and (77), we get that

$$\int_M (\Delta_g u_k)^2\, dv_g \le O(1)\int_M u_k^2\, dv_g + O(1) + o(1)\|u_k\|_{H_2^2} \le O(1) + o(1)\|u_k\|_{H_2^2}$$

when $k\to+\infty$. Taking (2.2) with $\epsilon = 1$ and using (77), we get that

$$\|u_k\|_{H_2^2}^2 = O(1) + o(1)\|u_k\|_{H_2^2}$$

when $\epsilon\to 0$, which implies (75). This proves the claim.

It follows from the weak compactness of the unit ball of $H_2^2(M)$ (see Theorem 1.2) that, up to a subsequence, we can assume that there exists $u_\infty\in H_2^2(M)$ such that

$$u_k \rightharpoonup u_\infty \text{ weakly in } H_2^2(M) \text{ in } \lim_{k\to+\infty} u_k = u_\infty \text{ strongly in } H_1^2(M). \tag{78}$$

In addition, still up to a subsequence, we can assume that $\lim_{k\to+\infty} u_k(x) = u_\infty(x)$ for a.e. $x\in M$. Let $\varphi\in C^\infty(M)$. Since $J'(u_k).\varphi = o(1)$ when $k\to+\infty$, we have that

$$\int_M \left(\Delta_g u_k \Delta_g\varphi\, dv_g + A(\nabla u_k^\#, \nabla\varphi^\#) + a u_k\varphi - |u_k|^{2^\sharp-2}u_k\varphi\right) dv_g = o(1)$$

when $k\to+\infty$. Since u_k goes weakly to u_∞ when $k\to+\infty$, we get that

$$\int_M \left(\Delta_g u_\infty \Delta_g\varphi\, dv_g + A(\nabla u_\infty^\#, \nabla\varphi^\#) + a u_\infty\varphi - |u_\infty|^{2^\sharp-2}u_\infty\varphi\right) dv_g = 0$$

for all $\varphi\in C^\infty(M)$. It then follows that u_∞ is a weak solution to (E), and then, by Proposition (3.1), $u_\infty\in C^4(M)$ satisfies

$$\Delta_g^2 u_\infty - \mathrm{div}_g(A\nabla u_\infty^\#) + a u_\infty = |u_\infty|^{2^\sharp-2}u_\infty. \tag{79}$$

Note that since u_k goes to u_∞ almost everywhere when $k\to+\infty$ and since $u_k\ge 0$, we have that $u_\infty\ge 0$.

6.5. **Proof of Theorem 6.1: Step 2.** We let $v_k = u_k - u_\infty$, where $u_\infty\in H_2^2(M)$ is the weak limit of $(u_k)_{k\in\mathbb{N}}$ defined in (78). In particular, we get that

$$\lim_{k\to+\infty}\|v_k\|_{H_1^2} = 0. \tag{80}$$

We define $I: H_2^2(M)\to\mathbb{R}$ such that

$$I(u) = \frac{1}{2}\int_M (\Delta_g u)^2\, dv_g - \frac{1}{2^\sharp}\int_M |u|^{2^\sharp}\, dv_g$$

for all $u\in H_2^2(M)$. We claim that

$$\begin{cases} (i) & (v_k)_{k\in\mathbb{N}} \text{ is a Palais-Smale sequence for } I, \text{ and} \\ (ii) & \lim_{k\to+\infty} I(v_k) = J(u_k) - J(u_\infty). \end{cases}$$

We prove the claim. We have that

$$\begin{aligned} J(u_k) &= J(u_\infty + v_k) \\ &= \frac{1}{2}\int_M (\Delta_g u_\infty)^2\, dv_g + \int_M \Delta_g u_\infty \Delta_g v_k\, dv_g + \frac{1}{2}\int_M (\Delta_g v_k)^2\, dv_g \\ &\quad + \int_M A(\nabla u_k^\#, \nabla u_k^\#)\, dv_g - \frac{1}{2^\sharp}\int_M |u_\infty + v_k|^{2^\sharp}\, dv_g. \end{aligned}$$

Since v_k goes weakly to 0 in $H_2^2(M)$ and strongly in $H_1^2(M)$, we get that

$$J(u_k) = J(u_\infty) + I(v_k) - \frac{1}{2^\sharp}\int_M \left(|u_\infty + v_k|^{2^\sharp} - |u_\infty|^{2^\sharp} - |v_k|^{2^\sharp}\right) dv_g + o(1) \quad (81)$$

when $k \to +\infty$. Then, as in (54), we get that

$$\lim_{k\to+\infty} I(u_k) = J(u_k) - J(u_\infty). \quad (82)$$

Let $\varphi \in C^\infty(M)$. We have that

$$\begin{aligned} J'(u_k).\varphi &= J'(u_\infty + v_k).\varphi \\ &= J'(u_\infty).\varphi + I'(v_k).\varphi + \int_M A(\nabla v_k^\#, \nabla \varphi^\#)\, dv_g + \int_M a v_k \varphi\, dv_g \\ &\quad - \int_M \left(|u_\infty + v_k|^{2^\sharp-2}(u_\infty + v_k) - |u_\infty|^{2^\sharp-2}u_\infty - |v_k|^{2^\sharp-2}v_k\right)\varphi\, dv_g \end{aligned} \quad (83)$$

With (79), we get that $J'(u_\infty).\varphi = 0$. With Hölder's inequality, we get that

$$\left|\int_M A(\nabla v_k^\#, \nabla\varphi^\#)\, dv_g + \int_M a v_k \varphi\, dv_g\right| \leq C\|v_k\|_{H_1^2}\|\varphi\|_{H_1^2} \quad (84)$$

where $C > 0$ is independant of k and φ. The following inequality will be useful here: for any $q > 2$, there exists $C(q) > 0$ such that

$$\left||x+y|^{q-2}(x+y) - |x|^{q-2}x - |y|^{q-2}y\right| \leq C(q)(|x|^{q-2}|y| + |y|^{q-2}|x|) \text{ for all } x, y \in \mathbb{R}. \quad (85)$$

With (85), we then get that

$$\begin{aligned} &\left|\int_M \left(|u_\infty + v_k|^{2^\sharp-2}(u_\infty + v_k) - |u_\infty|^{2^\sharp-2}u_\infty - |v_k|^{2^\sharp-2}v_k\right)\varphi\, dv_g\right| \\ &\leq C\int_M \left(|u_\infty|^{2^\sharp-2}|v_k| + |v_k|^{2^\sharp-2}|u_\infty|\right)|\varphi|\, dv_g \\ &\leq \||u_\infty|^{2^\sharp-2}|v_k| + |v_k|^{2^\sharp-2}|u_\infty|\|_{\frac{2^\sharp}{2^\sharp-1}}\|\varphi\|_{2^\sharp}. \end{aligned}$$

Since $u_\infty \in C^4(M)$, it is bounded in L^∞ and then

$$\begin{aligned} &\left|\int_M \left(|u_\infty + v_k|^{2^\sharp-2}(u_\infty + v_k) - |u_\infty|^{2^\sharp-2}u_\infty - |v_k|^{2^\sharp-2}v_k\right)\varphi\, dv_g\right| \\ &\leq C\left(\|v_k\|_{\frac{2^\sharp}{2^\sharp-1}} + \||v_k|^{2^\sharp-2}\|_{\frac{2^\sharp}{2^\sharp-1}}\right)\|\varphi\|_{2^\sharp} \end{aligned} \quad (86)$$

Pluging together (84) and (86) in (83) and using the Sobolev inequality (4), we get that

$$|J'(u_k).\varphi - I'(v_k).\varphi| \leq C\left(\|v_k\|_{H_1^2} + \|v_k\|_{\frac{2^\sharp}{2^\sharp-1}} + \||v_k|^{2^\sharp-2}\|_{\frac{2^\sharp}{2^\sharp-1}}\right)\|\varphi\|_{H_2^2} \quad (87)$$

for all $k \in \mathbb{N}$. Since $v_k \rightharpoonup 0$ weakly in $H_2^2(M)$ when $k \to +\infty$ and since the embeddings $H_2^2(M) \hookrightarrow L^q(M)$ is compact for $1 \leq q < 2^\sharp$, and also with (80), we get that

$$\lim_{k\to+\infty} \left(\|v_k\|_{H_1^2} + \|v_k\|_{\frac{2^\sharp}{2^\sharp-1}} + \||v_k|^{2^\sharp-2}\|_{\frac{2^\sharp}{2^\sharp-1}} \right) = 0.$$

Then, since (u_k) is a Palais-Smale sequence for J, we get with (87) that $(v_k)_{k\in\mathbb{N}}$ is a Palais-Smale sequence for I. This last assertion and (82) prove the claim.

6.6. **Proof of Theorem 6.1: Step 3.** We let

$$\beta := \lim_{k\to+\infty} I(v_k). \tag{88}$$

We claim that

$$\beta < \frac{2}{n} K_n^{-\frac{n}{4}} \Longrightarrow \lim_{k\to+\infty} \|v_k\|_{H_2^2(M)} = 0 \text{ and } \beta = 0.$$

We prove the claim. Definition (88) rewrites

$$\frac{1}{2}\int_M (\Delta_g v_k)^2 \, dv_g - \frac{1}{2^\sharp}\int_M |v_k|^{2^\sharp} \, dv_g = \beta + o(1) \tag{89}$$

when $k \to +\infty$. Since $(v_k)_{k\in\mathbb{N}}$ is a Palais-Smale sequence for I, computing $I'(v_k).v_k$, we get that

$$\int_M (\Delta_g v_k)^2 \, dv_g = \int_M |v_k|^{2^\sharp} \, dv_g + o(1) \tag{90}$$

when $k \to +\infty$. Equations (89) and (90) yield

$$\int_M (\Delta_g v_k)^2 \, dv_g = \int_M |v_k|^{2^\sharp} \, dv_g + o(1) = \frac{n\beta}{2} + o(1) \tag{91}$$

when $k \to +\infty$. The Sobolev inequality (42) and (80) yield

$$\left(\int_M |v_k|^{2^\sharp} \, dv_g \right)^{\frac{2}{2^\sharp}} \leq K_n \int_M (\Delta_g u_k)^2 \, dv_g + o(1)$$

when $k \to +\infty$. Plugging (91) in this inequality, and letting $k \to +\infty$, we get that

$$\left(\frac{n\beta}{2} \right)^{\frac{2}{2^\sharp}} \leq K_n \frac{n\beta}{2}.$$

This equality and $\frac{n\beta}{2} < K_n^{-\frac{n}{4}}$ yield $\beta = 0$. With (91) and (80), we get that $\lim_{k\to+\infty} \|v_k\|_{H_2^2} = 0$, and the claim is proved.

Exercice: Prove the claim using only (46) instead of (42).

6.7. **Proof of Theorem 6.1: Step 4.** We assume that $\beta > 0$. Then there exists a converging sequence $(x_k)_{k\in\mathbb{N}} \in M$, there exists a sequence $(\mu_k)_{k\in\mathbb{N}} \in \mathbb{R}_{>0}$ such that $\lim_{k\to+\infty} = 0$ such that, letting $w_k := v_k - u_{x_k,\mu_k}$, we have that

$$\begin{cases} (i) & (w_k)_{k\in\mathbb{N}} \text{ is a Palais-Smale sequence for } I, \\ (ii) & I(w_k) = I(v_k) - \frac{2}{n} K_n^{-\frac{n}{4}} + o(1) \text{ when } k \to +\infty \\ (iii) & w_k \rightharpoonup 0 \text{ weakly in } H_2^2(M) \text{ when } k \to +\infty. \end{cases}$$

This point is actually the crucial point: it says that with a certain amount of energy, one can substract a bubble without changing the nature of the sequence. We refer to [HeRo1] for the proof of this result.

6.8. **Proof of Theorem 6.1: Step 5.** The conclusion of the proof goes through an induction argument. Given $p \in \mathbb{N}^\star$, We say that $\mathcal{H}_p$ holds if for any $(v_k)_{k\in\mathbb{N}}$ Palais-Smale sequence for I such that (80) hold and

$$\limsup_{k\to+\infty} I(v_k) < p \cdot \frac{2}{n} K_n^{-\frac{n}{4}}$$

for all $k \in \mathbb{N}$, then there exists $N \in \mathbb{N}$ bubbles $(B_{k,1})_{k\in\mathbb{N}}$, ..., $(B_{k,N})_{k\in\mathbb{N}}$ such that

$$v_k = \sum_{i=1}^{N} B_{k,i} + R_k,$$

where $\lim_{k\to+\infty} R_k = 0$ in $H_2^2(M)$. Moreover, the energy splits, that is

$$I(v_k) = \sum_{i=1}^{N} I(B_{k,i}) + o(1) = \left(\frac{2}{n} K_n^{-\frac{n}{4}}\right) N + o(1)$$

where $\lim_{k\to+\infty} o(1) = 0$. We claim that $\mathcal{H}_p$ holds for all $p \in \mathbb{N}$. Step 3 yields that $\mathcal{H}_1$ holds. We let $p \in \mathbb{N}^\star$ such that $\mathcal{H}_p$ holds, and we let $(v_k)_{k\in\mathbb{N}}$ a Palais-Smale sequence for I such that $\limsup_{k\to+\infty} I(v_k) < (p+1) \cdot \frac{2}{n} K_n^{-\frac{n}{4}}$ for all $k \in \mathbb{N}$. If $v_k \to 0$ strongly in $H_2^2(M)$, then we are done. Otherwise, it follows from Step 4 that there exists a bubble $(B_{k,1})_{k\in\mathbb{N}}$ such that $w_k := v_k - B_{1,k}$ is a Palais-Smale sequence for I such that $I(w_k) = I(v_k) - \frac{2}{n} K_n^{-\frac{n}{4}} + o(1)$ when $k \to +\infty$, and then $\limsup_{k\to+\infty} I(w_k) < p \cdot \frac{2}{n} K_n^{-\frac{n}{4}}$. Since $\mathcal{H}_p$ holds, we get that there exists N bubbles $(B_{k,2})_{k\in\mathbb{N}}$, ..., $(B_{k,N+1})_{k\in\mathbb{N}}$ such that

$$w_k = \sum_{i=2}^{N+1} B_{k,i} + R_k,$$

where $\lim_{k\to+\infty} R_k = 0$ in $H_2^2(M)$, and such that

$$I(w_k) = \sum_{i=2}^{N} I(B_{k,i}) + o(1) = \left(\frac{2}{n} K_n^{-\frac{n}{4}}\right) N + o(1).$$

Coming back to v_k, we get that $\mathcal{H}_{p+1}$ holds. Theorem 6.1 follows from Step 2 and Step 5.

7. Appendix: Proof of Theorem 3.2

We let A a smooth symmetric $(2,0)-$tensor on M and $a \in C^\infty(M)$. We assume that the operator $P_g = \Delta_g^2 - \mathrm{div}_g(A(\nabla\cdot)^\#) + a$ is coercive, that is there exists $\lambda > 0$ such that

$$\int_M u P_g u \, dv_g \geq \lambda \int_M u^2 \, dv_g \tag{92}$$

for all $u \in H_2^2(M)$.

7.1. **Step 1.** Let $p > 1$. We claim that there exists $c > 0$ such that

$$\|u\|_p \leq c \|P_g u\|_p \tag{93}$$

for all $u \in H_4^p(M)$. We prove the claim by contradiction and assume that for all $i \in \mathbb{N}^\star$, there exists $u_i \in H_4^p(M)$ such that $\|u_i\|_p = 1$ and $\|P_g u_i\|_p \leq i^{-1}$. It follows from Theorem 1.7 that there exists $C > 0$ such that $\|u_i\|_{H_4^p} \leq C$. Since the embedding $H_4^p(M) \hookrightarrow H_2^p(M)$ is compact (see [Ada]), there exists a subsequence

$(u_{i'})$ of (u_i) such that $\lim_{i\to+\infty} u_{i'} = u$ strongly in $H_2^p(M)$. We let $f_i := P_g u_i$. For any $\varphi \in C^\infty(M)$, we have that

$$\int_M (\Delta_g u_{i'} \Delta_g \varphi + A((\nabla u_{i'})^\#, (\nabla\varphi)^\#) + a u_{i'}\varphi)\, dv_g = \int_M f_{i'}\varphi\, dv_g.$$

Letting $i \to +\infty$, we find that $P_g u = 0$ in the weak sense. It then follows from Theorem 1.8 that $u \in C^4(M)$. With (92), we get that $u \equiv 0$. A contradiction since $\|u\|_p = \lim_{i\to+\infty} \|u_{i'}\|_p = 1$. This proves the claim.

7.2. **Step 2.** Let $\alpha \in (0,1)$. We claim that for any $f \in C^{0,\alpha}(M)$, there exists $u \in C^4(M)$ such that $P_g u = f$. We prove the claim. We let the functional

$$\mathcal{F}(u) = \frac{1}{2}\int_M u P_g u\, dv_g - \int_M f u\, dv_g$$

for all $u \in H_2^2(M)$. Since P_g is coercive, we get that

$$\mathcal{F}(u) \ge \lambda \|u\|_2^2 - \|f\|_2 \|u\|_2 \ge -\frac{\|f\|_2^2}{4\lambda}. \tag{94}$$

Then $\mu := \inf\{\mathcal{F}(u) / u \in H_2^2(M)\} > -\infty$ is defined. Let $(u_i) \in H_2^2(M)$ be a minimizing sequence for μ, that is

$$\lim_{i\to+\infty} \mathcal{F}(u_i) = \mu. \tag{95}$$

With the first inequality of (94), we get that $\|u_i\|_2 \le C$ for all $i \in \mathbb{N}$. With (95) and Exercise [COER], we then get that $\|u_i\|_{H_2^2} = O(1)$ when $i \to +\infty$. It follows from Theorem 1.2 that there exists a subsequence $(u_{i'}) \in H_2^2(M)$ and there exists $u \in H_2^2(M)$ such that $u_{i'} \rightharpoonup u$ weakly in $H_2^2(M)$ when $i \to +\infty$. Up to extracting another subsequence, it follows from Theorem 1.6 that $\lim_{i\to+\infty} u_{i'} = u$ strongly in $H_1^2(M)$. We then get through easy calculations that

$$F(u_i) = \mathcal{F}(u) + \frac{1}{2}\int_M (\Delta_g(u_{i'} - u))^2\, dv_g + o(1) = \mu + o(1)$$

when $i \to +\infty$. Since μ is the infimum, we get that $\mu \le \mathcal{F}(u)$ and then

$$\lim_{i\to+\infty} \int_M (\Delta_g(u_{i'} - u))^2\, dv_g = 0,$$

and then $\mu = \mathcal{F}(u)$. Clearly $\mathcal{F} \in C^1(H_2^2(M), \mathbb{R})$, and then $\mathcal{F}'(u) = 0$, that is $P_g u = f$ in the weak sense. It then follows from Theorem 1.8 that $u \in C^4(M)$, and the claim is proved.

7.3. **Step 3.** Let $p > 1$. We claim that for any $f \in L^p(M)$, there exists a unique $u \in H_4^p(M)$ such that $P_g u = f$. We prove the claim. Let $(f_i)_{i\in\mathbb{N}} \in C^\infty(M)$ such that $\lim_{i\to+\infty} f_i = f$ strongly in $L^p(M)$. For any $i \in \mathbb{N}$, let $u_i \in C^4(M)$ such that $P_g u_i = f_i$ (this is s consequence of Step 2). With Theorem 1.7 and the coercivity of P_g, we get that for any $i, j \in \mathbb{N}$

$$\|u_i - u_j\|_{H_4^p(M)} \le C \cdot \left(\|f_i - f_j\|_{L^p(M)} + \|u_i - u_j\|_p\right) \le (1+c)C\|f_i - f_j\|_{L^p(M)}.$$

Then (u_i) is a Cauchy sequence for $H_4^p(M)$, and then there exists $u \in H_4^p(M)$ such that $\lim_{i\to+\infty} u_i = u$ in $H_4^p(M)$. Clearly we have that $P_g u = f$. Assume that $v \in H_4^p(M)$ satisfies $P_g v = f$, then $P_g(u - v) = 0$, and it follows from (92) that $u \equiv v$. This proves the claim.

The existence part of Theorem 3.2 is proved in Step 3 above. The apriori estimate of Theorem 3.2 is a consequence of (93) and Theorem 1.7.

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