ON THE EQUIVALENCE OF THE KAZDAN-WARNER AND THE POHOZÃEV IDENTITIES

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ABSTRACT. In this very short note, we enlighten a strong relation between the Kazdan-Warner identity on the standard sphere and the Pohozãev identity on the Euclidian space. As far as we know, such a relation has never been explicitly stated.

Let Ω be a smooth bounded open subset of $\mathbf{R}^n$ with $n \geq 3$ and let $\tilde{f} \in C^\infty(\overline{\Omega})$. If $\tilde{v} \in C^\infty(\overline{\Omega})$, $\tilde{v} > 0$ verifies

$$\Delta_\xi v = \tilde{f} v^{2^*-1} \qquad \text{in } \Omega \tag{1}$$

where ξ is the Euclidean metric, $\Delta_\xi v = -\partial_i^i v$ denotes the Euclidean Laplacian with the minus sign convention and $2^* = \frac{2n}{n-2}$, the Pohozãev identity [2] asserts that

$$\begin{aligned} \frac{n-2}{2n} \int_\Omega (x, \nabla \tilde{f})_\xi v^{2^*} dv_\xi \quad &= \quad \int_{\partial\Omega} (x, \nu)_\xi \left(\left(\frac{\partial v}{\partial \nu} \right)^2 - \frac{|\nabla v|_\xi^2}{2} + \frac{n-2}{2n} \tilde{f} v^{2^*} \right) d\sigma_\xi \\ &\quad + \frac{n-2}{2} \int_{\partial\Omega} v \frac{\partial v}{\partial \nu} d\sigma_\xi \end{aligned} \tag{P}$$

where ν denotes the outer normal vector of $\partial\Omega$. Independently, let (S^n, h) be the standard unit sphere of $\mathbf{R}^{n+1}$ and let $f \in C^\infty(S^n)$. If $u \in C^\infty(S^n)$, $u > 0$ verifies

$$\Delta_h u + \frac{n(n-2)}{4} u = f u^{2^*-1} \tag{2}$$

where $\Delta_h u = -div_h(\nabla u)$, the Kazdan-Warner identity [1] asserts that for any $\psi \in C^\infty(S^n)$ a first eigenfunction of Δ_h,

$$\int_{S^n} (\nabla f, \nabla \psi)_h \, u^{2^*} dv_h = 0 \tag{KW}$$

As it is well known, the first eigenvalue of Δ_h is $\lambda_1 = n$ and any eigenfunction associated to λ_1 is, up to a constant scale factor, of the form $\psi = (x_0, x)$ where $x_0 \in S^n$ and (x_0, x) denotes the scalar product in $\mathbf{R}^{n+1}$.

We prove here that (KW) is strictly equivalent to the limit of (P) as $\Omega \to \mathbf{R}^n$. For that purpose, we let as above f and $u \in C^\infty(S^n)$ verifying (2). We let also $x_0 \in S^n$ and $\psi(x) = (x_0, x)$. Then, by (KW),

$$\int_{S^n} (\nabla f, \nabla \psi)_h \, u^{2^*} dv_h = 0$$

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Let now $\pi : S^n \to \mathbf{R}^n$ be the stereographic projection of north pole x_0. We set $\tilde{f} = f \circ \pi^{-1}$, $\tilde{u} = u \circ \pi^{-1}$ and $\tilde{\psi} = \psi \circ \pi^{-1}$. Since $\left(\pi^{-1}\right)^* h = 4\left(1+|x|^2\right)^{-2}\xi$, we get

$$\int_{\mathbf{R}^n} (\nabla \tilde{f}, \nabla \tilde{\psi})_\xi \left(1+|x|^2\right)^{2-n} \tilde{u}^{2^*} dv_\xi = 0$$

A simple computation gives

$$\tilde{\psi}(x) = \frac{|x|^2 - 1}{1+|x|^2}$$

so that

$$\nabla \tilde{\psi}(x) = 4\left(1+|x|^2\right)^{-2} x$$

The identity (KW) then becomes

$$\int_{\mathbf{R}^n} (\nabla \tilde{f}, x)_\xi \left(1+|x|^2\right)^{-n} \tilde{u}^{2^*} dv_\xi = 0 \qquad (KWP)$$

We claim now that (KWP) is the limit of the Pohozãev identity (P) as we let Ω go to $\mathbf{R}^n$. Indeed, since u and f verify (2), if we set

$$v = \left(\frac{2}{1+|x|^2}\right)^{\frac{n}{2}-1} \tilde{u}$$

we have that v and $\tilde{f}$ verifies (1). So, by (P), for any $R > 0$,

$$\begin{aligned} \frac{n-2}{2n}\int_{B(0,R)} (x, \nabla \tilde{f})_\xi v^{2^*} dv_\xi &= \int_{\partial B(0,R)} (x,\nu)_\xi \left(\left(\frac{\partial v}{\partial \nu}\right)^2 - \frac{|\nabla v|_\xi^2}{2} + \frac{n-2}{2n}\tilde{f} v^{2^*}\right) d\sigma_\xi \\ &+ \frac{n-2}{2}\int_{\partial B(0,R)} v\frac{\partial v}{\partial \nu} d\sigma_\xi \end{aligned}$$

Now, since

$$|v(x)| \le C|x|^{2-n} \quad \text{and} \quad |\nabla v(x)| \le C|x|^{1-n}$$

the right-hand side term above goes to 0 and the left-hand side term is absolutely convergent as we let R go to $+\infty$ so that we obtain

$$\int_{\mathbf{R}^n} (\nabla \tilde{f}, x)_\xi v^{2^*} dv_\xi = 0$$

which is exactly (KWP). The above claim is proved.

References

[1] Jerry L. Kazdan and F. W. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, J. Differential Geometry **10** (1975), 113–134.

[2] S. I. Pohožaev, *On the eigenfunctions of the equation* $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR **165** (1965), 36–39.

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