

HOMOTOPY INVARIANCE OF COHOMOLOGY AND SIGNATURE OF A RIEMANNIAN FOLIATION

GEORGES HABIB AND KEN RICHARDSON

ABSTRACT. We prove that any smooth foliation that admits a Riemannian foliation structure has a well-defined basic signature, and this geometrically defined invariant is actually a foliated homotopy invariant. We also show that foliated homotopic maps between Riemannian foliations induce isomorphic maps on basic Lichnerowicz cohomology, and that the Álvarez class of a Riemannian foliation is invariant under foliated homotopy equivalence.

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1. INTRODUCTION

One of the interesting problems of the theory of foliations is to compute the basic index of a transverse Dirac-type operator in terms of topological invariants, a generalization of the Atiyah-Singer theorem. This question, which was first addressed by A. El Kacimi (see [8, Problem 2.8.9]) and by F.W. Kamber and J. Glazebrook (see [10]) in the 1980's, has attracted significant attention by researchers during the last decades and was open for many years. In order that such an index be well defined and finite, we restrict to the class of foliations where the normal bundle is endowed with a holonomy-invariant Riemannian structure, the setting of Riemannian foliations; these were first defined in [24], and good information on these foliations and their analytic and geometric properties can be found in [25] and [18].

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On any such Riemannian foliation, a so-called *bundle-like metric* can be chosen on the whole manifold that restricts to the given transverse metric on the normal bundle. For such a metric, the leaves of the foliation are locally equidistant.

We are particularly interested in the basic signature operator, a transversal version of the ordinary signature operator on even-dimensional manifolds. Several results have been obtained in this direction. J. Lott and A. Gorokhovsky in [11] state a formula under some conditions involving the stratification and leaf closures of the foliation. As a special case, they get an application to the basic signature operator, showing that the basic signature of the foliation is the same as the signature of the space of leaf closures of maximal orbit type, again under various conditions. In the paper [6] of J. Brüning, F.W. Kamber and K. Richardson, the authors obtain a general formula of the basic index of a transversally elliptic operator on a Riemannian foliation. Using these previous results, it is clear that the basic signature operator is defined as a Fredholm operator on the space of basic sections of a foliated vector bundle, and thus its index is dependent only on the homotopy class of the principal transverse symbol of the operator. However, this type of homotopy invariance, which is used in Proposition 3.5, is a weaker special case of foliated homotopy equivalence, which is simply a homotopy equivalence between foliations where leaves get mapped to leaves. In this paper we discuss a much more transparent description of the general homotopy invariance of the basic signature of a Riemannian foliation.

In what follows, we remark that we are studying properties of operators on basic forms, those differential forms that in a sense are constant on the leaves of the foliation. The basic forms are forms in the transverse variables alone when restricted to distinguished foliation charts. The exterior derivative maps basic forms to themselves, and from this differential we construct the basic cohomology groups. The basic signature pairing is a pairing on the half-dimensional cohomology, similar to the case of the ordinary signature of smooth manifolds.

In [5], M. Benameur and A. Rey-Alcantara proved directly that a foliated homotopy equivalence between two closed manifolds M and M' endowed respectively with taut Riemannian foliations \mathcal{F} and \mathcal{F}' implies that the corresponding basic signatures are the same. The tautness assumption (also called *homologically orientable* in some places) means that there exists a metric for which the leaves are immersed minimal submanifolds. One main idea of the proof was that any such equivalence induces an isomorphism between the corresponding basic cohomology groups of M and M' . One important observation is that in order to make this standard version of basic signature on cohomology well-defined, the tautness assumption is required, because in general the corresponding de Rham operator does not map self-dual to anti-self-dual basic forms. On a Riemannian manifold (M, g) endowed with a Riemannian foliation where the leaves are not necessarily minimal, the authors in [13] defined the basic signature operator in terms of the index of the so-called twisted Hodge – de Rham operator and the twisted basic Laplacian. These latter operators are formed using the twisted exterior differential $\tilde{d} = d - \frac{1}{2}\kappa_b \wedge$, where κ_b is the projection of the mean curvature one-form to basic forms (see Section 2 for details). From [2] it is well-known that κ_b is always closed and determines a class $[\kappa_b]$ (*the Álvarez class*) in basic cohomology that is independent of the bundle-like metric and of the Riemannian foliation structure. We point out here that our definition was not possible for the ordinary basic Laplacian since it does not commute with the transverse Hodge star operator. What is interesting here is that the whole bundle-like metric is used to form these operators and cohomology groups and classes, but as we will

soon see, the dimensions and indices coming from these groups and operators do not depend on the metric choices. In this paper, our aim is to understand and elucidate properties of the basic signature on all Riemannian foliations on closed manifolds, in fact on all foliations admitting such structures.

The paper is organized as follows. We first introduce the terminology concerning Riemannian foliations, basic cohomology, twisted basic cohomology and basic signature in Section 2. We discuss known results from [5] concerning the homotopy invariance of ordinary basic cohomology and the homotopy invariance of basic signature in the case of taut Riemannian foliations.

Several new ideas are presented in Section 3.1. In Theorem 3.1 we prove that there is a well-defined pairing in twisted basic cohomology $\tilde{H}^r(M, \mathcal{F}) \times \tilde{H}^s(M, \mathcal{F}) \rightarrow H_d^{q-r-s}(M, \mathcal{F})$ given by $([\alpha], [\beta]) \mapsto \bar{*}[\alpha \wedge \beta]$, and this feature allows us to define a signature pairing when $r = s = \frac{q}{2}$:

$$([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta \wedge \chi_{\mathcal{F}}.$$

Neither of these pairings would make any sense in ordinary basic cohomology unless (M, \mathcal{F}) is taut, but by using twisted basic cohomology, the pairing is well-defined on all Riemannian foliations. Note that the definitions of both twisted basic cohomology and the signature pairing require use of a given bundle-like metric, but in Proposition 3.5 we show that the invariants of the signature pairing are actually smooth foliation invariants. In [13] it was shown already that the dimensions of the twisted basic cohomology groups are independent of the metric or transverse structure of the foliation.

In Section 3.2, we prove properties of basic Lichnerowicz cohomology, which was studied previously in [26], [4], [15], [1], [21] and only uses the smooth structure of the foliation. Given a closed basic one-form θ , the map $d + \theta \wedge$ acts as a differential on basic forms and thus yields cohomology groups $H_{d+\theta \wedge}^*(M, \mathcal{F})$. The twisted basic cohomology discussed above is a special case of this with $\theta = -\frac{1}{2}\kappa_b$. In Corollary 3.10, we show that foliated homotopies induce equivalent maps on basic Lichnerowicz cohomology — by “equivalent” we mean the same map up to multiplication by a positive basic function. In Proposition 3.11, we prove that foliated homotopy equivalences induce isomorphisms on basic Lichnerowicz cohomology. The importance of using the Lichnerowicz cohomology is that we are able to use all possible closed one forms at once, and this insight leads to the results in Section 3.3.

In Proposition 3.12 we immediately use the Lichnerowicz cohomology to prove easily that the codimension and dimension of a foliation are foliated homotopy invariants. In Proposition 3.13, we show that for a transversely oriented Riemannian foliation (M, \mathcal{F}) of codimension q , basic Lichnerowicz cohomology satisfies twisted Poincaré duality, namely that for $0 \leq k \leq q$,

$$H_{d-\theta \wedge}^k(M, \mathcal{F}) \cong H_{d-(\kappa_b-\theta) \wedge}^{q-k}(M, \mathcal{F}).$$

We note that the twisted duality discovered by F. W. Kamber and Ph. Tondeur in [16] and the Poincaré duality for twisted basic cohomology, proved by the authors in [13], are the special cases $\theta = 0$ and $\theta = \frac{1}{2}\kappa_b$, respectively. Using this duality, we are able to prove in Proposition 3.16 that a foliated homotopy equivalence between transversely oriented Riemannian foliations (M, \mathcal{F}) and (M', \mathcal{F}') pulls back the Álvarez class $[\kappa'_b] \in H_d^1(M', \mathcal{F}')$ to the Álvarez class $[\kappa_b] \in H_d^1(M, \mathcal{F})$. We remark that it has been shown previously by H. Nozawa in [19], [20] that the Álvarez class is continuous with respect to smooth deformations of Riemannian foliations. Finally, in Theorem 3.19, we show that up to a sign depending

on orientation, the basic signature, now defined on all foliations admitting a Riemannian foliation structure, is a foliated homotopy invariant.

2. PRELIMINARIES

2.1. Riemannian foliations. In this section, we will recall some basic facts concerning Riemannian foliations that could be found in [25].

Let (M, \mathcal{F}) be a closed Riemannian manifold of dimension n endowed with a foliation \mathcal{F} given by an integrable subbundle $L \subset TM$ of rank p , with $n = p + q$. The subbundle $L = T\mathcal{F}$ is the tangent bundle to the foliation. Let $Q \cong TM/L$ denote the normal bundle, and let g_Q be a given metric on Q . The foliation (M, \mathcal{F}, g_Q) is called **Riemannian** if $\mathcal{L}_X g_Q = 0$ for any section $X \in \Gamma(L)$. In this paper, we will assume that we have chosen a metric g on M that is bundle-like, meaning through the isomorphism $Q \cong L^\perp$, $g_Q = g|_{L^\perp}$. Such bundle-like metrics always exist. One can show that there exists a unique metric connection ∇ (with respect to the induced metric) on the Q , called **transverse Levi-Civita connection**, which is torsion-free. Recall here that the torsion on Q is being defined as $T(X, Y) = \nabla_X \pi(Y) - \nabla_Y \pi(X) - \pi([X, Y])$, where X and Y are vector fields in $\Gamma(TM)$ and $\pi : TM \rightarrow Q$ is the projection. Such a connection ∇ can be expressed in terms of the Levi-Civita connection ∇^M on M as

$$\nabla_X Y = \begin{cases} \pi([X, Y]), & \forall X \in \Gamma(L), \\ \pi(\nabla_X^M Y), & \forall X \in \Gamma(Q). \end{cases}$$

One can also show that the curvature R^∇ associated with the connection ∇ satisfies $X \lrcorner R^\nabla = 0$ for all $X \in \Gamma L$, where the symbol “ \lrcorner ” denotes interior product.

Basic forms are differential forms on any foliation (M, \mathcal{F}) that locally depend only on the transverse variables. That is, they are forms $\alpha \in \Omega(M)$ satisfying the equations $X \lrcorner \alpha = X \lrcorner d\alpha = 0$ for all $X \in \Gamma(L)$. Let us denote by $\Omega(M, \mathcal{F}) \subset \Omega(M)$ the set of all basic forms. In fact, one can easily check that $\Omega(M, \mathcal{F})$ is preserved by the exterior derivative, and therefore one can associate to d the so-called **basic cohomology groups** $H_d^*(M, \mathcal{F})$ as

$$H_d^k(M, \mathcal{F}) = \frac{\ker d_k}{\text{image } d_{k-1}}$$

with

$$d_k = d : \Omega^k(M, \mathcal{F}) \rightarrow \Omega^{k+1}(M, \mathcal{F}).$$

The basic cohomology groups are finite-dimensional for Riemannian foliations, in which case they satisfy Poincaré duality if and only if the foliation is taut. Recall here that a foliation is said to be **taut** if there exists a metric on M so that the mean curvature of the leaves is zero. Given a bundle-like metric on (M, \mathcal{F}) , the mean curvature one-form κ is defined by

$$\kappa^\# = \sum_{i=1}^p \pi(\nabla_{f_i}^M f_i),$$

where $(f_i)_{1 \leq i \leq p}$ is a local orthonormal frame of $T\mathcal{F}$. The orthogonal projection $\kappa_b = P\kappa$ of κ , with $P : L^2(\Omega(M)) \rightarrow L^2(\Omega(M, \mathcal{F}))$, is a closed one-form whose cohomology class, called the **Álvarez class**, in $H_d^1(M, \mathcal{F})$ is independent of the choice of bundle-like metric (see [2]).

Finally, we denote by δ_b the L^2 -adjoint of d restricted to basic forms (see [25], [2], [22]). Then, for transversely oriented Riemannian foliations one has

$$\delta_b = P\delta = \pm \bar{*}d\bar{*} + \kappa_{b\perp} = \delta_T + \kappa_{b\perp},$$

where δ_T is the formal adjoint of d on the local quotients of the foliation charts and $\bar{*}$ is the pointwise transversal Hodge star operator defined on all k -forms γ by

$$\bar{*}\gamma = (-1)^{p(q-k)} * (\gamma \wedge \chi_{\mathcal{F}}), \quad (2.1)$$

with $\chi_{\mathcal{F}}$ being the leafwise volume form (or the **characteristic form**) and $*$ is the ordinary Hodge star operator.

2.2. Twisted basic cohomology. In this section, we shall review some results proved in the paper [13], where also the definitions of some of the terms below are given.

Given a bundle-like metric on a Riemannian foliation (M, \mathcal{F}) and a basic Clifford bundle $E \rightarrow M$, the basic Dirac operator is defined as the restriction

$$D_b = \sum_{i=1}^q e_i \cdot \nabla_{e_i}^E - \frac{1}{2} \kappa_b^\sharp,$$

to basic sections of E , where $\{e_i\}_{i=1, \dots, q}$ is a local orthonormal frame of Q . The basic Dirac operator preserves the set of basic sections and is transversally elliptic. From the expression of the basic Dirac operator applied to the basic Clifford bundle Λ^*Q^* , one may write on basic forms

$$\begin{aligned} D_{\text{tr}} &= d + \delta_T = d + \delta_b - \kappa_{b\perp} : \Omega^{\text{even}}(M, \mathcal{F}) \rightarrow \Omega^{\text{odd}}(M, \mathcal{F}) \\ D_b &= \frac{1}{2}(D_{\text{tr}} + D_{\text{tr}}^*)s = d - \frac{1}{2}\kappa_b \wedge + \delta_b - \frac{1}{2}\kappa_{b\perp}. \end{aligned} \quad (2.2)$$

In [12], we showed the invariance of the spectrum of D_b with respect to a change of metric on M in any way that leaves the transverse metric on the normal bundle intact (this includes modifying the subbundle $Q \subset TM$, as one must do in order to make the mean curvature basic, for example). That is,

Theorem 2.1. *(In [12]) Let (M, \mathcal{F}) be a compact Riemannian manifold endowed with a Riemannian foliation and basic Clifford bundle $E \rightarrow M$. The spectrum of the basic Dirac operator is the same for every possible choice of bundle-like metric that is associated to the transverse metric on the quotient bundle Q .*

In [13], the authors define the new cohomology $\tilde{H}^*(M, \mathcal{F}) = H_{d - \frac{1}{2}\kappa_b \wedge}^*(M, \mathcal{F})$ (called the **twisted basic cohomology**) of basic forms, using $\tilde{d} := d - \frac{1}{2}\kappa_b \wedge$ as a differential. Recall from (2.2) that the basic de Rham operator is $D_b = \tilde{d} + \tilde{\delta}$, where $\tilde{\delta} := \delta_b - \frac{1}{2}\kappa_{b\perp}$. Because κ_b is basic and closed, the twisted differential preserves $\Omega(M, \mathcal{F})$, $\tilde{d}^2 = 0$ and $\tilde{\delta}^2 = 0$. We show that the corresponding Betti numbers and eigenvalues of the twisted basic Laplacian $\tilde{\Delta} := \tilde{d}\tilde{\delta} + \tilde{\delta}\tilde{d}$ are independent of the choice of a bundle-like metric. In the remainder of this section, we assume that the foliation is transversely oriented so that the $\bar{*}$ operator is well-defined.

Definition 2.2. *We define the basic \tilde{d} -cohomology $\tilde{H}^*(M, \mathcal{F})$ by*

$$\tilde{H}^k(M, \mathcal{F}) = \frac{\ker \tilde{d}_k : \Omega^k(M, \mathcal{F}) \rightarrow \Omega^{k+1}(M, \mathcal{F})}{\text{image } \tilde{d}_{k-1} : \Omega^{k-1}(M, \mathcal{F}) \rightarrow \Omega^k(M, \mathcal{F})}.$$

Proposition 2.3. (in [13]) *The dimensions of $\tilde{H}^k(M, \mathcal{F})$ are independent of the choice of the bundle-like metric and independent of the transverse Riemannian foliation structure.*

2.3. The basic signature operator. We suppose that (M, \mathcal{F}, g_Q) is a transversally oriented Riemannian foliation of even codimension q , and let g_M be a bundle-like metric. Let

$$\star = i^{k(k-1) + \frac{q}{2}\bar{x}}$$

as an operator on basic k -forms, analogous to the involution used to identify self-dual and anti-self-dual forms on a manifold. Note that this endomorphism is symmetric, and

$$\star^2 = 1.$$

In the particular case when $q = 4m$ for an integer m , we have $\star = \bar{x}$ on $2m$ -forms.

Proposition 2.4. (In [13]) *We have $\star(\tilde{d} + \tilde{\delta}) = -(\tilde{d} + \tilde{\delta})\star$. In fact, $\star\tilde{d} = -\tilde{\delta}\star$ and $\star\tilde{\delta} = -\tilde{d}\star$.*

Let $\Omega^+(M, \mathcal{F})$ denote the $+1$ eigenspace of \star in $\Omega^*(M, \mathcal{F})$, and let $\Omega^-(M, \mathcal{F})$ denote the -1 eigenspace of \star in $\Omega^*(M, \mathcal{F})$. By the proposition above, $D_b = \tilde{d} + \tilde{\delta}$ maps $\Omega^\pm(M, \mathcal{F})$ to $\Omega^\mp(M, \mathcal{F})$. Therefore, we may define the basic signature operator as follows.

Definition 2.5. *On a transversally oriented Riemannian foliation of even codimension, let the **basic signature operator** be the operator $D_b : \Omega^+(M, \mathcal{F}) \rightarrow \Omega^-(M, \mathcal{F})$. We define the **basic signature** $\sigma(M, \mathcal{F})$ of the foliation to be the index*

$$\sigma(M, \mathcal{F}) = \dim \ker \left(\tilde{\Delta} \Big|_{\Omega^+(M, \mathcal{F})} \right) - \dim \ker \left(\tilde{\Delta} \Big|_{\Omega^-(M, \mathcal{F})} \right).$$

Remark 2.6. *We note that such a definition is not possible for the operator $d + \delta_b$, because the relationship in the proposition above does not hold for $d + \delta_b$.*

2.4. Known results on the homotopy invariance of the basic cohomology and signature in the taut case. In this section, we review the results in [5], where the smooth homotopy invariance of ordinary basic cohomology is proved and the basic signature is studied in the case where the foliation is taut. For this, given two foliated manifolds (M, \mathcal{F}) and (M', \mathcal{F}') , we say that a map $f : (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ is **foliated** if f maps the leaves of \mathcal{F} to the leaves of \mathcal{F}' , i.e. $f_*(T\mathcal{F}) \subset T\mathcal{F}'$. The following fact is well-known and easy to show.

Lemma 2.7. *If $f : (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ is foliated, then $f^*(\Omega(M', \mathcal{F}')) \subseteq \Omega(M, \mathcal{F})$.*

Definition 2.8. *Let (M, \mathcal{F}) and (M', \mathcal{F}') be two foliated manifolds. We say that the foliated maps $\phi : M \rightarrow M'$ and $\psi : M \rightarrow M'$ are **foliated homotopic** if there exists a continuous map $H : [0, 1] \times M \rightarrow M'$ such that $H(0, x) = \phi(x)$, $H(1, x) = \psi(x)$ and that for each $t \in [0, 1]$ the map $H(t, \cdot)$ is smooth and foliated.*

Definition 2.9. *We say that a foliated map $f : (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ is a **foliated homotopy equivalence** if there exists a foliated map $g : M' \rightarrow M$ such that $f \circ g$ is foliated homotopic to $\text{Id}_{M'}$ and $g \circ f$ is foliated homotopic to Id_M .*

Proposition 2.10. (In [5]; also in [9] for the case of foliated homeomorphisms) *If a map $f : M \rightarrow M'$ is a smooth foliated homotopy equivalence, then f^* induces an isomorphism between $H_d^*(M', \mathcal{F}')$ and $H_d^*(M, \mathcal{F})$.*

In what follows, for a taut Riemannian foliation (M, \mathcal{F}) of codimension 2ℓ , we let $A_{\mathcal{F}} : H_d^\ell(M, \mathcal{F}) \times H_d^\ell(M, \mathcal{F}) \rightarrow \mathbb{R}$ be the bilinear form

$$A_{\mathcal{F}}([\alpha], [\beta]) = \int_M \alpha \wedge \beta \wedge \chi_{\mathcal{F}},$$

which can be seen to be well-defined (for the taut case only). If ℓ is even, it is easy to see that

$$\begin{aligned} \sigma(M, \mathcal{F}) &= \dim \ker \left(\Delta|_{\Omega^+(M, \mathcal{F})} \right) - \dim \ker \left(\Delta|_{\Omega^-(M, \mathcal{F})} \right) \\ &= \text{signature}(A_{\mathcal{F}}(\cdot, \cdot)), \end{aligned}$$

because $\star d = -\delta \star$ and $\star \delta = -d \star$ and $\star = \bar{\star}$ on ℓ -forms. If ℓ is odd, it can be seen easily that $A_{\mathcal{F}}([\alpha], [\alpha]) \equiv 0$ for all $[\alpha] \in H_d^\ell(M, \mathcal{F})$, and also that $\sigma(M, \mathcal{F}) = 0$ since $\star = i\bar{\star}$ on ℓ -forms so that the kernels have the same complex dimension.

Theorem 2.11. *(In [5]) Let $f : M \rightarrow M'$ be a smooth foliated homotopy equivalence between two taut Riemannian foliations of codimension 2ℓ and transverse volume forms ν, ν' , respectively. Then $\sigma(M, \mathcal{F}) = \sigma(M', \mathcal{F}')$ if f preserves the transverse orientation and $\sigma(M, \mathcal{F}) = -\sigma(M', \mathcal{F}')$ otherwise.*

3. HOMOTOPY INVARIANCE OF TWISTED COHOMOLOGY AND BASIC SIGNATURE

3.1. Twisted basic cohomology and basic signature for general Riemannian foliations. Let q be the codimension of the transversally oriented foliation \mathcal{F} in M , and let $\bar{\star}$ denote the transversal Hodge star operator. This operator is defined by (2.1) but can then be extended to a map on cohomology classes. For example, from [13] we have formulas such as

$$\begin{aligned} \delta_b \bar{\star} &= (-1)^{k+1} \bar{\star} (d - \kappa_b \wedge), \\ d \bar{\star} &= (-1)^k \bar{\star} (\delta_b - \kappa_b \lrcorner), \end{aligned}$$

so that $\bar{\star}$ maps $(d - \kappa_b \wedge, \delta_b - \kappa_b \lrcorner)$ -harmonic forms to (d, δ_b) -harmonic forms. Using the corresponding Hodge theorems, this gives a map (actually an isomorphism) between the corresponding cohomology groups. That is, we can use $\bar{\star}$ to define the linear map

$$\bar{\star} : H_{d-\kappa_b \wedge}^k(M, \mathcal{F}) \xrightarrow{\cong} H_d^{q-k}(M, \mathcal{F}).$$

This was originally observed in [16] for the case of basic mean curvature and can be adjusted using the techniques in [2] and [22] for the general case.

Theorem 3.1. *Let (M, \mathcal{F}) be a Riemannian foliation of codimension q that is transversally oriented. There for integers $0 \leq r, s \leq q$, there is a pairing*

$$\tilde{H}^r(M, \mathcal{F}) \times \tilde{H}^s(M, \mathcal{F}) \rightarrow H_d^{q-r-s}(M, \mathcal{F}),$$

defined as follows. For $[\alpha] \in \tilde{H}^r(M, \mathcal{F})$ and $[\beta] \in \tilde{H}^s(M, \mathcal{F})$, $[\alpha \wedge \beta]$ defines a class in $H_{d-\kappa_b \wedge}^{r+s}(M, \mathcal{F})$, and thus $\bar{\star}[\alpha \wedge \beta]$ is a class in ordinary basic cohomology $H_d^{q-r-s}(M, \mathcal{F})$. In the particular case when $r = s = \frac{q}{2}$, this pairing is nondegenerate, and the result is

$$\bar{\star}[\alpha \wedge \beta] = \left[\int_M \alpha \wedge \beta \wedge \chi \right] \in H_d^0(M, \mathcal{F}) \cong \mathbb{R}.$$

Proof. We have

$$\begin{aligned} d(\alpha \wedge \beta) &= d(\alpha) \wedge \beta + (-1)^r \alpha \wedge d(\beta) \\ &= \frac{1}{2} \kappa_b \wedge \alpha \wedge \beta + \frac{(-1)^r}{2} \alpha \wedge \kappa_b \wedge \beta \\ &= \kappa_b \wedge \alpha \wedge \beta. \end{aligned}$$

Then $\alpha \wedge \beta$ defines a cohomology class in $H_{d-\kappa_b \wedge}^{r+s}(M, \mathcal{F})$. We then apply $\bar{*}$ to the associated basic harmonic form to get an element in $H_d^{q-r-s}(M, \mathcal{F})$. Note that the class $[\alpha \wedge \beta] \in H_{d-\kappa_b \wedge}^{r+s}(M, \mathcal{F})$ is well-defined. If $\alpha' = \alpha + \tilde{d}\gamma$ with $\gamma \in \Omega^{r-1}(M, \mathcal{F})$, then

$$\begin{aligned} \tilde{d}\gamma \wedge \beta &= d\gamma \wedge \beta - \frac{1}{2} \kappa_b \wedge \gamma \wedge \beta \\ &= d(\gamma \wedge \beta) - (-1)^{r-1} \gamma \wedge d\beta + \frac{1}{2} (-1)^{r-1} \gamma \wedge \kappa_b \wedge \beta - \kappa_b \wedge \gamma \wedge \beta \\ &= (d - \kappa_b \wedge)(\gamma \wedge \beta) - (-1)^{r-1} \gamma \wedge \left(d - \frac{1}{2} \kappa_b \wedge \right) \beta \\ &= (d - \kappa_b \wedge)(\gamma \wedge \beta). \end{aligned}$$

It follows that the result $[\alpha \wedge \beta]$ is independent of the representative of the class $[\alpha]$. By a similar argument, it is independent of the choice of β . It follows that $\bar{*}[\alpha \wedge \beta] \in H_d^{q-r-s}(M, \mathcal{F})$ is well-defined.

When $r = s = \frac{q}{2}$, we note that for any nonzero class $[\alpha] \in \tilde{H}^r(M, \mathcal{F})$, $[\bar{*}\alpha] \in \tilde{H}^r(M, \mathcal{F})$ by [13], and so $\alpha \wedge \bar{*}\alpha$ is a multiple of the transverse volume form, so

$$\bar{*}[\alpha \wedge \bar{*}\alpha] = \langle \alpha, \alpha \rangle \neq 0.$$

Repeating the argument in the second slot shows that the pairing is nondegenerate. \square

Suppose that (M, \mathcal{F}) is a Riemannian foliation of codimension 2ℓ , and we define the bilinear map $A_{\mathcal{F}} : \Omega^\ell(M, \mathcal{F}) \times \Omega^\ell(M, \mathcal{F}) \rightarrow \mathbb{R}$ by

$$A_{\mathcal{F}}(\alpha, \beta) = \int_M \alpha \wedge \beta \wedge \chi_{\mathcal{F}}.$$

Proposition 3.2. *The induced map $A_{\mathcal{F}} : \tilde{H}^\ell(M, \mathcal{F}) \times \tilde{H}^\ell(M, \mathcal{F}) \rightarrow \mathbb{R}$ is well-defined.*

Proof. This is a direct consequence of Theorem 3.1. \square

Lemma 3.3. *The basic signature $\sigma(M, \mathcal{F})$ of a Riemannian foliation of codimension 2ℓ is the same as the signature of the quadratic form $A_{\mathcal{F}}([\alpha], [\alpha])$ for $\alpha \in \tilde{H}^\ell(M, \mathcal{F})$.*

Proof. When ℓ is even, $\star \tilde{d} = -\tilde{d} \star$ and $\star \tilde{\delta} = -\tilde{d} \star$, so we compute for any $\tilde{\Delta}$ -harmonic ℓ -form $\alpha = \star \alpha = \bar{*}\alpha$ in $\Omega^+(M, \mathcal{F})$,

$$\begin{aligned} A_{\mathcal{F}}(\alpha, \alpha) &= \int_M \alpha \wedge \alpha \wedge \chi_{\mathcal{F}} \\ &= \int_M \alpha \wedge \bar{*}\alpha \wedge \chi_{\mathcal{F}} = \int_M |\alpha|^2. \end{aligned}$$

In the same way, we find $A_{\mathcal{F}}(\beta, \beta) = -\int_M |\beta|^2$ for any harmonic ℓ -form β in $\Omega^-(M, \mathcal{F})$.

Therefore,

$$\begin{aligned}\sigma(M, \mathcal{F}) &= \dim \ker \left(\tilde{\Delta} \Big|_{\Omega^+(M, \mathcal{F})} \right) - \dim \ker \left(\tilde{\Delta} \Big|_{\Omega^-(M, \mathcal{F})} \right) \\ &= \text{signature}(A_{\mathcal{F}}(\cdot, \cdot)).\end{aligned}$$

If ℓ is odd, it can be seen easily that $A_{\mathcal{F}}([\alpha], [\alpha]) \equiv 0$ for all $[\alpha]$, and since again $\star = i\bar{\star}$ on ℓ -forms in this case, the kernels have the same dimension so that $\sigma(M, \mathcal{F}) = 0$. \square

We will need the following lemma; this is known to experts but does not appear to be present in the literature.

Lemma 3.4. *If (M, \mathcal{F}) is a smooth foliation on a (not necessarily closed) manifold that admits a bundle-like metric, then any two such bundle-like metrics are homotopic through a smooth family of bundle-like metrics.*

Proof. Consider a smooth foliation (M, \mathcal{F}) of codimension q and dimension p on which a bundle-like metric is defined. Near any point we may choose a foliation chart with adapted coordinates $(x, y) \in \mathbb{R}^p \times \mathbb{R}^q$ on which an adapted local orthonormal frame $(b_1, b_2, \dots, b_p, e_1, \dots, e_q)$ is defined. Let $(b^1, b^2, \dots, b^p, e^1, \dots, e^q)$ be the corresponding coframe. Then the bundle-like metric takes the form

$$g = ds^2 = \sum_{j=1}^p (b^j)^2 + \sum_{k=1}^q (e^k)^2,$$

with

$$g_Q = \sum_{k=1}^q (e^k)^2 = \sum_{\alpha, \beta=1}^q h_{\alpha\beta} dy^\alpha dy^\beta$$

for a positive-definite symmetric matrix of functions $(h_{\alpha\beta})$ and

$$g_L = \sum_{j=1}^p (b^j)^2$$

positive definite when restricted to $L = T\mathcal{F}$. The bundle-like condition is equivalent to $h_{\alpha\beta}$ being a matrix of basic functions; that is, its restriction to $Q = T\mathcal{F}^\perp$ must be holonomy-invariant; see [23, Section IV, Proposition 4.2]. Now, suppose that \tilde{g} is another such bundle-like metric; therefore, in a possibly smaller foliation chart we have

$$\tilde{g} = \sum_{j=1}^p (\tilde{b}^j)^2 + \sum_{\alpha, \beta=1}^q \tilde{h}_{\alpha\beta}(y) dy^\alpha dy^\beta = \tilde{g}_L + \tilde{g}_Q,$$

noting that the normal bundle \tilde{Q} for \tilde{g} is typically different from that of g . Let $\Pi : TM \rightarrow L$ be the orthogonal projection defined by the first metric g . Since the tangential part of \tilde{g} also remains positive definite on L , $(\Pi^* \tilde{b}^1, \dots, \Pi^* \tilde{b}^p, \tilde{e}^1, \dots, \tilde{e}^q)$ forms a basis of T^*M , and so choosing them to be an orthonormal basis defines a new bundle-like metric

$$\bar{g} = \sum_{j=1}^p (\Pi^* \tilde{b}^j)^2 + \sum_{\alpha, \beta=1}^q \tilde{h}_{\alpha\beta}(y) dy^\alpha dy^\beta = (\Pi^* \otimes \Pi^*)(\tilde{g}_L) + \tilde{g}_Q,$$

with the feature that the bundles L and Q agree (and thus L^* and Q^* agree) for both \bar{g} and g . It is clear that \tilde{g} and \bar{g} are homotopic through a homotopy transforming \tilde{b}^j to $\Pi^* \tilde{b}^j$;

specifically, for $0 \leq t \leq 1$ we may set $b^j(t) = (1-t)\tilde{b}^j + t\Pi^*\tilde{b}^j$, and then the resulting metric homotopy is

$$g_t = \sum (b^j(t))^2 + \tilde{g}_Q; \quad g_0 = \tilde{g}, \quad g_1 = \bar{g}.$$

The homotopy is independent of the choice of coframe $\{\tilde{b}^j\}$, because if $U = (U_{\ell m})$ is any orthogonal matrix of functions and $\tilde{b}^{j'} = \sum_m U_{jm}\tilde{b}^m$, then

$$\begin{aligned} \sum_j (b^{j'}(t))^2 &= \sum_j \left(\sum_m (1-t)U_{jm}\tilde{b}^m + tU_{jm}\Pi^*\tilde{b}^m \right)^2 \\ &= \sum_j \left(\sum_m U_{jm}((1-t)\tilde{b}^m + t\Pi^*\tilde{b}^m) \right)^2 \\ &= \sum_j \sum_{\ell, m} U_{j\ell}((1-t)\tilde{b}^\ell + t\Pi^*\tilde{b}^\ell) U_{jm}((1-t)\tilde{b}^m + t\Pi^*\tilde{b}^m) \\ &= \sum_\ell ((1-t)\tilde{b}^\ell + t\Pi^*\tilde{b}^\ell)^2 = \sum_\ell (b^\ell(t))^2. \end{aligned}$$

Thus, this homotopy is independent of coordinates and choice of frame. Next, \bar{g} and g are homotopic through a convex combination of the respective metrics on L and Q ; specifically, letting $\bar{g} = \bar{g}_L + \bar{g}_Q$, for $t \in [0, 1]$, we have

$$h_t = (1-t)\bar{g}_L + tg_L + (1-t)\bar{g}_Q + tg_Q$$

is a family of metrics that satisfies the bundle-like condition for each t . We may now form the following smooth homotopy between \tilde{g} and g :

$$G_t = \begin{cases} g_{u(t)} & \text{for } 0 \leq t \leq \frac{1}{2}, \\ h_{v(t)} & \text{for } \frac{1}{2} \leq t \leq 1, \end{cases}$$

where $u, v : \mathbb{R} \rightarrow \mathbb{R}$ are smooth increasing functions such that $u \equiv 0$ on $(-\infty, 0]$, $u \equiv 1$ on $[\frac{1}{2}, \infty)$ and $v \equiv 0$ on $(-\infty, \frac{1}{2})$ and $v \equiv 1$ on $[1, \infty)$. \square

Proposition 3.5. *The basic signature $\sigma(M, \mathcal{F})$ of a Riemannian foliation does not depend on the transverse Riemannian structure or the bundle-like metric; it is a smooth invariant of the foliation.*

Proof. Observe that by the previous lemma, any two bundle-like metrics on (M, \mathcal{F}) are smoothly homotopic through bundle-like metrics, and it follows that the principal transverse symbols of the signature operators $\tilde{d} + \tilde{\delta}$ on $\Omega^\pm(M, \mathcal{F})$ with respect to those metrics are smoothly homotopic. Since the basic signature is the index of this operator on basic sections, there is a continuous path through Fredholm operators connecting the two operators, so that the index cannot change along that path. Thus, the basic signature is a smooth invariant of the foliation. See [8] and [6] for properties of the basic index.

Note that it is also possible to see this result through a long, detailed analysis of the differentials and bundle-like metrics and the effects on $\chi_{\mathcal{F}}$ and κ_b as in [2]. \square

3.2. Basic Lichnerowicz cohomology and foliated homotopy invariance . We start with any smooth foliation (M, \mathcal{F}) . In what follows, let θ be a closed basic one-form. Then $d + \theta \wedge$ is a differential on the space of basic forms. Let $H_{d+\theta \wedge}^*(M, \mathcal{F})$ denote the resulting cohomology, which is sometimes called basic Lichnerowicz cohomology or basic Morse-Novikov cohomology; see [26], [4], [15], [1], [21].

Lemma 3.6. ([1, Proposition 3.0.11]) *If $[\alpha] = [\beta] \in H_d^1(M, \mathcal{F})$, then $H_{d+\alpha\wedge}^*(M, \mathcal{F}) \cong H_{d+\beta\wedge}^*(M, \mathcal{F})$.*

Lemma 3.7. *Let $f : (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ be a foliated map, and let θ be a closed basic one form. Then $f^* : \Omega(M', \mathcal{F}') \rightarrow \Omega(M, \mathcal{F})$ induces a linear map from $H_{d+\theta\wedge}^*(M', \mathcal{F}')$ to $H_{d+f^*\theta\wedge}^*(M, \mathcal{F})$.*

Proof. By Lemma 2.7, $f^*\Omega(M', \mathcal{F}') \subseteq \Omega(M, \mathcal{F})$. We must prove that the linear map f^* maps closed and exact forms to closed and exact forms, respectively. Let $[\alpha] \in H_{d+\theta\wedge}^k(M', \mathcal{F}')$, then

$$\begin{aligned} (d + f^*\theta\wedge)(f^*\alpha) &= d(f^*\alpha) + f^*\theta \wedge f^*\alpha = f^*(d\alpha) + f^*(\theta \wedge \alpha) \\ &= f^*((d + \theta\wedge)\alpha) = 0. \end{aligned}$$

Thus closed forms on M' are mapped to closed forms on M . Next, for any $\beta \in \Omega^{k-1}(M', \mathcal{F}')$,

$$\begin{aligned} f^*((d + \theta\wedge)\beta) &= d(f^*\beta) + f^*\theta \wedge f^*\beta \\ &= (d + f^*\theta\wedge)f^*\beta, \end{aligned}$$

so that exact forms map to exact forms. \square

Let us consider two manifolds (M, \mathcal{F}) and (M', \mathcal{F}') endowed with a Riemannian foliations \mathcal{F} and \mathcal{F}' . We denote by κ (resp. κ') the mean curvature of the foliation \mathcal{F} (resp. \mathcal{F}'), with metrics chosen so that both mean curvatures forms are basic. By [7], this can always be done.

Proposition 3.8. *Let $f : (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ be a foliated map. Suppose that a bundle-like metric $g_{M'}$ on M' is given such that the mean curvature κ' is basic. Suppose that the basic cohomology class $[f^*(\kappa')] \in H_d^1(M, \mathcal{F})$ contains the mean curvature one-form for some bundle-like metric on M . Then f^* induces a linear map from $\tilde{H}^*(M', g_{M'})$ to $\tilde{H}^*(M, g_M)$ with respect to some bundle-like metric g_M on M such that $\kappa = f^*(\kappa')$.*

Proof. By Lemma 2.7, $f^*\Omega(M', \mathcal{F}') \subseteq \Omega(M, \mathcal{F})$. We are given that for some given \tilde{g}_M , $\tilde{\kappa} = f^*\kappa' + d\eta$ for some basic function η . We then choose multiply the metric in the leaf direction by $\exp(\eta)$ and obtain a bundle-like metric g_M such that $\kappa = f^*\kappa'$. Lemma 3.7 completes the proof with $\theta = -\frac{1}{2}\kappa'$. \square

Lemma 3.9. *Let $H : I \times M \rightarrow M'$ be a smooth foliated homotopy from (M, \mathcal{F}) to (M', \mathcal{F}') , and let θ' be a closed basic one-form on M' . Let*

$$a_t = \exp\left(\int_0^t j_s^*(\partial_t \lrcorner H^* \theta') ds\right) \in \Omega^0(M, \mathcal{F}),$$

where $0 \leq t \leq 1$ and $j_s : M \rightarrow I \times M$ is defined by $j_s(\cdot) = (s, \cdot)$. Let $h : \Omega^k(M', \mathcal{F}') \rightarrow \Omega^{k-1}(M, \mathcal{F})$ be defined by

$$h(\sigma) = \int_0^1 a_s j_s^*(\partial_t \lrcorner H^* \sigma) ds.$$

Then

$$a_1(\cdot) H(1, \cdot)^* - H(0, \cdot)^* = (d + H(0, \cdot)^* \theta' \wedge) h + h(d + \theta' \wedge)$$

as operators on $\Omega^*(M', \mathcal{F}')$.

Proof. The proof is exactly the same as the proof of Lemma 1.1 in [14], but with basic functions and forms. With the definitions given, we just use the fact that $H(t, \cdot)^* \theta' - H(0, \cdot)^* \theta' = d(\log |a_t|)$ and calculate the derivatives. \square

Using the chain homotopy h in the Lemma above, we get the following with $H(0, \cdot) = \phi$, $H(1, \cdot) = \psi$, $a_1(\cdot) = f$. Also, because of [3, Corollary 13.3], if two smooth foliated maps are (continuously) homotopic, then there exists a smooth homotopy between them.

Corollary 3.10. (*Homotopy invariance of basic Lichnerowicz cohomology*) *Let ϕ and ψ be two smooth maps that are foliated homotopic from (M, \mathcal{F}) to (M', \mathcal{F}') , and let θ' be a closed basic one-form on M' . Then there exists a positive basic function μ on M such that $\phi^* = \mu\psi^* : H_{d+\theta'\wedge}^*(M', \mathcal{F}') \rightarrow H_{d+(\phi^*\theta')\wedge}^*(M, \mathcal{F})$.*

Proposition 3.11. (*Also in [9] for the case of foliated homeomorphisms and $\theta = 0$*) *If a map $f : (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ is a foliated homotopy equivalence and θ' is a closed basic one-form on M' , then f^* induces an isomorphism between $H_{d+\theta'\wedge}^*(M', \mathcal{F}')$ and $H_{d+f^*\theta'\wedge}^*(M, \mathcal{F})$.*

Proof. Given f and g as in the definition, by Lemma 3.7, we have linear maps

$$H_{d+\theta\wedge}^*(M', \mathcal{F}') \xrightarrow{f^*} H_{d+f^*\theta\wedge}^*(M, \mathcal{F}) \xrightarrow{g^*} H_{d+g^*f^*\theta\wedge}^*(M', \mathcal{F}').$$

Since $f \circ g$ is foliated homotopic to the identity, by Corollary 3.10, there exists a positive basic function μ such that

$$\text{id} = \mu g^* f^* = \mu (f \circ g)^* : H_{d+\theta'\wedge}^*(M', \mathcal{F}') \rightarrow H_{d+\theta'\wedge}^*(M', \mathcal{F}').$$

In particular, $g^* f^* \theta' = \theta' + \frac{d\mu}{\mu}$. After considering the map $g \circ f$, we then see that we must have that f^* and g^* are isomorphisms, since multiplication by μ is also an isomorphism on basic Lichnerowicz cohomology. \square

3.3. Invariance of the basic signature.

Proposition 3.12. *If $f : (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ is a foliated homotopy equivalence between Riemannian foliations, then (M, \mathcal{F}) and (M', \mathcal{F}') have the same codimension and dimension.*

Proof. Every class in $H_d^1(M, \mathcal{F})$ is represented by $f^*\theta$ for some closed basic one-form θ on M' , and f^* is an isomorphism from $H_{d+\theta\wedge}^k(M', \mathcal{F}')$ to $H_{d+f^*\theta\wedge}^k(M, \mathcal{F})$. Since the largest k such that $H_{d+\theta\wedge}^k(M', \mathcal{F}') \neq 0$ over all possible θ is the codimension of \mathcal{F}' (with $\theta = -\kappa'_b$), and the codimension of \mathcal{F} is computed similarly, the two codimensions of the foliations must match. Since f is in particular a homotopy equivalence of the manifolds M and M' , f^* induces isomorphisms on ordinary cohomology of M , and therefore the dimensions of M and M' are also the same. The result follows. \square

We need the following results to prove connections between foliated homotopy equivalences and cohomology.

Proposition 3.13. (*Twisted Poincaré duality for basic Lichnerowicz cohomology*) *If (M, \mathcal{F}) is a transversely oriented Riemannian foliation of codimension q on a closed manifold and θ is a closed basic one-form, then the transverse Hodge star operator $\overline{*} : \Omega^k(M, \mathcal{F}) \rightarrow \Omega^{q-k}(M, \mathcal{F})$ induces the isomorphism*

$$H_{d-\theta\wedge}^k(M, \mathcal{F}) \cong H_{d-(\kappa_b-\theta)\wedge}^{q-k}(M, \mathcal{F}).$$

Proof. From [2], [22], [12] we have the following identities for operators acting on $\Omega^k(M, \mathcal{F})$:

$$\begin{aligned}
 \bar{*}^2 &= (-1)^{k(q-k)} \\
 (\theta \lrcorner) &= (-1)^{q(k+1)} \bar{*} (\theta \wedge) \bar{*} \\
 \delta_b &= (-1)^{q(k+1)+1} \bar{*} (d - \kappa_b \wedge) \bar{*} \\
 (\theta \lrcorner) \bar{*} &= (-1)^k \bar{*} (\theta \wedge) \\
 \bar{*} (\theta \lrcorner) &= (-1)^{k+1} (\theta \wedge) \bar{*} \\
 \delta_b \bar{*} &= (-1)^{k+1} \bar{*} (d - \kappa_b \wedge) \\
 \bar{*} \delta_b &= (-1)^k (d - \kappa_b \wedge) \bar{*}.
 \end{aligned}$$

Then, letting the raised $*$ denote the L^2 adjoint with respect to basic forms,

$$(d - \theta \wedge)^* = \delta_b - \theta \lrcorner,$$

and the associated Laplacian is

$$\Delta_\theta = (\delta_b - \theta \lrcorner) (d - \theta \wedge) + (d - \theta \wedge) (\delta_b - \theta \lrcorner).$$

Then from the formulas above, if $\beta \in \Omega^k(M, \mathcal{F})$,

$$\begin{aligned}
 \bar{*} \Delta_\theta \beta &= \bar{*} (\delta_b - \theta \lrcorner) (d - \theta \wedge) \beta + \bar{*} (d - \theta \wedge) (\delta_b - \theta \lrcorner) \beta \\
 &= (-1)^{k+1} (d - \kappa_b \wedge + \theta \wedge) \bar{*} (d - \theta \wedge) \beta \\
 &\quad + (-1)^k (\delta_b + (\theta - \kappa_b) \lrcorner) \bar{*} (\delta_b - \theta \lrcorner) \beta \\
 &= (-1)^{k+1} (d - \kappa_b \wedge + \theta \wedge) (-1)^{k+1} (\delta_b - (\kappa_b - \theta) \lrcorner) \bar{*} \beta \\
 &\quad + (-1)^k (\delta_b + (\theta - \kappa_b) \lrcorner) (d - \kappa_b \wedge + \theta \wedge) \bar{*} \beta \\
 &= \Delta_{\kappa_b - \theta} \bar{*} \beta.
 \end{aligned}$$

Thus, the operator $\bar{*}$ maps Δ_θ -harmonic forms to $\Delta_{\kappa_b - \theta}$ -harmonic forms and vice versa, so from the basic Hodge theorem for basic Lichnerowicz cohomology (see [15, Section 3.3]), $\bar{*}$ induces the required isomorphism. \square

Lemma 3.14. *(In [4], [1], [15]) If (M, \mathcal{F}) is Riemannian foliation of codimension q on a closed, connected manifold and θ is a closed basic one-form, then $H_{d-\theta \wedge}^0(M, \mathcal{F}) \cong \mathbb{R}$ if and only if θ is exact. Otherwise, $H_{d-\theta \wedge}^0(M, \mathcal{F}) \cong \{0\}$.*

Proposition 3.15. *If (M, \mathcal{F}) is a transversely oriented Riemannian foliation of codimension q on a closed, connected manifold and θ is a closed basic one-form, then $H_{d-\theta \wedge}^q(M, \mathcal{F}) \cong \mathbb{R}$ if and only if $[\theta] = [\kappa_b] \in H_d^1(M, \mathcal{F})$.*

Proof. By Proposition 3.13, $H_{d-\theta \wedge}^q(M, \mathcal{F}) \cong H_{d-(\kappa_b - \theta) \wedge}^0(M, \mathcal{F})$. By Lemma 3.14, this group is \mathbb{R} if and only if $\kappa_b - \theta$ is exact and is zero otherwise. \square

It has been shown previously by H. Nozawa in [19], [20] that the Álvarez class $[\kappa_b]$ is continuous with respect to smooth deformations of Riemannian foliations. The following proposition extends these results further.

Proposition 3.16. *If $f : M \rightarrow M'$ is a foliated homotopy equivalence between transversely oriented Riemannian foliations with basic mean curvatures κ_b and κ'_b , respectively, then $[f^* \kappa'_b] = [\kappa_b] \in H_d^1(M, \mathcal{F})$.*

Proof. By Proposition 3.11, $f^* : H_{d-\kappa'_b \wedge}^q(M', \mathcal{F}') \rightarrow H_{d-f^*\kappa'_b \wedge}^q(M, \mathcal{F})$ is an isomorphism, so $H_{d-f^*\kappa'_b \wedge}^q(M, \mathcal{F}) \cong \mathbb{R}$ by the previous proposition. Therefore, by the same proposition, $[f^*\kappa'_b] = [\kappa_b] \in H_d^1(M, \mathcal{F})$. \square

Corollary 3.17. *If $f : M \rightarrow M'$ is a foliated homotopy equivalence between Riemannian foliations, then there exist bundle-like metrics on (M, \mathcal{F}) and (M', \mathcal{F}') such that $f^* : \widetilde{H}^k(M', \mathcal{F}') \rightarrow \widetilde{H}^k(M, \mathcal{F})$ is defined and is an isomorphism.*

Proof. By Proposition 3.11, $f^* : \widetilde{H}^k(M', \mathcal{F}') \rightarrow H_{d-\frac{1}{2}f^*\kappa'_b \wedge}^k(M, \mathcal{F})$ is an isomorphism, with $\theta' = \frac{1}{2}\kappa'_b$. By Proposition 3.16, $[f^*\kappa'_b] = [\kappa_b]$. We then modify the bundle-like metric so that $\kappa = \kappa_b = f^*\kappa'$ exactly by using [7] to make the mean curvature basic and then by multiplying the leafwise metric by a conformal factor to set the element of $[\kappa_b]$. \square

Recall that given a Riemannian foliation (M, \mathcal{F}) , the leafwise volume form $\chi_{\mathcal{F}}$ is related to the mean curvature κ by Rummeler's formula [25] as $d\chi_{\mathcal{F}} = -\kappa \wedge \chi_{\mathcal{F}} + \varphi_0$, where φ_0 is a $(p+1)$ -form ($\dim \mathcal{F} = p$) on M such that $X_1 \lrcorner \cdots \lrcorner (X_p \lrcorner \varphi_0) = 0$ for all $X_1, \dots, X_p \in \Gamma(L)$. We have

Lemma 3.18. *Let $f : (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ be a foliated homotopy equivalence between Riemannian foliations of codimension 2ℓ . Let ν and ν' be the transverse volume forms for these metrics on M and M' , respectively. Then there is a real nonzero constant λ such that*

$$\langle f^*(\alpha), \nu \rangle_M = \lambda \frac{\text{Vol}(M)}{\text{Vol}(M')} \langle \alpha, \nu' \rangle_{M'}.$$

for all $\alpha \in \Omega^{2\ell}(M', \mathcal{F}')$.

Proof. We choose a metric on (M', \mathcal{F}') such that the mean curvature κ' is basic. Then $[f^*\kappa'] = [\kappa_b]$ for any metric on M by Proposition 3.16. As in the last proof, we then modify the bundle-like metric so that $\kappa = \kappa_b = f^*\kappa'$ exactly. Since α is a basic 2ℓ -form, it defines a class in $H_{d-\kappa' \wedge}^{2\ell}(M', \mathcal{F}') \simeq H_b^0(M', \mathcal{F}') \simeq \mathbb{R}[\nu']$. We need to check that $[\nu'] \neq 0$ in the cohomology group $H_{d-\kappa' \wedge}^{2\ell}(M', \mathcal{F}')$. Assume by contradiction that it is zero, then there exists some basic form $\gamma \in \Omega^{2\ell-1}(M', \mathcal{F}')$ of degree $2\ell-1$ such that $\nu' = d\gamma - \kappa' \wedge \gamma$. Then, we compute

$$\begin{aligned} \int_{M'} \nu' \wedge \chi_{\mathcal{F}'} &= \int_{M'} d\gamma \wedge \chi_{\mathcal{F}'} - \kappa' \wedge \gamma \wedge \chi_{\mathcal{F}'} \\ &= \int_{M'} d(\gamma \wedge \chi_{\mathcal{F}'}) + \gamma \wedge d\chi_{\mathcal{F}'} - \kappa' \wedge \gamma \wedge \chi_{\mathcal{F}'} \\ &= \int_{M'} -\gamma \wedge \kappa' \wedge \chi_{\mathcal{F}'} - \kappa' \wedge \gamma \wedge \chi_{\mathcal{F}'} = 0. \end{aligned}$$

This would imply $\text{Vol}(M') = 0$, a contradiction. Therefore $[\nu'] \neq 0$. Hence $\alpha = \beta\nu' + (d - \kappa' \wedge)\phi$ for some $\beta \in \mathbb{R}$ and $\phi \in \Omega^{2\ell-1}(M', \mathcal{F}')$. Thus,

$$\langle \alpha, \nu' \rangle_{M'} = \int_{M'} \alpha \wedge \chi_{\mathcal{F}'} = \int_{M'} (\beta\nu' \wedge \chi_{\mathcal{F}'} + (d\phi - \kappa' \wedge \phi) \wedge \chi_{\mathcal{F}'} = \beta \text{Vol}(M').$$

The last term vanishes as a consequence of the previous computation (just replace γ by ϕ). Also, we have

$$\begin{aligned} \langle f^*(\alpha), \nu \rangle_M &= \int_M \beta f^*(\nu') \wedge \chi_{\mathcal{F}} + f^*(d\phi - \kappa' \wedge \phi) \wedge \chi_{\mathcal{F}} \\ &= \int_M \beta f^*(\nu') \wedge \chi_{\mathcal{F}} + (d(f^*\phi) - f^*(\kappa') \wedge f^*\phi) \wedge \chi_{\mathcal{F}} \\ &= \int_M \beta f^*(\nu') \wedge \chi_{\mathcal{F}} + (d - \kappa \wedge)(f^*\phi) \wedge \chi_{\mathcal{F}} = \int_M \beta f^*(\nu') \wedge \chi_{\mathcal{F}} = \beta \lambda \text{Vol}(M). \end{aligned}$$

In the last equality, we used the fact that $f^*\nu'$ defines a cohomology class $[f^*\nu']$ in $H_{d-\kappa\wedge}^q(M, \mathcal{F}) \cong \mathbb{R}$ and therefore we can write that $f^*\nu' = \lambda\nu + (d^M - \kappa\wedge)\varphi$ for some real number λ . Furthermore, since f is a foliated homotopy equivalence, by Proposition 3.11, it induces an isomorphism from $H_{d-\kappa'\wedge}^q(M', \mathcal{F}')$ to $H_{d-\kappa\wedge}^q(M, \mathcal{F})$, so $[f^*\nu']$ is a nonzero class. Thus, the constant λ is nonzero. \square

Theorem 3.19. *Let $f : M \rightarrow M'$ be a foliated homotopy equivalence between two Riemannian foliations of codimension 2ℓ and transverse volume forms ν, ν' respectively. Then $\sigma(M, \mathcal{F}) = \sigma(M', \mathcal{F}')$ if f preserves the transverse orientation and $\sigma(M, \mathcal{F}) = -\sigma(M', \mathcal{F}')$ otherwise.*

Proof. We assume that ℓ is even, because the result is trivial when ℓ is odd. By Corollary 3.17, we may choose metrics such that $f^* : \tilde{H}^\ell(M', \mathcal{F}') \rightarrow \tilde{H}^\ell(M, \mathcal{F})$ is an isomorphism and is well-defined. For any $\alpha'_1, \alpha'_2 \in \tilde{H}^\ell(M', \mathcal{F}')$, we compute

$$\begin{aligned} A_{\mathcal{F}}(f^*\alpha'_1, f^*\alpha'_2) &= \int_M f^*\alpha'_1 \wedge f^*\alpha'_2 \wedge *\nu \\ &= \langle f^*(\alpha'_1 \wedge \alpha'_2), \nu \rangle_M \\ &= \lambda \frac{\text{Vol}(M)}{\text{Vol}(M')} \langle \alpha'_1 \wedge \alpha'_2, \nu' \rangle_{M'} = \lambda \frac{\text{Vol}(M)}{\text{Vol}(M')} A_{\mathcal{F}'}(\alpha'_1, \alpha'_2). \end{aligned}$$

Then by computing signatures of these quadratic forms, the conclusion follows. \square

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LEBANESE UNIVERSITY, FACULTY OF SCIENCES II, DEPARTMENT OF MATHEMATICS, P.O. BOX 90656
FANAR-MATN, LEBANON

E-mail address, G. Habib: ghabib@ul.edu.lb

DEPARTMENT OF MATHEMATICS, TEXAS CHRISTIAN UNIVERSITY, FORT WORTH, TEXAS 76129, USA

E-mail address, K. Richardson: k.richardson@tcu.edu