COMPACTNESS AND GLOBAL ESTIMATES FOR A FOURTH ORDER EQUATION OF CRITICAL SOBOLEV GROWTH ARISING FROM CONFORMAL GEOMETRY

EMMANUEL HEBEY, FRÉDÉRIC ROBERT, AND YULIANG WEN

ABSTRACT. Given (M, g) a smooth compact Riemannian manifold of dimension $n \geq 5$, we investigate compactness for fourth order critical equations like

$$P_g u = u^{2^\sharp - 1} \ ,$$

where $P_g u = \Delta_g^2 u + b\Delta_g u + cu$ is a Paneitz-Branson operator with constant coefficients b and c, u is required to be positive, and $2^\sharp = \frac{2n}{n-4}$ is critical from the Sobolev viewpoint. We prove that such equations are compact on locally conformally flat manifolds, unless b lies in some closed interval associated to the spectrum of the smooth symmetric $(2, 0)$-tensor field involved in the definition of the geometric Paneitz-Branson operator.

CONTENTS

In 1983, Paneitz [**?**] introduced a conformally fourth order operator defined on 4-dimensional Riemannian manifolds. Branson [**?**] generalized the definition to n-dimensional Riemannian manifolds, $n \geq 5$. While the conformal Laplacian is associated to the scalar curvature, the geometric Paneitz-Branson operator is associated to a notion of Q-curvature. The Q-curvature in dimension 4, and for locally conformally flat manifolds, turns out to be the integrand in the Gauss-Bonnet formula for the Euler characteristic. We let in this article (M, g) be a smooth compact locally conformally flat Riemannian n-manifold, $n \geq 5$, and consider fourth order equations of critical Sobolev growth like

$$\Delta_g^2 u + b_\alpha \Delta_g u + c_\alpha u = u^{2^\sharp - 1} \ , \tag{0.1}$$

where $\Delta_g = -\mathrm{div}_g \nabla$, α is an integer, (b_α) and (c_α) are converging sequences of positive real numbers with positive limits, $c_\alpha \leq b_\alpha^2/4$ for all α, u is required to be positive, and $2^\sharp = \frac{2n}{n-4}$ is critical from the Sobolev viewpoint. The family of equations (**??**) may of course reduce to one equation when the sequences consisting of the b_α's and c_α's are constant sequences. Equations like (**??**) are modelized on the conformal equation associated to the Paneitz-Branson operator when the

Date: July 20, 2004. Revised January 25, 2005.

1991 *Mathematics Subject Classification.* 58E30, 58J05.

Key words and phrases. Blow-up, compactness, fourth order operator, critical exponent.

The research of the third author was supported by the National Science Foundation of China and the Shanghai Priority Academic Discipline.

background metric g is Einstein. In the case of an arbitrary manifold, the conformal equation associated to the Paneitz-Branson operator reads as

$$\Delta_g^2 u - div_g\left(A_g du\right) + \frac{n-4}{2} Q_g u = \frac{n-4}{2} Q_{\hat{g}} u^{2^\sharp - 1} \ , \tag{0.2}$$

where Q_g and $Q_{\hat{g}}$ are the Q-curvature of g and $\hat{g} = u^{4/(n-4)} g$,

$$A_g = \frac{(n-2)^2 + 4}{2(n-1)(n-2)} S_g g - \frac{4}{n-2} Rc_g \ , \tag{0.3}$$

and Rc_g and S_g are respectively the Ricci curvature and scalar curvature of g. When g is Einstein, equation (**??**) becomes

$$\Delta_g^2 u + \alpha_n S_g \Delta_g u + a_n S_g^2 u = \frac{n-4}{2} Q_{\hat{g}} u^{2^\sharp - 1} \ ,$$

where α_n and a_n are positive dimensional constants such that $a_n < \alpha_n^2/4$, and S_g is constant since g is Einstein. In particular, when we ask for $Q_{\hat{g}}$ to be constant, we recover an equation like (**??**). More material on the Paneitz-Branson operator can be found in the very nice survey articles by Chang [**?**] and Chang-Yang [**?**].

In what follows we let $H_2^2(M)$ be the Sobolev space consisting of functions u in $L^2(M)$ which are such that $|\nabla u|$ and $|\nabla^2 u|$ are also in $L^2(M)$. Thanks to the Bochner-Lichnerowicz-Weitzenböck formula, a possible norm on $H_2^2(M)$ is

$$\|u\|_{H_2^2}^2 = \int_M (\Delta_g u)^2 dv_g + \sum_{i=0}^{1} \int_M |\nabla^i u|^2 dv_g \ .$$

A weak nonnegative solution $u \in H_2^2(M)$ of one of the equations in (**??**) is smooth and either is the zero function or is everywhere positive. A sequence (u_α) in $H_2^2(M)$ of positive functions is then said to be a sequence of solutions of the family (**??**) if for any α, u_α is a solution of (**??**). Examples of compact manifolds, including locally conformally flat manifolds, for which equations like (**??**) have nonconstant solutions for abitrarily large b_α's and c_α's are in Felli, Hebey and Robert [**?**].

In what follows we say that the family of equations (**??**) is pseudo-compact if for any bounded sequence (u_α) in $H_2^2(M)$ of positive solutions of (**??**) which converges weakly in $H_2^2(M)$, the weak limit u^0 of the u_α's is not zero. Pseudo-compactness is of traditional interest since it provides nontrivial solutions of the limit equation we get from (**??**) by letting $\alpha \to +\infty$. In contrast to pseudo-compactness, we say that the family of equations (**??**) is compact if any bounded sequence (u_α) in $H_2^2(M)$ of positive solutions of (**??**) is actually bounded in $C^{4,\theta}(M)$, $0 < \theta < 1$, and thus converges, up to a subsequence, in $C^4(M)$ to some function u^0. Compactness is a stronger notion than pseudo-compactness since by the Sobolev inequality, and by (**??**), $\|u_\alpha\|_{H_2^2} \geq C$ for some $C > 0$ independent of α. With respect to blow-up terminology, see Section **??** for details, pseudo-compactness allows bubbles in the H_2^2-decomposition of sequences of solutions of (**??**), while compactness does not.

For A_g the smooth symmetric $(2,0)$-tensor field in (**??**), we denote by $\lambda_i(A_g)_x$, $i = 1, \ldots, n$, the g-eigenvalues of $A_g(x)$, and define λ_1 to be the infimum over i and x of the $\lambda_i(A_g)_x$'s, and λ_2 to be the supremum over i and x of the $\lambda_i(A_g)_x$'s. Then we let $\mathcal{S}_c$ be the critical set (or wild spectrum of A_g) defined by

$$\mathcal{S}_c = \left\{ \lambda \in \mathbb{R} \text{ s.t. } \lambda_1 \leq \lambda \leq \lambda_2 \right\} . \tag{0.4}$$

Pseudo compactness for second order elliptic equations of Yamabe type have been intensively studied. Compactness for second order equations of Yamabe type goes back to the remarkable work of Schoen on the Yamabe equation [**?**, **?**, **?**, **?**]. Further results were then obtained by Druet [**?**, **?**]. Motivations for our work were Schoen [**?**] and Druet [**?**]. Possible related references on second and fourth order equations are Brendle [**?**], Chang [**?**], Chang and Yang [**?**, **?**], Chen and Lin [**?**], Devillanova and Solimini [**?**], Djadli, Hebey and Ledoux [**?**], Djadli, Malchiodi and Ould Ahmedou [**?**, **?**], Druet and Hebey [**?**], Druet, Hebey and Robert [**?**], Han and Li [**?**], Hebey and Robert [**?**], Li and Zhu [**?**], Lin [**?**], Lions [**?**], Lu, Wei and Xu [**?**], Marques [**?**], Robert [**?**], Robert and Struwe [**?**], and Struwe [**?**].

We prove in this article that the following general results hold. We state Theorems **??** and **??** for families of equations like (**??**), but recall that, of course, this includes the more traditional viewpoint of one single equation when the b_α's and c_α's are independent of α.

Theorem 0.1. *Let (M,g) be a smooth compact locally conformally flat manifold of dimension n, and (b_α), (c_α) be converging sequences of positive real numbers with positive limits and such that $c_\alpha \leq b_\alpha^2/4$ for all α. We consider equations like*

$$\Delta_g^2 u + b_\alpha \Delta_g u + c_\alpha u = u^{2^\sharp - 1} \qquad (E_\alpha)$$

*and assume that $b_\infty \notin \mathcal{S}_c$, where b_∞ is the limit of the b_α's and $\mathcal{S}_c$ is the critical set given by (**??**). Then the family (E_α) is pseudo-compact when $n \geq 6$, and compact when $n \geq 9$.*

Theorem **??** is a complement to the compactness assertion in Theorem **??** when the dimension $n = 6, 7, 8$ and b_∞ lies below the lower bound λ_1 of $\mathcal{S}_c$.

Theorem 0.2. *Let (M,g) be a smooth compact locally conformally flat manifold of dimension $n = 6, 7, 8$, and (b_α), (c_α) be converging sequences of positive real numbers with positive limits and such that $c_\alpha \leq b_\alpha^2/4$ for all α. We consider equations like*

$$\Delta_g^2 u + b_\alpha \Delta_g u + c_\alpha u = u^{2^\sharp - 1} \qquad (E_\alpha)$$

*and assume that $b_\infty < \min \mathcal{S}_c$, where b_∞ is the limit of the b_α's and $\mathcal{S}_c$ is the critical set given by (**??**). Then the family (E_α) is compact.*

A major stress in proving Theorems **??** and **??** is to understand large solutions. Namely, solutions with large energies which, in studying their possible blow-up, involve multi-bubbles. Specific examples of blowing-up sequences of solutions of equations like (**??**) are discussed in Section **??**. These examples respectively indicate that the case $n = 8$ with respect to compactness is most likely to be special, that a condition like $b_\infty \notin \mathcal{S}_c$ is sharp, and that there are equations like (**??**) which possess unbounded sequences of solutions in H_2^2. Section **??** is devoted to preliminary material on blow-up theory. We discuss in this section the H_2^2-decomposition and pointwise estimates for sequences of solutions of equations like (**??**). Relative concentrations for sequences (u_α) of solutions of equations like (**??**) are discussed in Sections **??** and **??** when the weak limit u^0 of the u_α's is zero. The proof of the pseudo-compactness part of Theorem **??** in Section **??** relies on these concentrations. Sections **??** to **??** are devoted to refined estimates on sequences (u_α) of solutions of equations like (**??**) when we do not assume anything on u^0. The proof of the compactness part of Theorem **??** and of Theorem **??** in Section **??**

rely on these estimates. Section **??** is devoted to the existence of a Green's function, and estimates on this function, for second order operators with nondifferentiable coefficients – a technical result we need in Section **??**. The assumption that our manifolds have to be locally conformally flat is required only in Sections **??** and **??**.

In the sequel, in order to fix notations, the limit equation we get from (**??**) by letting $\alpha \to +\infty$ is the equation

$$\Delta_g^2 u + b_\infty \Delta_g u + c_\infty u = u^{2^\sharp - 1} \ , \tag{0.5}$$

where b_∞ and c_∞ are the limits of (b_α) and (c_α). We let H_k^q be the Sobolev space of functions in L^q with k derivatives in L^q, and $2^\star = 2n/(n-2)$ be the critical Sobolev exponent for the embeddings of H_1^2 in L^p-spaces.

1. Examples and comments on the Theorems

We discuss three specific examples which respectively indicate that the case $n = 8$ with respect to compactness is most likely to be special, that a condition like $b_\infty \notin \mathcal{S}_c$ is sharp, and that there are equations like (**??**) which possess unbounded sequences of solutions in H_2^2. For that purpose, we let (S^n, g_0) be the unit n-sphere. The geometric equation (**??**) on the sphere reads as

$$\Delta_{g_0}^2 u + \overline{\alpha}_n \Delta_{g_0} u + \overline{a}_n u = u^{2^\sharp - 1} \ , \tag{1.1}$$

where $\overline{\alpha}_n = \frac{n^2-2n-4}{2}$ and $\overline{a}_n = \frac{n(n-4)(n^2-4)}{16}$. In particular, for $\mathcal{S}_c$ as in (**??**), $\mathcal{S}_c = \{\overline{\alpha}_n\}$. Given $\beta > 1$ and $x_0 \in S^n$, we let $U_{x_0,\beta}$ be the function on S^n defined by

$$U_{x_0,\beta}(x) = \overline{a}_n^{\frac{n-4}{8}} \left(\frac{\sqrt{\beta^2 - 1}}{\beta - \cos d_{g_0}(x_0, x)} \right)^{\frac{n-4}{2}} \ . \tag{1.2}$$

As is well known, for any $\beta > 1$ and any $x_0 \in S^n$, the $U_{x_0,\beta}$'s are solutions of (**??**). This can be checked directly, or using conformal invariance and the Lin's result we discuss in Section **??**. The $L^{2^\sharp}$-norm of $U_{x_0,\beta}$ is a positive constant independent of β and x_0. Moreover, $U_{x_0,\beta}(x) \to 0$ as $\beta \to 1$ if $x \neq x_0$, while $U_{x_0,\beta}(x_0) \to +\infty$ as $\beta \to 1$. In particular, (**??**) is not compact, neither pseudo-compact. This is coherent with Theorems **??** and **??** since in this situation the b_α's are constant and all in $\mathcal{S}_c$ (so that, in particular, $b_\infty \in \mathcal{S}_c$).

The first example we really want to discuss in this section is as follows. We fix $\lambda > 1$, $\beta > 1$, and $x_0 \in S^n$. We let also (β_α) be a sequence such that $\beta_\alpha > 1$ for all α, and $\beta_\alpha \to 1$ as $\alpha \to +\infty$. We define the u_α's by

$$u_\alpha = \lambda U_{x_0,\beta} + U_{x_0,\beta_\alpha} \ . \tag{1.3}$$

Then the u_α's are solutions of equations like (**??**). More precisely, if we let L_{g_0} be the operator $L_{g_0} u = \Delta_{g_0} u + \frac{\overline{\alpha}_n}{2} u$, the u_α's are such that

$$\Delta_{g_0}^2 u_\alpha + b_\alpha \Delta_{g_0} u_\alpha + c_\alpha u_\alpha = u_\alpha^{2^\sharp - 1} \tag{1.4}$$

for all α, where the b_α's and c_α's are given by $b_\alpha = \overline{\alpha}_n + h_\alpha$, $c_\alpha = \overline{a}_n + \frac{\overline{\alpha}_n}{2} h_\alpha$, and

$$h_\alpha = \frac{\left(\lambda U_{x_0,\beta} + U_{x_0,\beta_\alpha}\right)^{2^\sharp - 1} - \lambda U_{x_0,\beta}^{2^\sharp - 1} - U_{x_0,\beta_\alpha}^{2^\sharp - 1}}{\lambda L_{g_0} U_{x_0,\beta} + L_{g_0} U_{x_0,\beta_\alpha}} \ . \tag{1.5}$$

Noting that for $u > 0$ a solution of (**??**),

$$L^2_{g_0} u = u^{2^\sharp - 1} + \big(\frac{\overline{\alpha}_n^2}{4} - \overline{a}_n\big)u$$

and that $\overline{a}_n < \overline{\alpha}_n^2/4$, it follows from the maximum principle that $L_{g_0} u > 0$ so that h_α in (**??**) is well defined. Easy computations give that the sequence consisting of the h_α's given by (**??**) is bounded in $L^\infty(S^n)$ when $n \geq 8$. Moreover, if we assume that $n = 8$, then the b_α's and c_α's converge in $L^p(S^8)$ for all $p \geq 1$ as $\alpha \to +\infty$, with the property (which stops to hold when $n \geq 9$) that

$$\liminf_{\alpha \to +\infty} \inf_{B_{x_0}(R\sqrt{\beta_\alpha - 1})} b_\alpha > \overline{\alpha}_n$$

for all $R > 0$. In particular, the pertinent quantities $b_\alpha(x_0)$ are such that

$$\liminf_{\alpha \to +\infty} b_\alpha(x_0) > \overline{\alpha}_n$$

while, by construction, $u_\alpha(x_0) \to +\infty$ as $\alpha \to +\infty$. Summarizing, when $n = 8$, the u_α's are solutions of (**??**), an equation like (**??**), the b_α's and c_α's in (**??**) are bounded in $L^\infty(S^8)$, they converge in $L^p(S^8)$ for all p, and the u_α's blow up at x_0 with $b_\infty \notin \mathcal{S}_c$ where, here, b_∞ is the limit of the $b_\alpha(x_0)$'s. Even if the b_α's and c_α's are not constant functions, and the convergence of the b_α's and c_α's is only in L^p, this example gives strong indications that, with respect to the assertion on compactness in Theorems **??** and **??**, a particular phenomenon is most likely to happen when the dimension $n = 8$. For second order equations of critical growth, see Druet [**?**], the critical dimension is $n = 6$.

Concerning the second example we discuss in this section, we let $k \in \mathbb{N}$, where $k \geq 1$, we let (x^i_α), $i = 1, \dots, k$, be k converging sequences of points in S^n, and let (β_α) be a sequence of real numbers such that $\beta_\alpha > 1$ for all α, and $\beta_\alpha \to 1$ as $\alpha \to +\infty$. Then we define the function u_α by

$$u_\alpha = \sum_{i=1}^k U_{x^i_\alpha, \beta_\alpha} \ . \tag{1.6}$$

As is easily checked, the u_α's are such that for any α,

$$\Delta^2_{g_0} u_\alpha + \overline{\alpha}_n \Delta_{g_0} u_\alpha + c_\alpha u_\alpha = u_\alpha^{2^\sharp - 1} \ , \tag{1.7}$$

where $\overline{\alpha}_n$ is as in (**??**), $c_\alpha = \overline{a}_n + h_\alpha$, $\overline{a}_n$ is as in (**??**), and

$$h_\alpha = \frac{\big(\sum_{i=1}^k U_{x^i_\alpha, \beta_\alpha}\big)^{2^\sharp - 1} - \sum_{i=1}^k U^{2^\sharp - 1}_{x^i_\alpha, \beta_\alpha}}{\sum_{i=1}^k U_{x^i_\alpha, \beta_\alpha}} \ . \tag{1.8}$$

We assume that $n \geq 12$ and choose the x^i_α's and β_α's such that for any α, $x^i_\alpha \neq x^j_\alpha$, and such that for instance, $d^{14}_\alpha \geq \beta_\alpha - 1$ where $d_\alpha = \inf_{i \neq j} d_{g_0}(x^i_\alpha, x^j_\alpha)$. Similar arguments to those used in Druet and Hebey [**?**] in the second order case (see also Druet and Hebey [**?**]) give that $h_\alpha \to 0$ in $C^1(S^n)$ as $\alpha \to +\infty$. In particular,

$$c_\alpha \to \overline{a}_n \ \text{ in } C^1(S^n)$$

as $\alpha \to +\infty$, and the u_α's blow up with k bubbles in their H^2_2-decomposition (see Section **??** for the terminology). Moreover, as is easily checked, we can choose the u_α's in such a way that for any $1 \leq m \leq k$, the u_α's have m arbitrary geometrical blow-up points $x_1, \dots, x_m$ (the limits of the x^i_α's as $\alpha \to +\infty$), and such that the u_α's have an arbitrary number $k(j)$ of bubbles (B_α) in their H^2_2-decomposition with

centers x_α^i converging to x_j (as long as m and the $k(j)$'s satisfy $\sum_{j=1}^m k(j) = k$). Noting that in this example, $b_\alpha = \overline{a}_n$ is in $\mathcal{S}_c$, and even if the c_α's are not constant functions, this provides another illustration (in addition to the solutions (**??**) of (**??**) on S^n) of the fact that a condition like $b_\infty \notin \mathcal{S}_c$ in Theorem **??** is sharp. This example extends to the projective space, and more generally to any quotient of the sphere.

Concerning the third and last example we discuss in this section, the idea is to let $k \to +\infty$ in the above example (**??**). We still assume that $n \geq 12$ and let (k_α) be a sequence of integers such that $k_\alpha \to +\infty$ as $\alpha \to +\infty$. For any α, we let $x_\alpha^1, \ldots, x_\alpha^{k_\alpha}$ be k_α distinct points in S^n, and let d_α be the infimum over $i \neq j \in \{1, \ldots, k_\alpha\}$ of the distances $d_{g_0}(x_\alpha^i, x_\alpha^j)$. We let (β_α) be a sequence of real numbers such that $\beta_\alpha > 1$ for all α and such that $\beta_\alpha \to 1$ as $\alpha \to +\infty$. We assume, for instance, that $d_\alpha^{14} \geq k_\alpha(\beta_\alpha - 1)$ for all α, and that $k_\alpha^{20}(\beta_\alpha - 1) \to 0$ as $\alpha \to +\infty$. We define u_α by

$$u_\alpha = \sum_{i=1}^{k_\alpha} U_{x_\alpha^i, \beta_\alpha} \ . \tag{1.9}$$

Then the u_α's are solution of (**??**) and (**??**) with $k = k_\alpha$, and here again, similar arguments to those used in Druet and Hebey [**?**] in the second order case (see also Druet and Hebey [**?**]) give that $c_\alpha \to \overline{a}_n$ in $C^1(S^n)$ as $\alpha \to +\infty$. Independently, we easily get that $\|u_\alpha\|_{H_2^2} \to +\infty$ as $\alpha \to +\infty$. The u_α's are solutions of (**??**), an equation like (**??**), the c_α's in (**??**) are such that $c_\alpha \to \overline{a}_n$ in $C^1(S^n)$ as $\alpha \to +\infty$, and $\|u_\alpha\|_{H_2^2} \to +\infty$ as $\alpha \to +\infty$. In particular, there are equations like (**??**) for which we do not have an a priori H_2^2-bound on the energy of the solutions (and, for such general equations, the assumption on the H_2^2-norm in the definition of pseudo-compactness or compactness is necessary). As above, this example extends to the projective space, and more generally to any quotient of the sphere.

By the work of Lin [**?**], where smooth positive solutions in the Euclidean space $\mathbb{R}^n$ of the critical equation $\Delta^2 u = u^{2^\sharp - 1}$ are classified, we easily get that the $U_{x_0,\beta}$'s in (**??**), together with the constant solution $\overline{a}_n^{(n-4)/8}$, are the only positive solutions of (**??**) in S^n. Their energy, defined as the $L^{2^\sharp}$-norm of the solution, is a dimensional constant and, in particular, (**??**) has one and only one admissible level of energy $\overline{a}_n^{n/4}\omega_n$. On the other hand, we just saw that there are sequences (u_α) of equations like (**??**) in S^n such that $c_\alpha \to \overline{a}_n$ in $C^1(S^n)$ as $\alpha \to +\infty$, so that, in some sense, (**??**) converges C^1 to (**??**), and such that $\|u_\alpha\|_{2^\sharp} \to +\infty$ as $\alpha \to +\infty$. If necessary, this illustrates how much equations like (**??**) are unstable with respect to their lower order terms.

As a general remark we mention that a reasonable guess on Theorems **??** and **??** is that Theorem **??** remains true if we only ask that $b_\infty \neq \frac{1}{n} tr_g(A_g)_x$ for all $x \in M$, and that Theorem **??** remains true if we only ask that $b_\infty < \frac{1}{n} tr_g(A_g)_x$ for all $x \in M$, where $tr_g(A_g)$ is the trace with respect to g of A_g. This would be true if we could develop a C^0-theory for critical fourth order equations like the one developed for critical second order equations by Druet, Hebey and Robert [**?**]. When g is Einstein, and hence (M, g) is a space form since we also assumed that g is locally conformally flat, $tr_g(A_g)$ is constant and $\mathcal{S}_c = \{\frac{1}{n} tr_g(A_g)\}$ so that we are back to what we proved.

2. Preliminary material

Let $D_2^2(\mathbb{R}^n)$ be the Beppo-Levi space defined as the completion of the space of smooth functions with compact support in $\mathbb{R}^n$ w.r.t. the norm $\|u\| = \|\Delta u\|_2$. Nonnegative solutions $u \in D_2^2(\mathbb{R}^n)$ of the critical Euclidean equation

$$\Delta^2 u = u^{2^\sharp - 1} \tag{2.1}$$

have been classified by Lin [**?**] (see also Hebey-Robert [**?**] for a slight additional remark on Lin's result). They all are of the form

$$u_{\lambda,x_0}(x) = \left(\frac{\lambda}{\lambda^2 + \frac{|x-x_0|^2}{\sqrt{\lambda_n}}} \right)^{\frac{n-4}{2}} , \tag{2.2}$$

where $\lambda > 0$, $x_0 \in \mathbb{R}^n$, and $\lambda_n = n(n-4)(n^2-4)$. Let K_n be the sharp constant for the Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{2^\sharp} dx \right)^{2/2^\sharp} \leq K_n \int_{\mathbb{R}^n} (\Delta u)^2 \, dx \, . \tag{2.3}$$

The sharp inequality (**??**) has been intensively studied. In particular by Beckner [**?**], Edmunds-Fortunato-Janelli [**?**], Lieb [**?**], and Lions [**?**]. As a consequence of their work,

$$K_n^{-1} = \pi^2 \lambda_n \Gamma\left(\frac{n}{2}\right)^{4/n} \Gamma\left(n\right)^{-4/n} \, ,$$

where Γ is the Euler function, and the u_{λ,x_0}'s in (**??**) are extremal functions for the sharp inequality (**??**). The extension of (**??**) to Riemannian manifolds is studied in Hebey [**?**] (following previous work by Hebey and Vaugon [**?**] in the second order case).

In what follows we let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, and we discuss the Sobolev decomposition and pointwise estimates for sequences of solutions of (**??**). If (x_α) is a converging sequence in M, and (μ_α) is such that $\mu_\alpha > 0$ and $\mu_\alpha \to 0$ as $\alpha \to +\infty$, we define the standard bubble (B_α) with respect to the x_α's and μ_α's by

$$B_\alpha(x) = \eta(r_\alpha) \left(\frac{\mu_\alpha}{\mu_\alpha^2 + \frac{d_g(x_\alpha,x)^2}{\sqrt{\lambda_n}}} \right)^{\frac{n-4}{2}} , \tag{2.4}$$

where d_g is the distance with respect to g, $r_\alpha = d_g(x_\alpha, x)$, λ_n is as above, and $\eta : \mathbb{R} \to \mathbb{R}$ is a smooth nonnegative cutoff function with small support (less than the injectivity radius of the manifold) around 0. The x_α's are referred to as the centers of (B_α), and the μ_α's as the weights of (B_α). It is easily checked that

$$\|B_\alpha\|_{H_2^2}^2 = K_n^{-n/4} + o(1) \, ,$$

where K_n is as above, and $o(1) \to 0$ as $\alpha \to +\infty$. Up to $o(1)$, the H_2^2-norm of a bubble is a dimensional constant independent of the bubble. As a remark, for any $R > 0$,

$$\lim_{\alpha \to +\infty} \int_{B_{x_\alpha}(R\mu_\alpha)} (\Delta_g B_\alpha)^2 \, dv_g = K_n^{-n/4} + \varepsilon_R \, ,$$

where the sequence (ε_R) is such that $\varepsilon_R \to 0$ as $R \to +\infty$, while the integral of $(\Delta_g B_\alpha)^2$ over $B_{x_\alpha}(\delta_\alpha \mu_\alpha)$ goes to zero as $\alpha \to +\infty$ if $\delta_\alpha \to 0$ as $\alpha \to +\infty$. We say the H_2^2-range of interaction of (B_α) is of the order μ_α. On the other hand, for any $R > 0$,

$$\inf_{x \in B_{x_\alpha}(R\sqrt{\mu_\alpha})} B_\alpha(x) = \left(\frac{\sqrt{\lambda_n}}{R^2}\right)^{\frac{n-4}{2}} + \varepsilon_\alpha \ ,$$

where the sequence (ε_α) is such that $\varepsilon_\alpha \to 0$ as $\alpha \to +\infty$, while the supremum over $M \backslash B_{x_\alpha}(R_\alpha \sqrt{\mu_\alpha})$ of B_α goes to zero as $\alpha \to +\infty$ if $R_\alpha \to +\infty$ as $\alpha \to +\infty$. We say the C^0-range of interaction of (B_α) is of the order $\sqrt{\mu_\alpha}$.

Lemma **??** below was proved in Hebey-Robert [**?**]. It extends to fourth order equations of critical Sobolev growth the well-known result of Struwe [**?**] proved in the case of second order equations of critical Sobolev growth. We state Lemma **??** with no proof and refer to Hebey-Robert [**?**] for more details.

Lemma 2.1. *Let (u_α) be a bounded sequence in $H_2^2(M)$ of nonnegative solutions of (**??**). Then there exists $k \in \mathbb{N}$, $u^0 \geq 0$ a nonnegative solution of (**??**), and k bubbles (B_α^i), $i = 1, \ldots, k$, such that, up to a subsequence,*

$$u_\alpha = u^0 + \sum_{i=1}^k B_\alpha^i + R_\alpha \ , \ \text{and}$$

$$\|u_\alpha\|_{H_2^2}^2 = \|u^0\|_{H_2^2}^2 + \sum_{i=1}^k \|B_\alpha^i\|_{H_2^2}^2 + o(1) \ ,$$

where $R_\alpha \to 0$ in $H_2^2(M)$ as $\alpha \to +\infty$, and $o(1) \to 0$ as $\alpha \to +\infty$.

Lemma **??** is what we refer to as the H_2^2-decomposition of the u_α's. When $k \geq 1$ in Lemma **??**, we say that the u_α's blow up. As an illustration of Lemma **??**, let (x_α) be a converging sequence of points in S^n, and (β_α) be a sequence of real numbers such that $\beta_\alpha > 1$ for all α and $\beta_\alpha \to 1$ as $\alpha \to +\infty$. Then,

$$U_{x_\alpha, \beta_\alpha} = B_\alpha + R_\alpha \ ,$$

where the $U_{x_\alpha,\beta_\alpha}$'s, solutions of (**??**) on the sphere, are given by (**??**), where (B_α) is the bubble of center the x_α's and weights the μ_α's given by

$$\mu_\alpha = \sqrt{\frac{4(\beta_\alpha - 1)}{\sqrt{\lambda_n}(\beta_\alpha + 1)}}$$

and where $R_\alpha \to 0$ in $H_2^2(S^n)$ as $\alpha \to +\infty$. Moreover, in this example, there exists $C > 1$ such that $\frac{1}{C} B_\alpha(x) \leq U_{x_\alpha,\beta_\alpha}(x) \leq C B_\alpha(x)$ for all α and all x for which $r_\alpha = d_g(x_\alpha, x)$ is such that $\eta(r_\alpha) = 1$. In the general case, for arbitrary sequences of solutions of equations like (**??**) on arbitrary manifolds, and multi-bubbles, pointwise estimates are given by Lemma **??**. Such estimates go back to Schoen [**?**] (see also Schoen and Zhang [**?**]) when dealing with second order operators. They have been intensively used by Druet [**?**] (still in the case of second order operators). We refer also to Robert [**?**].

Lemma 2.2. *In addition to the estimates in Lemma **??**, there exists $C > 0$, such that, up to a subsequence,*

$$\Big(\min_{1 \leq i \leq k} d_g(x_\alpha^i, x)\Big)^{\frac{n-4}{2}} \big|u_\alpha(x) - u^0(x)\big| \leq C$$

*for all α and all x, where u^0 is as in Lemma **??**, and the x^i_α's, $i = 1, \dots, k$, are the centers of the bubbles in the decomposition of the u_α's given by Lemma **??**.*

*Proof of Lemma **??**.* Let Φ_α be the function defined at x as the minimum over i in $\{1, \dots, k\}$ of the $d_g(x^i_\alpha, x)$'s where the x^i_α's are the centers of the bubbles in the decomposition of the u_α's given by Lemma **??**, and let v_α be the function given by

$$v_\alpha(x) = \Phi_\alpha(x)^{\frac{n-4}{2}} u_\alpha(x) \ .$$

Let also $y_\alpha \in M$ be such that v_α is maximum at y_α. We prove Lemma **??** by contradiction and assume that $v_\alpha(y_\alpha) \to +\infty$ as $\alpha \to +\infty$. We let $\mu_\alpha = u_\alpha(y_\alpha)^{-2/(n-4)}$ so that $\mu_\alpha \to 0$ as $\alpha \to +\infty$. Then, by the definition of y_α,

$$\lim_{\alpha \to +\infty} \frac{d_g(x^i_\alpha, y_\alpha)}{\mu_\alpha} = +\infty \tag{2.5}$$

for all $i = 1, \dots, k$. Let $\delta > 0$ be less than the injectivity radius of (M, g). We define the function w_α in $B_0(\delta \mu_\alpha^{-1})$ by

$$w_\alpha(x) = \mu_\alpha^{\frac{n-4}{2}} u_\alpha \left(\exp_{y_\alpha}(\mu_\alpha x) \right) \ , \tag{2.6}$$

where $B_0(\delta \mu_\alpha^{-1})$ is the Euclidean ball of center 0 and radius $\delta \mu_\alpha^{-1}$, and where $\exp_{y_\alpha}$ is the exponential map at y_α. Given $R > 0$, for any $i = 1, \dots, k$, and $x \in B_0(R)$,

$$\begin{aligned} d_g\left(x^i_\alpha, \exp_{y_\alpha}(\mu_\alpha x)\right) &\geq d_g\left(x^i_\alpha, y_\alpha\right) - R\mu_\alpha \\ &\geq \left(1 - \frac{R\mu_\alpha}{\Phi_\alpha(y_\alpha)}\right) \Phi_\alpha(y_\alpha) \end{aligned}$$

and the right hand side of the last equation is positive by (**??**). Coming back to (**??**), thanks to the definition of y_α, we then get that

$$w_\alpha(x) \leq \left(1 - \frac{R\mu_\alpha}{\Phi_\alpha(y_\alpha)}\right)^{-\frac{n-4}{2}}$$

for all $x \in B_0(R)$. In particular, the w_α's are uniformly bounded on any compact subset of $\mathbb{R}^n$. It is easily checked that

$$\Delta^2_{g_\alpha} w_\alpha + b_\alpha \mu_\alpha^2 \Delta_{g_\alpha} w_\alpha + c_\alpha \mu_\alpha^4 w_\alpha = w_\alpha^{2^\sharp - 1} \ , \tag{2.7}$$

where $g_\alpha(x) = \left(\exp^\star_{y_\alpha} g\right)(\mu_\alpha x)$. Let ξ be the Euclidean metric. Clearly, for any compact subset K of $\mathbb{R}^n$, $g_\alpha \to \xi$ in $C^2(K)$ as $\alpha \to +\infty$. Moreover, equation (**??**) can be written as

$$\left[\left(\Delta_{g_\alpha} + d_{1,\alpha}(\mu_\alpha)^2\right) \circ \left(\Delta_{g_\alpha} + d_{2,\alpha}(\mu_\alpha)^2\right)\right] w_\alpha = w_\alpha^{2^\sharp - 1} \ , \tag{2.8}$$

where $d_{1,\alpha}$ and $d_{2,\alpha}$ are given by

$$d_{1,\alpha} = \frac{b_\alpha}{2} + \sqrt{\frac{b_\alpha^2}{4} - c_\alpha} \quad \text{and} \quad d_{2,\alpha} = \frac{b_\alpha}{2} - \sqrt{\frac{b_\alpha^2}{4} - c_\alpha} \ . \tag{2.9}$$

Thanks to standard elliptic theory and (**??**) we then get that the w_α's are bounded in $C^{4,\theta}_{loc}(\mathbb{R}^n)$, $0 < \theta < 1$. In particular, up to a subsequence, we can assume that $w_\alpha \to w$ in $C^4_{loc}(\mathbb{R}^n)$ as $\alpha \to +\infty$. Here w is a nonnegative function of $C^4(\mathbb{R}^n)$ such that $w \leq w(0) = 1$. Moreover, $w \in D^2_2(\mathbb{R}^n)$ and $w \in L^{2^\sharp}(\mathbb{R}^n)$. Clearly, we have that

$$\int_{B_{y_\alpha}(R\mu_\alpha)} u_\alpha^{2^\sharp} dv_g = \int_{\mathbb{R}^n} w^{2^\sharp} dx + \varepsilon_R(\alpha) \ , \tag{2.10}$$

where $\varepsilon_R(\alpha)$ is such that

$$\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\varepsilon_R(\alpha)=0\ .$$

Thanks to the decomposition of Lemma **??**,

$$\int_{B_{y_\alpha}(R\mu_\alpha)}u_\alpha^{2^\sharp}dv_g=\int_{B_{y_\alpha}(R\mu_\alpha)}\left(u^0+\sum_{i=1}^k B_\alpha^i+R_\alpha\right)^{2^\sharp}dv_g\ .$$

Hence,

$$\int_{B_{y_\alpha}(R\mu_\alpha)}u_\alpha^{2^\sharp}dv_g\le C\sum_{i=1}^k\int_{B_{y_\alpha}(R\mu_\alpha)}(B_\alpha^i)^{2^\sharp}dv_g+o(1)\ , \tag{2.11}$$

where $o(1)\to 0$ as $\alpha\to+\infty$ and $C>0$ is independent of α and R. By (**??**) we can write that

$$\lim_{\alpha\to+\infty}\int_{B_{y_\alpha}(R\mu_\alpha)}(B_\alpha^i)^{2^\sharp}dv_g=0 \tag{2.12}$$

for all $R>0$ and all $i=1,\dots,k$. Coming back to (**??**) and (**??**), we then get that $\int_{\mathbb{R}^n}w^{2^\sharp}dx=\varepsilon_\alpha(R)$, where $\varepsilon_R(\alpha)$ is such that $\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\varepsilon_R(\alpha)=0$. Letting $\alpha\to+\infty$, and then $R\to+\infty$, this implies that

$$\int_{\mathbb{R}^n}w^{2^\sharp}dx=0$$

and since w is continuous, nonnegative, and such that $w(0)=1$, we get our contradiction. Lemma **??** is proved. □

Let S be the subset of M given by

$$S=\left\{\lim_{\alpha\to+\infty}x_\alpha^i,i=1,\dots,k\right\}\ , \tag{2.13}$$

where the x_α^i's, $i=1,\dots,k$, are the centers of the bubbles (B_α^i) in the decomposition of the u_α's given by Lemma **??** (and $S=\emptyset$ if the u_α's do not blow up). We refer to the point in S as geometrical blow-up points. By Lemma **??**, $u_\alpha\to u^0$ in $H^2_{2,loc}(M\backslash S)$ as $\alpha\to+\infty$. By Lemma **??**, the u_α's are bounded in any compact subset of $M\backslash S$. Standard elliptic theory and the splitting

$$P_\alpha=[(\Delta_g+d_{1,\alpha})\circ(\Delta_g+d_{2,\alpha})]\ ,$$

where P_α is the operator in the left hand side of (**??**), and $d_{1,\alpha}$ and $d_{2,\alpha}$ are given by (**??**), then give that, up to a subsequence,

$$u_\alpha\to u^0\ \text{ in } C^4_{loc}(M\backslash S) \tag{2.14}$$

as $\alpha\to+\infty$. Assuming that the u_α's blow up, we let Φ_α be the function in Lemma **??** given by

$$\Phi_\alpha(x)=\min_{1\le i\le k}d_g(x_\alpha^i,x)\ , \tag{2.15}$$

where the x_α^i's are the centers of the bubbles (B_α^i) in Lemma **??**. An important complement to Lemma **??** is the following.

Lemma 2.3. *In addition to the estimate in Lemma* **??** *we also have that*

$$\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\sup_{x\in M\backslash\Omega_\alpha(R)}\Phi_\alpha(x)^{\frac{n-4}{2}}\left|u_\alpha(x)-u^0(x)\right|=0\ ,$$

*where u^0 is as in Lemma **??**, Φ_α is given by (**??**), the x^i_α's and μ^i_α's are the centers and weights of the bubbles (B^i_α) in Lemma **??**, and, for $R > 0$, $\Omega_\alpha(R)$ is given by $\Omega_\alpha(R) = \bigcup_{i=1}^k B_{x^i_\alpha}(R\mu^i_\alpha)$.*

*Proof of Lemma **??**.* We prove Lemma **??** by contradiction and assume that there exists a sequence (y_α) of points in M, and that there exists $\delta_0 > 0$ such that for any $i = 1, \dots, k$,

$$\frac{d_g(x^i_\alpha, y_\alpha)}{\mu^i_\alpha} \to +\infty \tag{2.16}$$

as $\alpha \to +\infty$, and such that for any α,

$$\Phi_\alpha(y_\alpha)^{\frac{n-4}{2}} \big|u_\alpha(y_\alpha) - u^0(y_\alpha)\big| \geq \delta_0 \ . \tag{2.17}$$

Clearly, $\Phi_\alpha(y_\alpha) \to 0$ as $\alpha \to +\infty$ by (**??**). We let $\mu_\alpha = u_\alpha(y_\alpha)^{-2/(n-4)}$. Then we can rewrite (**??**) as

$$\frac{\Phi_\alpha(y_\alpha)}{\mu_\alpha} \geq \delta_1 \ , \tag{2.18}$$

where $\delta_1^{(n-4)/2} = \delta_0/2$. In particular, $\mu_\alpha \to 0$ as $\alpha \to +\infty$. Given $\delta > 0$ less than the injectivity radius of (M, g), we define the function w_α in the Euclidean ball $B_0(\delta\mu_\alpha^{-1})$ by

$$w_\alpha(x) = \mu_\alpha^{\frac{n-4}{2}} u_\alpha\left(\exp_{y_\alpha}(\mu_\alpha x)\right)$$

and let g_α be the metric given by $g_\alpha(x) = \left(\exp^\star_{y_\alpha} g\right)(\mu_\alpha x)$. For any compact subset K of $\mathbb{R}^n$, and if ξ stands for the Euclidean metric, we have that $g_\alpha \to \xi$ in $C^2(K)$ as $\alpha \to +\infty$. By (**??**) we can write that if (x_α) is a sequence in $B_0(\delta_1/2)$, then

$$\begin{aligned} d_g\big(x^i_\alpha, \exp_{y_\alpha}(\mu_\alpha x_\alpha)\big) &\geq d_g\big(y_\alpha, x^i_\alpha\big) - d_g\big(y_\alpha, \exp_{y_\alpha}(\mu_\alpha x_\alpha)\big) \\ &\geq \delta_1\mu_\alpha - d_{g_\alpha}(0, x_\alpha)\mu_\alpha \end{aligned}$$

for all i and all α. In particular, $d_g\big(x^i_\alpha, \exp_{y_\alpha}(\mu_\alpha x_\alpha)\big) \geq C\mu_\alpha$ for some $C > 0$ independent of α, and up to a subsequence, we get with the estimate of Lemma **??** that

$$w_\alpha(x) \leq C \tag{2.19}$$

for all $x \in B_0(\delta_1/2)$ and all α, where $C > 0$ is independent of α and x. Now we may follow the arguments of the proof of Lemma **??**. On one hand, the w_α's are solutions of (**??**) in $B_0(\delta_1/2)$, where $d_{1,\alpha}$ and $d_{2,\alpha}$ are given by (**??**). On the other hand, they are bounded in $B_0(\delta_1/2)$ by (**??**). Then it follows from standard elliptic theory that the w_α's are bounded in $C^{4,\theta}(B_0(\delta_1/4))$, $0 < \theta < 1$. In particular, up to a subsequence, we can assume that $w_\alpha \to w$ in $C^4(B_0(\delta_1/8))$ as $\alpha \to +\infty$. Moreover, $w(0) = 1$ since $w_\alpha(0) = 1$ for all α. Let $\delta_2 = \delta_1/8$. We have that

$$\begin{aligned} \int_{B_{y_\alpha}(\delta_2\mu_\alpha)} u_\alpha^{2^\sharp} dv_g &= \int_{B_0(\delta_2)} w_\alpha^{2^\sharp} dv_{g_\alpha} \\ &= \int_{B_0(\delta_2)} w^{2^\sharp} dx + o(1) \ , \end{aligned} \tag{2.20}$$

where $o(1) \to 0$ as $\alpha \to +\infty$, while, by Lemma **??**,

$$\int_{B_{y_\alpha}(\delta_2\mu_\alpha)} u_\alpha^{2^\sharp} dv_g \leq C \sum_{i=1}^k \int_{B_{y_\alpha}(\delta_2\mu_\alpha)} (B^i_\alpha)^{2^\sharp} dv_g + o(1) \ , \tag{2.21}$$

where $C > 0$ is independent of α, and the (B^i_α)'s are the bubbles in Lemma **??**. Independently, here again, we can write that

$$\int_{B_{y_\alpha}(\delta_2\mu_\alpha)} (B^i_\alpha)^{2^\sharp} dv_g = o(1) \tag{2.22}$$

for all i. Then, combining (**??**)-(**??**), we get that w satisfies

$$\int_{B_0(\delta_2)} w^{2^\sharp} dx = 0$$

and this is impossible since w is continuous, nonnegative, and such that $w(0) = 1$. This proves Lemma **??**. □

3. Relative concentrations when $n \geq 8$

We let (M, g) be a smooth compact Riemannian manifold of dimension n. We assume in what follows that $n \geq 8$ and, for the reader's convenience, we discuss the notion of L^2-concentration. We let (u_α) be a bounded sequence in $H^2_2(M)$ of nonnegative solutions of (**??**). The material below, and in the following section, is concerned with pseudo-compactness. We may therefore assume by contradiction that the u_α's converge weakly in $H^2_2(M)$ to the zero function. If S is the set consisting of the geometrical blow-up points of the u_α's, as defined in (**??**), we write that $S = \{x_1, \dots, x_p\}$. Given $\delta > 0$, we define

$$\mathcal{R}_{L^2}(\alpha, \delta) = \frac{\int_{\mathcal{B}_\delta} u^2_\alpha dv_g}{\int_M u^2_\alpha dv_g} \, , \tag{3.1}$$

where $\mathcal{B}_\delta$ is the union of the $B_{x_i}(\delta)$'s, $i = 1, \dots, p$. Since we assumed that $u^0 \equiv 0$, the two quantities in this ratio go to zero as $\alpha \to +\infty$. Then L^2-concentration states as follows.

Lemma 3.1. *Assume $u^0 \equiv 0$. When $n \geq 8$, up to a subsequence, and for any $\delta > 0$, $\mathcal{R}_{L^2}(\alpha, \delta) \to 1$ as $\alpha \to +\infty$.*

Lemma **??** is easy to prove when $n \geq 9$. The proof is slightly more delicate when $n = 8$. When $n \leq 7$, as is easily checked, bubbles as in (**??**) do not concentrate in L^2 and L^2-concentration fails in this case to be the right key notion for concentration. The cases of dimensions $n = 6$ and $n = 7$ are treated in Section **??**.

Proof of Lemma **??**. Let $\Lambda > 0$ be such that $E(u_\alpha) \leq \Lambda$ for all α. For convenience, we set $\tilde{u}_\alpha = \|u_\alpha\|^{-1}_{2^\sharp} u_\alpha$ so that $\int_M \tilde{u}^{2^\sharp}_\alpha dv_g = 1$. Then

$$\Delta^2_g \tilde{u}_\alpha + b_\alpha \Delta_g \tilde{u}_\alpha + c_\alpha \tilde{u}_\alpha = \lambda_\alpha \tilde{u}^{2^\sharp - 1}_\alpha \, , \tag{3.2}$$

where $\lambda_\alpha = \|u_\alpha\|^{8/(n-4)}_{2^\sharp}$. Noting that the operator in the left hand side of (**??**) is uniformly coercive as $\alpha \to +\infty$ (the coefficients are positive and converge to positive limits), there exist $\Lambda_1, \Lambda_2 > 0$ such that $\Lambda_1 \leq \lambda_\alpha \leq \Lambda_2$ for all α. Up to a subsequence, thanks to the compactness of the embedding of H^2_2 in H^2_1, we may assume that $\|\tilde{u}_\alpha\|_{H^2_1} \to 0$ as $\alpha \to +\infty$. We let also $\tilde{v}_\alpha$ be given by

$$\tilde{v}_\alpha = \Delta_g \tilde{u}_\alpha + d_{2,\alpha} \tilde{u}_\alpha \, ,$$

where $d_{2,\alpha}$ is as in (**??**). We have that

$$\Delta^2_g u + b_\alpha \Delta_g u + c_\alpha u = \big(\Delta_g + d_{1,\alpha}\big)\big(\Delta_g + d_{2,\alpha}\big) u$$

for all functions u, where $d_{1,\alpha}$ is as in (**??**). Hence $\Delta_g\tilde{v}_\alpha + d_{1,\alpha}\tilde{v}_\alpha \geq 0$, and $\tilde{v}_\alpha$ is nonnegative. Let $\delta > 0$ be given. Thanks to (**??**) with $u^0 \equiv 0$, $\lambda_\alpha \tilde{u}_\alpha^{2^\sharp-1} - c_\alpha \tilde{u}_\alpha \leq 0$ in $M\backslash\mathcal{B}_\delta$ when α is sufficiently large. It follows that

$$\Delta_g\Big(\Delta_g\tilde{u}_\alpha + b_\alpha\tilde{u}_\alpha\Big) \leq 0 \tag{3.3}$$

in $M\backslash\mathcal{B}_\delta$ when α is sufficiently large. Also, we have that $\Delta_g\tilde{u}_\alpha + b_\alpha\tilde{u}_\alpha \geq \tilde{v}_\alpha$ since $\tilde{u}_\alpha \geq 0$ and $d_{2,\alpha} \leq b_\alpha$. By the De Giorgi-Nash-Moser iterative scheme, that we apply to (**??**), we then get that

$$\begin{aligned}\sup_{M\backslash\mathcal{B}_{\delta/2}} \tilde{v}_\alpha &\leq \sup_{M\backslash\mathcal{B}_{\delta/2}} \big(\Delta_g\tilde{u}_\alpha + b_\alpha\tilde{u}_\alpha\big)\\ &\leq C\int_{M\backslash\mathcal{B}_{\delta/3}} \big(\Delta_g\tilde{u}_\alpha + b_\alpha\tilde{u}_\alpha\big)dv_g\ ,\end{aligned}$$

where $C > 0$ is independent of α. Let η be a smooth function such that $0 \leq \eta \leq 1$, $\eta = 0$ in $\mathcal{B}_{\delta/4}$, and $\eta = 1$ in $M\backslash\mathcal{B}_{\delta/3}$. Then, integrating by parts,

$$\begin{aligned}\int_{M\backslash\mathcal{B}_{\delta/3}} \big(\Delta_g\tilde{u}_\alpha + b_\alpha\tilde{u}_\alpha\big)dv_g &\leq \int_M \eta\big(\Delta_g\tilde{u}_\alpha + b_\alpha\tilde{u}_\alpha\big)dv_g\\ &\leq C\int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha dv_g\ ,\end{aligned}$$

where $C > 0$ is independent of α. It follows that

$$\sup_{M\backslash\mathcal{B}_{\delta/2}} \tilde{v}_\alpha \leq C\int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha dv_g \tag{3.4}$$

when α is sufficiently large, where $C > 0$ is independent of α. Applying the De Giorgi-Nash-Moser iterative scheme to the equation $\Delta_g\tilde{u}_\alpha + d_{2,\alpha}\tilde{u}_\alpha = \tilde{v}_\alpha$, it follows from (**??**) that

$$\sup_{M\backslash\mathcal{B}_{\delta}} \tilde{u}_\alpha \leq C\int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha dv_g \tag{3.5}$$

when α is sufficiently large, where $C > 0$ is independent of α. In particular, thanks to (**??**),

$$\int_{M\backslash\mathcal{B}_{\delta}} \tilde{u}_\alpha^2 dv_g \leq C\int_M \tilde{u}_\alpha dv_g \int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha dv_g$$

and integrating (**??**) we get that

$$\int_{M\backslash\mathcal{B}_{\delta}} \tilde{u}_\alpha^2 dv_g \leq C\int_M \tilde{u}_\alpha^{2^\sharp-1} dv_g \Big(\int_M \tilde{u}_\alpha^2 dv_g\Big)^{1/2} \tag{3.6}$$

when α is sufficiently large, where $C > 0$ is independent of α. First we assume that $n \geq 12$. Then $1 < 2^\sharp - 1 \leq 2$, and it follows from Hölder's inequality that

$$\int_M \tilde{u}_\alpha^{2^\sharp-1} dv_g \leq C\Big(\int_M \tilde{u}_\alpha^2 dv_g\Big)^{(2^\sharp-1)/2}\ ,$$

where $C > 0$ is independent of α. Thanks to (**??**) we then get that

$$\int_{M\backslash\mathcal{B}_{\delta}} \tilde{u}_\alpha^2 dv_g \leq C\Big(\int_M \tilde{u}_\alpha^2\Big)^{2^\sharp/2}$$

and that

$$1 - \mathcal{R}_{L^2}(\alpha,\delta) \leq C\Big(\int_M \tilde{u}_\alpha^2\Big)^{\frac{2^\sharp}{2}-1}\ .$$

Noting that $\tilde{u}_\alpha \to 0$ in L^2 as $\alpha \to +\infty$, it follows that $\mathcal{R}_{L^2}(\alpha,\delta) \to 1$ as $\alpha \to +\infty$. Lemma **??** is proved when $n \geq 12$. Now we assume that $9 \leq n < 12$. Then $2 < 2^\sharp - 1 < 2^\sharp$, and it follows from Hölder's inequality that

$$\begin{aligned}\int_M \tilde{u}_\alpha^{2^\sharp-1} dv_g &\leq \big(\int_M \tilde{u}_\alpha^2 dv_g\big)^{\frac{n-4}{8}} \big(\int_M \tilde{u}_\alpha^{2^\sharp} dv_g\big)^{\frac{12-n}{8}} \\ &\leq \big(\int_M \tilde{u}_\alpha^2 dv_g\big)^{\frac{n-4}{8}}\end{aligned}$$

since $\|\tilde{u}_\alpha\|_{2^\sharp} = 1$. Thanks to (**??**) we then get that

$$\int_{M\setminus\mathcal{B}_\delta} \tilde{u}_\alpha^2 dv_g \leq C\big(\int_M \tilde{u}_\alpha^2\big)^{n/8}$$

and that

$$1 - \mathcal{R}_{L^2}(\alpha,\delta) \leq C\big(\int_M \tilde{u}_\alpha^2\big)^{\frac{n}{8}-1} .$$

Here again, $\tilde{u}_\alpha \to 0$ in L^2 as $\alpha \to +\infty$. It follows that $\mathcal{R}_{L^2}(\alpha,\delta) \to 1$ as $\alpha \to +\infty$ when $9 \leq n < 12$. This proves Lemma **??** for such n's, and we are left with the case when $n = 8$. It easily follows from (**??**) that

$$\int_{M\setminus\mathcal{B}_\delta} u_\alpha^2 dv_g \leq C\int_M u_\alpha^{2^\sharp-1} dv_g \big(\int_M u_\alpha^2 dv_g\big)^{1/2} \tag{3.7}$$

when α is sufficiently large, where $C > 0$ is independent of α. Given $\delta > 0$, we write that

$$\begin{aligned}\int_M u_\alpha^{2^\sharp-1} dv_g &\leq \int_{M\setminus\mathcal{B}_\delta} u_\alpha^{2^\sharp-1} dv_g + \int_{\mathcal{B}_\delta} u_\alpha^{2^\sharp-1} dv_g \\ &\leq \big(\max_{M\setminus\mathcal{B}_\delta} u_\alpha\big)\int_{M\setminus\mathcal{B}_\delta} u_\alpha^{2^\sharp-2} dv_g + \int_{\mathcal{B}_\delta} u_\alpha^{2^\sharp-1} dv_g .\end{aligned}$$

Coming back to (**??**), and since $2^\sharp = 4$ when $n = 8$, we get that

$$\mathcal{R}_{L^2}(\alpha,\delta) \leq \big(\max_{M\setminus\mathcal{B}_\delta} u_\alpha\big)\|u_\alpha\|_2 + \mathcal{R}_\delta(\alpha) , \tag{3.8}$$

where

$$\mathcal{R}_\delta(\alpha) = \frac{\int_{\mathcal{B}_\delta} u_\alpha^{2^\sharp-1} dv_g}{\sqrt{\int_M u_\alpha^2 dv_g}} . \tag{3.9}$$

Clearly, see for instance (**??**) with $u^0 \equiv 0$,

$$\lim_{\alpha\to+\infty} \big(\max_{M\setminus\mathcal{B}_\delta} u_\alpha\big)\|u_\alpha\|_2 = 0 \tag{3.10}$$

and we are left with getting estimates for $\mathcal{R}_\delta(\alpha)$. We come back here to the H_2^2-decomposition of the u_α's given by Lemma **??**. We let the x_α^i's and the μ_α^i's be the centers and weights of the bubbles involved in this decomposition. Given $R > 0$, and for k as in Lemma **??**, we let also $\Omega_\alpha(R)$ be the union from $i = 1$ to k of the geodesic balls centered at x_α^i and of radii $R\mu_\alpha^i$. Since $2^\sharp = 4$ when $n = 8$, we can write by Hölder's inequality that

$$\int_{\mathcal{B}_\delta} u_\alpha^{2^\sharp-1} dv_g \leq \int_{\Omega_\alpha(R)} u_\alpha^{2^\sharp-1} dv_g + \sqrt{\int_{\mathcal{B}_\delta\setminus\Omega_\alpha(R)} u_\alpha^{2^\sharp} dv_g}\sqrt{\int_M u_\alpha^2 dv_g} .$$

Then,

$$\mathcal{R}_\delta(\alpha) \leq \frac{\int_{\Omega_\alpha(R)} u_\alpha^{2^\sharp-1} dv_g}{\sqrt{\int_M u_\alpha^2 dv_g}} + \sqrt{\int_{\mathcal{B}_\delta\setminus\Omega_\alpha(R)} u_\alpha^{2^\sharp} dv_g}\ , \tag{3.11}$$

where $\mathcal{R}_\delta(\alpha)$ is as in (**??**). As is easily checked, we get with the H_2^2-decomposition of Lemma **??** that

$$\begin{aligned}
&\int_{\mathcal{B}_\delta\setminus\Omega_\alpha(R)} u_\alpha^{2^\sharp} dv_g = \varepsilon_R(\alpha)\ ,\\
&\int_{\Omega_\alpha(R)} u_\alpha^{2^\sharp-1} dv_g \leq C\big(\max_i \mu_\alpha^i\big)^{\frac{n-4}{2}} \left(\int_{B_0(R)} u^{2^\sharp-1} dx + o(1)\right)\ ,\ \text{and}\\
&\int_M u_\alpha^2 dv_g \geq \big(\max_i \mu_\alpha^i\big)^{n-4} \left(\int_{B_0(R)} u^2 dx + o(1)\right)\ ,
\end{aligned} \tag{3.12}$$

where $\lim_{R\to+\infty}\lim_{\alpha\to+\infty}\varepsilon_R(\alpha) = 0$, where $o(1)\to 0$ as $\alpha\to+\infty$, $C>0$ is independent of α and R, and $u = u_{1,0}$ is given by (**??**). By (**??**) and (**??**) we then get that

$$\limsup_{\alpha\to+\infty} \mathcal{R}_\delta(\alpha) \leq \varepsilon_R + C\frac{\int_{B_0(R)} u^{2^\sharp-1} dx}{\sqrt{\int_{B_0(R)} u^2 dx}}\ , \tag{3.13}$$

where $\varepsilon_R\to 0$ as $R\to+\infty$, and $C>0$ does not depend on R. We have that

$$\int_{B_0(R)} u^{2^\sharp-1} dx < \int_{\mathbb{R}^n} u^{2^\sharp-1} dx$$

for all R, so that the integrals in the left hand side of this equation are uniformly bounded with respect to R. On the other hand, when $n=8$, we have that $\int_{B_0(R)} u^2 dx\to+\infty$ as $R\to+\infty$. Hence, we get with (**??**) that $\mathcal{R}_\delta(\alpha)\to 0$ for all $\delta>0$ as $\alpha\to+\infty$. Coming back to (**??**), and by (**??**), it follows that $\mathcal{R}_{L^2}(\alpha,\delta)\to 1$ as $\alpha\to+\infty$ for all $\delta>0$, and this ends the proof of Lemma **??**. □

We still write that $S = \{x_1,\dots,x_p\}$, where S is the set consisting of the geometrical blow-up points of (u_α), and, for $\delta>0$, we define the ratio

$$\mathcal{R}_{\nabla L^2}(\alpha,\delta) = \frac{\int_{\mathcal{B}_\delta} |\nabla u_\alpha|^2 dv_g}{\int_M |\nabla u_\alpha|^2 dv_g}\ , \tag{3.14}$$

where $\mathcal{B}_\delta$ is the union of the $B_{x_i}(\delta)$'s, $i=1,\dots,p$. Since we assumed that $u^0\equiv 0$, the two quantities in this ratio go to zero as $\alpha\to+\infty$. We claim here that, as it was the case for L^2-concentration, the ratio itself goes to 1 as $\alpha\to+\infty$. We refer to this property as ∇L^2-concentration. We obtain ∇L^2-concentration in Lemma **??** below as a corollary of L^2-concentration. The cases $n=6$ and $n=7$ with respect to this concentration are treated in the following section.

Lemma 3.2. *Assume $u^0\equiv 0$. When $n\geq 8$, up to a subsequence, and for any $\delta>0$, $\mathcal{R}_{\nabla L^2}(\alpha,\delta)\to 1$ as $\alpha\to+\infty$.*

Proof of Lemma **??**. We let the $\tilde u_\alpha$'s be as in (**??**), and let $\tilde v_\alpha = \Delta_g\tilde u_\alpha + d_{2,\alpha}\tilde u_\alpha$, where $d_{2,\alpha}$ is as in (**??**). Given $\delta>0$, we let also η be a smooth function such that

$0 \leq \eta \leq 1$, $\eta = 0$ in $\mathcal{B}_{\delta/2}$, and $\eta = 1$ in $M \backslash \mathcal{B}_\delta$. Then, thanks to (**??**),

$$\begin{aligned} \int_M \eta\big(\Delta_g \tilde{u}_\alpha + d_{2,\alpha}\tilde{u}_\alpha\big)\tilde{u}_\alpha dv_g &\leq C \int_M \eta \tilde{u}_\alpha dv_g \int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha dv_g \\ &\leq C \int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha^2 dv_g \end{aligned}$$

when α is sufficiently large, where $C > 0$ is independent of α. Integrating by parts, it follows that

$$\int_M \eta |\nabla \tilde{u}_\alpha|^2 dv_g \leq C \int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha^2 dv_g$$

when α is sufficiently large, where $C > 0$ is independent of α. In particular,

$$\int_{M\backslash\mathcal{B}_\delta} |\nabla \tilde{u}_\alpha|^2 dv_g \leq C \int_{M\backslash\mathcal{B}_{\delta/4}} \tilde{u}_\alpha^2 dv_g \ ,$$

and writing that

$$1 - \mathcal{R}_{\nabla L^2}(\alpha,\delta) = \frac{\int_{M\backslash\mathcal{B}_\delta} |\nabla \tilde{u}_\alpha|^2 dv_g}{\int_M \tilde{u}_\alpha^2 dv_g} \times \frac{\int_M \tilde{u}_\alpha^2 dv_g}{\int_M |\nabla \tilde{u}_\alpha|^2 dv_g} \ ,$$

we get that ∇L^2-concentration follows from Lemma **??** and Lemma **??** below. □

Another estimate we need to prove the assertion on pseudo-compactness in Theorem **??**, which we also used in the proof of Lemma **??**, is the global balance $L^2 - \nabla L^2$. Here again we obtain this balance, as stated in Lemma **??** below, as a corollary of L^2-concentration. The cases $n = 6$ and $n = 7$ with respect to this balance are treated in the following section.

Lemma 3.3. *Assume $u^0 \equiv 0$. When $n \geq 8$, up to a subsequence,*

$$\int_M u_\alpha^2 dv_g = o(1) \int_M |\nabla u_\alpha|^2 dv_g \ , \tag{3.15}$$

where $o(1) \to 0$ as $\alpha \to +\infty$.

Proof of Lemma **??**. We let $\delta > 0$. By Hölder's inequalities,

$$\int_{\mathcal{B}_\delta} u_\alpha^2 dv_g \leq Vol_g(\mathcal{B}_\delta)^{(2^\star-2)/2^\star} \|u_\alpha\|_{2^\star}^2 \ ,$$

where $Vol_g(\mathcal{B}_\delta)$ stands for the volume of $\mathcal{B}_\delta$ with respect to g. Independently, we can write with the Sobolev inequality corresponding to the embedding of the second order Sobolev space H_1^2 into $L^{2^\star}$ that

$$\|u_\alpha\|_{2^\star}^2 \leq A\left(\|\nabla u_\alpha\|_2^2 + \|u_\alpha\|_2^2\right) \ ,$$

where $A > 0$ is independent of α. Noting that

$$\int_M u_\alpha^2 dv_g = \int_{\mathcal{B}_\delta} u_\alpha^2 dv_g + \int_{M\backslash\mathcal{B}_\delta} u_\alpha^2 dv_g$$

and since $Vol_g(\mathcal{B}_\delta) \to 0$ as $\delta \to 0$, we then get that

$$\int_M u_\alpha^2 dv_g \leq C_1 \int_{M\backslash\mathcal{B}_\delta} u_\alpha^2 dv_g + C_2 Vol_g(\mathcal{B}_\delta)^{(2^\star-2)/2^\star} \int_M |\nabla u_\alpha|^2 dv_g$$

for all $\delta > 0$ small, where $C_1, C_2 > 0$ are independent of α and δ. In particular, if $\mathcal{R}_{L^2}(\alpha,\delta) \to 1$ as $\alpha \to +\infty$, we get (**??**) by letting first $\alpha \to +\infty$, and then $\delta \to 0$. This proves Lemma **??**. □

As a remark, it follows from Lemma **??** and Lemma **??** that when $n \geq 8$, and for any $\delta > 0$,

$$\int_{M\setminus\mathcal{B}_\delta} |\nabla^2 u_\alpha|^2 dv_g = o(1) \int_M |\nabla u_\alpha|^2 dv_g \ , \tag{3.16}$$

where $o(1) \to 0$ as $\alpha \to +\infty$. In order to prove (**??**), we fix $\delta > 0$ and let η be a smooth function such that $0 \leq \eta \leq 1$, $\eta = 0$ in $\mathcal{B}_{\delta/2}$, and $\eta = 1$ in $M\setminus\mathcal{B}_\delta$. We consider (**??**) with $u = u_\alpha$, multiply the equation by $\eta^2 u_\alpha$, and integrate over M. Then,

$$\begin{aligned} &\int_M \Delta_g u_\alpha \Delta_g(\eta^2 u_\alpha) dv_g + b_\alpha \int_M \big(\nabla u_\alpha \nabla(\eta^2 u_\alpha)\big) dv_g \\ &+ c_\alpha \int_M \eta^2 u_\alpha^2 dv_g = \int_M \eta^2 u_\alpha^{2^\sharp} dv_g \ . \end{aligned} \tag{3.17}$$

As is easily checked,

$$\int_M \Delta_g u_\alpha \Delta_g(\eta^2 u_\alpha) dv_g = \int_M \left(\Delta_g(\eta u_\alpha)\right)^2 dv_g + O\left(\|u_\alpha\|_{H_1^2(\mathcal{B}_\delta\setminus\mathcal{B}_{\delta/2})}\right) \ ,$$

where $\|u\|_{H_1^2(A)} = \int_A (|\nabla u|^2 + u^2) dv_g$, and

$$\int_M \big(\nabla u_\alpha \nabla(\eta^2 u_\alpha)\big) dv_g = \int_M |\nabla(\eta u_\alpha)|^2 dv_g + O\left(\|u_\alpha\|_{L^2(\mathcal{B}_\delta\setminus\mathcal{B}_{\delta/2})}\right) \ .$$

Independently, thanks to (**??**),

$$\int_M \eta^2 u_\alpha^{2^\sharp} dv_g = o\left(\int_M \eta^2 u_\alpha^2 dv_g\right) \ .$$

Coming back to (**??**), it follows that

$$\begin{aligned} &\int_M \left(\Delta_g(\eta u_\alpha)\right)^2 dv_g + b_\alpha \int_M |\nabla(\eta u_\alpha)|^2 dv_g + \big(c_\alpha + o(1)\big) \int_M \eta^2 u_\alpha^2 dv_g \\ &= O\left(\|u_\alpha\|_{L^2(\mathcal{B}_\delta\setminus\mathcal{B}_{\delta/2})}\right) + O\left(\|\nabla u_\alpha\|_{L^2(\mathcal{B}_\delta\setminus\mathcal{B}_{\delta/2})}\right) \ , \end{aligned} \tag{3.18}$$

where $o(1) \to 0$ as $\alpha \to +\infty$. By the Bochner-Lichnerowicz-Weitzenböck formula,

$$\begin{aligned} \int_M \left(\Delta_g(\eta u_\alpha)\right)^2 dv_g &= \int_M |\nabla^2(\eta u_\alpha)|^2 dv_g + \int_M Rc_g\left(\nabla(\eta u_\alpha), \nabla(\eta u_\alpha)\right) dv_g \\ &= \int_M |\nabla^2(\eta u_\alpha)|^2 dv_g + O\left(\int_M |\nabla(\eta u_\alpha)|^2 dv_g\right) \ , \end{aligned}$$

where Rc_g is the Ricci curvature of g. By (**??**) we then get that

$$\int_{M\setminus\mathcal{B}_\delta} |\nabla^2 u_\alpha|^2 dv_g \leq C_1 \int_{M\setminus\mathcal{B}_{\delta/2}} |\nabla u_\alpha|^2 dv_g + C_2 \int_{M\setminus\mathcal{B}_{\delta/2}} u_\alpha^2 dv_g \ ,$$

where $C_1, C_2 > 0$ do not depend on α, and (**??**) follows from Lemma **??** and Lemma **??**.

4. Relative concentrations when $n = 6, 7$

We let (M, g) be a smooth compact Riemannian manifold of dimension n. As in Section **??**, we are concerned with pseudo-compactness. We let (u_α) be a bounded sequence in $H_2^2(M)$ of nonnegative solutions of (**??**), and we assume by contradiction that the u_α's converge weakly in $H_2^2(M)$ to the zero function. We prove that Lemma **??** and Lemma **??** of the preceding section still hold when $n = 6, 7$. In the sequel the notations are those of Section **??**. In particular, $\mathcal{R}_{\nabla L^2}(\alpha, \delta)$ is defined in (**??**). We claim that the following result holds.

Lemma 4.1. *Assume $u^0 \equiv 0$. When $n = 6, 7$, ∇L^2-concentration holds so that, up to a subsequence, and for any $\delta > 0$, $\mathcal{R}_{\nabla L^2}(\alpha, \delta) \to 1$ as $\alpha \to +\infty$. Moreover, the global balance $L^2 - \nabla L^2$ holds also so that*

$$\int_M u_\alpha^2 dv_g = o(1) \int_M |\nabla u_\alpha|^2 dv_g \ ,$$

where $o(1) \to 0$ as $\alpha \to +\infty$.

Proof of Lemma **??**. We assume $n = 6, 7$, and let $\delta > 0$ be given. We claim that

$$\int_{M \setminus \mathcal{B}_\delta} u_\alpha^2 dv_g = o(1) \left(\int_M u_\alpha^{2^\star} dv_g \right)^{2/2^\star} \ , \tag{4.1}$$

where $o(1) \to 0$ as $\alpha \to +\infty$. In order to prove (**??**), we first note that similar arguments to those used in the proof of Lemma **??** give that

$$\sup_{M \setminus \mathcal{B}_\delta} \tilde{u}_\alpha \leq C \int_{M \setminus \mathcal{B}_{\delta/4}} \tilde{u}_\alpha dv_g \tag{4.2}$$

when α is sufficiently large, where $C > 0$ is independent of α, and $\tilde{u}_\alpha = \|u_\alpha\|_{2^\sharp}^{-1} u_\alpha$. In particular, we can write with (**??**) that

$$\begin{aligned} \int_{M \setminus \mathcal{B}_\delta} \tilde{u}_\alpha^2 dv_g &\leq C \|\tilde{u}_\alpha\|_{L^1(M)}^2 \\ &\leq C \|\tilde{u}_\alpha\|_{L^1(M)} \|\tilde{u}_\alpha\|_{L^{2^\star}(M)} \end{aligned} \tag{4.3}$$

and then, with (**??**), we can write that

$$\int_{M \setminus \mathcal{B}_\delta} \tilde{u}_\alpha^2 dv_g \leq C \frac{\int_M \tilde{u}_\alpha^{2^\sharp - 1} dv_g}{\|\tilde{u}_\alpha\|_{L^{2^\star}(M)}} \left(\int_M \tilde{u}_\alpha^{2^\star} dv_g \right)^{2/2^\star} \ , \tag{4.4}$$

where $C > 0$ is independent of α, since, integrating equation (**??**) satisfied by the u_α's, we get that $c_\alpha \int_M u_\alpha dv_g = \int_M u_\alpha^{2^\sharp - 1} dv_g$. If we assume now that $n = 7$, then $2^\star < 2^\sharp - 1 < 2^\sharp$, and we can write by Hölder's inequality that

$$\int_M \tilde{u}_\alpha^{2^\sharp - 1} dv_g \leq C \|\tilde{u}_\alpha\|_{L^{2^\star}(M)}^{2^\star/(2^\sharp - 2^\star)} \ . \tag{4.5}$$

Since $2^\star/(2^\sharp - 2^\star) > 1$ when $n = 7$, and $\|\tilde{u}_\alpha\|_{L^{2^\star}(M)} \to 0$ as $\alpha \to +\infty$, we get with (**??**) and (**??**) that (**??**) is true when $n = 7$. Now we assume that $n = 6$. We let the x_α^i's and the μ_α^i's be the centers and weights of the bubbles involved in the decomposition of Lemma **??**. Given $R > 0$, and for k as in Lemma **??**, we let, as in the proof of Lemma **??**, $\Omega_\alpha(R)$ be the union from $i = 1$ to k of the geodesic

balls centered at x_α^i and of radii $R\mu_\alpha^i$. Then, coming back to the u_α's, by Hölder's inequality as above, and since $n = 6$, we can write that

$$\begin{aligned}\int_M u_\alpha^{2^\sharp-1}dv_g &= \int_{M\setminus\Omega_\alpha(R)} u_\alpha^{2^\sharp-1}dv_g + \int_{\Omega_\alpha(R)} u_\alpha^{2^\sharp-1}dv_g \\ &\leq \left(\int_{M\setminus\Omega_\alpha(R)} u_\alpha^{2^\sharp} dv_g\right)^{2/3} \|u_\alpha\|_{L^{2^\star}(M)} + \int_{\Omega_\alpha(R)} u_\alpha^{2^\sharp-1}dv_g \ .\end{aligned} \tag{4.6}$$

By the H_2^2-decomposition of Lemma **??** we have that, when $n = 6$,

$$\begin{aligned}&\int_{M\setminus\Omega_\alpha(R)} u_\alpha^{2^\sharp} dv_g = \varepsilon_R(\alpha) \ , \\ &\int_{\Omega_\alpha(R)} u_\alpha^{2^\sharp-1}dv_g \leq C\big(\max_i \mu_\alpha^i\big) \ , \text{ and} \\ &\int_M u_\alpha^{2^\star} dv_g \geq \big(\max_i \mu_\alpha^i\big)^{2^\star} \left(\int_{B_0(R)} u^{2^\star} dx + o(1)\right) \ ,\end{aligned} \tag{4.7}$$

where

$$\lim_{R\to+\infty} \lim_{\alpha\to+\infty} \varepsilon_R(\alpha) = 0 \ ,$$

where $o(1) \to 0$ as $\alpha \to +\infty$, $C > 0$ is independent of α and R, and $u = u_{1,0}$ is given by (**??**). By (**??**) and (**??**) we can then write that when $n = 6$,

$$\limsup_{\alpha\to+\infty} \frac{\int_M u_\alpha^{2^\sharp-1}dv_g}{\|u_\alpha\|_{L^{2^\star}(M)}} \leq \varepsilon_R + \frac{C}{\left(\int_{B_0(R)} u^{2^\star}dx\right)^{1/2^\star}} \ , \tag{4.8}$$

where $\varepsilon_R \to 0$ as $R \to +\infty$, and $C > 0$ does not depend on R. Noting that $\int_{B_0(R)} u^{2^\star}dx \to +\infty$ as $R \to +\infty$ when $n = 6$, it follows from (**??**) and (**??**) that (**??**) is also true when $n = 6$. Now that we have (**??**) for all $\delta > 0$, similar arguments to those developed in the proof of Lemma **??** give that

$$\int_M u_\alpha^2 dv_g = o(1)\left(\int_M u_\alpha^{2^\star} dv_g\right)^{2/2^\star} \ , \tag{4.9}$$

and then that

$$\int_M u_\alpha^2 dv_g = o(1)\int_M |\nabla u_\alpha|^2 dv_g \ . \tag{4.10}$$

In particular, the global balance $L^2 - \nabla L^2$ holds when $n = 6, 7$. We obtain ∇L^2-concentration as in the proof of Lemma **??**. This ends the proof of Lemma **??**. □

As in Section **??**, it follows from Lemma **??** that for any $\delta > 0$,

$$\int_{M\setminus\mathcal{B}_\delta} |\nabla^2 u_\alpha|^2 dv_g = o(1)\int_M |\nabla u_\alpha|^2 dv_g \ , \tag{4.11}$$

where $o(1) \to 0$ as $\alpha \to +\infty$.

5. A splitting estimate

We let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$. We are concerned in this section with getting estimates to prove the compactness assertion of Theorem **??** and Theorem **??**. We borrow material developed for second order equations by Devillanova and Solimini [**?**]. We let (u_α) be a bounded sequence in $H_2^2(M)$ of nonnegative solutions of (**??**), and assume that the u_α's blow up. In this section, u^0 may be nonzero. Up to renumbering and up to a subsequence, with the notations of Section **??**, we can assume that

$$\mu_\alpha^1 = \max_{1 \leq i \leq k} \mu_\alpha^i \ , \tag{5.1}$$

where the μ_α^i's are the weights of the bubbles (B_α^i) of Lemma **??**. We let

$$x_\alpha = x_\alpha^1 \ \text{ and } \ \mu_\alpha = \mu_\alpha^1 \ , \tag{5.2}$$

where the x_α^1's are the centers of (B_α^1). The main purpose of this section is to prove the following splitting type estimate.

Lemma 5.1. *Let p_1, p_2 be arbitrary real numbers such that $2^\sharp/2 < p_2 < 2^\sharp < p_1$. Then there exists $C > 0$, and sequences (u_α^1) and (u_α^2) of nonnegative functions such that, up to a subsequence, $u_\alpha \leq u_\alpha^1 + u_\alpha^2$, $\|u_\alpha^1\|_{p_1} \leq C$, and*

$$\|u_\alpha^2\|_{p_2} \leq C \mu_\alpha^{\frac{n}{p_2} - \frac{n}{2^\sharp}}$$

*for all α, where μ_α is given by (**??**) and (**??**).*

We prove Lemma **??** thanks to Steps **??** to **??** below. Note that a basic model for Lemma **??** is $u_\alpha = u^0 + B_\alpha^1$, $u_\alpha^1 = u^0$, and $u_\alpha^2 = B_\alpha^1$. For p_1 and p_2 such that $2^\sharp/2 < p_2 < 2^\sharp < p_1$, and $\sigma > 0$, we define the norm $\| \cdot \|_{p_1,p_2,\sigma}$ on $L^\infty(M)$, the space of bounded functions in M, by

$$\|u\|_{p_1,p_2,\sigma} = \inf \Big\{ C > 0 \text{ s.t. } (I^\sigma_{p_1,p_2}) \text{ holds for } u \Big\} \ ,$$

where $(I^\sigma_{p_1,p_2})$ holds for u if there exist nonnegative functions $u^1, u^2 \in L^\infty(M)$ such that $|u| \leq u^1 + u^2$,

$$\|u^1\|_{p_1} \leq C \ \text{ and } \ \|u^2\|_{p_2} \leq C \sigma^{\frac{n}{2^\sharp} - \frac{n}{p_2}} \ .$$

Step **??** states as follows.

Step 5.1. *Let $u, v \in H_2^2(M) \cap L^\infty(M)$ and $K \in L^\infty(M)$ be nonnegative functions such that*

$$\Big(\Delta_g + \frac{a}{2} \Big)^2 u \leq K v \tag{5.3}$$

for some $a \in [\Lambda_1, \Lambda_2]$ where $\Lambda_1 < \Lambda_2$ are positive. Let p_1, p_2 be arbitrary real numbers such that $2^\sharp/2 < p_2 < 2^\sharp < p_1$, and $\sigma > 0$ arbitrary. Then

$$\|u\|_{p_1,p_2,\sigma} \leq C \|K\|_{n/4} \|v\|_{p_1,p_2,\sigma} \ ,$$

where $C > 0$ depends only on the manifold, p_1, p_2, Λ_1, and Λ_2.

Proof of Step **??**. Let $\Lambda > \|v\|_{p_1,p_2,\sigma}$, Λ arbitrary. Then there exist $v^1, v^2 \geq 0$ in $L^\infty(M)$ such that $v \leq v^1 + v^2$, $\|v^1\|_{p_1} \leq \Lambda$, and $\|v^2\|_{p_2} \leq \Lambda \sigma^{(n/2^\sharp) - (n/p_2)}$. We let

$u^i \in H^2_2(M)$ and $\tilde u^i \in H^2_2(M)$ be such that

$$\begin{aligned} &\left(\Delta_g + \frac{a}{2}\right)\tilde u^i = Kv^i \ , \text{ and} \\ &\left(\Delta_g + \frac{a}{2}\right) u^i = \tilde u^i \ . \end{aligned} \tag{5.4}$$

Then $\tilde u^i \in H^p_2(M)$, and $u^i \in H^p_4(M)$ for all $p > 1$. In particular $u^i \in L^\infty(M)$, $i = 1, 2$, and, of course, it follows from (**??**) that

$$\left(\Delta_g + \frac{a}{2}\right)^2 u^i = Kv^i \ .$$

By the maximum principle that we apply to the two equations in (**??**), $u^i \geq 0$ for all $i = 1, 2$. Now we let $q_i > 1$, $i = 1, 2$, be such that $\frac{1}{q_i} = \frac{4}{n} + \frac{1}{p_i}$. Noting that $Kv^i \in L^{q_i}(M)$, we get that $\tilde u^i \in H^{q_i}_2(M)$ and then that $u^i \in H^{q_i}_4(M)$. In particular, $u^i \in H'^{q_i}_4(M)$ where, for k integer and $q > 1$, $H'^q_k(M)$ is the (reduced) Sobolev space defined as the completion of $C^\infty(M)$ with respect to the norm

$$\|u\|_{H'^q_k} = \sum_{i=0}^{E(k/2)} \|\Delta^i_g u\|_q + \sum_{i=0}^{E((k-1)/2)} \|\nabla\Delta^i_g u\|_q$$

and where $E(s)$ is the greatest integer not exceeding s. By Step **??** below, that we apply to the second equation in (**??**), we easily get that $\|u^i\|_{H'^{q_i}_4} \leq C\|\tilde u^i\|_{H^{q_i}_2}$ where $C > 0$ depends only on the manifold, p_i, Λ_1, and Λ_2. Applying now Step **??** to the first equation in (**??**), we can write that for any $i = 1, 2$,

$$\begin{aligned} \|u^i\|_{H'^{q_i}_4} &\leq C\|Kv^i\|_{q_i} \\ &\leq C\|K\|_{n/4}\|v^i\|_{p_i} \ , \end{aligned}$$

where $C > 0$ depends only on the manifold, p_1, p_2, Λ_1, and Λ_2. By the Sobolev embedding theorem for H'^q_k-spaces, see for instance Aubin [**?**], $H'^{q_i}_4(M) \subset L^{p_i}(M)$, $i = 1, 2$. Hence, for any $i = 1, 2$,

$$\|u^i\|_{p_i} \leq C\|K\|_{n/4}\|v^i\|_{p_i} \ ,$$

where $C > 0$ depends only on the manifold, p_1, p_2, Λ_1, and Λ_2. By (**??**), and since $v \leq v^1 + v^2$,

$$\left(\Delta_g + \frac{a}{2}\right)^2 u \leq \left(\Delta_g + \frac{a}{2}\right)^2 u^1 + \left(\Delta_g + \frac{a}{2}\right)^2 u^2 \ .$$

Then, by the maximum principle that we apply again twice, $u \leq u^1 + u^2$. It follows that $\|u\|_{p_1,p_2,\sigma} \leq C\|K\|_{n/4}\Lambda$, and since $\Lambda > \|v\|_{p_1,p_2,\sigma}$ is arbitrary, this proves Step **??**. □

Step **??** (inspired from Gilbarg and Trudinger [**?**]) is standard. We state it with no proof.

Step 5.2. *Let $u \in H^p_2(M)$ and $f \in L^p(M)$, $p > 1$, be two functions such that $L_a u = f$ where $L_a = \Delta_g + a$, $a \in [\Lambda_1, \Lambda_2]$, and $\Lambda_1 < \Lambda_2$ are positive. Then $\|u\|_{H^p_2} \leq C\|f\|_p$ where $C > 0$ depends only on the manifold, p, Λ_1, and Λ_2.*

The next step in the proof, Step **??** below, is a bootstrap argument to improve the values of p_1 and p_2 we get from Step **??**. We let $\theta(n) = \frac{n(n+4)}{4(n-4)}$. Step **??** states as follows.

Step 5.3. *Let $u, v \in H^2_2(M) \cap L^\infty(M)$ be nonnegative functions such that*

$$\left(\Delta_g + \frac{a}{2}\right)^2 u = v^{2^\sharp - 1} + Av$$

for some $a, A \in [\Lambda_1, \Lambda_2]$ where $0 < \Lambda_1 < \Lambda_2$. Let p_1, p_2 be arbitrary real numbers such that $2^\sharp - 1 < p_2 < 2^\sharp < p_1 < \theta(n)$, and $q_1, q_2 > 1$ be such that $\frac{1}{q_i} = \frac{2^\sharp - 1}{p_i} - \frac{4}{n}$, $i = 1, 2$. Then, for any $\sigma > 0$,

$$\|u\|_{q_1,q_2,\sigma} \leq C \left(\|v\|^{2^\sharp - 1}_{p_1,p_2,\sigma} + 1 \right) ,$$

where $C > 0$ depends only one the manifold, p_1, p_2, Λ_1, and Λ_2.

*Proof of Step **??**.* Let $\Lambda > \|v\|_{p_1,p_2,\sigma}$, Λ arbitrary. Then there exist $v^1, v^2 \geq 0$ in $L^\infty(M)$ such that $v \leq v^1 + v^2$, $\|v^1\|_{p_1} \leq \Lambda$, and $\|v^2\|_{p_2} \leq \Lambda \sigma^{(n/2^\sharp) - (n/p_2)}$. We let u^1 and u^2 be such that

$$\left(\Delta_g + \frac{a}{2}\right)^2 u^1 = (1 + A)\, 2^{2^\sharp - 1} (v^1)^{2^\sharp - 1} + A \ , \text{ and}$$
$$\left(\Delta_g + \frac{a}{2}\right)^2 u^2 = (1 + A)\, 2^{2^\sharp - 1} (v^2)^{2^\sharp - 1} \ .$$

Then $u^i \in H^p_4(M) \cap L^\infty(M)$, $i = 1, 2$, for all $p > 1$, and it follows from the maximum principle applied twice that $u^1, u^2 \geq 0$. Since $p_i > 2^\sharp - 1$, $i = 1, 2$, we can write with Step **??**, as in the proof of Step **??**, that

$$\|u^1\|_{H'^{p_1/(2^\sharp - 1)}_4} \leq C \| (1 + A)\, 2^{2^\sharp - 1} (v^1)^{2^\sharp - 1} + A \|_{p_1/(2^\sharp - 1)} \ , \text{ and}$$
$$\|u^2\|_{H'^{p_2/(2^\sharp - 1)}_4} \leq C \| (1 + A)\, 2^{2^\sharp - 1} (v^2)^{2^\sharp - 1} \|_{p_2/(2^\sharp - 1)} \ ,$$

where $C > 0$ depends only on the manifold, p_1, p_2, Λ_1, and Λ_2. Independently, since $2^\sharp - 1 < p_i < \theta(n)$, we can write with the Sobolev embedding theorem for H'^q_k-spaces that $H'^{p_i/(2^\sharp - 1)}_4(M) \subset L^{q_i}(M)$, $i = 1, 2$. It follows that

$$\|u^1\|_{q_1} \leq C \left(\|v^1\|^{2^\sharp - 1}_{p_1} + 1 \right) \ , \text{ and}$$
$$\|u^2\|_{q_2} \leq C \|v^2\|^{2^\sharp - 1}_{p_2} \ ,$$

where $C > 0$ depends only on the manifold, p_1, p_2, Λ_1, and Λ_2. Noting that

$$\begin{aligned} \left(\Delta_g + \frac{a}{2}\right)^2 u &= v^{2^\sharp - 1} + Av \\ &\leq (1 + A) v^{2^\sharp - 1} + A \\ &\leq 2^{2^\sharp - 1} (1 + A) (v^1)^{2^\sharp - 1} + A + 2^{2^\sharp - 1} (1 + A) (v^2)^{2^\sharp - 1} \\ &= \left(\Delta_g + \frac{a}{2}\right)^2 u^1 + \left(\Delta_g + \frac{a}{2}\right)^2 u^2 \end{aligned}$$

we then get with the maximum principle applied twice that $u \leq u^1 + u^2$. In particular, since $\left(2^\sharp - 1\right) \left((n/2^\sharp) - (n/p_2) \right) = (n/2^\sharp) - (n/q_2)$, we get that

$$\|u\|_{q_1,q_2,\sigma} \leq C \left(\Lambda^{2^\sharp - 1} + 1 \right) ,$$

where $C > 0$ depends only on the manifold, p_1, p_2, Λ_1, and Λ_2. Since $\Lambda > \|v\|_{p_1,p_2,\sigma}$ is arbitrary, this proves Step **??**. □

The initialisation step in the proof, Step **??** below, states as follows. We recall that if $G : M \times M \backslash \Delta \to \mathbb{R}$, Δ being the diagonal in $M \times M$, is the Green function of $L = \Delta_g + a$, $a > 0$, then

$$\tilde{G}(x,y) = \int_M G(x,z)G(z,y)dv_g(z)$$

is the Green function of $L^2 = L \circ L$. The integral makes sense and estimates on $\tilde{G}$ follow from material in Druet, Hebey and Robert [**?**].

Step 5.4. *Let (u_α) be a bounded sequence in $H_2^2(M)$ of nonnegative solutions of (**??**). There exists $p_0(n) = \max\left(2^\sharp/(2^\sharp - 1), 2^\sharp/2\right)$ and $p(n) > 2^\sharp$ with the property that for any p_1, p_2 satisfying $p_0(n) < p_2 < 2^\sharp < p_1 < p(n)$ there exists $C > 0$ such that, up to a subsequence,*

$$\|u_\alpha\|_{p_1,p_2,\mu_\alpha^{-1}} \leq C$$

*for all α, where μ_α is given by (**??**) and (**??**).*

*Proof of Step **??**.* We let G_α be the Green function of the operator $\left(\Delta_g + \frac{b_\alpha}{2}\right)^2$. Then,

$$u_\alpha(x) = \int_M G_\alpha(x,y)\left(u_\alpha^{2^\sharp-2}(y) + \left(\frac{b_\alpha^2}{4} - a_\alpha\right)\right)u_\alpha(y)dv_g(y)$$

for all $x \in M$. By Lemma **??**, up to a subsequence, it follows that

$$u_\alpha(x) \leq C\int_M G_\alpha(x,y)\left(1 + \sum_{i=1}^k (B_\alpha^i)^{2^\sharp-2}(y) + |R_\alpha(y)|^{2^\sharp-2}\right)u_\alpha(y)dv_g(y)\ , \quad (5.5)$$

where $C > 0$ is independent of α, the (B_α^i)'s are bubbles, and $R_\alpha \to 0$ in $H_2^2(M)$ as $\alpha \to +\infty$. We let v_α and w_α^i, $i = 1, \dots, k$, be given by

$$\begin{aligned} v_\alpha(x) &= \int_M G_\alpha(x,y)u_\alpha(y)dv_g(y) \\ w_\alpha^i(x) &= \int_M G_\alpha(x,y)(B_\alpha^i)^{2^\sharp-2}(y)u_\alpha(y)dv_g(y)\ . \end{aligned} \quad (5.6)$$

From the equation $\left(\Delta_g + \frac{b_\alpha}{2}\right)^2 v_\alpha = u_\alpha$, from Step **??** and arguments as in the proof of Step **??**, and since the u_α's are bounded in $H_2^2(M)$, there exists $p(n) > 2^\sharp$, depending only on n, such that for any $2^\sharp < p_1 < p(n)$, and any α,

$$\|v_\alpha\|_{p_1} \leq C\ , \quad (5.7)$$

where $C > 0$ does not depend on α. In a similar way, we get with the equations

$$\left(\Delta_g + \frac{b_\alpha}{2}\right)^2 w_\alpha^i = (B_\alpha^i)^{2^\sharp-2}u_\alpha$$

that for any $p_0(n) < p_2 < 2^\sharp$, there exists $C, C' > 0$ such that for any i and any α,

$$\begin{aligned} \|w_\alpha^i\|_{p_2} &\leq C\|(B_\alpha^i)^{2^\sharp-2}\|_r\|u_\alpha\|_{2^\sharp} \\ &\leq C'\|(B_\alpha^i)^{2^\sharp-2}\|_r\ , \end{aligned}$$

where $r \in \left(\frac{n}{8}, \frac{n}{4}\right)$ is such that $\frac{1}{r} = \frac{1}{p_2} + \frac{4}{n} - \frac{1}{2^\sharp}$. From equation (**??**), from (**??**) and (**??**), and since $\frac{n}{8} < r < \frac{n}{4}$, we can write that for any i and any α,

$$\begin{aligned} \|(B_\alpha^i)^{2^\sharp-2}\|_r &\leq C(\mu_\alpha^i)^{\frac{n}{r}-4} \\ &\leq C\mu_\alpha^{\frac{n}{r}-4}\ , \end{aligned}$$

where $C > 0$ is independent of α and i. It follows that for any $p_0(n) < p_2 < 2^\sharp$, for any α, and any i,

$$\|w_\alpha^i\|_{p_2} \leq C(\mu_\alpha^{-1})^{\frac{n}{2^\sharp}-\frac{n}{p_2}} \ , \tag{5.8}$$

where $C > 0$ is independent of α and i. Now we let $\hat{v}_\alpha$ be given by

$$\hat{v}_\alpha(x) = \int_M G_\alpha(x,y)|R_\alpha(y)|^{2^\sharp-2}u_\alpha(y)dv_g(y) \ .$$

Then

$$\left(\Delta_g + \frac{b_\alpha}{2}\right)^2 \hat{v}_\alpha = |R_\alpha|^{2^\sharp-2}u_\alpha$$

and it follows from Step **??** that for any $\frac{2^\sharp}{2} < p_2 < 2^\sharp < p_1$,

$$\|\hat{v}_\alpha\|_{p_1,p_2,\mu_\alpha^{-1}} = o\left(\|u_\alpha\|_{p_1,p_2,\mu_\alpha^{-1}}\right) \ . \tag{5.9}$$

By (**??**) and (**??**), if $p_0(n) < p_2 < 2^\sharp < p_1 < p(n)$, then

$$\left\|v_\alpha + \sum_{i=1}^k w_\alpha^i\right\|_{p_1,p_2,\mu_\alpha^{-1}} \leq C \ , \tag{5.10}$$

where $C > 0$ does not depend on α. Noting that if $0 \leq u \leq v$, then for any p_1, p_2, and σ, $\|u\|_{p_1,p_2,\sigma} \leq \|v\|_{p_1,p_2,\sigma}$, and that by (**??**),

$$u_\alpha \leq C\left(v_\alpha + \sum_{i=1}^k w_\alpha^i + \hat{v}_\alpha\right)$$

we get with (**??**) and (**??**) that for any $p_0(n) < p_2 < 2^\sharp < p_1 < p(n)$, there exists $C > 0$ such that, for any α, $\|u_\alpha\|_{p_1,p_2,\mu_\alpha^{-1}} \leq C$. This proves Step **??**. □

With Steps **??** to **??** we are in position to prove Lemma **??**. The proof of Lemma **??** proceeds as follows.

Proof of Lemma **??**. We proceed by induction, starting from Step **??**, using Step **??**. An easy remark is that

$$\|u\|_{\tilde{p}_1,p_2,\sigma} \leq \|u\|_{p_1,p_2,\sigma} \tag{5.11}$$

if $\tilde{p}_1 \leq p_1$. We fix p_1, p_2 such that $\frac{2^\sharp}{2} < p_2 < 2^\sharp < p_1$. We let $p_1^0 > 2^\sharp$ be close to $2^\sharp$, and let $k_0 \geq 1$ be such that the increasing sequence (p_1^k) given by

$$\frac{1}{p_1^{k+1}} = \frac{2^\sharp-1}{p_1^k} - \frac{4}{n}$$

satisfies $p_1^k < \theta(n)$ for all $k \leq k_0$, and $p_1^{k_0+1} \geq \theta(n)$, where $\theta(n)$ is as in Step **??**. Similarly, for $p_2^0 < 2^\sharp$ we construct the decreasing sequence (p_2^k) by

$$\frac{1}{p_2^{k+1}} = \frac{2^\sharp-1}{p_2^k} - \frac{4}{n} \ .$$

We choose p_2^0 such that $p_2^{k_0+2} = p_2$. Then, since $p_2 > 2^\sharp/2$, $p_2^k > 2^\sharp - 1$ for all $k \leq k_0 + 1$. The closer $p_1^0 > 2^\sharp$ is to $2^\sharp$, the larger k_0 is, and the larger k_0 is, the closer $p_2^0 < 2^\sharp$ has to be to $2^\sharp$. In particular, we can assume that $p_2^0 > 2^\sharp/(2^\sharp - 1)$.

Then, by Steps **??** and **??**, we get that there exists $C > 0$ such that, up to a subsequence, and for any α,

$$\|u_\alpha\|_{p_1^{k_0+1},p_2^{k_0+1},\mu_\alpha^{-1}} \leq C \ .$$

In particular, by (**??**), $\|u_\alpha\|_{\tilde{p}_1,p_2^{k_0+1},\mu_\alpha^{-1}} \leq C$ for $\tilde{p}_1 < \theta(n)$ as close as we want to $\theta(n)$. We then apply Step **??** once more and get that

$$\|u_\alpha\|_{\hat{p}_1,p_2^{k_0+2},\mu_\alpha^{-1}} \leq C \ ,$$

where $\hat{p}_1 \to +\infty$ as $\tilde{p}_1 \to \theta(n)$. Choosing $\tilde{p}_1$ sufficiently close to $\theta(n)$, we can assume that $\hat{p}_1 \geq p_1$, and, thanks to (**??**), this proves Lemma **??**. □

6. An integral estimate

We let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$. Here also we are concerned with getting estimates to prove the compactness assertion of Theorem **??** and Theorem **??**. We let (u_α) be a bounded sequence in $H_2^2(M)$ of nonnegative solutions of (**??**), and assume that the u_α's blow up. As in Section **??**, u^0 may be nonzero. Up to renumbering and up to a subsequence, as done in Section **??**, we can assume that

$$\mu_\alpha^1 = \max_{1\leq i\leq k} \mu_\alpha^i \ , \tag{6.1}$$

where the μ_α^i's are the weights of the bubbles (B_α^i) of Lemma **??**. Then, as in (**??**), we let $x_\alpha = x_\alpha^1$ and $\mu_\alpha = \mu_\alpha^1$, where the x_α^1's are the centers of (B_α^1). The main purpose of this section is to prove the following integral estimate.

Lemma 6.1. *There exists $C_1, C_2 > 0$ such that, up to a subsequence,*

$$\frac{1}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g \leq C_1 + C_2 \frac{\mu_\alpha^{\frac{n-4}{2}}}{r^{n-4}} \ , \text{ and}$$
$$\frac{1}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} |\Delta_g u_\alpha| \, d\sigma_g \leq C_1 + C_2 \frac{\mu_\alpha^{\frac{n-4}{2}}}{r^{n-2}}$$

for all α and all $r > 0$ sufficiently small, independent of α, where $\partial B_{x_\alpha}(r)$ is the boundary of the geodesic ball $B_{x_\alpha}(r)$, and $d\sigma_g$ is the measure induced on $\partial B_{x_\alpha}(r)$ by g.

We prove Lemma **??** thanks to Steps **??** and **??** below. As a preliminary remark, given $x_0 \in M$, we let β_{x_0} be the smooth function around x_0 such that for u smooth in M, and $r > 0$ small (less than the injectivity radius of the manifold),

$$\begin{aligned} &\frac{d}{dr}\left(\frac{1}{r^{n-1}} \int_{\partial B_{x_0}(r)} u d\sigma_g\right) \\ &= \frac{1}{r^{n-1}} \int_{\partial B_{x_0}(r)} \left(\frac{\partial u}{\partial \nu}\right) d\sigma_g + \frac{1}{r^{n-1}} \int_{\partial B_{x_0}(r)} \beta_{x_0} u d\sigma_g \ , \end{aligned} \tag{6.2}$$

where $\partial B_{x_0}(r)$ is the boundary of the geodesic ball $B_{x_0}(r)$, where $d\sigma_g$ is the volume element on $\partial B_{x_0}(r)$ induced by g, and $\frac{\partial}{\partial \nu}$ is the normal derivative with respect to the outward unit normal vector ν. As is well known, see for instance Sakai [**?**],

$$d_g(x_0, x) \, |\beta_{x_0}(x)| = O''\left(d_g(x_0, x)^2\right) \ , \tag{6.3}$$

where the notation in the right hand side of (**??**) stands for a C^3-function such that the kth derivatives of this function, $k = 0, 1, 2$, are bounded by $Cd_g(x_0, x)^{2-k}$ where $C > 0$ does not depend on x_0 and x. We also have for the function β_{x_0} that $\beta_{x_0}(x) = O'\left(d_g(x_0, x)\right)$ where the notation in the right hand side of this equation stands for a C^1-function such that the kth derivatives of this function, $k = 0, 1$, are bounded by $Cd_g(x_0, x)^{1-k}$ where $C > 0$ does not depend on x_0 and x. In what follows, for $r > 0$ small, we let

$$\varphi_\alpha(r) = \frac{1}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} (\Delta_g u_\alpha)\, d\sigma_g \ , \tag{6.4}$$

where x_α is given by (**??**), and set $\beta_\alpha = \beta_{x_\alpha}$. We let also $F_{1,\alpha}$, $F_{2,\alpha}$, and $F_{3,\alpha}$ be the functions given by

$$\begin{aligned} F_{1,\alpha}(r) &= \frac{1}{r^{n-1}} \int_{B_{x_\alpha}(r)} \left(u_\alpha^{2^\sharp - 1} - c_\alpha u_\alpha\right) dv_g \\ F_{2,\alpha}(r) &= \frac{1}{r^{n-1}} \int_{B_{x_\alpha}(r)} \left(1 + u_\alpha^{2^\sharp - 1}\right) dv_g \end{aligned} \tag{6.5}$$

and

$$F_{3,\alpha}(r) = \frac{1}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g \ . \tag{6.6}$$

We regard the φ_α's alternatively as functions of the variable r or functions of the variable x in $\mathbb{R}^n$ such that $r = |x|$. The first step in the proof of Lemma **??** is as follows.

Step 6.1. *For $r > 0$ small, the φ_α's in (**??**) are solutions of*

$$\begin{aligned} &\Delta\varphi_\alpha + \frac{B_\alpha(r)x^i}{r}\partial_i\varphi_\alpha + C_\alpha(r)\varphi_\alpha \\ &\quad = F'_{1,\alpha}(r) + \frac{n-1}{r}F_{1,\alpha}(r) + \Theta^1_\alpha(r)F_{2,\alpha}(r) + \Theta^2_\alpha(r)F_{3,\alpha}(r) \ , \end{aligned} \tag{6.7}$$

*where Δ is the Euclidean Laplacian, where $F_{1,\alpha}$, $F_{2,\alpha}$, and $F_{3,\alpha}$ are given by (**??**) and (**??**), and where the B_α's, C_α's, Θ^1_α's, and Θ^2_α's are bounded functions both with respect to r and α.*

*Proof of Step **??**.* By (**??**),

$$\begin{aligned} \left(\frac{d\varphi_\alpha}{dr}\right)(r) = \frac{1}{r^{n-1}} &\int_{\partial B_{x_\alpha}(r)} \left(\frac{\partial \Delta_g u_\alpha}{\partial \nu}\right) d\sigma_g \\ &+ \frac{1}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} \beta_\alpha \Delta_g u_\alpha d\sigma_g \end{aligned} \tag{6.8}$$

and, by (**??**), it easily follows that

$$\begin{aligned} \left(\frac{d\varphi_\alpha}{dr}\right)(r) = -F_{1,\alpha}(r) + \frac{b_\alpha}{r^{n-1}} &\int_0^r t^{n-1}\varphi_\alpha(t)dt \\ &+ \frac{1}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} \beta_\alpha \Delta_g u_\alpha d\sigma_g \ . \end{aligned} \tag{6.9}$$

Then we get that

$$\begin{aligned}\Delta\varphi_\alpha + b_\alpha\varphi_\alpha = F'_{1,\alpha}(r) + \frac{n-1}{r}F_{1,\alpha}(r)\\ - \frac{1}{r^{n-1}}\frac{d}{dr}\left(\int_{\partial B_{x_\alpha}(r)}\beta_\alpha\Delta_g u_\alpha d\sigma_g\right)\ ,\end{aligned} \tag{6.10}$$

where Δ is the Euclidean Laplacian (so that, if u is radially symmetrical, then Δu is given by $-\Delta u = u'' + \frac{n-1}{r}u'$). Independently, see for instance (**??**), we can write that

$$\begin{aligned}\frac{d}{dr}\left(\int_{\partial B_{x_\alpha}(r)}\beta_\alpha\Delta_g u_\alpha d\sigma_g\right) = \int_{\partial B_{x_\alpha}(r)}\beta_\alpha\left(\frac{\partial\Delta_g u_\alpha}{\partial\nu}\right)d\sigma_g\\ + \int_{\partial B_{x_\alpha}(r)}\left(\frac{\partial\beta_\alpha}{\partial\nu} + \beta_\alpha^2 + \frac{n-1}{r}\beta_\alpha\right)\Delta_g u_\alpha d\sigma_g\ .\end{aligned} \tag{6.11}$$

From now on we define the functions $\overline{\beta}_\alpha : (0,+\infty)\times M \to \mathbb{R}$ of the variables (r,x) by

$$\overline{\beta}_\alpha(r,x) = \frac{1}{r}\tilde{\beta}_\alpha(x) + \frac{\Delta_g\tilde{\beta}_\alpha(x_\alpha)}{2nr}r_\alpha^2\ , \tag{6.12}$$

where $r_\alpha = d_g(x_\alpha,x)$, and $\tilde{\beta}_\alpha = O''\left(r_\alpha^2\right)$ is the function in the right hand side of (**??**). Then

$$\overline{\beta}_\alpha(r,x) = O\left(\frac{r_\alpha^2}{r}\right)\ \text{ and }\ \Delta_g\overline{\beta}_\alpha(r,x) = O\left(\frac{r_\alpha}{r}\right)\ . \tag{6.13}$$

Moreover, we can write that

$$\begin{aligned}\int_{\partial B_{x_\alpha}(r)}\beta_\alpha\left(\frac{\partial\Delta_g u_\alpha}{\partial\nu}\right)d\sigma_g = \int_{\partial B_{x_\alpha}(r)}\overline{\beta}_\alpha(r,x)\left(\frac{\partial\Delta_g u_\alpha}{\partial\nu}\right)(x)d\sigma_g(x)\\ - \frac{\Delta_g\tilde{\beta}_\alpha(x_\alpha)}{2n}r\int_{\partial B_{x_\alpha}(r)}\left(\frac{\partial\Delta_g u_\alpha}{\partial\nu}\right)d\sigma_g\ .\end{aligned} \tag{6.14}$$

Integrating by parts, using (**??**), we have that

$$\begin{aligned}&\int_{\partial B_{x_\alpha}(r)}\overline{\beta}_\alpha(r,x)\left(\frac{\partial\Delta_g u_\alpha}{\partial\nu}\right)(x)d\sigma_g(x)\\ &= \int_{B_{x_\alpha}(r)}\left(\Delta_g\overline{\beta}_\alpha(r,x) + b_\alpha\overline{\beta}_\alpha(r,x)\right)\Delta_g u_\alpha(x)dv_g(x)\\ &+ \int_{\partial B_{x_\alpha}(r)}\frac{\partial\overline{\beta}_\alpha(r,x)}{\partial\nu}\Delta_g u_\alpha(x)d\sigma_g(x)\\ &- \int_{B_{x_\alpha}(r)}\overline{\beta}_\alpha(r,x)\left(u_\alpha^{2^\sharp-1}(x) - c_\alpha u_\alpha(x)\right)dv_g(x)\ .\end{aligned} \tag{6.15}$$

Let (h_α) be a sequence of functions such that $|h_\alpha(x)| \leq C$ for all α, all x, and some $C > 0$ independent of α and x. Since $\left(\Delta_g + \frac{b_\alpha}{2}\right) u_\alpha \geq 0$, we can write that

$$\begin{aligned}
&\int_{B_{x_\alpha}(r)} h_\alpha \Delta_g u_\alpha dv_g \\
&= \int_{B_{x_\alpha}(r)} h_\alpha \left(\Delta_g u_\alpha + \frac{b_\alpha}{2} u_\alpha\right) dv_g - \frac{b_\alpha}{2} \int_{B_{x_\alpha}(r)} h_\alpha u_\alpha dv_g \\
&= H_\alpha(r) \int_{B_{x_\alpha}(r)} \left(\Delta_g u_\alpha + \frac{b_\alpha}{2} u_\alpha\right) dv_g - \frac{b_\alpha}{2} \int_{B_{x_\alpha}(r)} h_\alpha u_\alpha dv_g \\
&= H_\alpha(r) \int_{B_{x_\alpha}(r)} \Delta_g u_\alpha dv_g + \frac{b_\alpha}{2} \int_{B_{x_\alpha}(r)} \left(H_\alpha(r) - h_\alpha\right) u_\alpha dv_g \ ,
\end{aligned} \tag{6.16}$$

where H_α is such that $|H_\alpha(r)| \leq C$ for all r and all α. With this remark (**??**), with (**??**), and with (**??**) we can then write that

$$\begin{aligned}
&\int_{\partial B_{x_\alpha}(r)} \overline{\beta}_\alpha(r,x) \left(\frac{\partial \Delta_g u_\alpha}{\partial \nu}\right)(x) d\sigma_g(x) \\
&= H_{1,\alpha}(r) \int_{B_{x_\alpha}(r)} \Delta_g u_\alpha dv_g + H_{2,\alpha}(r) \int_{\partial B_{x_\alpha}(r)} \Delta_g u_\alpha d\sigma_g \\
&\quad + H_{3,\alpha}(r) \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g + H_{4,\alpha}(r) \int_{B_{x_\alpha}(r)} \left(1 + u_\alpha^{2^\sharp - 1}\right) dv_g \ ,
\end{aligned} \tag{6.17}$$

where the $H_{i,\alpha}$'s, $1 \leq i \leq 4$, are such that $|H_{i,\alpha}(r)| \leq C$ for all r and all α. Clearly, thanks to the properties of β_α, we also have that

$$\begin{aligned}
&\int_{\partial B_{x_\alpha}(r)} \left(\frac{\partial \beta_\alpha}{\partial \nu} + \beta_\alpha^2 + \frac{n-1}{r}\beta_\alpha\right) \Delta_g u_\alpha d\sigma_g \\
&= H_{5,\alpha}(r) \int_{\partial B_{x_\alpha}(r)} \Delta_g u_\alpha d\sigma_g + H_{6,\alpha}(r) \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g
\end{aligned} \tag{6.18}$$

and, by (**??**), we have that

$$\begin{aligned}
\int_{\partial B_{x_\alpha}(r)} \left(\frac{\partial \Delta_g u_\alpha}{\partial \nu}\right) d\sigma_g &= H_{7,\alpha}(r) \int_{B_{x_\alpha}(r)} \Delta_g u_\alpha dv_g \\
&\quad + H_{8,\alpha}(r) \int_{B_{x_\alpha}(r)} \left(1 + u_\alpha^{2^\sharp - 1}\right) dv_g \ ,
\end{aligned} \tag{6.19}$$

where the $H_{i,\alpha}$'s, $5 \leq i \leq 8$, are such that $|H_{i,\alpha}(r)| \leq C$ for all r and all α. Combining (**??**)–(**??**), (**??**), and (**??**)–(**??**), it follows that

$$\begin{aligned}
\Delta \varphi_\alpha + b_\alpha \varphi_\alpha &= F'_{1,\alpha}(r) + \frac{n-1}{r} F_{1,\alpha}(r) \\
&+ \frac{H_{9,\alpha}(r)}{r^{n-1}} \int_{B_{x_\alpha}(r)} \Delta_g u_\alpha dv_g + \frac{H_{10,\alpha}(r)}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} \Delta_g u_\alpha d\sigma_g \\
&+ \frac{H_{11,\alpha}(r)}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g + \frac{H_{12,\alpha}(r)}{r^{n-1}} \int_{B_{x_\alpha}(r)} \left(1 + u_\alpha^{2^\sharp - 1}\right) dv_g \ ,
\end{aligned} \tag{6.20}$$

where, as above, the $H_{i,\alpha}$'s, $8 \leq i \leq 12$, are such that $|H_{i,\alpha}(r)| \leq C$ for all r and all α. Independently, by (**??**) and (**??**),

$$\begin{aligned}\frac{d\varphi_\alpha}{dr} &= -F_{1,\alpha}(r) + \frac{b_\alpha}{r^{n-1}} \int_{B_{x_\alpha}(r)} \Delta_g u_\alpha dv_g \\ &\quad + \frac{1}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} \beta_\alpha \Delta_g u_\alpha d\sigma_g\end{aligned} \tag{6.21}$$

and, since we have that $\left(\Delta_g + \frac{b_\alpha}{2}\right) u_\alpha \geq 0$, we can write that

$$\begin{aligned}\int_{\partial B_{x_\alpha}(r)} \beta_\alpha \Delta_g u_\alpha d\sigma_g &= H_{13,\alpha}(r) \int_{\partial B_{x_\alpha}(r)} \Delta_g u_\alpha d\sigma_g \\ &\quad + H_{14,\alpha}(r) \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g \ ,\end{aligned} \tag{6.22}$$

where the the $H_{i,\alpha}$'s, $i = 13, 14$, are such that $|H_{i,\alpha}(r)| \leq C$ for all r and all α. As a supplementary remark, we can also write that

$$F_{1,\alpha}(r) = O\left(\frac{1}{r^{n-1}}\right) \int_{B_{x_\alpha}(r)} \left(1 + u_\alpha^{2^\sharp - 1}\right) dv_g \ . \tag{6.23}$$

Combining (**??**)–(**??**) we then get that

$$\begin{aligned}&\Delta\varphi_\alpha + H_{15,\alpha}(r)\frac{d\varphi_\alpha}{dr} + H_{16,\alpha}(r)\varphi_\alpha = F'_{1,\alpha}(r) + \frac{n-1}{r}F_{1,\alpha}(r) \\ &+\frac{H_{17,\alpha}(r)}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g + \frac{H_{18,\alpha}(r)}{r^{n-1}} \int_{B_{x_\alpha}(r)} \left(1 + u_\alpha^{2^\sharp - 1}\right) dv_g \ ,\end{aligned}$$

where the $H_{i,\alpha}$'s, $15 \leq i \leq 18$, are such that $|H_{i,\alpha}(r)| \leq C$ for all r and all α. Noting that such an equation reads also as

$$\begin{aligned}&\Delta\varphi_\alpha + \frac{H_{15,\alpha}(r)x^i}{r}\partial_i\varphi_\alpha + H_{16,\alpha}(r)\varphi_\alpha \\ &= F'_{1,\alpha}(r) + \frac{n-1}{r}F_{1,\alpha}(r) + H_{17,\alpha}(r)F_{3,\alpha}(r) + H_{18,\alpha}(r)F_{2,\alpha}(r) \ ,\end{aligned}$$

this ends the proof of Step **??**. □

In what follows we let L_α be the operator of Step **??**. Namely,

$$L_\alpha u = \Delta u + \frac{B_\alpha(r)x^i}{r}\partial_i u + C_\alpha(r)u \ , \tag{6.24}$$

where Δ is the Euclidean Laplacian and the B_α's and C_α's are bounded functions both with respect to r and α. We write that $2xy \leq \varepsilon^2 x^2 + \varepsilon^{-2} y^2$ for two real numbers x and y, and that

$$\int_{B_0(\delta)} r^{-1} \left|B_\alpha(r)x^i u\partial_i u\right| dx \leq C \int_{B_0(\delta)} |u||\nabla u| dx$$

for all $u \in C_0^\infty(B_0(\delta))$, the space of smooth functions with compact support in the Euclidean ball centered at 0 and of radius δ. Then we easily get that for $\delta > 0$ small, and any $u \in C_0^\infty(B_0(\delta))$,

$$\int_{B_0(\delta)} (L_\alpha u) u dx \geq \frac{1}{2} \int_{B_0(\delta)} |\nabla u|^2 dx - A \int_{B_0(\delta)} u^2 dx \ ,$$

where $A > 0$ is independent of u and α. If λ_1 is the first eigenvalue of the Laplacian for the Dirichlet problem in $B_0(1)$, we then get that for any $u \in C_0^\infty(B_0(\delta))$,

$$\int_{B_0(\delta)} (L_\alpha u) u dx \geq \frac{1}{4} \int_{B_0(\delta)} |\nabla u|^2 dx + \left(\frac{\lambda_1}{4\delta^2} - A \right) \int_{B_0(\delta)} u^2 dx$$

so that for $\delta > 0$ sufficiently small, there exists $C_\delta > 0$ with the property that

$$\int_{B_0(\delta)} (L_\alpha u) u dx \geq C_\delta \|u\|_{H_1^2}^2$$

for all $u \in C_0^\infty(B_0(\delta))$, where $\| \cdot \|_{H_1^2}$ is the usual norm on H_1^2. In particular, the operators L_α are uniformly coercive on balls $B_0(\delta)$ when $\delta > 0$ (independent of α) is sufficiently small. Then the second step in the proof of Lemma **??** is as follows.

Step 6.2. *There exists $C > 0$ such that, up to a subsequence, for any α and $r > 0$ sufficiently small, independent of α,*

$$|\varphi_\alpha(r)| \leq C + C \frac{\mu_\alpha^{\frac{n-4}{2}}}{r^{n-2}} \, , \tag{6.25}$$

*where the μ_α's are given by (**??**), and the φ_α's are given by (**??**).*

*Proof of Step **??**.* Let x_0 be the limit of the x_α's in (**??**). Let also $\delta > 0$ be such that the L_α's are uniformly coercive on $B_0(\delta)$, and $B_{x_0}(2\delta) \cap S = \{x_0\}$, where S, the set of geometrical blow-up points, is as in (**??**). By (**??**), the φ_α's converge in $C^2_{loc}(B_0(2\delta)\backslash\{0\})$. We let η smooth be such that $\eta = 0$ in $B_0(s)$ and $\eta = 1$ in $M\backslash B_0(2s)$ where $s \in (0, \delta/2)$. By the Lax-Milgram theorem we can solve the equation $L_\alpha \tilde{\varphi}_\alpha = -L_\alpha(\eta\varphi_\alpha)$ in $B_0(\delta)$, $\tilde{\varphi}_\alpha = 0$ on $\partial B_0(\delta)$, where L_α is given by (**??**). Letting $\hat{\varphi}_\alpha = \tilde{\varphi}_\alpha + \eta\varphi_\alpha$ we then get that $\hat{\varphi}_\alpha$ solves the equation

$$\begin{aligned} &L_\alpha \hat{\varphi}_\alpha = 0 \ \text{ in } B_0(\delta) \, , \text{ and} \\ &\hat{\varphi}_\alpha = \varphi_\alpha \ \text{ on } \partial B_0(\delta) \, . \end{aligned} \tag{6.26}$$

By standard elliptic theory, and the above remark on the uniform coercivity of the L_α's, the $\hat{\varphi}_\alpha$'s are in $H_2^p(B_0(\delta))$ for all p, and we have that

$$\|\hat{\varphi}_\alpha\|_{C^1\left(\overline{B_0(\delta)}\right)} \leq C \tag{6.27}$$

for all α, where $C > 0$ is independent of α. Now we let $F_{4,\alpha}$ be the right hand side in equation (**??**) so that

$$F_{4,\alpha}(x) = F'_{1,\alpha}(r) + \frac{n-1}{r} F_{1,\alpha}(r) + \Theta^1_\alpha(r) F_{2,\alpha}(r) + \Theta^2_\alpha(r) F_{3,\alpha}(r) \, , \tag{6.28}$$

where $r = |x|$, $F_{1,\alpha}$, $F_{2,\alpha}$, and $F_{3,\alpha}$ are given by (**??**) and (**??**), and the Θ^1_α's, and Θ^2_α's are bounded functions both with respect to r and α. Letting $\overline{\varphi}_\alpha = \varphi_\alpha - \hat{\varphi}_\alpha$, it follows that

$$\begin{aligned} &L_\alpha \overline{\varphi}_\alpha = F_{4,\alpha} \ \text{ in } B_0(\delta) \, , \text{ and} \\ &\overline{\varphi}_\alpha = 0 \ \text{ on } \partial B_0(\delta) \, . \end{aligned} \tag{6.29}$$

Moreover, by (**??**), we have that

$$\|\overline{\varphi}_\alpha\|_{C^1\left(\overline{B_0(\delta)}\backslash B_0(\delta/2)\right)} \leq C \tag{6.30}$$

for all α, where $C > 0$ is independent of α. Of course we also have that the $\overline{\varphi}_\alpha$'s are in $H_2^p(B_0(\delta))$ for all p. Computing $F'_{1,\alpha}$ we easily find that

$$F'_{1,\alpha}(r) + \frac{n-1}{r}F_{1,\alpha}(r) = O\left(\frac{1}{r^{n-1}}\int_{\partial B_{x_\alpha}(r)}|f_\alpha|d\sigma_g\right) + O\left(\frac{1}{r^{n-2}}\int_{B_{x_\alpha}(r)}|f_\alpha|dv_g\right) ,$$

where $f_\alpha = u_\alpha^{2^\sharp-1} - c_\alpha u_\alpha$. It follows that

$$\begin{aligned} F_{4,\alpha}(r) \quad &\leq \quad \frac{C_1}{r^{n-1}}\int_{\partial B_{x_\alpha}(r)}\left(1 + u_\alpha^{2^\sharp-1}\right)d\sigma_g \\ &\quad + \frac{C_2}{r^{n-1}}\int_{B_{x_\alpha}(r)}\left(1 + u_\alpha^{2^\sharp-1}\right)dv_g \end{aligned}$$

and we can also write that

$$\begin{aligned} F_{4,\alpha}(r) \leq C_3\int_{\partial B_0(1)}\left(1 + \tilde{u}_\alpha^{2^\sharp-1}(rx)\right)d\sigma(x) \\ + \frac{C_4}{r^{n-1}}\int_{B_0(r)}\left(1 + \tilde{u}_\alpha^{2^\sharp-1}(x)\right)dx , \end{aligned} \tag{6.31}$$

where the C_i's are positive constants independent of r and α, where the function $\tilde{u}_\alpha$ is given by $\tilde{u}_\alpha(x) = u_\alpha\left(\exp_{x_\alpha}(x)\right)$, and where $d\sigma$ is with respect to the Euclidean measure dx. Now we let G_α be the Green's function of L_α for the Dirichlet problem in $B_0(\delta)$ (as discussed in Section **??**). Then there exists $C > 0$ such that for any α, and any $x, y \in B_0(\delta)$,

$$|G_\alpha(x,y)| \leq \frac{C}{|y-x|^{n-2}} \tag{6.32}$$

and we also have that for any α,

$$\overline{\varphi}_\alpha(x) = \int_{B_0(\delta)} G_\alpha(x,y)F_{4,\alpha}(y)dy + \int_{\partial B_0(\delta)} G_\alpha(x,y)\partial_\nu\overline{\varphi}_\alpha(y)d\sigma(y) .$$

We fix x in $B_0(\delta/2)$. By (**??**) and (**??**),

$$\overline{\varphi}_\alpha(x) \leq C\int_{B_0(\delta)}\frac{F_{4,\alpha}(y)}{|y-x|^{n-2}}dy + C , \tag{6.33}$$

where $C > 0$ is independent of x and α. Let K_α be the function given by

$$K_\alpha(x) = \int_{B_0(\delta)}\frac{1}{|y-x|^{n-2}}\left(\int_{\partial B_0(1)}\left(1 + \tilde{u}_\alpha^{2^\sharp-1}(|y|\theta)\right)d\sigma(\theta)\right)dy \tag{6.34}$$

and let ψ_α be the function given by

$$\psi_\alpha(r) = \int_{B_0(r)}\left(1 + \tilde{u}_\alpha^{2^\sharp-1}(x)\right)dx . \tag{6.35}$$

Noting that

$$\psi'_\alpha(r) = r^{n-1}\int_{\partial B_0(1)}\left(1 + \tilde{u}_\alpha^{2^\sharp-1}(r\theta)\right)d\sigma(\theta) ,$$

and integrating by parts, we easily get that

$$|K_\alpha(x)| \leq C_5 + C_6\int_{B_0(\delta)}\frac{\psi_\alpha(|y|)}{|y-x|^{n-1}|y|^{n-1}}dy , \tag{6.36}$$

where $C_5, C_6 > 0$ do not depend on x and α. Combining (**??**), (**??**), and (**??**), we then get that for $x \in B_0(\delta/2)$,

$$|\overline{\varphi}_\alpha(x)| \leq C_7 + C_8 \int_{B_0(\delta)} \frac{\psi_\alpha(|y|)}{|y-x|^{n-1}|y|^{n-1}} dy \ , \tag{6.37}$$

where $C_7, C_8 > 0$ do not depend on x and α, and where ψ_α is given by (**??**). Now we let $p > n/2$, and set $p_1 = (2^\sharp - 1)p$, $p_2 = 2^\sharp - 1$. By Lemma **??**, there exist sequences (u^1_α) and (u^2_α) of nonnegative functions such that $u_\alpha \leq u^1_\alpha + u^2_\alpha$, $\|u^1_\alpha\|_{p_1} \leq C$, and

$$\|u^2_\alpha\|_{p_2} \leq C \mu_\alpha^{\frac{n}{p_2} - \frac{n}{2^\sharp}} \ ,$$

where $C > 0$ is independent of α. It follows that

$$|\psi_\alpha(r)| \leq C_9 r^{n(1-\frac{1}{p})} + C_{10} \mu_\alpha^{\frac{n-4}{2}} \ , \tag{6.38}$$

where $C_9, C_{10} > 0$ do not depend on r and α. Then, combining (**??**) and (**??**), we get that

$$\begin{aligned} |\overline{\varphi}_\alpha(x)| \ &\leq \ C_{11} + C_{12} \int_{B_0(\delta)} \frac{|y|^{1-\frac{n}{p}}}{|y-x|^{n-1}} dy \\ &\quad + C_{13} \mu_\alpha^{\frac{n-4}{2}} \int_{B_0(\delta)} \frac{1}{|y-x|^{n-1}|y|^{n-1}} dy \end{aligned}$$

and, since $p > n/2$, it follows from Giraud's lemma [**?**] that

$$|\overline{\varphi}_\alpha(x)| \leq C_{14} + C_{15} \frac{\mu_\alpha^{\frac{n-4}{2}}}{r^{n-2}} \ ,$$

where the C_i's, $i = 11, \ldots, 15$, are independent of x and α. Since $\varphi_\alpha = \overline{\varphi}_\alpha + \hat{\varphi}_\alpha$, and (**??**) holds, this proves Step **??**. □

With Steps **??** and **??** we are now in position to prove Lemma **??**.

Proof of Lemma **??**. Let Φ_α be the function $F_{3,\alpha}$ in (**??**). Then

$$\Phi_\alpha(r) = \frac{1}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g \ .$$

By (**??**), and thanks to the definition (**??**) of φ_α, we can write that

$$\Phi'_\alpha(r) = -\frac{1}{r^{n-1}} \int_0^r t^{n-1} \varphi_\alpha(t) dt + h_\alpha(r) \Phi_\alpha(r) \ , \tag{6.39}$$

where the h_α's are bounded functions both with respect to r and α. Integrating (**??**) between r and $\delta/2$, where $\delta > 0$ is small, we get that

$$\begin{aligned} &e^{-\int_0^r h_\alpha(t)dt} \Phi_\alpha(r) - e^{-\int_0^{\delta/2} h_\alpha(t)dt} \Phi_\alpha(\delta/2) \\ &= \int_r^{\delta/2} \frac{\int_0^t s^{n-1} \varphi_\alpha(s) ds}{t^{n-1}} e^{-\int_0^t h_\alpha(s)ds} dt \ . \end{aligned}$$

By Step **??** we then get that

$$|\Phi_\alpha(r)| \leq C_1 + C_2 \frac{\mu_\alpha^{\frac{n-4}{2}}}{r^{n-4}} \tag{6.40}$$

for all $r < \delta/2$, where $C_1, C_2 > 0$ are independent of α and r. On the other hand, since $c_\alpha \leq b_\alpha^2/4$, we can write that $\Delta_g u_\alpha + \frac{b_\alpha}{2} u_\alpha \geq 0$. By (**??**) of Step **??**, by (**??**), and since $u_\alpha \geq 0$, we can then write that

$$\begin{aligned}
&\frac{1}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} |\Delta_g u_\alpha| \, d\sigma_g \\
&\leq \frac{1}{r^{n-1}} \left| \int_{\partial B_{x_\alpha}(r)} \Delta_g u_\alpha d\sigma_g \right| + \frac{b_\alpha}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g \\
&\leq C_3 + C_4 \frac{\mu_\alpha^{\frac{n-4}{2}}}{r^{n-2}} \, ,
\end{aligned}$$

where $C_3, C_4 > 0$ are independent of α and r. Together with (**??**), this proves Lemma **??**. □

7. Asymptotic estimates

As in the previous sections, we let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, and we are concerned with getting estimates to prove the compactness assertion of Theorem **??** and Theorem **??**. We let (u_α) be a bounded sequence in $H_2^2(M)$ of nonnegative solutions of (**??**), and assume that the u_α's blow up. As in Sections **??** and **??**, u^0 may be nonzero. Up to renumbering and up to a subsequence, as done in Sections **??** and **??**, we can assume that

$$\mu_\alpha^1 = \max_{1 \leq i \leq k} \mu_\alpha^i \, , \tag{7.1}$$

where the μ_α^i's are the weights of the bubbles (B_α^i) of Lemma **??**. Then, as in (**??**), we let $x_\alpha = x_\alpha^1$ and $\mu_\alpha = \mu_\alpha^1$, where the x_α^1's are the centers of (B_α^1). We let also $\overline{u}_\alpha$ be the function defined in the Euclidean space by

$$\overline{u}_\alpha(x) = u_\alpha \left(\exp_{x_\alpha}(\sqrt{\mu_\alpha} x) \right) \, , \tag{7.2}$$

where $\exp_{x_\alpha}$ is the exponential map at x_α. We use the terminology biharmonic in the sequel for functions u such that $\Delta^2 u = 0$, where Δ is the Euclidean Laplacian. We prove in this section that the following estimate holds.

Lemma 7.1. *There exist $\delta > 0$, $A > 0$, and a biharmonic function $\varphi \in C^4(B_0(2\delta))$ such that, up to a subsequence,*

$$\overline{u}_\alpha(x) \to \frac{A}{|x|^{n-4}} + \varphi(x)$$

*in $C^3_{loc}(B_0(2\delta) \backslash \{0\})$ as $\alpha \to +\infty$, where $\overline{u}_\alpha$ is given by (**??**). Moreover, φ is positive in $B_0(2\delta)$ if $u^0 \not\equiv 0$, where u^0 is as in Lemma **??**.*

We prove Lemma **??** using Steps **??** to **??**. Up to a subsequence we may assume that for any given i, either $d_g(x_\alpha, x_\alpha^i) \leq C\sqrt{\mu_\alpha}$ for some $C > 0$ and all α, or $d_g(x_\alpha, x_\alpha^i)/\sqrt{\mu_\alpha} \to +\infty$ as $\alpha \to +\infty$, where the x_α^i's are the centers of the bubbles (B_α^i) given by Lemma **??**. If I is the subset of $\{1, \dots, k\}$ consisting of the i's for which $d_g(x_\alpha, x_\alpha^i) \leq C\sqrt{\mu_\alpha}$ for some $C > 0$ and all α, we then let

$$\hat{S} = \left\{ \lim_{\alpha \to +\infty} \frac{1}{\sqrt{\mu_\alpha}} \exp_{x_\alpha}^{-1}(x_\alpha^i) \, , \; i \in I \right\} \, , \tag{7.3}$$

where exp_{x_α} is the exponential map at x_α, and the limits in $\hat{S}$ are assumed to exist up to passing to a subsequence. Clearly, $0 \in \hat{S}$. Step **??** in the proof of Lemma **??** is as follows.

Step 7.1. *There exists $\overline{u} \in C^4\big(\mathbb{R}^n \backslash \hat{S}\big)$ such that, up to a subsequence, $\overline{u}_\alpha \to \overline{u}$ in $C^3_{loc}\big(\mathbb{R}^n \backslash \hat{S}\big)$, where $\overline{u}_\alpha$ is given by (**??**). Moreover $\overline{u}$ is biharmonic in $\mathbb{R}^n \backslash \hat{S}$ with the property that $\overline{u}$ and $\Delta \overline{u}$ are both nonnegative in $\mathbb{R}^n \backslash \hat{S}$.*

Proof of Step **??**. By (**??**),

$$\Delta^2_{g_\alpha} \overline{u}_\alpha + b_\alpha \mu_\alpha \Delta_{g_\alpha} \overline{u}_\alpha + c_\alpha \mu^2_\alpha \overline{u}_\alpha = h_\alpha \overline{u}_\alpha \ , \tag{7.4}$$

where $g_\alpha(x) = \big(\exp^\star_{x_\alpha} g\big)\big(\sqrt{\mu_\alpha} x\big)$, and $h_\alpha = \mu^2_\alpha \overline{u}^{2^\sharp - 2}_\alpha$. If ξ stands for the Euclidean metric, $g_\alpha \to \xi$ in $C^2(K)$ for any compact subset K of $\mathbb{R}^n$. Given $R > 0$ and $\delta > 0$, we let $K = \overline{B_0(R)} \backslash \bigcup_{x \in \hat{S}} B_x(\delta)$. By Lemma **??**, the h_α's are uniformly bounded in K. By Lemma **??**,

$$h_\alpha \to 0 \text{ in } \ L^\infty(K) \tag{7.5}$$

as $\alpha \to +\infty$. Now we claim that for any $\delta_1 < \delta_2$ positive, and any $p \in \big(1, \frac{n}{n-2}\big)$, there exists $C = C(\delta_1, \delta_2, p)$ positive such that

$$\int_{R^{\delta_2}_{\delta_1}} \overline{u}^p_\alpha dv_{g_\alpha} \leq C \tag{7.6}$$

for all α, where $R^{\delta_2}_{\delta_1}$ is the Euclidean annulus centered at 0 and of radii δ_1 and δ_2. In order to prove (**??**) we use Lemma **??**. We let $A^{\delta_2}_{\delta_1} = A^{\delta_2}_{\delta_1}(\alpha)$ be the annulus centered at x_α and of radii $\delta_1 \sqrt{\mu_\alpha}$ and $\delta_2 \sqrt{\mu_\alpha}$. Integrating the two equations in Lemma **??** over this annulus we get that

$$\begin{aligned} &\frac{1}{Vol_g\left(A^{\delta_2}_{\delta_1}\right)} \int_{A^{\delta_2}_{\delta_1}} u_\alpha dv_g \leq C \ \ , \text{ and} \\ &\frac{1}{Vol_g\left(A^{\delta_2}_{\delta_1}\right)} \int_{A^{\delta_2}_{\delta_1}} |\Delta_g u_\alpha| \, dv_g \leq C \mu^{-1}_\alpha \ , \end{aligned} \tag{7.7}$$

where $C > 0$ is independent of α, and $Vol_g\left(A^{\delta_2}_{\delta_1}\right)$ is the volume of $A^{\delta_2}_{\delta_1}$ with respect to g. Then (**??**) gives that

$$\int_{R^{\delta_2}_{\delta_1}} \overline{u}_\alpha dv_{g_\alpha} \leq C \quad \text{and} \quad \int_{R^{\delta_2}_{\delta_1}} |\Delta_{g_\alpha} \overline{u}_\alpha| \, dv_{g_\alpha} \leq C \ . \tag{7.8}$$

We let F_α be such that $F_\alpha = \Delta_{g_\alpha} \overline{u}_\alpha$ in $R^{\delta_2}_{\delta_1}$ and $F_\alpha = 0$ outside $R^{\delta_2}_{\delta_1}$. Given $\delta > \delta_2$ we let also G_α be the Green's function of Δ_{g_α} in $B_0(\delta)$ with zero Dirichlet boundary condition, and set

$$v_\alpha(x) = \int_{B_0(\delta)} G_\alpha(x, y) F_\alpha(y) dv_{g_\alpha}(y) \ .$$

By standard properties of the Green's function, there exists $C > 0$ such that

$$G_\alpha(x, y) \leq \frac{C}{|y - x|^{n-2}} \tag{7.9}$$

for all $x \in R_{\delta_1}^{\delta_2}$, all $y \in B_0(\delta)$, and all α. For $p \in \left(1, \frac{n}{n-2}\right)$ we let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. For $\varphi \in L^q(R_{\delta_1}^{\delta_2})$, by (**??**), we can write that

$$\left| \int_{R_{\delta_1}^{\delta_2}} v_\alpha \varphi dx \right| \leq C \int_{R_{\delta_1}^{\delta_2}} \left(\int_{R_{\delta_1}^{\delta_2}} \frac{\varphi(x)}{|y-x|^{n-2}} dx \right) |F_\alpha(y)| \, dy \, .$$

This implies that

$$\begin{aligned} \left| \int_{R_{\delta_1}^{\delta_2}} v_\alpha \varphi dx \right| &\leq C \|\varphi\|_{L^q(R_{\delta_1}^{\delta_2})} \int_{R_{\delta_1}^{\delta_2}} \left(\int_{R_{\delta_1}^{\delta_2}} \frac{dx}{|y-x|^{p(n-2)}} \right)^{1/p} |F_\alpha(y)| \, dy \\ &\leq C \|\varphi\|_{L^q(R_{\delta_1}^{\delta_2})} \|F_\alpha\|_{L^1(R_{\delta_1}^{\delta_2})} \end{aligned}$$

and then, by (**??**), that

$$\left| \int_{R_{\delta_1}^{\delta_2}} v_\alpha \varphi dx \right| \leq C \|\varphi\|_{L^q(R_{\delta_1}^{\delta_2})} \, ,$$

where $C > 0$ does not depend on α and φ. By duality, taking $\varphi = v_\alpha^{p-1}$, we get that

$$\int_{R_{\delta_1}^{\delta_2}} v_\alpha^p dv_{g_\alpha} \leq C \, , \tag{7.10}$$

where $C > 0$ is independent of α. Since $\Delta_{g_\alpha} (v_\alpha - \overline{u}_\alpha) = 0$ in $R_{\delta_1}^{\delta_2}$, it follows from standard elliptic theory (the De Giorgi-Nash-Moser iterative scheme) that if $\Omega \subset\subset R_{\delta_1}^{\delta_2}$, then

$$\sup_\Omega |v_\alpha - \overline{u}_\alpha| \leq C \, \|v_\alpha - \overline{u}_\alpha\|_{L^1(R_{\delta_1}^{\delta_2})} \, ,$$

where $C > 0$ is independent of α. By (**??**) and (**??**), and since $\delta_1 < \delta_2$ are arbitrary, this implies (**??**). In particular, with similar ideas to those developed in Agmon-Douglis-Nirenberg [**?**, **?**] (see also Section **??** for the global version of the local estimates in [**?**, **?**] we use here), we get with (**??**), (**??**), and (**??**) that for any $p \in \left(1, \frac{n}{n-2}\right)$, and any $\Omega \subset\subset \mathbb{R}^n \backslash \hat{S}$, the $\overline{u}_\alpha$'s are uniformly bounded in $H_4^p(\Omega)$. By standard bootstrap arguments, it follows that the $\overline{u}_\alpha$'s are uniformly bounded in $H_4^p(\Omega)$ for all $p > 1$. Then, by the Sobolev embedding theorem, we get that, up to a subsequence, the $\overline{u}_\alpha$'s converge in $C^3_{loc}(\mathbb{R}^n \backslash \hat{S})$ to some nonnegative function $\overline{u}$ as $\alpha \to +\infty$. By (**??**) and (**??**), $\overline{u}$ is biharmonic. In particular, $\overline{u}$ is smooth in $\mathbb{R}^n \backslash \hat{S}$. Independently, since $c_\alpha \leq b_\alpha^2/4$, we can also write that $(L'_\alpha)^2 \overline{u}_\alpha \geq 0$, so that $L'_\alpha \overline{u}_\alpha \geq 0$, where $L'_\alpha = \Delta_{g_\alpha} + (b_\alpha \mu_\alpha)/2$. It follows by passing to the limit as $\alpha \to +\infty$ that $\Delta \overline{u} \geq 0$, and this proves Step **??**. □

In what follows we write that $\hat{S} = \{x_1, \dots, x_p\}$ with $x_1 = 0$. Step **??** in the proof of Lemma **??** is as follows.

Step 7.2. *There exist $a_i, b_i \in \mathbb{R}$, $i = 1, \dots, p$, and a smooth biharmonic function φ in $\mathbb{R}^n$ such that*

$$\overline{u}(x) = \sum_{i=1}^p \frac{b_i}{|x - x_i|^{n-4}} + \sum_{i=1}^p \frac{a_i}{|x - x_i|^{n-2}} + \varphi(x) \tag{7.11}$$

for all x in $\mathbb{R}^n \backslash \hat{S}$.

Proof of Step **??**. We fix $i = 1, \dots, p$. Since $\Delta\overline{u}$ is harmonic and nonnegative in $B_{x_i}(\delta_0)\backslash\{x_i\}$, for some $\delta_0 > 0$, classical results in harmonic analysis (see for instance Veron [**?**]) give that

$$\Delta\overline{u}(x) = \frac{A}{|x - x_i|^{n-2}} + \psi(x) \ ,$$

where $A \in \mathbb{R}$ and ψ is harmonic in $B_{x_i}(\delta_0)$. Let $\tilde{\psi}$ be such that $\Delta\tilde{\psi} = \psi$ in $B_{x_i}(\delta_0)$, and let $\hat{u}$ be the function in $B_{x_i}(\delta_0)\backslash\{x_i\}$ given by

$$\hat{u}(x) = \overline{u}(x) - \frac{A}{2(n-4)|x|^{n-4}} - \tilde{\psi}(x) \ .$$

Then $\hat{u}$ is harmonic in $B_{x_i}(\delta_0)\backslash\{x_i\}$. Clearly, for $B \in \mathbb{R}$, the function $\hat{u}_B$ given by

$$\hat{u}_B(x) = \hat{u}(x) + \frac{B}{|x - x_i|^{n-2}}$$

is still harmonic in $B_{x_i}(\delta_0)\backslash\{x_i\}$, while $\hat{u}_B$ is nonnegative in $B_{x_i}(\delta_0/2)\backslash\{x_i\}$ if we choose $B > 0$ sufficiently large. Then (see again Veron [**?**]), for $B > 0$ large, $\hat{u}_B$ writes as

$$\hat{u}_B(x) = \frac{C}{|x - x_i|^{n-2}} + \hat{\psi}(x) \ ,$$

where $C \in \mathbb{R}$ and $\hat{\psi}$ is harmonic in $B_{x_i}(\delta_0)$. In particular,

$$\overline{u}(x) = \frac{C_1}{|x - x_i|^{n-4}} + \frac{C_2}{|x - x_i|^{n-2}} + \varphi_i(x)$$

in $B_{x_i}(\delta_0)\backslash\{x_i\}$, where $C_1, C_2 \in \mathbb{R}$ and φ_i is biharmonic in $B_{x_i}(\delta_0)$. A local result from which we easily get that Step **??** holds. □

Since $\overline{u} \geq 0$ and $\Delta\overline{u} \geq 0$, it follows from (**??**) and equation (**??**) below that $a_i \geq 0$ and $b_i \geq 0$ for all i. Step **??** in the proof of Lemma **??** is as follows.

Step 7.3. *The biharmonic function φ in (**??**) is nonnegative and constant, while $a_1 = 0$ in (**??**).*

Proof of Step **??**. It follows from (**??**) that

$$\Delta\overline{u}(x) = \sum_{i=1}^{p} \frac{2(n-4)b_i}{|x - x_i|^{n-2}} + \Delta\varphi(x) \tag{7.12}$$

for all $x \in \mathbb{R}^n\backslash\hat{S}$. By (**??**) and (**??**) we then get that φ and $\Delta\varphi$ are uniformly bounded from below since $\overline{u} \geq 0$ and $\Delta\overline{u} \geq 0$. By Liouville's theorem, since $\Delta\varphi$ is harmonic, $\Delta\varphi = K_0$ is constant. Noting that by (**??**), K_0 is the limit of the $\Delta\overline{u}(x)$'s as $x \to +\infty$, we get that $K_0 \geq 0$. Writing that

$$\Delta\left(\varphi + \frac{K_0}{2n}|x|^2\right) = 0$$

and noting that $\varphi + \frac{K_0}{2n}|x|^2$ is bounded from below since φ is bounded from below, another application of Liouville's theorem gives that $\varphi + \frac{K_0}{2n}|x|^2 = K_0'$ is constant. By (**??**), and since $\overline{u} \geq 0$, $\varphi(x)$ has to be nonnegative for x large. This implies that $K_0 = 0$ and thus that φ is a nonnegative constant. This proves the first assertion

in Step **??**. Concerning the second assertion, we know from Lemma **??** that there exists $C_1, C_2 > 0$ such that

$$\frac{1}{r^{n-1}} \int_{\partial B_{x_\alpha}(r)} u_\alpha d\sigma_g \leq C_1 + C_2 \frac{\mu_\alpha^{\frac{n-4}{2}}}{r^{n-4}}$$

for all α and all $r > 0$ sufficiently small. By Step **??**, letting $r = \delta\sqrt{\mu_\alpha}$, with $\delta > 0$ small, we then get that

$$\frac{1}{\delta^{n-1}} \int_{\partial B_0(\delta)} \overline{u} d\sigma \leq C_3 + \frac{C_4}{\delta^{n-4}} \ ,$$

where $d\sigma$ is the measure on $\partial B_0(\delta)$ induced by the Euclidean metric, and $C_3, C_4 > 0$ are independent of δ and α. By (**??**), letting $\delta \to 0$, it follows that $a_1 = 0$. This proves Step **??**. □

By Step **??** and (**??**) we can now write that

$$\overline{u}(x) = \frac{A}{|x|^{n-4}} + \sum_{i=2}^{p} \frac{b_i}{|x - x_i|^{n-4}} + \sum_{i=2}^{p} \frac{a_i}{|x - x_i|^{n-2}} + K_0 \tag{7.13}$$

for all $x \in \mathbb{R}^n \backslash \hat{S}$, where A, the a_i's and b_i's, and K_0 are nonnegative constants. Then Step **??** in the proof of Lemma **??** is as follows.

Step 7.4. *The constant A in (**??**) is positive.*

*Proof of Step **??**.* For $d_{1,\alpha}$ and $d_{2,\alpha}$ as in (**??**), we can write the fourth order operator $P_\alpha = \Delta_g^2 + b_\alpha \Delta_g + c_\alpha$ as the product $L_\alpha^1 L_\alpha^2$ where L_α^1 and L_α^2 are the second order operators given by $L_\alpha^1 = \Delta_g + d_{1,\alpha}$ and $L_\alpha^2 = \Delta_g + d_{2,\alpha}$. If G_α^1 stands for the Green function of L_α^1, and G_α^2 for the Green function of L_α^2, we then get that

$$\tilde{G}_\alpha(x, y) = \int_M G_\alpha^1(x, z) G_\alpha^2(z, y) dv_g(z)$$

is the Green function of P_α. By standard properties of G_α^1 and G_α^2, as studied for instance in the appendix of Druet, Hebey and Robert [**?**], there exists $C > 0$ such that $G_\alpha^1(x, y)$ and $G_\alpha^2(x, y)$ are both controlled from below by $C/d_g(x, y)^{n-2}$ for all $x \neq y$. Then it follows that there exists $C > 0$ such that for any $x \neq y$ in M, and any α,

$$\tilde{G}_\alpha(x, y) \geq \frac{C}{d_g(x, y)^{n-4}} \ . \tag{7.14}$$

We assume from now on that the ratios $d_g(x_\alpha, x_\alpha^i)/\mu_\alpha$ converge (with a limit possibly $+\infty$) for all i as $\alpha \to +\infty$. This holds up to passing to a subsequence. We let $\delta_1 < \delta_2$ positive be such that the closed interval $[\delta_1, \delta_2]$ does not contain any of such limits. Then, for $x \in B_0(\delta_2) \backslash B_0(\delta_1)$, $d_g\big(x_\alpha^i, \exp_{x_\alpha}(\mu_\alpha x)\big) \geq C\mu_\alpha$ where $C > 0$ is independent of α and x, and if we let v_α be the function given by

$$v_\alpha(x) = \mu_\alpha^{\frac{n-4}{2}} u_\alpha\big(\exp_{x_\alpha}(\mu_\alpha x)\big)$$

it follows from the above equation and Lemma **??** that there exists $C > 0$ such that $v_\alpha(x) \leq C$ for all α and all $x \in B_0(\delta_2) \backslash B_0(\delta_1)$. We let $\tilde{g}_\alpha$ be the metric given by $\tilde{g}_\alpha(x) = \big(\exp_{x_\alpha}^\star g\big)(\mu_\alpha x)$. If ξ stands for the Euclidean metric, $\tilde{g}_\alpha \to \xi$ in

$C^2(K)$ as $\alpha \to +\infty$ for all compact subsets K of $\mathbb{R}^n$. Since the v_α's are bounded in $B_0(\delta_2)\backslash B_0(\delta_1)$,

$$\int_{B_0(\delta_2)\backslash B_0(\delta_1)} v_\alpha^{2^\sharp} dv_{\tilde{g}_\alpha} \leq C \int_{B_0(\delta_2)\backslash B_0(\delta_1)} v_\alpha^{2^\sharp-1} dv_{\tilde{g}_\alpha} \ ,$$

where $C > 0$ is independent of α. Independently, by Lemma **??**, we can write that

$$\begin{aligned} \int_{B_0(\delta_2)\backslash B_0(\delta_1)} v_\alpha^{2^\sharp} dv_{\tilde{g}_\alpha} &= \int_{B_{x_\alpha}(\delta_2\mu_\alpha)\backslash B_{x_\alpha}(\delta_1\mu_\alpha)} u_\alpha^{2^\sharp} dv_g \\ &\geq C \int_{B_{x_\alpha}(\delta_2\mu_\alpha)\backslash B_{x_\alpha}(\delta_1\mu_\alpha)} B_\alpha^{2^\sharp} dv_g + o(1) \ , \end{aligned}$$

where (B_α) is the bubble of centers the x_α's and weights the μ_α's, $C > 0$ is independent of α, and $o(1) \to 0$ as $\alpha \to +\infty$. Noting that

$$\int_{B_{x_\alpha}(\delta_2\mu_\alpha)\backslash B_{x_\alpha}(\delta_1\mu_\alpha)} B_\alpha^{2^\sharp} dv_g = \int_{B_0(\delta_2)\backslash B_0(\delta_1)} u_{1,0}^{2^\sharp} dv_{\tilde{g}_\alpha} \ ,$$

where $u_{1,0}$ is the positive function given by (**??**), it follows that there exists $C > 0$ such that

$$\int_{B_0(\delta_2)\backslash B_0(\delta_1)} v_\alpha^{2^\sharp-1} dv_{\tilde{g}_\alpha} \geq C \tag{7.15}$$

for all α. Now we fix $x \in B_0(\delta)\backslash\{0\}$, where $\delta > 0$ is such that $B_0(\delta) \cap \hat{S}$ contains only 0, and, for $y \in B_0(\delta_2)\backslash B_0(\delta_1)$, $x \neq \sqrt{\mu_\alpha} y$, we let $\hat{G}_\alpha$ be the function given by $\hat{G}_\alpha(x,y) = \tilde{G}_\alpha\big(\exp_{x_\alpha}(\sqrt{\mu_\alpha}x), \exp_{x_\alpha}(\mu_\alpha y)\big)$. Then, by the Green's representation formula, we write that

$$\begin{aligned} \overline{u}_\alpha(x) &= \int_M \tilde{G}_\alpha\big(\exp_{x_\alpha}(\sqrt{\mu_\alpha}x), y\big) u_\alpha^{2^\sharp-1}(y) dv_g(y) \\ &\geq \int_{B_{x_\alpha}(\delta_2\mu_\alpha)\backslash B_{x_\alpha}(\delta_1\mu_\alpha)} \tilde{G}_\alpha\big(\exp_{x_\alpha}(\sqrt{\mu_\alpha}x), y\big) u_\alpha^{2^\sharp-1}(y) dv_g(y) \\ &\geq \mu_\alpha^{\frac{n-4}{2}} \int_{B_0(\delta_2)\backslash B_0(\delta_1)} \hat{G}_\alpha(x,y) v_\alpha^{2^\sharp-1}(y) dv_{\tilde{g}_\alpha}(y) \ . \end{aligned} \tag{7.16}$$

Noting that by (**??**), there exists $C > 0$ such that

$$\begin{aligned} \mu_\alpha^{\frac{n-4}{2}} \hat{G}_\alpha(x,y) &\geq \frac{C\mu_\alpha^{\frac{n-4}{2}}}{\big|\sqrt{\mu_\alpha}x - \mu_\alpha y\big|^{n-4}} \\ &\geq \frac{C}{\big|x - \sqrt{\mu_\alpha} y\big|^{n-4}} \end{aligned} \tag{7.17}$$

for all $x \in B_0(\delta)$ and all $y \in B_0(\delta_2)\backslash B_0(\delta_1)$ with $x \neq \sqrt{\mu_\alpha} y$, it follows from (**??**), (**??**), (**??**), and Step **??** that there exists $C > 0$ such that

$$\overline{u}(x) \geq \frac{C}{|x|^{n-4}} \tag{7.18}$$

for all $x \in B_0(\delta)\backslash\{0\}$. Coming back to (**??**), we get with (**??**) that $A > 0$. This proves Step **??**. □

The last step we need in the proof of Lemma **??** is as follows.

Step 7.5. *If $u^0 \not\equiv 0$, the constant K_0 in (**??**) is positive.*

Proof of Step **??**. First if $u^0 \not\equiv 0$ then, since u^0 is a nonnegative solution of (**??**), it is smooth and positive everywhere. Now we let $\hat{x}_0$ be the limit of the x_α's, and let $\delta > 0$ small. As in the proof of Step **??**, given $x \in \mathbb{R}^n \backslash \hat{S}$, we write that

$$\overline{u}_\alpha(x) = \int_M \tilde{G}_\alpha\big(\exp_{x_\alpha}(\sqrt{\mu_\alpha}x), y\big) u_\alpha^{2^\sharp - 1}(y) dv_g(y) \ ,$$

where $\tilde{G}_\alpha$ is the Green function of $\Delta_g^2 + b_\alpha \Delta_g + c_\alpha$. In particular,

$$\overline{u}_\alpha(x) \geq \int_{M \backslash B_{\hat{x}_0}(\delta)} \tilde{G}_\alpha\big(\exp_{x_\alpha}(\sqrt{\mu_\alpha}x), y\big) u_\alpha^{2^\sharp - 1}(y) dv_g(y) \ .$$

Letting $\alpha \to +\infty$, it follows that

$$\overline{u}(x) \geq \int_{M \backslash B_{\hat{x}_0}(\delta)} G(\hat{x}_0, y) (u^0)^{2^\sharp - 1} dv_g(y) \ ,$$

where G is the Green function of the limit operator $\Delta_g^2 + b_\infty \Delta_g + c_\infty$. Letting δ tend to zero, we get that

$$\begin{aligned} \overline{u}(x) &\geq \int_M G(\hat{x}_0, y) (u^0)^{2^\sharp - 1} dv_g(y) \\ &= u^0(\hat{x}_0) \end{aligned} \tag{7.19}$$

since u^0 is a solution of (**??**). By (**??**), $\overline{u}(x) \to K_0$ as $|x| \to +\infty$. By assumption, $u^0(\hat{x}_0) > 0$. It follows from these remarks and equation (**??**) that $K_0 > 0$. This proves Step **??**. □

With Steps **??** to **??** we are now in position to prove Lemma **??**. The proof of Lemma **??** proceeds as follows.

Proof of Lemma **??**. We let $\delta > 0$ be such that $B_0(3\delta)$ and $\hat{S}$ intersect only at 0, where $\hat{S}$ is given by (**??**). By Step **??**, the $\overline{u}_\alpha$'s converge, up to a subsequence, to $\overline{u}$ in $C^3_{loc}(B_0(2\delta)\backslash\{0\})$ as $\alpha \to +\infty$. By Steps **??** to **??**, we can write that

$$\overline{u}(x) = \frac{A}{|x|^{n-4}} + \varphi(x)$$

for all $x \in B_0(2\delta)\backslash\{0\}$, where $A > 0$, and φ is biharmonic and nonnegative in $B_0(2\delta)$. The explicit equation for φ is

$$\varphi(x) = \sum_{i=2}^p \frac{b_i}{|x - x_i|^{n-4}} + \sum_{i=2}^p \frac{a_i}{|x - x_i|^{n-2}} + K_0 \ ,$$

where a_i, b_i, and K_0 are nonnegative constants. By Step **??**, K_0, and thus φ in $B_0(2\delta)$, is positive if $u^0 \not\equiv 0$. This proves Lemma **??**. □

Lemma **??** below is the infinitesimal analogue of the global balance $L^2 - \nabla L^2$ stated in Lemmas **??** and **??**. Since, here, u^0 may not be zero, the proof is more involved.

Lemma 7.2. *Let $\delta > 0$ be as in Lemma* **??**. *Then, for any α,*

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} u_\alpha^2 dv_g = o(1) \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g \ ,$$

where $o(1) \to 0$ as $\alpha \to +\infty$. Moreover, $\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g \geq C\mu_\alpha^2$ for all α, where $C > 0$ is independent of α.

Proof of Lemma **??**. By Lemma **??**, the Euclidean Sobolev inequality for the embedding $H_1^2 \subset L^{2^\star}$ that we apply in $B_0(\delta)$, and Hölder's inequality,

$$\int_{B_0(\delta)} \overline{u}_\alpha^2 dx \leq C_1 \int_{B_0(\delta)} |\nabla \overline{u}_\alpha|^2 dx + C_2 \ , \tag{7.20}$$

where $\delta > 0$ is as in Lemma **??**, and $C_1, C_2 > 0$ are independent of α. In order to get (**??**), we write that $B_0(\delta) = B_0(r) \bigcup (B_0(\delta) \backslash B_0(r))$, that the L^2-norm of $\overline{u}_\alpha$ in $B_0(\delta)\backslash B_0(r)$ is bounded by Lemma **??**, that the L^2-norm of $\overline{u}_\alpha$ in $B_0(r)$ is controled by r times the $L^{2^\star}$-norm of $\overline{u}_\alpha$ in $B_0(\delta)$ by Hölder, and then we choose $r > 0$ small. Coming back to the u_α's, it follows from (**??**) that

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} u_\alpha^2 dv_g \leq C_3 \mu_\alpha \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g + C_4 \mu_\alpha^2 \mu_\alpha^{\frac{n-4}{2}} \ , \tag{7.21}$$

where $C_3, C_4 > 0$ are independent of α. Now we let $\varphi \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ in $B_0(1/2)$, and $\varphi \equiv 0$ in $\mathbb{R}^n \backslash B_0(1)$. Then we define φ_α by

$$\varphi_\alpha(x) = \mu_\alpha^{-\frac{n-2}{2}} \varphi\left(\frac{1}{\mu_\alpha} \exp_{x_\alpha}^{-1}(x)\right) \ .$$

Given $r > 0$, we can write that for α large,

$$\begin{aligned} \int_{B_{x_\alpha}(r\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g &\geq \int_{B_{x_\alpha}(\mu_\alpha)} |\nabla u_\alpha|^2 dv_g \\ &\geq \mu_\alpha^2 \int_{B_{x_\alpha}(\mu_\alpha)} |\nabla u_\alpha|^2 \varphi_\alpha^{2^\star - 2} dv_g \ . \end{aligned}$$

Thanks to the decomposition in Lemma **??**, noting that $H_2^2(M) \subset H_1^{2^\star}(M)$, and by Hölder's inequalities, we can also write that

$$\int_{B_{x_\alpha}(\mu_\alpha)} |\nabla u_\alpha|^2 \varphi_\alpha^{2^\star - 2} dv_g = \int_{B_{x_\alpha}(\mu_\alpha)} |\nabla B_\alpha|^2 \varphi_\alpha^{2^\star - 2} dv_g + o(1) \ ,$$

where (B_α) is the bubble of centers the x_α's and weights the μ_α's, and where $o(1) \to 0$ as $\alpha \to +\infty$. Then, noting that

$$\int_{B_{x_\alpha}(\mu_\alpha)} |\nabla B_\alpha|^2 \varphi_\alpha^{2^\star - 2} dv_g = \int_{B_0(1)} |\nabla u|^2 \varphi^{2^\star - 2} dv_{g_\alpha} \ ,$$

where $u = u_{1,0}$ is given by (**??**), and $g_\alpha(x) = \left(\exp_{x_\alpha}^\star g\right)(\mu_\alpha x)$, we easily get that for any $r > 0$, there exists $C > 0$, independent of α, such that for α large,

$$\int_{B_{x_\alpha}(r\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g \geq C\mu_\alpha^2 \ . \tag{7.22}$$

Taking $r = \delta$, coming back to (**??**), we get with (**??**) that

$$\int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} u_\alpha^2 dv_g \leq C_5\big(\mu_\alpha + \mu_\alpha^{\frac{n-4}{2}}\big) \int_{B_{x_\alpha}(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dv_g \ ,$$

where $C_5 > 0$ is independent of α. This ends the proof of Lemma **??**. □

8. The Green's function of L_α

We let Ω be a smooth bounded domain of $\mathbb{R}^n$. For k integer and $p > 1$, we let $H^p_k(\Omega)$ be the standard Sobolev space of functions in $L^p(\Omega)$ with k derivatives in L^p. Then we let $H^p_{k,0}(\Omega)$ be the completion of $C^\infty_0(\Omega)$ in $H^p_k(\Omega)$, where $C^\infty_0(\Omega)$ is the space of smooth functions with compact support in Ω. We let c and the b^i's be functions in $L^\infty(\Omega)$, $1 \le i \le n$, and let K_0 be such that

$$|c(x)| + \sum_{i=1}^n \left|b^i(x)\right| \le K_0$$

for all $x \in \Omega$. We also assume that the operator $Lu = \Delta u + b^i \partial_i u + cu$ is coercive in the sense that there exists $\lambda > 0$ such that

$$\int_\Omega \left(|\nabla u|^2 + u b^i \partial_i u + c u^2\right) dx \ge \lambda \int_\Omega |\nabla u|^2 dx \tag{8.1}$$

for all $u \in H^2_{1,0}(\Omega)$. Then we claim that there exists a Green function for L which satisfies uniform bound with respect to the coefficients b^i and c. More precisely, we claim that there exists $G : \Omega \times \overline{\Omega} \backslash D \to \mathbb{R}$, where D is the diagonal in $\Omega \times \Omega$, such that G satisfies the three propositions:

(G1) for any $x \in \Omega$, the function $y \to G(x,y)$ is in $L^1(\Omega)$ and in $L^\infty(\partial\Omega)$,

(G2) for any $u \in H^q_2(\Omega) \cap H^q_{1,0}(\Omega)$, $q > n$,

$$u(x) = \int_\Omega G(x,y) \left(\Delta u + b^i \partial_i u + cu\right)(y) dy + \int_{\partial\Omega} G(x,y) \partial_\nu u(y) d\sigma(y)$$

for all $x \in \Omega$, where ν is the outward unit normal vector of $\partial\Omega$, and

(G3) there exists $C > 0$, depending only on Ω, K_0, and λ, such that

$$|G(x,y)| \le \frac{C}{|y-x|^{n-2}}$$

for all $x \in \Omega$ and all $y \in \overline{\Omega}$ such that $x \ne y$,

where, concerning (G2), it should be noted that by the Sobolev embedding theorem, $H^q_2(\Omega) \subset C^1(\overline{\Omega})$. The existence of G (for the operator L_α of Section **??**) was used in Section **??**. The difficult point here is that the coefficients b^i (and c) are not assumed to be differentiable functions (the situation we face with L_α). In order to prove (G1)-(G3) we proceed as follows. For $x, y \in \mathbb{R}^n$, $x \ne y$, we let

$$H(x,y) = \frac{1}{(n-2)\omega_{n-1}|y-x|^{n-2}}$$

and for $i = 1, \dots, n$, we let also $H_i(x,y) = \partial_{i,x} H(x,y)$ so that

$$H_i(x,y) = \frac{y^i - x^i}{\omega_{n-1}|y-x|^n} \ .$$

It is easily checked that for $u \in H^q_2(\Omega) \cap H^q_{1,0}(\Omega)$, $q > n$, and $x \in \Omega$,

$$\int_\Omega H(x,y) \Delta u(y) dy = u(x) - \int_{\partial\Omega} H(x,y) \partial_\nu u(y) d\sigma(y) \tag{8.2}$$

and that for $u \in H^q_2(\Omega) \cap H^q_{1,0}(\Omega)$, $q > n$, for $i = 1, \dots, n$, and for $x \in \Omega$,

$$\int_\Omega H_i(x,y) \Delta u(y) dy = \partial_i u(x) - \int_{\partial\Omega} H_i(x,y) \partial_\nu u(y) d\sigma(y) \ . \tag{8.3}$$

For $x, y \in \overline{\Omega}$, $x \neq y$, and $i = 1, \dots, n$, we define Γ_1 and the Γ_1^i's by the equations

$$\Gamma_1(x,y) = -c(y)H(x,y) \ , \text{ and}$$
$$\Gamma_1^i(x,y) = -b^i(y)H(x,y) \ .$$

Then, by induction, we define the Γ_j's and Γ_j^i's, where $j \geq 1$ is integer, by the equations

$$\Gamma_{j+1}(x,y) = -c(y) \int_\Omega \left(\Gamma_j(x,z)H(z,y) + \sum_{k=1}^n \Gamma_j^k(x,z)H_k(z,y) \right) dz \ , \text{ and}$$
$$\Gamma_{j+1}^i(x,y) = -b^i(y) \int_\Omega \left(\Gamma_j(x,z)H(z,y) + \sum_{k=1}^n \Gamma_j^k(x,z)H_k(z,y) \right) dz \ .$$

It follows from Giraud's lemma [**?**] that for $j \geq 1$ there exists $C_j(\Omega, K_0) > 0$ such that

$$|\Gamma_j(x,y)| + \sum_{i=1}^n \left|\Gamma_j^i(x,y)\right| \leq \frac{C_j(\Omega,K_0)}{|y-x|^{n-j-1}} \ \text{ if } n > j+1$$
$$|\Gamma_j(x,y)| + \sum_{i=1}^n \left|\Gamma_j^i(x,y)\right| \leq C_j(\Omega,K_0)\left(1 + \left|\ln|y-x|\right|\right) \ \text{ if } n = j+1 \tag{8.4}$$
$$|\Gamma_j(x,y)| + \sum_{i=1}^n \left|\Gamma_j^i(x,y)\right| \leq C_j(\Omega,K_0) \ \text{ if } n < j+1 \ .$$

For $x \in \Omega$ and $y \in \Omega \backslash \{x\}$ we let

$$\begin{aligned} G(x,y) &= H(x,y) \\ &\quad + \sum_{j=1}^n \int_\Omega \left(\Gamma_j(x,z)H(z,y) + \sum_{k=1}^n \Gamma_j^k(x,z)H_k(z,y) \right) dz + u_x(y) \end{aligned} \tag{8.5}$$

and for $y \in \partial\Omega$, we let

$$G(x,y) = H(x,y) + \sum_{j=1}^n \int_\Omega \left(\Gamma_j(x,z)H(z,y) + \sum_{k=1}^n \Gamma_j^k(x,z)H_k(z,y) \right) dz \ , \tag{8.6}$$

where $u_x \in H^2_{1,0}(\Omega)$ will be fixed later on. By (**??**), the function $y \to G(x,y)$ is in $L^p(\Omega)$ for all $1 \leq p < \frac{n}{n-2}$ and also in $L^\infty(\partial\Omega)$. In particular, $y \to G(x,y)$ satisfies (G1). Independently, by (**??**) and (**??**), and thanks to the definition of the Γ_j's and Γ_j^i's, we easily get that for $u \in H_2^q(\Omega) \cap H^q_{1,0}(\Omega)$, and $x \in \Omega$,

$$\begin{aligned} &\int_\Omega G(x,y)\left(\Delta u + b^i\partial_i u + cu\right)(y)dy \\ &= u(x) - \int_\Omega \Gamma_{n+1}(x,y)u(y)dy - \int_\Omega \Gamma_{n+1}^k(x,y)\partial_k u(y)dy \\ &\quad + \int_\Omega \left((\nabla u_x \nabla u) + u_x b^i \partial_i u + cu_x u\right)dy - \int_{\partial\Omega} G(x,y)\partial_\nu u(y)d\sigma(y) \ . \end{aligned} \tag{8.7}$$

By (**??**) we have that

$$|\Gamma_{n+1}(x,y)| + \sum_{i=1}^n \left|\Gamma_{n+1}^i(x,y)\right| \leq C(\Omega,K_0)$$

for $x, y \in \Omega$, $x \neq y$. Now we let $u_x \in H^2_{1,0}(\Omega)$ be such that

$$\begin{aligned} &\int_\Omega \left((\nabla u_x \nabla \varphi) + u_x b^i \partial_i \varphi + c u_x \varphi\right) dy \\ &= \int_\Omega \Gamma_{n+1}(x,y)\varphi(y)dy + \int_\Omega \Gamma^k_{n+1}(x,y)\partial_k \varphi(y) dy \end{aligned} \tag{8.8}$$

for all $\varphi \in H^2_{1,0}(\Omega)$. The existence of u_x easily follows from the Lax-Milgram theorem and the coercivity assumption (**??**). Moreover, we get by (**??**) and (**??**) that for any $u \in H^q_2(\Omega) \cap H^q_{1,0}(\Omega)$, and any $x \in \Omega$,

$$u(x) = \int_\Omega G(x,y)\left(\Delta u + b^i \partial_i u + cu\right)(y)dy + \int_{\partial\Omega} G(x,y)\partial_\nu u(y) d\sigma(y)\ .$$

In particular, (G2) is satisfied and we are left with the proof of (G3). By standard elliptic theory, and (**??**), there exists $C(\Omega, K_0, \lambda) > 0$ such that

$$\sup_{y\in\Omega} |u_x(y)| \leq C(\Omega, K_0, \lambda) \tag{8.9}$$

for all $x \in \Omega$. Then, by the definition of G, by (**??**), and by (**??**), we get that

$$|G(x,y)| \leq \frac{C}{|y-x|^{n-2}}$$

for all $x, y \in \Omega$, with $x \neq y$, where $C > 0$ depends only on Ω, K_0, and λ. This proves (G3) and the above claim.

9. Proof of pseudo-compactness

We prove the pseudo-compactness assertion of Theorem **??** in this section. We let (M, g) be a smooth compact locally conformally flat Riemannian manifold of dimension $n \geq 5$, and let (u_α) be a bounded sequence of nonnegative solutions of (**??**). By contradiction we can assume that the u_α's blow up and that the weak limit u^0 in $H^2_2(M)$ of the u_α's is zero. Roughly speaking, the argument in this section consists in applying a Pohozaev type identity to the u_α's in small balls of the type $B_{x_i}(\delta)$, where the x_i's stand for the geometrical blow-up points of the u_α's, and then to get the contradiction by conformal invariance and the estimates we proved in Sections **??** and **??**. We start with conformal invariance. As already mentioned in the introduction, the geometric Paneitz-Branson operator and the Q-curvature satisfy conformal transormation laws. The same holds for the conformal Laplacian and the scalar curvature. Let $\hat g$ be a conformal metric to g. We write that $g = \varphi^{4/(n-4)}\hat g$. Let also $\hat u_\alpha = u_\alpha \varphi$. Then, by conformal invariance,

$$\begin{aligned} &\Delta^2_{\hat g}\hat u_\alpha + b_\alpha \varphi^{\frac{4}{n-4}} \Delta_{\hat g}\hat u_\alpha - B_\alpha(\nabla\varphi, \nabla \hat u_\alpha) + h_\alpha \hat u_\alpha + \varphi^{\frac{n+4}{n-4}} div_g(\varphi^{-1} A_g d\hat u_\alpha) \\ &\quad = div_{\hat g}(A_{\hat g} d\hat u_\alpha) - \frac{n-4}{2} Q^n_{\hat g}\hat u_\alpha - \frac{n-2}{4(n-1)} b_\alpha \varphi^{\frac{4}{n-4}} S_{\hat g}\hat u + \hat u_\alpha^{2^\sharp - 1}\ , \end{aligned} \tag{9.1}$$

where A_g is given by (**??**), B_α is given by

$$B_\alpha = \frac{4b_\alpha}{n-4}\varphi^{\frac{8-n}{n-4}}\hat g + \varphi^{\frac{12-n}{n-4}} A_g$$

and where

$$h_\alpha = b_\alpha \varphi^{\frac{2}{n-4}} \Delta_{\hat g} \varphi^{\frac{2}{n-4}} - \frac{n-2}{4(n-1)} b_\alpha \varphi^{\frac{8}{n-4}} S_g + c_\alpha \varphi^{\frac{8}{n-4}} - \frac{n-4}{2} Q_g \varphi^{\frac{8}{n-4}} + \varphi^{\frac{n+4}{n-4}} div_g(A_g d\varphi^{-1}) \, .$$

We assumed here, as in Theorem **??**, that our manifold is locally conformally flat. We let $x_0 \in S$ where S is the set of geometrical blow-up points given by (**??**). Then there exists $\delta > 0$ and a conformal metric $\hat g$ to g such that $\hat g$ is flat in $B_{x_0}(4\delta)$. According to what we just said, see in particular equation (**??**), we can write that

$$\begin{aligned} &\Delta^2 \hat u_\alpha + b_\alpha \varphi^{\frac{4}{n-4}} \Delta \hat u_\alpha - B_\alpha(\nabla\varphi, \nabla \hat u_\alpha) + h_\alpha \hat u_\alpha \\ &\quad + \varphi^{\frac{n+4}{n-4}} div_g(\varphi^{-1} A_g d\hat u_\alpha) = \hat u_\alpha^{2^\sharp - 1} \end{aligned} \tag{9.2}$$

in $B_{x_0}(4\delta)$, where A_g, B_α, and h_α are as above, and $\Delta = \Delta_{\hat g}$ is the Euclidean Laplacian. We choose $\delta > 0$ sufficiently small such that $S \cap B_{x_0}(4\delta) = \{x_0\}$. Also, we let η be a smooth function in $\mathbb{R}^n$ such that $\eta = 1$ in $B_0(\delta)$ and $\eta = 0$ in $\mathbb{R}^n \backslash B_0(2\delta)$, where $B_0(r)$ stands for the Euclidean ball of center 0 and radius r. We regard $\eta \hat u_\alpha$ as a function in the Euclidean space. Also, we regard φ and A_g as defined in the Euclidean space. By Lemmas **??**, **??**, and **??**, and by (**??**) and (**??**), we can write that when $n \geq 6$, and for any $j = 0, 1, 2$,

$$\int_{B_0(2\delta)\backslash B_0(\delta)} |\nabla^j \hat u_\alpha|^2 dx = o\left(\int_M |\nabla u_\alpha|^2 dv_g \right) \, . \tag{9.3}$$

Now we apply to the $\eta \hat u_\alpha$'s the Pohozaev type identity

$$\begin{aligned} &\int_\Omega \left(x^k \partial_k u \right) \Delta^2 u dx + \frac{n-4}{2} \int_\Omega u \Delta^2 u dx \\ &= \frac{n-4}{2} \int_{\partial\Omega} \left(-u \frac{\partial \Delta u}{\partial \nu} + \frac{\partial u}{\partial \nu} \Delta u \right) d\sigma \\ &+ \int_{\partial\Omega} \left(\frac{1}{2}(x,\nu)(\Delta u)^2 - (x, \nabla u) \frac{\partial \Delta u}{\partial \nu} + \frac{\partial (x, \nabla u)}{\partial \nu} \Delta u \right) d\sigma \end{aligned} \tag{9.4}$$

which holds for all smooth bounded domains Ω in $\mathbb{R}^n$ and all $u \in C^4(\overline{\Omega})$, where ν is the outward unit normal of $\partial\Omega$, and $d\sigma$ is the Euclidean volume element on $\partial\Omega$. We let in what follows $\Omega = B_0(2\delta)$ and $u = \eta \hat u_\alpha$. By (**??**), integrating by parts, we easily get that

$$\int_{\mathbb{R}^n} \eta^2 x^k \partial_k \hat u_\alpha \Delta^2 \hat u_\alpha dx + \frac{n-4}{2} \int_{\mathbb{R}^n} \eta^2 \hat u_\alpha \Delta^2 \hat u_\alpha dx = o\left(\int_M |\nabla u_\alpha|^2 dv_g \right) \, . \tag{9.5}$$

Multiplying equation (**??**) by $\eta^2 \hat u_\alpha$, and integrating by parts, we can write by Lemmas **??**, **??**, and **??** that

$$\begin{aligned} &\int_{\mathbb{R}^n} \eta^2 \hat u_\alpha \Delta^2 \hat u_\alpha dx + b_\alpha \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{4}{n-4}} |\nabla \hat u_\alpha|^2 dx \\ &= \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{8}{n-4}} A_g(\nabla \hat u_\alpha, \nabla \hat u_\alpha) dx + \int_{\mathbb{R}^n} \eta^2 \hat u_\alpha^{2^\sharp} dx + o\left(\int_M |\nabla u_\alpha|^2 dv_g \right) \, . \end{aligned} \tag{9.6}$$

In a similar way, multiplying equation (**??**) by $\eta^2 x^k \partial_k \hat{u}_\alpha$, and integrating by parts, we can write by Lemmas **??**, **??**, and **??** that

$$\begin{aligned}&\int_{\mathbb{R}^n} \eta^2 (\Delta^2 \hat{u}_\alpha) x^k \partial_k \hat{u}_\alpha dx - \frac{(n-2)}{2} b_\alpha \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|^2 dx \\ &+ \frac{n-2}{2} \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{8}{n-4}} A_g(\nabla \hat{u}_\alpha, \nabla \hat{u}_\alpha) dx + \frac{(n-4)}{2} \int_{\mathbb{R}^n} \eta^2 \hat{u}_\alpha^{2^\sharp} dx \\ &= \varepsilon_\delta O\left(\int_M |\nabla u_\alpha|^2 dv_g \right) + o\left(\int_M |\nabla u_\alpha|^2 dv_g \right) ,\end{aligned} \tag{9.7}$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. The proofs of equations (**??**), (**??**), and (**??**) involve only straightforward computations. Now, plugging (**??**) and (**??**) into (**??**), it comes that

$$\begin{aligned}&b_\alpha \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{4}{n-4}} |\nabla \hat{u}_\alpha|^2 dx - \int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{8}{n-4}} A_g(\nabla \hat{u}_\alpha, \nabla \hat{u}_\alpha) dx \\ &= \varepsilon_\delta O\left(\int_M |\nabla u_\alpha|^2 dv_g \right) + o\left(\int_M |\nabla u_\alpha|^2 dv_g \right) ,\end{aligned} \tag{9.8}$$

where $\varepsilon_\delta \to 0$ as $\delta \to 0$. The norm of $\nabla \hat{u}_\alpha$ in the first term of (**??**) is with respect to the Euclidean metric $\hat{g} = \xi$. Noting that $|\nabla u|^2_{\hat{g}} = \varphi^{4/(n-4)} |\nabla u|^2_g$, it follows from (**??**) that

$$\begin{aligned}&\int_{\mathbb{R}^n} \eta^2 \varphi^{\frac{8}{n-4}} \Big(A_g - b_\alpha g \Big)(\nabla \hat{u}_\alpha, \nabla \hat{u}_\alpha) dx \\ &= \varepsilon_\delta O\left(\int_M |\nabla u_\alpha|^2 dv_g \right) + o\left(\int_M |\nabla u_\alpha|^2 dv_g \right) .\end{aligned} \tag{9.9}$$

Since $b_\infty \notin S_c$, where S_c is as in (**??**), $A_g - b_\alpha g$ has a sign when α is sufficiently large. In particular, coming back to our manifold, it follows from (**??**) and Lemmas **??** and **??** that there exists $t > 0$, independent of α and δ, such that

$$\int_{B_{x_0}(t\delta)} |\nabla u_\alpha|^2 dv_g = \varepsilon_\delta O\left(\int_M |\nabla u_\alpha|^2 dv_g \right) + o\left(\int_M |\nabla u_\alpha|^2 dv_g \right) \tag{9.10}$$

for all $\delta > 0$ sufficiently small, and all α sufficiently large. Summing (**??**) over the $x_0 \in S$, and thanks to Lemmas **??** and **??**, we then get that for any $\delta > 0$,

$$\int_M |\nabla u_\alpha|^2 dv_g = \varepsilon_\delta O\left(\int_M |\nabla u_\alpha|^2 dv_g \right) + o\left(\int_M |\nabla u_\alpha|^2 dv_g \right)$$

for all α sufficiently large. The contradiction easily follows since $\varepsilon_\delta \to 0$ as $\delta \to 0$. This proves the pseudo-compactness part of Theorem **??**.

10. Proof of compactness

We prove the compactness assertion of Theorem **??** and Theorem **??** in this section. We let (M, g) be a smooth compact locally conformally flat Riemannian manifold of dimension $n \geq 5$, and (u_α) be a bounded sequence of nonnegative solutions of (**??**). By contradiction we can assume that the u_α's blow up, since if not we get Theorem **??** and Theorem **??** by (**??**). Up to renumbering and up to a subsequence, as in Sections **??**–**??**, we can assume that

$$\mu_\alpha^1 = \max_{1 \leq i \leq k} \mu_\alpha^i , \tag{10.1}$$

where the μ_α^i's are the weights of the bubbles (B_α^i) of Lemma **??**. Then, as in the preceding sections, we let $x_\alpha = x_\alpha^1$ and $\mu_\alpha = \mu_\alpha^1$, where the x_α^1's are the centers of (B_α^1). Roughly speaking, the argument in this section consists in applying the Pohozaev type identity (**??**) to the u_α's in small balls $B_{x_\alpha}(\delta\sqrt{\mu_\alpha})$, and then to get the contradiction by conformal invariance and the estimates we proved in Sections **??** to **??**. As a remark, we need to consider smaller balls than in the preceding section, of radii $\delta\sqrt{\mu_\alpha}$ instead of δ, because of the weak limit u^0 which, when nonzero, dominates the other terms in the Pohozaev identity on balls of fixed radii. A similar phenomenon (with the limit of the u_α's after rescaling) appears on balls of radii $\delta\mu_\alpha$. The sharp quantity in this argument turns out to be the C^0-range of interaction $\delta\sqrt{\mu_\alpha}$. We need also to be more precise than in the preceding section and compute the boundary terms in the right hand side of (**??**). As in Section **??**, we start with conformal invariance. We let $x_0 \in S$ be the limit of the x_α's, and let $\delta_0 > 0$ and $\hat{g}$ be such that $\hat{g}$ is flat in $B_{x_0}(4\delta_0)$. We write that $g = \varphi^{4/(n-4)}\hat{g}$, with $\varphi(x_0) = 1$, and let $\hat{u}_\alpha = u_\alpha\varphi$. Then equation (**??**) holds in $B_{x_0}(4\delta_0)$. Now, as already mentioned, we apply the Pohozaev identity (**??**) of Section **??** to the $\hat{u}_\alpha$'s with $\Omega = B_0(\delta\sqrt{\mu_\alpha})$ where $\delta > 0$ is given by Lemmas **??** and **??**. In the process we assimilate x_α with 0 (thanks to the exponential map $\exp_{x_\alpha}$ with respect to g) and regard $\hat{u}_\alpha$ as a function in the Euclidean space. With an abusive use of notations, we still denote by φ the function $\varphi \circ \exp_{x_\alpha}$, by A_g the tensor field $(\exp_{x_\alpha})^\star A_g$, and by $\hat{g}$ the metric $(\exp_{x_\alpha})^\star \hat{g}$. Applying the Pohozaev identity (**??**) to the $\hat{u}_\alpha$'s in $B_0(\delta\sqrt{\mu_\alpha})$ we get that

$$\begin{aligned}
&\int_{B_0(\delta\sqrt{\mu_\alpha})} \left(x^k\partial_k\hat{u}_\alpha\right)\Delta^2\hat{u}_\alpha dx + \frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha\Delta^2\hat{u}_\alpha dx \\
&= \frac{n-4}{2}\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \left(-\hat{u}_\alpha\frac{\partial\Delta\hat{u}_\alpha}{\partial\nu} + \frac{\partial\hat{u}_\alpha}{\partial\nu}\Delta\hat{u}_\alpha\right) d\sigma \\
&+ \int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \left(\frac{1}{2}(x,\nu)(\Delta\hat{u}_\alpha)^2 - (x,\nabla\hat{u}_\alpha)\frac{\partial\Delta\hat{u}_\alpha}{\partial\nu} + \frac{\partial(x,\nabla\hat{u}_\alpha)}{\partial\nu}\Delta\hat{u}_\alpha\right) d\sigma\ .
\end{aligned} \tag{10.2}$$

Integrating by parts, using (**??**), we can also write that

$$\begin{aligned}
&\int_{B_0(\delta\sqrt{\mu_\alpha})} \left(x^k\partial_k\hat{u}_\alpha\right)\Delta^2\hat{u}_\alpha dx + \frac{n-4}{2}\int_{B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha\Delta^2\hat{u}_\alpha dx \\
&= b_\alpha\int_{B_0(\delta\sqrt{\mu_\alpha})} \varphi^{\frac{4}{n-4}}|\nabla\hat{u}_\alpha|^2 dx - \int_{B_0(\delta\sqrt{\mu_\alpha})} \varphi^{\frac{8}{n-4}}A_g(\nabla\hat{u}_\alpha,\nabla\hat{u}_\alpha)dx \\
&\quad + o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})} |\nabla\hat{u}_\alpha|^2 dx\right) + O\left(\int_{B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha^2 dx\right) \\
&\quad + O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} \hat{u}_\alpha^2(1+\hat{u}_\alpha^{2^\sharp-2})dx\right) + O\left(\int_{\partial B_0(\delta\sqrt{\mu_\alpha})} |\nabla\hat{u}_\alpha|^2 dx\right)\ ,
\end{aligned} \tag{10.3}$$

where, in this equation, as already mentioned, we regard φ and A_g as defined in the Euclidean space. The proof of (**??**) involves only straightforward computations.

By Lemma **??**,

$$\begin{aligned}&\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2(1+\hat{u}_\alpha^{2^\sharp-2})dx=o\left(\mu_\alpha^{\frac{n-4}{2}}\right)\ ,\ \text{and}\\&\int_{\partial B_0(\delta\sqrt{\mu_\alpha})}|\nabla\hat{u}_\alpha|^2dx=o\left(\mu_\alpha^{\frac{n-4}{2}}\right)\end{aligned}\tag{10.4}$$

while, by Lemma **??**,

$$\int_{B_0(\delta\sqrt{\mu_\alpha})}\hat{u}_\alpha^2dx=o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})}|\nabla\hat{u}_\alpha|^2dx\right)\ .\tag{10.5}$$

Independently, we can also write with the change of variables $x=\sqrt{\mu_\alpha}y$ and Lemma **??** that if R_α stands for the right hand side in (**??**), then

$$\begin{aligned}\mu_\alpha^{-\frac{n-4}{2}}R_\alpha\to{}&\frac{n-4}{2}\int_{\partial B_0(\delta)}\left(-\tilde{u}\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial\tilde{u}}{\partial\nu}\Delta\tilde{u}\right)d\sigma\\&+\int_{\partial B_0(\delta)}\left(\frac{1}{2}(x,\nu)(\Delta\tilde{u})^2-(x,\nabla\tilde{u})\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial(x,\nabla\tilde{u})}{\partial\nu}\Delta\tilde{u}\right)d\sigma\end{aligned}\tag{10.6}$$

as $\alpha\to+\infty$, where

$$\tilde{u}(x)=\frac{A}{|x|^{n-4}}+\hat{\varphi}(x)\tag{10.7}$$

is given by Lemma **??** (so that $\Delta^2\hat{\varphi}=0$). Coming back to the Pohozaev identity (**??**) of Section **??**, taking $\Omega=B_0(\delta)\backslash B_0(r)$, and since $\Delta^2\tilde{u}=0$ in Ω, it comes that

$$\begin{aligned}&\frac{n-4}{2}\int_{\partial B_0(\delta)}\left(-\tilde{u}\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial\tilde{u}}{\partial\nu}\Delta\tilde{u}\right)d\sigma\\&+\int_{\partial B_0(\delta)}\left(\frac{1}{2}(x,\nu)(\Delta\tilde{u})^2-(x,\nabla\tilde{u})\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial(x,\nabla\tilde{u})}{\partial\nu}\Delta\tilde{u}\right)d\sigma\\&=\frac{n-4}{2}\int_{\partial B_0(r)}\left(-\tilde{u}\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial\tilde{u}}{\partial\nu}\Delta\tilde{u}\right)d\sigma\\&+\int_{\partial B_0(r)}\left(\frac{1}{2}(x,\nu)(\Delta\tilde{u})^2-(x,\nabla\tilde{u})\frac{\partial\Delta\tilde{u}}{\partial\nu}+\frac{\partial(x,\nabla\tilde{u})}{\partial\nu}\Delta\tilde{u}\right)d\sigma\end{aligned}\tag{10.8}$$

for all $r>0$. Combining (**??**), (**??**), and (**??**), letting $r\to0$, we then get that

$$\mu_\alpha^{-\frac{n-4}{2}}R_\alpha\to K_0\tag{10.9}$$

as $\alpha\to+\infty$, where $K_0=(n-2)(n-4)^2\omega_{n-1}A\hat{\varphi}(0)$. By Lemma **??**, $A>0$, and we can assume that $\hat{\varphi}(0)>0$ (since if not $u^0\equiv0$ and we are back to Section **??**). In particular, $K_0>0$, and we get by combining (**??**)–(**??**), and (**??**), that

$$\begin{aligned}&b_\alpha\int_{B_0(\delta\sqrt{\mu_\alpha})}\varphi^{\frac{4}{n-4}}|\nabla\hat{u}_\alpha|^2dx-\int_{B_0(\delta\sqrt{\mu_\alpha})}\varphi^{\frac{8}{n-4}}A_g(\nabla\hat{u}_\alpha,\nabla\hat{u}_\alpha)dx\\&=o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})}|\nabla\hat{u}_\alpha|^2dx\right)+\big(K_0+o(1)\big)\mu_\alpha^{\frac{n-4}{2}}\ ,\end{aligned}\tag{10.10}$$

where $o(1)\to0$ as $\alpha\to+\infty$. The norm of $\nabla\hat{u}_\alpha$ in the first term of (**??**) is with respect to the Euclidean metric $\hat{g}=\xi$. Noting that $|\nabla u|_{\hat{g}}^2=\varphi^{4/(n-4)}|\nabla u|_g^2$, it

follows from (**??**) that

$$\int_{B_0(\delta\sqrt{\mu_\alpha})} \varphi^{\frac{8}{n-4}}\big(A_g - b_\alpha g\big)(\nabla\hat{u}_\alpha, \nabla\hat{u}_\alpha)dx$$
$$= o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})} |\nabla\hat{u}_\alpha|^2 dx\right) - \big(K_0 + o(1)\big)\mu_\alpha^{\frac{n-4}{2}}$$

an equation from which we easily get with Lemma **??** that

$$\int_{B_0(\delta\sqrt{\mu_\alpha})} \varphi^{\frac{2n}{n-4}}\big(A_g - b_\alpha g\big)(\nabla u_\alpha, \nabla u_\alpha)dx$$
$$= o\left(\int_{B_0(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dx\right) - \big(K_0 + o(1)\big)\mu_\alpha^{\frac{n-4}{2}}\ . \tag{10.11}$$

If $b_\infty > \max \mathcal{S}_c$, where $\mathcal{S}_c$ is given by (**??**), then (**??**) implies that

$$\big(\lambda + o(1)\big)\int_{B_0(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dx = \big(K_0 + o(1)\big)\mu_\alpha^{\frac{n-4}{2}} \tag{10.12}$$

for some $\lambda > 0$ independent of α. By Lemma **??**, we can also write that

$$\int_{B_0(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dx \geq C\mu_\alpha^2 \tag{10.13}$$

for some $C > 0$ independent of α. The contradiction follows from (**??**) and (**??**) when $n \geq 9$ since, in this case, $\frac{n-4}{2} > 2$. This proves the assertion on compactness in Theorem **??**. If, on the contrary, $b_\infty < \min \mathcal{S}_c$, then (**??**) gives that

$$\big(\lambda + o(1)\big)\int_{B_0(\delta\sqrt{\mu_\alpha})} |\nabla u_\alpha|^2 dx + \big(K_0 + o(1)\big)\mu_\alpha^{\frac{n-4}{2}} = 0 \tag{10.14}$$

for some $\lambda > 0$ independent of α. In particular, (**??**) would give that $K_0 \leq 0$, and, since $K_0 > 0$, the contradiction follows here again. This proves Theorem **??**.

Aknowledgements: The authors thank the referee for useful remarks on the manuscript.

References

[1] Agmon, S., Douglis, A. and Nirenberg, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, *Comm. Pure Appl. Math.*, 12, 623-727, 1959.

[2] Agmon, S., Douglis, A. and Nirenberg, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II, *Comm. Pure Appl. Math.*, 17, 35-92, 1964.

[3] Aubin, T., Espaces de Sobolev sur les variétés riemanniennes, *Bull. Sc. Math.*, 100, 149-173, 1976.

[4] Beckner, W., Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, *Ann. of Math.*, 138, 213-242, 1993.

[5] Branson, T.P., Group representations arising from Lorentz conformal geometry, *J. Funct. Anal.*, 74, 199-291, 1987.

[6] Brendle, S., Prescribing a higher order conformal invariant on S^n, *Comm. Anal. Geom.*, 11, 837-858, 2003.

[7] Chang, S.Y.A., On Paneitz operator - a fourth order differential operator in conformal geometry, Harmonic Analysis and Partial Differential Equations, Essays in honor of Alberto P. Calderon, Eds: M. Christ, C. Kenig and C. Sadorsky, Chicago Lectures in Mathematics, 1999, 127-150.

[8] Chang, S.Y.A. and Yang, P.C., Extremal metrics of zeta Functional Determinants on 4-manifolds, *Ann. of Math.*, 142, 171-212, 1995.
[9] Chang, S.Y.A. and Yang, P.C., On a fourth order curvature invariant, Comp. Math. 237, Spectral Problems in Geometry and Arithmetic, Ed: T. Branson, AMS, 9-28, 1999.
[10] Chen, C.C., and Lin, C.S., Prescribing scalar curvature on S^N. I. A priori estimates, *J. Differential Geom.*, 57, 67-171, 2001.
[11] Devillanova, G. and Solimini, S., Concentration estimates and multiple solutions to elliptic problems at critical growth, *Adv. Differential Equations*, 7, 1257-1280, 2002.
[12] Djadli, Z., Hebey, E., and Ledoux, M., Paneitz-type operators and applications, *Duke Math. J.*, 104, 129-169, 2000.
[13] Djadli, Z., Malchiodi, A., and Ould Ahmedou, M., Prescribing a fourth order conformal invariant on the standard sphere. I. A perturbation result, *Commun. Contemp. Math.*, 4, 375-408, 2002.
[14] Djadli, Z., Malchiodi, A., and Ould Ahmedou, M., Prescribing a fourth order conformal invariant on the standard sphere. II. Blow up analysis and applications, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 1, 387-434, 2002.
[15] Druet, O., The best constants problem in Sobolev inequalities, *Math. Ann.* , 314, 327-346, 1999.
[16] Druet, O., From one bubble to several bubbles: the low-dimensional case, *J. Differential Geom.*, 63, 399-473, 2003.
[17] Druet, O., Compactness for Yamabe metrics in low dimensions, *Internat. Math. Res. Notices*, 23, 1143-1191, 2004.
[18] Druet, O., and Hebey, E., Blow-up examples for second order elliptic PDEs of critical Sobolev growth, *Trans. Amer. Math. Soc.*, to appear.
[19] Druet, O., and Hebey, E., Elliptic equations of Yamabe type, *Internat. Math. Res. Surveys*, To appear.
[20] Druet, O., Hebey, E., and Robert, F., Blow-up theory for elliptic PDEs in Riemannian geometry, *Mathematical Notes*, Vol. 45, Princeton University Press, 2004.
[21] Edmunds, D.E., Fortunato, F., Janelli, E., Critical exponents, critical dimensions, and the biharmonic operator, *Arch. Rational Mech. Anal.*, 112, 269-289, 1990.
[22] Felli, V., Hebey, E., and Robert, R., Fourth order equations of critical Sobolev growth. Energy function and solutions of bounded energy in the conformally flat case, *Nonlinear Differential Equations and Applications*, To appear.
[23] Gilbarg, G., and Trudinger, N. S., *Elliptic partial differential equations of second order*, Second edition, Grundlehren der Mathematischen Wissenschaften, 224, Springer, Berlin-New York, 1983.
[24] Giraud, G., Sur le problème de Dirichlet généralisé, *Ann. Sc. Ecole Norm. Sup.*, 46, 131-245, 1929.
[25] Han, Z.C., Li, Y., The Yamabe problem on manifolds with boundary: existence and compactness results, *Duke Math. J.*, 99, 489-542, 1999.
[26] Hebey, E., Sharp Sobolev inequalities of second order, *J. Geom. Anal.*, 13, 145-162, 2003.
[27] Hebey, E., and Robert, F., Coercivity and Struwe's compactness for Paneitz type operators with constant coefficients, *Calc. Var. Partial Differential Equations*, 13, 491-517, 2001.
[28] Hebey, E., and Vaugon, M., The best constant problem in the Sobolev embedding theorem for complete Riemannian manifolds, *Duke Math.. J.*, 79, 235-279, 1995.
[29] Li, Y., Zhu, M., Yamabe type equations on three-dimensional Riemannian manifolds. *Commun. Contemp. Math.*, 1, 1-50, 1999.
[30] Lieb, E.H., Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, *Ann. of Math.*, 118, 349-374, 1983.
[31] Lin, C.S., A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$, *Comment. Math. Helv.*, 73, 206-231, 1998.
[32] Lions, P.L., The concentration-compactness principle in the calculus of variations I, II, *Rev. Mat. Iberoamericana*, 1, 145-201 and 45-121, 1985.
[33] Lu, G., Wei, J., Xu, X., On conformally invariant equation $(-\Delta)^p u - K(x)u^{(N+2p)/(N-2p)} = 0$ and its generalizations, *Ann. Mat. Pura Appl.*, 4, 179, 309-329, 2001.
[34] Marques, F.C., A priori estimates for the Yamabe problem in the non-locally conformally flat case, *Preprint*, 2004.

[35] Paneitz, S., A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds, *Preprint*, 1983.
[36] Robert, F., Positive solutions for a fourth order equation invariant under isometries, *Proc. Amer. Math. Soc.*, 131, 1423-1431, 2003.
[37] Robert, F., and Strwue, M., Asymptotic profile for a fourth order pde with critical exponential growth in dimension four, *Adv. Nonlinear Stud.*, 4, 397-415, 2004.
[38] Sakai, T., *Riemannian geometry*, Translations of Mathematical Monographs, 149, American Mathematical Society, Providence, R.I., 1996.
[39] Schoen, R., Lecture notes from courses at Stanford, written by D.Pollack, *Preprint*, 1988.
[40] Schoen, R., Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, Topics in Calculus of Variations (Montecatini Terme, 1987), *Lecture Notes in Math.*, vol. 1365, Springer-Verlag, Berlin, pp. 120-154, 1989.
[41] Schoen, R., On the number of constant scalar curvature metrics in a conformal class, Differential Geometry: A symposium in honor of Manfredo do Carmo, Proc. Int. Conf. (Rio de Janeiro, 1988). *Pitman Monogr. Surveys Pure Appl. Math.*, vol. 52, Longman Scientific and Technical, Harlow, pp. 311-320, 1991.
[42] Schoen, R., A report on some recent progress on nonlinear problems in geometry, *Surveys in Differential Geometry* (Cambridge, Mass, 1990), Suppl. J. Diff. Geom., vol. 1, Lehigh University, Pennsylvania, pp. 201-241, 1991.
[43] Schoen, R., and Zhang, D., Prescribed scalar curvature on the n-sphere, *Calc. Var. Partial Differential Equations*, 4, 1-25, 1996.
[44] Struwe, M., A global compactness result for elliptic boundary problems involving limiting nonlinearities, *Math. Z.*, 187, 511-517, 1984.
[45] Veron, L., *Singularities of solutions of second order quasilinear equations*, Pitman Research Notes in Mathematical Series, Addison Wesley Longman Lim., Harlow, 1996.

Emmanuel Hebey, Université de Cergy-Pontoise, Département de Mathématiques, Site de Saint-Martin, 2 avenue Adolphe Chauvin, 95302 Cergy-Pontoise cedex, France
E-mail address: Emmanuel.Hebey@math.u-cergy.fr

Frédéric Robert, Laboratoire J.A.Dieudonné, Université de Nice Sophia-Antipolis, Parc Valrose, 06108 Nice cedex 2, France
E-mail address: frobert@math.unice.fr

Yuliang Wen, East China Normal University, Department of Mathematics, 200062 Shanghai, P.R.China
E-mail address: ylwen@math.ecnu.edu.cn