Eigenvalue Estimate for the basic Laplacian on manifolds with foliated boundary

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Abstract

On a compact Riemannian manifold whose boundary is endowed with a Riemannian flow, we give a sharp lower bound for the first eigenvalue of the basic Laplacian acting on basic 1-forms. The equality case gives rise to a particular geometry of the flow and of the boundary. Namely, we prove that the flow is a local product and the boundary is η-umbilical. This allows to characterize the quotient of $\mathbb{R}\times B'$ by some group Γ as the limiting manifold. Here B' denotes the unit closed ball. Finally, we deduce several rigidity results describing the product $\mathbb{S}^1\times\mathbb{S}^n$ as the boundary of a manifold.

Key words: Riemannian flow, manifolds with boundary, basic Laplacian, eigenvalue, second fundamental form, O'Neill tensor, basic Killing forms, rigidity results.

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1 Introduction

The Reilly formula on compact manifolds with smooth boundary has been used to estimate the eigenvalues of the Laplacian acting on functions. The aim is to state rigidity results that arise from the optimality of those estimates [3, 18, 27]. For example, C. Xia [27] gave a lower bound for the first eigenvalue in terms of the lowest bound of the principal curvatures, assumed to be positive. The equality case is mainly characterized by the closed ball. As a direct application, he proved the following rigidity result (see [27, Thm. 2]): *Assume that on a compact domain N with a non-negative Ricci curvature, the boundary M is isometric to the round sphere with a non-negative mean curvature and that the sectional curvature of N vanishes along planes tangent to M, the domain is then isometric to the unit closed ball.* This result answered partially a question proposed by Schroeder and Strake in [23, Thm. 1].

In [16], S. Raulot and A. Savo generalized the Reilly formula to differential p-forms on M (see Equation (6)) by integrating the Bochner formula and using the Stokes' theorem. Among the terms in the generalized Reilly formula, an extension of the second fundamental form S to differential p-forms is defined in a canonical way so that its eigenvalues, called the p-curvatures, depend mainly on the principal curvatures (see Section 2 for the definition). Therefore, with the use of the Reilly formula, they obtained a lower bound for the first eigenvalue $\lambda'_{1,p}$ of the Laplacian acting on exact

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p-forms in terms of those p-curvatures. In fact, they proved that on a compact Riemannian manifold (N^{n+2}, g) with non-negative curvature operator such that the p-curvatures of its boundary M are bounded from below by $\sigma_p(M) > 0$, we have for any p, $1 \leq p \leq \frac{n+2}{2}$,

$$\lambda'_{1,p} \geq \sigma_p(M)\sigma_{n-p+2}(M). \tag{1}$$

The equality holds if and only if N is isometric to the Euclidean ball. In particular, they showed that when the boundary is isometric to the round sphere and the p-curvatures are bounded from below by p, this forces N to be isometric to the Euclidean unit ball. We notice that the above estimate generalized the one in [27] since for $p = 1$ the first eigenvalue on exact 1-forms is also the first eigenvalue on functions.

In this paper, we will consider a different geometric situation. Assume that on a given Riemannian manifold N the boundary M is endowed with a Riemannian flow, that is a Riemannian foliation of 1-dimensional leaves (see Section 2). It is then natural to ask whether the above results can be generalized to the basic Laplacian defined on basic forms in order to deduce a foliated version of the mentioned rigidity results. Recall that basic forms are differential forms on M that are constants along the leaves of the flow. In this context, the geometry of the transverse structure will clearly play an essential role since many additional terms will be involved in the estimate such as the O'Neill tensor (see Section 2 for the definition).

We state our main estimate:

Theorem 1.1 *Let (N^{n+2}, g) be a Riemannian manifold with non-negative curvature operator whose boundary M has a positive n-curvature $\sigma_n(M)$. Assume that M is endowed with a minimal Riemannian flow given by a unit vector field ξ. Then*

$$\lambda'_{1,1} \geq \sigma_{n+1}(M)\Big(\sigma_2(M) - \sup_M g(S(\xi), \xi)\Big), \tag{2}$$

where $\lambda'_{1,1}$ denotes the first positive eigenvalue of the basic Laplacian restricted to basic closed 1-forms.

The key point in the proof of Theorem 1.1 is to show, under some curvature assumptions, that any basic closed p-form on M can be extended to a unique p-form which is closed and co-closed on the whole manifold (see Lemma 3.1). This can be done with the help of some boundary problem established in [22, Lemma 3.4.7]. We point out here that the case $p > 1$ will be studied in a forthcoming paper. This is due to the difference in the estimation (more terms will be involved) and later in the equality case.

The equality case in (2) is of special interest since it implies that the flow is a local product and the boundary is η-umbilical. This means the second fundamental form vanishes in the direction of the vector field ξ and is a multiple of the identity in the orthogonal direction to the leaves (see Lemmas 4.1 and 4.2). Mainly, we get:

Theorem 1.2 *Under the assumptions of Theorem 1.1 with $\sigma_1(M) \geq 0$, if the equality case is realized in (2), the manifold M is then isometric to the quotient $\Gamma \backslash \mathbb{R} \times \mathbb{S}^n$ and N is isometric to $\Gamma \backslash \mathbb{R} \times B'$, for some fixed-point-free cocompact discrete subgroup $\Gamma \subset \mathbb{R} \times SO_{n+1}$, where B' is the unit closed ball in $\mathbb{R}^{n+1}$.*

The problematic part in the last result is to prove that the extension $\hat{\xi}$ of the vector field ξ (which is then parallel) to the whole manifold, coming from the boundary problem in [22], is also parallel on N. Therefore, we show that any connected integral submanifold of $(\mathbb{R}\hat{\xi})^\perp$ is isometric to the unit closed ball by using the result of Raulot and Savo. At the end of Section 4, we prove through examples that, depending on the dimension n, the converse of Theorem 1.2 is not true in general.

Using the limiting case of Inequality (2), we end the paper by stating several rigidity results on manifolds with foliated boundary (see Section 5). Indeed, we consider the situation where the boundary of a compact manifold carries a solution of the so-called basic special Killing form. In particular, we prove that, depending on the sign of the lowest principal curvature, the following result holds:

Corollary 1.3 *Let N be a $(n+2)$-dimensional compact manifold with non-negative curvature operator. Assume that the boundary M is $\mathbb{S}^1 \times \mathbb{S}^n$ and $g(S(\xi), \xi) \leq 0$. If the inequality $\sigma_2(M) \geq 1$ holds, the manifold N is isometric to $\mathbb{S}^1 \times B'$.*

The vector field ξ is in this case the unit vector field tangent to $\mathbb{S}^1$. Also, we prove the following:

Corollary 1.4 *Let N be a $(n+2)$-dimensional compact manifold with non-negative curvature operator. Assume that the boundary M is $\mathbb{S}^1 \times \mathbb{S}^n$ and $g(S(\xi), \xi) \geq 0$. If the inequality $\sigma_2(M) \geq 1 + \sup\limits_M g(S(\xi), \xi)$ holds, the manifold N is isometric to $\mathbb{S}^1 \times B'$.*

Finally, we point out that rigidity results on manifolds with foliated boundary have been studied in the context of spin geometry in [4] which could be seen as a foliated version of the results in [9, 10, 17].

2 Preliminaries

In the first part of this section, we recall some basic ingredients on Riemannian flows defined on a Riemannian manifold. For more details, we refer to [2, 26].

Let (M^{n+1}, g) be a Riemannian manifold and let ξ be a smooth unit vector field defined on M. We say that ξ defines a Riemannian flow on M if the following relation holds $(\mathcal{L}_\xi g)(Y, Z) = 0$ [2, 21] for any vector fields Y, Z orthogonal to ξ. In other words, the vector field ξ defines a foliation (by its integral curves) on M such that the metric is constant along the leaves. It is not difficult to check that, on a Riemannian flow, the endomorphism $h := \nabla^M \xi$ (known as the O'Neill tensor) defines a skew-symmetric tensor field on the normal bundle $Q = \xi^\perp \to M$. Therefore, one can associate to h a differential 2-form Ω given for all sections $Y, Z \in \Gamma(Q)$ by $\Omega(Y, Z) = g(h(Y), Z)$. On the other hand, by inducing the metric g from the manifold M to the normal bundle Q, one can define a covariant derivative ∇ (called transversal Levi-Civita connection) on sections of Q compatible with such a metric. Namely, it is defined for any section Y on Q by

$$\nabla_X Y =: \begin{cases} \pi[X, Y] & \text{if } X = \xi, \\ \\ \pi(\nabla^M_X Y) & \text{if } X \perp \xi, \end{cases}$$

where $\pi : TM \to Q$ denotes the orthogonal projection (see [26]). The curvature of the normal bundle (as a vector bundle) associated with the connection ∇ satisfies the property $\xi \lrcorner R^\nabla = 0$. Moreover, one can easily check that the corresponding Levi-Civita connections on M and Q are related for all sections Z, W in $\Gamma(Q)$ via the Gauss-type formulas:

$$\begin{cases} \nabla^M_Z W = \nabla_Z W - g(h(Z), W)\xi, \\ \\ \nabla^M_\xi Z = \nabla_\xi Z + h(Z) - \kappa(Z)\xi, \end{cases} \tag{3}$$

where $\kappa := \nabla^M_\xi \xi$ is the mean curvature of the flow. A flow is said to be minimal if its mean curvature κ is zero.

The set of basic forms $\Omega_B(M)$ is the set of all differential forms φ on M such that $\xi \lrcorner \varphi = 0$ and $\xi \lrcorner d^M \varphi = 0$. This can be seen locally as the set of all forms on M which just depend on the transverse variables. Clearly from the definition, these forms are preserved by the exterior derivative d^M and hence one may set $d_b := d^M|_{\Omega_B(M)}$. When M is compact, we denote by δ_b the L^2-adjoint of d_b and define the basic Laplacian as being $\Delta_b = d_b\delta_b + \delta_b d_b$. The operator Δ_b is an essentially self-adjoint operator and transversally elliptic and therefore it has a discrete spectrum by the spectral theory of transversal elliptic operators [5, 6].

In the next part, we will give a brief overview on manifolds with boundary and the Reilly formula that could be found in details in [16].

Let (N^{n+2}, g) be a Riemannian manifold of dimension $n+2$ with boundary M and let ν be the inward unit nomal vector field on M. The shape operator (or the Weingarten tensor) is a symmetric tensor field on TM defined for all $X \in \Gamma(TM)$ by $S(X) = -\nabla^N_X \nu$ where ∇^N is the Levi-Civita connection of N. At a point $x \in M$, we denote by $\eta_1(x), \cdots, \eta_{n+1}(x)$ the eigenvalues of S (that are called principal curvatures) and we arrange them in a way such that $\eta_1(x) \leq \eta_2(x) \leq \cdots \leq \eta_{n+1}(x)$. For any integer number p in $\{1, \cdots, n+1\}$; the p-curvatures σ_p of M are defined by $\sigma_p(x) = \eta_1(x) + \cdots + \eta_p(x)$. It is a clear fact that for any two numbers p and q such that $p \leq q$, the inequality $\frac{\sigma_p(x)}{p} \leq \frac{\sigma_q(x)}{q}$ holds where the equality is achieved if and only if $\eta_1(x) = \eta_2(x) = \cdots = \eta_q(x)$ or $p = q$.

The p-curvatures are the lowest eigenvalues of the tensor $S^{[p]}$, the canonical extension of S, to differential p-forms on M. Given any p-form φ on M, this tensor is defined as

$$S^{[p]}(\varphi)(X_1, \cdots, X_p) = \sum_{i=1}^{p} \varphi(X_1, \cdots, S(X_i), \cdots, X_p),$$

where X_i are vector fields on M for $i = 1, \cdots, p$. By convention, we set $S^{[0]} = 0$. One then can easily check that the following inequality

$$\langle S^{[p]}(\varphi), \varphi \rangle \geq \sigma_p(M) |\varphi|^2 \tag{4}$$

holds, where $\sigma_p(M)$ is the infimum over M of the p-curvatures σ_p. Moreover, we have the following property

$$S^{[p+1]}(X \wedge \varphi) = S(X) \wedge \varphi + X \wedge S^{[p]}(\varphi), \tag{5}$$

for all $X \in \Gamma(TM)$. Indeed, for any vector fields $X_1, \ldots, X_{p+1}$, we write

$$\begin{aligned} S^{[p+1]}(X \wedge \varphi)(X_1, \ldots, X_{p+1}) &= \sum_{j=1}^{p+1} (X \wedge \varphi)(X_1, \ldots, S(X_j), \ldots, X_{p+1}) \\ &= \sum_{i=1}^{p+1} (-1)^{i+1} g(X, S(X_i)) \varphi(X_1, \ldots, \hat{X}_i, \ldots, X_{p+1}) \\ &\quad + \sum_{\substack{i,j=1 \\ i \neq j}}^{p+1} (-1)^{i+1} g(X, X_i) \varphi(X_1, \ldots, \hat{X}_i, \ldots, S(X_j), \ldots, X_{p+1}) \\ &= (S(X) \wedge \varphi)(X_1, \ldots, X_{p+1}) + (X \wedge S^{[p]}(\varphi))(X_1, \ldots, X_{p+1}), \end{aligned}$$

where we identify vector fields with the corresponding 1-forms.

We finish this section by stating the Reilly formula established in [16, Thm. 3]. As we already mentioned in the introduction, this formula comes from the integration of the Bochner-Weitzenböck formula over N (assumed to be compact) of the Hodge Laplacian and the use of the Stokes' formula.

More precisely, we let J^* the restriction of differential forms on N to the boundary M. For any p-form α on N, the Reilly formula is then

$$\begin{aligned}\int_N |d^N\alpha|^2 v_g + \int_N |\delta^N\alpha|^2 v_g &= \int_N |\nabla^N\alpha|^2 v_g + \int_N \langle W_N^{[p]}(\alpha), \alpha\rangle v_g \\ &+2\int_M \langle \nu\lrcorner\alpha, \delta^M(J^*\alpha)\rangle v_g + \int_M \mathcal{B}(\alpha,\alpha) v_g,\end{aligned} \tag{6}$$

where v_g is the volume element of g on N and on its boundary M,

$$\begin{aligned}\mathcal{B}(\alpha,\alpha) &= \langle S^{[p]}(J^*\alpha), J^*\alpha\rangle + \langle S^{[n+2-p]}(J^*(*_N\alpha)), J^*(*_N\alpha)\rangle \\ &= \langle S^{[p]}(J^*\alpha), J^*\alpha\rangle + (n+1)H|\nu\lrcorner\alpha|^2 - \langle S^{[p-1]}(\nu\lrcorner\alpha), \nu\lrcorner\alpha\rangle,\end{aligned}$$

and $W_N^{[p]} = \sum_{i,j=1}^{n+2} e_j \wedge e_i \lrcorner R(e_i, e_j)$ is the curvature term in the Bochner formula with $\{e_i\}_{i=1,\cdots,n+2}$ is a local orthonormal frame of TN and "$*_N$" is the Hodge star operator on N. One can prove that $J^*(*_N\alpha)$ is equal (up to a sign) to $*_M(\nu\lrcorner\alpha)$ where "$*_M$" denotes the Hodge star operator on M. We also have $|J^*\alpha|^2 + |\nu\lrcorner\alpha|^2 = |\alpha|^2$ at any point $x \in M$. We point out that if the curvature operator of N is non-negative, the term $W_N^{[p]}$ is also non-negative for all p (see [7]).

In the last part of this section (see [22] for more details), we define the space of harmonic fields $H^p(N)$ as being the set of closed and co-closed p-forms on N. When the manifold N is compact, the space $H^p(N)$ decomposes as [22, Thm. 2.4.8]

$$H^p(N) = H_D^p(N) \oplus H_{\mathrm{co}}^p(N),$$

where $H_D^p(N) := \{\alpha \in H^p(N) |\, J^*\alpha = 0\}$ is the Dirichlet harmonic space and $H_{\mathrm{co}}^p(N) := \{\alpha \in H^p(N) |\, \alpha = \delta^N\beta\}$ is the space of co-exact harmonic field. This splitting, known as the first Friedrich's decomposition, is L^2-orthogonal. Finally, we recall that given any p-form φ on M, the boundary problem

$$\begin{cases} \Delta^N\hat{\varphi} = 0 & \text{in } N, \\ J^*\hat{\varphi} = \varphi,\ J^*(\delta^N\hat{\varphi}) = 0 & \text{on } M \end{cases} \tag{7}$$

admits a solution $\hat{\varphi}$ (which is also a p-form) on N (see Lemma 3.4.7 in [22]). The solution is unique, up to a Dirichlet harmonic field. In particular, it is proved in [1, Lemma 3.1] that any solution $\hat{\varphi}$ is co-closed on N and that $d^N\hat{\varphi} \in H^{p+1}(N)$.

3 Eigenvalue estimate for the basic Laplacian on manifolds with foliated boundary

In this section, we will prove Theorem 1.1. For this, we consider a compact Riemannian manifold N whose boundary M carries a Riemannian flow. We start by establishing two lemmas that will be crucial for the estimate. In the first one and by using the boundary value problem (7), we will see that any basic closed p-form on M can be extended to a closed and co-closed p-form on N. In the second lemma, we will prove that Inequality (4) can be expressed in another way when one restricts to basic differential forms. First, we have:

Lemma 3.1 *Let (N^{n+2}, g) be a Riemannian manifold with $W_N^{[p+1]} \geq 0$ for some $1 \leq p \leq n$. Assume that the boundary M carries a Riemannian flow given by a unit vector field ξ. Let φ be any basic closed p-form. If $\sigma_{n+1-p}(M) > 0$, then $\hat{\varphi}$ is closed and co-closed on N.*

Proof. As we mentioned in Section 2 the form $\hat{\varphi}$, solution of the problem (7), is co-closed on N and the $(p+1)$-form $\hat{\omega} := d^N\hat{\varphi}$ is also closed and co-closed on N. We then apply the Reilly formula for the form $\hat{\omega}$ to get

$$0 = \int_N |\nabla^N \hat{\omega}|^2 v_g + \int_N \langle W_N^{[p+1]}(\hat{\omega}), \hat{\omega}\rangle v_g + \int_M \langle S^{[n+1-p]}(J^* *_N \hat{\omega}), J^* *_N \hat{\omega}\rangle v_g.$$

Here we used the fact that $J^*\hat{\omega} = d^M\varphi = d_b\varphi = 0$, since the form φ is a basic closed form. Now using Inequality (4), we have at any point of M that

$$\begin{aligned}\langle S^{[n+1-p]}(J^* *_N \hat{\omega}), J^* *_N \hat{\omega}\rangle &\geq \sigma_{n+1-p}(M)|J^* *_N \hat{\omega}|^2 \\ &= \sigma_{n+1-p}(M)|\nu\lrcorner\hat{\omega}|^2 = \sigma_{n+1-p}(M)|\hat{\omega}|^2.\end{aligned}$$

Therefore from the fact that $W_N^{[p+1]} \geq 0$ and $\sigma_{n+1-p}(M) > 0$, we deduce that $\hat{\omega}$ vanishes on M and is also parallel on N. Hence it vanishes everywhere and thus $\hat{\varphi}$ is closed. □

Next, we state the second lemma:

Lemma 3.2 *Let (N^{n+2}, g) be a Riemannian manifold with boundary M. Assume that M carries a Riemannian flow given by a unit vector field ξ. For any basic p-form φ on M, we have*

$$\langle S^{[p]}\varphi, \varphi\rangle \geq (\sigma_{p+1}(M) - g(S(\xi), \xi))|\varphi|^2.$$

Proof. By using Equality (5) for $X = \xi$, we can write

$$\begin{aligned}\langle S^{[p+1]}(\xi\wedge\varphi), \xi\wedge\varphi\rangle &= \langle S(\xi)\wedge\varphi, \xi\wedge\varphi\rangle + \langle \xi\wedge S^{[p]}\varphi, \xi\wedge\varphi\rangle \\ &= \langle \xi\lrcorner(S(\xi)\wedge\varphi), \varphi\rangle + \langle \xi\lrcorner(\xi\wedge S^{[p]}\varphi), \varphi\rangle \\ &= g(S(\xi), \xi)|\varphi|^2 + \langle S^{[p]}\varphi, \varphi\rangle - \langle \xi\wedge(\xi\lrcorner S^{[p]}\varphi), \varphi\rangle \\ &= g(S(\xi), \xi)|\varphi|^2 + \langle S^{[p]}\varphi, \varphi\rangle,\end{aligned}$$

where the term $\langle \xi\wedge(\xi\lrcorner S^{[p]}\varphi), \varphi\rangle$ vanishes because φ is basic. We finally finish the proof of the lemma by using Inequality (4) and the fact that $|\xi\wedge\varphi| = |\varphi|$. □

Since Δ_b commutes with d_b and δ_b, the space of closed (resp. co-closed) basic forms is also preserved. For this, we denote $\lambda'_{1,p}$ (resp. $\lambda''_{1,p}$) as the first eigenvalue of the basic Laplacian operator restricted to closed (resp. co-closed) p-forms. We then have that the first eigenvalue $\lambda_{1,p}$ of Δ_b is equal to $\lambda_{1,p} = \min(\lambda'_{1,p}, \lambda''_{1,p})$ and that $\lambda''_{1,p} = \lambda'_{1,n-p}$ [7]. These two facts come from the basic Hodge-de Rham decomposition and basic Poincaré duality (the flow is assumed to be minimal) [12, 13, 19]. In [11], the authors established an estimate for the first eigenvalues $\lambda'_{1,p}$ and $\lambda''_{1,p}$ *à la Gallot-Meyer* for Riemannian foliations; for this purpose they assume that the normal curvature (i.e. the one of the normal bundle) is bounded from below by some constant. They also prove that if the estimate is attained for $\lambda'_{1,1}$ (i.e. on closed 1-forms), the foliation is transversally isometric to the action of a discrete subgroup of $\mathrm{O}(n)$ acting on the sphere (see [14] for the definition) where n is the rank of the normal bundle. Now, we are ready to prove Theorem 1.1:

Proof of Theorem 1.1. Let φ be any basic 1-eigenform for the basic Laplacian that is closed. From Lemma 3.1, it admits an extension $\hat{\varphi}$ which is closed and co-closed on N. The Reilly formula applied then to the 1-form $\hat{\varphi}$ gives, under the curvature assumption and the use of Lemma 3.2 for the eigenform φ, that

$$0 \geq 2\int_M \langle \nu\lrcorner\hat{\varphi}, \delta^M\varphi\rangle v_g + \sigma_2(M)\int_M |\varphi|^2 v_g - \int_M g(S(\xi), \xi)|\varphi|^2 v_g + \sigma_{n+1}(M)\int_M |\nu\lrcorner\hat{\varphi}|^2 v_g.$$

Now from the pointwise inequality $|\nu\lrcorner\hat{\varphi} + \frac{1}{\sigma_{n+1}(M)}\delta^M\varphi|^2 \geq 0$, the above one can be reduced to the following

$$\int_M |\delta^M\varphi|^2 v_g \geq \sigma_2(M)\sigma_{n+1}(M)\int_M |\varphi|^2 v_g - \sigma_{n+1}(M)\sup_M g(S(\xi), \xi)\int_M |\varphi|^2 v_g. \quad (8)$$

We notice here that $\sigma_{n+1}(M)$ is positive since we have pointwise $\sigma_{n+1} \geq \sigma_n > 0$. Using the relation $\delta_b = \delta_M$ on basic 1-forms [19, Prop.2.4], Inequality (8) implies

$$\lambda'_{1,1} \int_M |\varphi|^2 v_g = \int_M |\delta_b \varphi|^2 v_g \geq \sigma_{n+1}(M)\big(\sigma_2(M) - \sup_M g(S(\xi),\xi)\big) \int_M |\varphi|^2 v_g,$$

which finishes the proof of the theorem. $\square$

4 The equality case

In this section, we study the limiting case of Inequality (2). We will prove that, under the condition that all the principal curvatures are non-negative, the boundary has to be η-umbilical (see Lemma 4.1). This means that the second fundamental form vanishes along ξ and is equal to a multiple of the identity in the direction of Q. We will also show that the O'Neill tensor defining the flow vanishes; this is equivalent to say that the normal bundle is integrable (see Lemma 4.2). Moreover, we will see that the extension $\hat{\xi}$ of the vector field ξ (coming from the problem (7)) is parallel on the whole manifold. This will allow us to classify all manifolds for which the estimate (2) is optimal.

First, we begin to prove that the boundary is η-umbilical.

Lemma 4.1 *Under the assumptions of Theorem 1.1 with $\sigma_1(M) \geq 0$, if the equality is attained in* (2)*, then M is η-umbilical in N, i.e. $S(\xi) = 0$ and $S(X) = \eta X$ for X orthogonal to ξ.*

Proof. The equality is attained in (2) if and only if $\lambda'_{1,1} = \sigma_{n+1}(M)\big(\sigma_2(M) - g(S(\xi),\xi)\big)$. In this case $\hat{\varphi}$ is parallel on N and the functions σ_2, σ_{n+1} and $g(S(\xi),\xi)$ are constant. Moreover, we have (see [16], formulas (15) and (23)),

$$\begin{cases} \nabla^M_X \varphi = g(\hat{\varphi},\nu) S(X), \quad \text{for all} \quad X \in \Gamma(TM), \\ \\ \delta^M \varphi = -(n+1) H \nu \lrcorner \hat{\varphi} \\ \\ d^M(\nu \lrcorner \hat{\varphi}) = -S(\varphi). \end{cases} \tag{9}$$

Hence we can write

$$g(\hat{\varphi},\nu) g(S(\xi),\xi) \overset{(9)}{=} g(\nabla^M_\xi \varphi, \xi) = -g(\varphi,\kappa) = 0.$$

Recall here that the flow is assumed to be minimal. As $g(S(\xi),\xi)$ is constant, it is either zero or $g(\hat{\varphi},\nu) = 0$. But if $g(\hat{\varphi},\nu) = 0$, we obtain from the first equation in (9) that $\nabla^M_X \varphi = 0$ for all $X \in \Gamma(TM)$ and therefore $\Delta_b \varphi = \lambda'_{1,1}\varphi = 0$, which gives a contradiction. Thus we deduce that $g(S(\xi),\xi) = 0$. Now, since $0 \leq \eta_1 = \sigma_1 \leq g(S(\xi),\xi) = 0$ one gets from the one hand that $\eta_1 = 0$. On the other hand and in order to prove that $S(\xi) = 0$, we consider an orthonormal basis $\{f_1, \ldots, f_{n+1}\}$ of eigenvectors of S associated with the eigenvalues $\eta_1, \ldots, \eta_{n+1}$. By writing the vector field ξ in this orthonormal frame as $\xi = \sum_{i=1}^{n+1} a_i f_i$ for some real functions a_i, one gets that

$$0 = g(S(\xi),\xi) = \sum_{i=2}^{n+1} a_i^2 \eta_i.$$

Recall that all the η_i's are non-negative for $i = 2, \cdots, n+1$. If $\eta_2 = 0$, then $\lambda'_{1,1} = \sigma_2\sigma_{n+1} = 0$ which is impossible. Finally, we deduce that all the a_i's are zero for $i \geq 2$ which means that $S(\xi) = 0$.

To prove the other part of the lemma, we use (9) to write

$$S(\varphi) = \frac{1}{(n+1)H} d^M(\delta^M \varphi) = \frac{1}{\sigma_{n+1}} \lambda'_{1,1}\varphi = \big(\sigma_2 - g(S(\xi),\xi)\big)\varphi = \sigma_2 \varphi.$$

By differentiating this equation along any vector field $X \in \Gamma(TM)$ and using again the first equation in (9), we obtain from the one hand

$$\nabla^M_X S(\varphi) = \sigma_2 g(\hat{\varphi}, \nu) S(X). \tag{10}$$

On the other hand, we write

$$\begin{aligned} \nabla^M_X S(\varphi) &= (\nabla^M_X S)(\varphi) + S(\nabla^M_X \varphi) \\ &= (\nabla^M_\varphi S)(X) + R^N(\varphi, X)\nu + g(\hat{\varphi}, \nu) S^2(X) \\ &= \nabla^M_\varphi (S(X)) - S(\nabla^M_\varphi X) + R^N(\varphi, X)\nu + g(\hat{\varphi}, \nu) S^2(X), \end{aligned} \tag{11}$$

where in the second equality we used the Gauss-Codazzi equation. Now by combining (10) and (11), we find the following identity

$$\nabla^M_\varphi S(X) - S(\nabla^M_\varphi X) + R^N(\varphi, X)\nu + g(\hat{\varphi}, \nu) S^2(X) = \sigma_2 g(\hat{\varphi}, \nu) S(X).$$

Taking the scalar product of the last equation with X and tracing over an orthonormal frame of TM, we obtain after using the fact that $\sigma_{n+1} = (n+1)H$ is constant

$$-g(\hat{\varphi}, \nu)\mathrm{Ric}^N(\varphi, \nu) = (\sigma_2\sigma_{n+1} - |S|^2) g(\hat{\varphi}, \nu)^2. \tag{12}$$

It is easy to see that the inequality $\sigma_2\sigma_{n+1} - |S|^2 \leq 0$ holds with equality if and only if $\eta_2 = \cdots = \eta_{n+1}$. To check the sign of the l.h.s. of (12), we use the fact that $\mathrm{Ric}^N(\hat{\varphi}, \hat{\varphi}) = 0$ (the vector field $\hat{\varphi}$ is parallel) with the decomposition $\hat{\varphi} = g(\hat{\varphi}, \nu)\nu + \varphi$ along the boundary. This allows to deduce that

$$g(\hat{\varphi}, \nu)^2 \mathrm{Ric}^N(\nu, \nu) + 2g(\hat{\varphi}, \nu)\mathrm{Ric}^N(\varphi, \nu) + \mathrm{Ric}^N(\varphi, \varphi) = 0. \tag{13}$$

Hence, the second term in the l.h.s. of (13) is non-positive as a consequence of the non-negativity of the curvature. That mainly means both terms in (12) vanish, which ends the proof. $\square$

In the following, we will prove that when the equality of the estimate is realized, the O'Neill tensor vanishes. The proof relies on the fact that the eigenform φ (up to its norm) defines a geodesic vector field and therefore the O'Neill tensor satisfies a differential equation along those geodesics. It turns out that the solution of such differential equation, that we find explicitly, blows up at some limit. This contradicts the compactness of the boundary.

Lemma 4.2 *Under the assumptions of Theorem 1.1 with $\sigma_1(M) \geq 0$, if the equality is attained in* (2)*, the flow is then a local product, that is the O'Neill tensor h is equal to* 0.

Proof. First, note that using (9), we have

$$X(g(\hat{\varphi}, \nu)) = -g(S(\varphi), X) = -\eta g(\varphi, X), \quad \text{for } X \in \Gamma(TM). \tag{14}$$

From the one side, the curvature on M of the eigenform φ when applied to ξ and $Y \in \Gamma(Q)$ gives after using the η-umbilicity of the boundary and the first identity in (9) that

$$\begin{aligned}
R^M(\xi,Y)\varphi &= \nabla^M_\xi\nabla^M_Y\varphi - \nabla^M_Y\nabla^M_\xi\varphi - \nabla^M_{[\xi,Y]}\varphi \\
&= \nabla^M_\xi(g(\hat{\varphi},\nu)\eta Y) - \sum_{i=1}^n g([\xi,Y],e_i)g(\hat{\varphi},\nu)\eta e_i \\
&= \eta\xi(g(\hat{\varphi},\nu))Y + \eta g(\hat{\varphi},\nu)\nabla^M_\xi Y - \sum_{i=1}^n g(\nabla^M_\xi Y - \nabla^M_Y\xi, e_i)g(\hat{\varphi},\nu)\eta e_i \\
&= -\eta g(\varphi, S(\xi))Y + \eta g(\hat{\varphi},\nu)\nabla^M_\xi Y - \sum_{i=1}^n g(\nabla^M_\xi Y, e_i)g(\hat{\varphi},\nu)\eta e_i \\
&\quad + \sum_{i=1}^n g(h(Y),e_i)g(\hat{\varphi},\nu)\eta e_i \\
&= \eta g(\hat{\varphi},\nu)g(\nabla^M_\xi Y,\xi)\xi + \eta g(\hat{\varphi},\nu)h(Y) \\
&= \eta g(\hat{\varphi},\nu)h(Y), \qquad (15)
\end{aligned}$$

where $\{e_1,\dots,e_n\}$ is a local orthonormal frame of $\Gamma(Q)$. From the other side, by differentiating the relation $h(\varphi)=0$, that we can get from (3) and (9), along any vector field Y orthogonal to ξ we obtain

$$\nabla^M_\varphi\nabla^M_Y\xi + R^M(Y,\varphi)\xi + \nabla^M_{[Y,\varphi]}\xi = 0.$$

This implies that

$$\nabla^M_\varphi h(Y) + R^M(Y,\varphi)\xi + \sum_{i=1}^n g(\nabla^M_Y\varphi - \nabla^M_\varphi Y, e_i)\nabla^M_{e_i}\xi = 0.$$

Taking now the scalar product of the last identity with $h(Y)$ and tracing over an orthonormal frame of $\Gamma(Q)$, we deduce after using again (9) that

$$\frac{1}{2}\varphi(|h|^2) - \sum_{j=1}^n R^M(\xi, h(e_j),\varphi,e_j) + \eta g(\hat{\varphi},\nu)|h|^2 = 0.$$

Combining the last equation with (15), we finally get the relation

$$\varphi(|h|^2) = -4\eta g(\hat{\varphi},\nu)|h|^2. \qquad (16)$$

In the following, we will prove that the vector field $Y = \frac{\varphi}{|\varphi|}$ is geodesic and that the O'Neill tensor satisfies a differential equation along the geodesic curves of this vector field. For this, we set $U = \{x\in M|\ \varphi(x)\neq 0\}$. Clearly, the set U is open and dense in M. To show that Y is geodesic, we compute

$$\nabla^M_Y Y = \left(\varphi(\frac{1}{|\varphi|})\varphi + \frac{1}{|\varphi|}\nabla^M_\varphi\varphi\right)\frac{1}{|\varphi|} = \Big(\frac{-\varphi(|\varphi|)}{|\varphi|^2}\varphi + \frac{1}{|\varphi|}g(\hat{\varphi},\nu)\eta\varphi\Big)\frac{1}{|\varphi|}.$$

Since $\varphi(|\varphi|^2) = 2g(\nabla^M_\varphi\varphi,\varphi) = 2g(\hat{\varphi},\nu)\eta|\varphi|^2$, then

$$|\varphi|\varphi(|\varphi|) = g(\hat{\varphi},\nu)\eta|\varphi|^2. \qquad (17)$$

That means $\nabla^M_Y Y = 0$. Let us denote by $c(t)$ a maximal geodesic of Y starting at some point $x\in U$ and consider the function $f = g(\hat{\varphi},\nu)\circ c$. Then a direct computation of the first derivative gives that $\dot{f}(t) = g_{c(t)}(df, Y) = -\eta g_{c(t)}(\varphi, Y) = -\eta|\varphi|_{c(t)} < 0$. The second derivative is equal to

$$\ddot{f} = g(\nabla^M_Y df, Y) = -\eta g(\hat{\varphi},\nu)g(SY,Y) = -\eta^2 f.$$

Solving this last differential equation, we get $f(t) = A\cos(\eta t) + B\sin(\eta t)$, where A and B are arbitrary constant. In the following, we will find the differential equation satisfied by $l := |h|^2 \circ c$ and will try to solve it explicitly. In fact, the first derivative of l is equal to

$$\dot{l} = Y(|h|^2) = \frac{\varphi}{|\varphi|}(|h|^2) \stackrel{(16)}{=} -4\eta \frac{g(\hat{\varphi},\nu)}{|\varphi|} l, \tag{18}$$

and its second derivative is

$$\ddot{l} = g(\nabla^M_Y dl, Y) = \frac{1}{|\varphi|}\varphi(\frac{1}{|\varphi|}\varphi(|h|^2)). \tag{19}$$

But, we have

$$\begin{aligned}
\varphi\Big(\frac{1}{|\varphi|}\varphi(|h|^2)\Big) &= \varphi\Big(\frac{1}{|\varphi|}\Big)\varphi(|h|^2) + \frac{1}{|\varphi|}\varphi\Big(\varphi(|h|^2)\Big) \\
&\stackrel{(16)}{=} \Big(\frac{-\varphi(|\varphi|)}{|\varphi|^2}\Big)\Big(-4\eta g(\hat{\varphi},\nu)|h|^2\Big) + \frac{1}{|\varphi|}(-4\eta)\varphi(g(\hat{\varphi},\nu)|h|^2) \\
&\stackrel{(17)}{=} \frac{4\eta^2 g(\hat{\varphi},\nu)^2}{|\varphi|}|h|^2 - \frac{4\eta}{|\varphi|}\Big(\varphi(g(\hat{\varphi},\nu))|h|^2 + g(\hat{\varphi},\nu)\varphi(|h|^2)\Big) \\
&\stackrel{(14),(16)}{=} \frac{4\eta^2 g(\hat{\varphi},\nu)^2}{|\varphi|}|h|^2 - \frac{4\eta}{|\varphi|}\Big(-\eta|\varphi|^2|h|^2 - 4\eta g(\hat{\varphi},\nu)^2|h|^2\Big) \\
&= \frac{20\eta^2 g(\hat{\varphi},\nu)^2}{|\varphi|}|h|^2 + 4\eta^2|\varphi||h|^2.
\end{aligned}$$

Hence we deduce that

$$\ddot{l} = \left(\frac{20\eta^2 g(\hat{\varphi},\nu)^2}{|\varphi|^2} + 4\eta^2\right) l. \tag{20}$$

Combining now equations (18) and (20), we deduce $\frac{\ddot{l}}{l} = \frac{5\dot{l}^2}{4l^2} + 4\eta^2$. Solving this last equation, we obtain $\dfrac{\dot{l}}{l} = 4\eta\tan(\eta t + E)$ and thus $l = \dfrac{D}{\cos^4(\eta t + E)}$ where E is an arbitrary constant and D is a non-negative constant. In conclusion, we find that:

$$\begin{cases} f(t) = & A\cos(\eta t) + B\sin(\eta t) \quad \text{with} \quad \dot{f} < 0 \\ \\ l(t) = & \dfrac{D}{\cos^4(\eta t + E)}, \quad D \geq 0. \end{cases}$$

Note also that

$$\frac{\dot{l}}{4\eta l} = \tan(\eta t + E) \stackrel{(18)}{=} \frac{\eta f(t)}{\dot{f}(t)} = -\frac{A\cos(\eta t) + B\sin(\eta t)}{A\sin(\eta t) - B\cos(\eta t)}. \tag{21}$$

Those equations are defined over the interval $I = c^{-1}(U)$. Let $\tilde{c} : \mathbb{R} \to M$ be the geodesic extension of the curve c to all $\mathbb{R}$ (which exists because M is compact) and consider the function $\tilde{f} = g(\hat{\varphi},\nu)\circ\tilde{c}$. Clearly, we have that $c^{-1}(U) = \tilde{c}^{-1}(U)$ and that $\tilde{f} = f$ on I. At all points outside I, the function $\tilde{f}$ is equal to a constant, because of the formula $|\hat{\varphi}|^2 = |\varphi|^2 + \tilde{f}^2$. Recall here that the form $\hat{\varphi}$ is parallel and thus is of constant norm. Therefore, we get that

$$\tilde{f}(t) = \begin{cases} A\mathrm{cos}(\eta t) + B\mathrm{sin}(\eta t) & t \in I \\ \\ \text{const.} & \text{otherwise.} \end{cases}$$

But it is easy to check that the function $\tilde{f}$ is not of class C^2. Hence $\varphi(\tilde{c}(t)) \neq 0$ and we deduce that $c = \tilde{c} : \mathbb{R} \to U$.

Let us consider J the maximal interval containing 0 on which $\dot{f} < 0$ (it will be the same as for the function l). We shall prove that on this interval, the function l tends to $+\infty$ which contradicts the fact that it is a bounded function, as M is compact. We first notice that $\tan E = \dfrac{A}{B}$ (replace $t = 0$ in Equation (21)) and that $B < 0$ (take $t = 0$ in $\dot{f}$). Depending on the sign of A, we distinguish the following cases:

- **Case where** $A = 0$: In this case, the interval J is equal to $]\frac{-\pi}{2\eta}, \frac{\pi}{2\eta}[$ and $E = k\pi$. Hence, as $t \to \pm\frac{\pi}{2\eta}$, the function $l(t) = \frac{D}{\cos^4(\eta t)}$ tends to $+\infty$ which is a contradiction.
- **Case where** $A > 0$: The interval J is $]\alpha, \beta[$, where α and β are given by

  $$\alpha = \frac{1}{\eta}\arctan\frac{B}{A},\ \frac{-\pi}{4\eta} < \alpha < 0 \quad \text{and} \quad \beta = \frac{1}{\eta}(\arctan\frac{B}{A} + \pi),\ \frac{3\pi}{4\eta} < \beta < \frac{\pi}{\eta}.$$

  At these values of α and β, the derivative of f vanishes while f does not vanish. From (21), we get $\tan(\eta t + E) \to \infty$ as t goes to α or to β, i.e. $l(t)$ tends to infinity which is also a contradiction.
- **Case where** $A < 0$: We can use a similar argument as in the previous case.

□

Now, we have all the materials to prove Theorem 1.2:

Proof of Theorem 1.2. In order to prove that the manifold N is a local product, we need first to show that the vector field ξ defining the flow can be extended to a unique parallel vector field $\hat{\xi}$ on N which is orthogonal to ν. For this purpose, we proceed as in [4].

Let $\hat{\xi}$ be the solution of the boundary problem (7) associated with the vector field ξ on M. Let us consider the 2-form $\hat{\omega} := d^N\hat{\xi}$ which is clearly closed and co-closed on N. Its restriction to the boundary is $J^*\hat{\omega} = d^M\xi = 0$, since ξ is parallel on M. Therefore, the Reilly formula applied to $\hat{\omega}$ reduces to

$$0 = \int_N \left[|\nabla^N\hat{\omega}|^2 + \langle W_N^{[2]}(\hat{\omega}), \hat{\omega}\rangle\right]v_g + (n+1)\int_M H|\nu\lrcorner\hat{\omega}|^2 v_g - \int_M \langle S(\nu\lrcorner\hat{\omega}), \nu\lrcorner\hat{\omega}\rangle v_g. \qquad (22)$$

As the second fundamental form is equal to zero in the direction of ξ and to $\eta\mathrm{Id}$ in the orthogonal direction to ξ, we find for any 1-form β on M that

$$S(\beta)(X) = \eta\beta(X) \text{ for } X \perp \xi \text{ and } S(\beta)(\xi) = 0.$$

Hence, $\langle S(\nu\lrcorner\hat{\omega}), \nu\lrcorner\hat{\omega}\rangle = \eta\sum_{i=1}^n (\nu\lrcorner\hat{\omega})(e_i)^2$, where $\{\xi, e_1, \cdots, e_n\}$ is a local orthonormal frame of T_xM. Using the fact that $(n+1)H = n\eta$, Equation (22) gives

$$0 = \int_N \left[|\nabla^N\hat{\omega}|^2 + \langle W_N^{[2]}(\hat{\omega}), \hat{\omega}\rangle\right]v_g + (n-1)\eta\int_M |\nu\lrcorner\hat{\omega}|^2 v_g + \eta\int_M (\nu\lrcorner\hat{\omega})(\xi)^2 v_g.$$

Using the curvature assumption on N, we deduce that $\hat{\omega}$ is parallel on N and $\nu\lrcorner\hat{\omega} = 0$ on M. But at each point of M, we have $|\hat{\omega}|^2 = |\nu\lrcorner\hat{\omega}|^2 + |J^*\hat{\omega}|^2 = 0$. Therefore $\hat{\omega}$ vanishes on N; which means that $\hat{\xi}$ is closed. Applying now the Stokes' formula

$$\int_N \langle d^N\alpha, \beta\rangle v_g = \int_N \langle \alpha, \delta^N\beta\rangle v_g - \int_M \langle J^*\alpha, \nu\lrcorner\beta\rangle$$

to $\alpha = \delta^N\hat{\xi}$ and $\beta = \hat{\xi}$, we deduce that $\hat{\xi}$ is co-closed on N. Finally, we use again the Reilly formula to the vector field $\hat{\xi}$ to get

$$0 = \int_N |\nabla^N\hat{\xi}|^2 v_g + \langle \mathrm{Ric}^N(\hat{\xi}), \hat{\xi}\rangle v_g + (n+1)\int_M H|\nu\lrcorner\hat{\xi}|^2 v_g.$$

This implies that $\hat{\xi}$ is parallel on N and $\nu\lrcorner\hat{\xi} = 0$ on M.

In the sequel, we follow the same proof as in [4]. As $\hat{\xi}$ is parallel on N, we consider a connected integral submanifold N_1 of the bundle $(\mathbb{R}\hat{\xi})^\perp$, where the orthogonal is taken in N. It is also straightforward to see that N_1 is the quotient of a totally geodesic hypersurface $\widetilde{N}_1$ of the universal cover $\widetilde{N}$ (which is complete) of N. In particular, the manifold $\widetilde{N}_1$ is complete as being a level hypersurface of the function f defined on $\widetilde{N}$ by $d^{\widetilde{N}} f = \widetilde{\hat{\xi}}$ (recall here that $d^N\hat{\xi} = 0$). From the fact that the universal cover is a local isometry, we deduce that N_1 is complete. On the other hand, the boundary of N_1 is a totally geodesic hypersurface in M with a normal vector field ξ and it is totally umbilical in N_1 since the second fundamental form is equal to $-\nabla^{N_1}_X \nu = -\nabla^N_X \nu = \eta X$. Hence by the Gauss formula, and for any $X \in \Gamma(T\partial N_1)$, the Ricci curvature of N_1 is equal to

$$
\begin{aligned}
g(\mathrm{Ric}^{\partial N_1} X, X) &= \sum_{i=1}^n g(R^M(X, e_i)e_i, X) \\
&= \sum_{i=1}^n g(R^N(X, e_i)e_i, X) - g(S^2(X), X) + (n+1)Hg(S(X), X) \\
&= (n-1)\eta^2|X|^2 + \sum_{i=1}^n g(R^N(X, e_i)e_i, X) \geq (n-1)\eta^2|X|^2,
\end{aligned}
$$

where the last inequality comes from the fact that the curvature operator on N is positive. Here $\{e_i\}_{i=1,\cdots,n}$ is a local orthonormal frame of $\Gamma(Q)$. Therefore by Myers's theorem, we deduce that ∂N_1 is compact which with the rigidity result in [15, Thm. 1.1] allows to say that N_1 is compact.

On the other hand, we have from Equations (9) that $\varphi = -\frac{1}{\eta} d^M(\nu\lrcorner\hat{\varphi})$ which means that it is d^M-exact and thus $d^{\partial N_1}$-exact, as ∂N_1 is totally geodesic in M and both φ and $\nu\lrcorner\hat{\varphi}$ are basic. Moreover, the basic form φ is an eigenform of the Laplacian on ∂N_1, that is $\Delta^{\partial N_1}\varphi = \Delta_b\varphi = \lambda'_{1,1}\varphi$. Therefore, if we denote by $\lambda^{\partial N_1}_{1,1}$ the first eigenvalue of $\Delta^{\partial N_1}$ restricted to exact forms on N_1 and by $\sigma_p(\partial N_1)$ the p-curvatures of ∂N_1 into the compact manifold N_1, we get from the main estimate in [16, Thm. 5] (see Inequality (1)) that

$$
n\eta^2 = \sigma_1(\partial N_1)\sigma_n(\partial N_1) \leq \lambda^{\partial N_1}_{1,1} \leq \lambda'_{1,1} = \sigma_2(M)\sigma_{n+1}(M) = n\eta^2.
$$

Hence the equality is attained in the estimate of Raulot and Savo [16, Thm. 1] and therefore N_1 is isometric to the Euclidean closed ball B'. Finally, by the de Rham theorem, the manifold $\widetilde{N}$ is isometric to $\mathbb{R} \times B'$ and N is the quotient of the Riemannian product $\mathbb{R} \times B'$ by its fundamental group. As $\pi_1(N)$ embeds into $\pi_1(M)$ (any isometry of B' fixing pointwise $\mathbb{S}^n$ is the identity map), N is isometric to $_\Gamma\backslash^{\mathbb{R} \times B'}$, for some fixed-point-free cocompact discrete subgroup $\Gamma \subset \mathbb{R} \times \mathrm{SO}_{n+1}$. $\square$

Examples 4.1 *First, we point out that when the transversal Ricci curvature is positive, the first basic cohomology group is zero [8]. In this case, the first positive eigenvalue on basic closed forms is equal to the first positive eigenvalue on basic exact forms which is also equal to the first positive eigenvalue of the basic Laplacian on functions. This is the case for the manifold* $M = {}_\Gamma\backslash^{\mathbb{R} \times \mathbb{S}^n}$ *with the foliation given by the vector field* ∂t.

In general, the limiting case of Inequality (2) is not characterized by M *being isometric to* ${}_\Gamma\backslash^{\mathbb{R} \times \mathbb{S}^n}$ *and* N *isometric to* ${}_\Gamma\backslash^{\mathbb{R} \times B'}$, *where* B' *is the unit closed ball in* $\mathbb{R}^{n+1}$. *To see this, we consider the action of* Γ *on* $\mathbb{R} \times \mathbb{S}^3$ *by* $(0, p) \to (1, -p)$. *For this example, the lower bound of Inequality* (2) *is equal to* 3 *which is the first eigenvalue on* $\mathbb{S}^3$ *associated to a homogeneous harmonic polynomial on* $\mathbb{R}^4$ *of degree 1, as an eigenfunction. But a polynomial of degree 1 is not invariant by the action of* Γ. *That means the first eigenvalue is strictly bigger than* 3 *(it is in fact equal to* 8*) and hence the equality case in (2) is not attained. Recall here that the eigenvalues on the round sphere* $\mathbb{S}^n$ *are* *given by $l(l+n-1)$ for $l \geq 0$. This example can be generalized to any quotient ${}_{\Gamma}\backslash^{\mathbb{R}\times\mathbb{S}^n}$ where n is odd because for the odd-dimensional sphere $\mathbb{S}^n$ we don't necessarily have any fixed directions by an element of $\mathrm{SO}(n+1)$, so the first invariant eigenvalue could be higher than n.*

However, for the even-dimensional sphere, any element of $\mathrm{SO}(n+1)$ must always have a direction that is fixed (eigenvector for the eigenvalue 1), say v. The function $f(x) = v\cdot x$ is a linear function with eigenvalue n that is invariant by that element of $\mathrm{SO}(n+1)$. Thus, for n even and for this special discrete group Γ, the equality case is characterized by M.

Let us give another example. Consider the quotient of $M = {}_{\Gamma}\backslash^{\mathbb{R}\times\mathbb{S}^2}$ where Γ acts on $\mathbb{R}\times\mathbb{S}^2$ by $(z,\theta,0)\to(z,\theta+\alpha,1)$. Here α denotes an irrational multiple of 2π. In this case, the basic Laplacian is given by $\Delta_b = -(1-z^2)\partial_z^2 + 2z\partial_z$ and its eigenvalues are given by $l(l+1)$ for $l\geq 0$ [20]. In this case the first positive eigenvalue of Δ_b is 2 and hence the equality case in (2) is satisfied.

5 Rigidity results on manifolds with foliated boundary

In this section, we study the case where the Riemannian flow carries some special forms (that we shall call them special Killing forms [24, 25]) in order to derive rigidity results on manifolds with foliated boundary arising from the equality case of our main estimate. On a Riemannian manifold (M,g), a special Killing p-form ω is a co-closed form satisfying for all $X\in\Gamma(TM)$ the relations

$$\nabla_X\omega = \frac{1}{p+1}X\lrcorner d\omega \qquad \text{and} \qquad \nabla_X d\omega = -\lambda(p+1)X\wedge\omega,$$

where λ is a non-negative constant (see also [11]). Killing forms have been studied in many papers. For example, we quote the work of [24, Thm. 4.8] where it is shown that there is a one-to-one correspondence between those forms and parallel forms constructed on the cone of the underlying manifold. This result has led to a complete classification of compact simply connected manifolds carrying such forms. For Riemannian flows (or in general Riemannian foliations), one can adapt the same definition of special Killing forms to the set-up of basic forms. Here, we recall that ∇ is the transversal Levi-Civita connection defined in Section 2. As for the general case, one can show that a basic special Killing p-form is a co-closed eigenform of the basic Laplacian corresponding to the eigenvalue $\lambda(p+1)(n-p)$ where n is the rank of Q.

In the following, we will consider a compact manifold N whose boundary carries a basic special Killing p-form. Depending on the sign of the term $g(S(\xi),\xi)$, we will characterize the boundary as being the product $\mathbb{S}^1\times\mathbb{S}^n$ (since it admits basic special Killing forms) and will see that, under some assumptions on the principal curvature, the manifold N is isometric to the product of $\mathbb{S}^1$ with the unit closed ball. This comes in fact from the equality case of the estimate (2).

We first have:

Corollary 5.1 *Let N be a $(n+2)$-dimensional compact manifold with non-negative curvature operator. Assume that the boundary M carries a minimal Riemannian flow such that $g(S(\xi),\xi)\leq 0$ and also admits a basic special Killing $(n-1)$-form. If the inequality $\sigma_2(M)\geq 1$ holds, the manifold N is isometric to ${}_{\Gamma}\backslash^{\mathbb{R}\times B'}$.*

Proof. Let φ be a basic special Killing $(n-1)$-form on M. Then $*_b\varphi$ is a basic closed 1-eigenform for the basic Laplacian, that is $\Delta_b(*_b\varphi) = n(*_b\varphi)$. Here " $*_b$ " is the basic Hodge star operator [26] which commutes with the basic Laplacian as the flow is minimal. Hence, we get the estimate $\lambda'_{1,1}\leq n$. In order to get the lower bound from Theorem 1.1, we need to check the condition on σ_n. For this, we consider the functions $\theta_i = \sigma_{i+1}-\eta_1$ for all $i = 1,\cdots,n$. We then have

$$\sigma_n = \theta_{n-1}+\eta_1 \geq (n-1)\theta_1+\eta_1 \geq (n-1)(1-\eta_1)+\eta_1 = (n-1)-(n-2)\eta_1 > 0,$$

since $\eta_1 \leq g(S(\xi),\xi) \leq 0$. Moreover, we compute

$$\sigma_{n+1} = \theta_n + \eta_1 \geq n\theta_1 + \eta_1 \geq n - \eta_1(n-1) \geq n.$$

Therefore, we deduce that $\lambda'_{1,1} \geq n$ from the main estimate in Theorem 1.1 and the equality case is realized ($\sigma_1 = 0$ because $\eta_1 = 0$). This finishes the proof as a consequence of the equality case. $\square$

Using this last result, we provide a proof of Corollary 1.3:

Proof of Corollary 1.3. As a consequence of Corollary 5.1, the manifold N must be isometric to the quotient of $\mathbb{R} \times B'$ by some discrete free fixed point cocompact subgroup Γ. Since M is assumed to be isometric to $\mathbb{S}^1 \times \mathbb{S}^n$, then Γ's action is trivial on the $\mathbb{S}^n$ factor, that means that N is isometric to $\mathbb{S}^1 \times B'$. $\square$

In the following, we will relax the condition on σ_2 by assuming that the sectional curvature of N vanishes along planes on TM.

Corollary 5.2 *Let N be a $(n+2)$-dimensional compact manifold with non-negative curvature operator. Assume that $M = \mathbb{S}^1 \times \mathbb{S}^n$ with $n \geq 3$, the sectional curvature K^N of N vanishes on M, the mean curvature $H > 0$ and $g(S(\xi),\xi) \leq 0$. Then, the manifold N is isometric to $\mathbb{S}^1 \times B'$.*

Proof. For $X, Y \in \Gamma(TM)$ with $|X| = |Y| = 1$ and $g(X,Y) = 0$, the sectional curvatures of M and N are related by

$$K^M(X,Y) = K^N(X,Y) + g(S(X),X)g(S(Y),Y) - g(S(X),Y)^2.$$

By applying this formula to $X = \xi \in T\mathbb{S}^1$ and $Y = e_i \in T\mathbb{S}^n$, where $\{\xi, e_1, \ldots, e_n\}$ is an orthonormal frame of TM, we obtain

$$(n+1)Hg(S(\xi),\xi) - |S(\xi)|^2 = 0.$$

From the assumption on $g(S(\xi),\xi)$, we deduce that from the one hand that $S(\xi) = 0$. On the other hand, we consider an orthonormal basis $\{\xi, f_2, \ldots, f_{n+1}\}$ of eigenvectors of S associated to the eigenvalues $0, \eta_2, \ldots, \eta_{n+1}$. For $i \neq j$, we get

$$K^M(f_i, f_j) = K^N(f_i, f_j) + \eta_i\eta_j,$$

which gives $\eta_i\eta_j = 1$ for all $i \neq j$. We thus conclude that all η_i's are constants, equal to 1 or -1. Since $H > 0$ then $\eta_i = 1$ and hence $\sigma_2 = 1$. The result is deduced by using Corollary 1.3. $\square$

This last result could be compared to Theorem 2 in [27] where the author characterized the round sphere as the boundary of a compact manifold under similar assumptions. We finally end this section by stating similar results as before:

Corollary 5.3 *Let N be a $(n+2)$-dimensional compact manifold with non-negative curvature operator. Assume that the boundary M carries a minimal Riemannian flow such that $g(S(\xi),\xi) \geq 0$ and also admits a basic special Killing $(n-1)$-form. If the inequality $\sigma_2(M) \geq 1 + \sup_M g(S(\xi),\xi)$ holds, the manifold N is isometric to $\Gamma\backslash\mathbb{R} \times B'$.*

Proof. We follow the same proof as in Corollary 5.1. It is sufficient to prove that $\sigma_n > 0$ and that $\sigma_{n+1} \geq n$. As before, we consider the functions $\theta_i = \sigma_{i+1} - \eta_1$ to compute

$$\begin{aligned}
\sigma_n = \theta_{n-1} + \eta_1 &\geq (n-1)\theta_1 + \eta_1 \\
&\geq (n-1)(1 + g(S(\xi),\xi) - \eta_1) + \eta_1 \\
&= (n-1) + (n-1)g(S(\xi),\xi) - (n-2)\eta_1 \\
&\geq (n-1) + g(S(\xi),\xi) > 0.
\end{aligned}$$

Here we used the fact that $\eta_1 \leq g(S(\xi),\xi)$. Moreover,

$$\begin{aligned}\sigma_{n+1} = \theta_n + \eta_1 &\geq n\theta_1 + \eta_1\\ &\geq n(1 + g(S(\xi),\xi) - \eta_1) + \eta_1\\ &= n + ng(S(\xi),\xi) - (n-1)\eta_1\\ &\geq n + g(S(\xi),\xi) \geq n.\end{aligned}$$

This finishes the proof of the corollary. □

Finally, a similar proof can be done to deduce Corollary 1.4 as the one of Corollary 1.3.

We end this paper by the following comment: As we mentioned in the introduction, the case where $p > 1$ will be treated in another work, since the estimate would not be of the same type. Even though Lemmas 3.1 and 3.2 in Section 3 are true for any p, the relations between the codifferentials δ_M and δ_b involves terms that mainly depend on the 2-form Ω for $p > 1$ [19]. In this situation, the equality case would not be of the same technique.

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