COERCIVITY AND STRUWE'S COMPACTNESS FOR PANEITZ TYPE OPERATORS WITH CONSTANT COEFFICIENTS

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The Paneitz operator discovered in [11] is the fourth order operator defined on a 4-dimensional Riemannian manifold (M, g) by

$$P_g^4 u = \Delta_g^2 u - div_g\Big(\frac{2}{3}S_g g - 2Rc_g\Big)du$$

where $\Delta_g u = -div_g \nabla u$ is the Laplacian of u with respect to g, S_g is the scalar curvature of g, and Rc_g is the Ricci curvature of g. An extension to manifolds of dimension $n \geq 5$, due to Branson [2], is the fourth order operator defined by

$$P_g^n u = \Delta_g^2 u - div_g\left(\frac{(n-2)^2+4}{2(n-1)(n-2)}S_g g - \frac{4}{n-2}Rc_g\right)du + \frac{n-4}{2}Q_g^n u$$

where

$$Q_g^n = \frac{1}{2(n-1)}\Delta_g S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}S_g^2 - \frac{2}{(n-2)^2}|Rc_g|^2$$

Both P_g^4 and P_g^n have conformal properties: for all $u \in C^\infty(M)$, $P_{\tilde{g}}^4 u = e^{-4\varphi}P_g^4 u$ when $n = 4$ and $\tilde{g} = e^{2\varphi}g$, while $P_g^n(u\varphi) = \varphi^{(n+4)/(n-4)}P_{\tilde{g}}^n u$ when $n \geq 5$ and $\tilde{g} = \varphi^{4/(n-4)}g$. With respect to these relations, P_g^4 in dimension 4 is a natural analogue of Δ_g in dimension 2, while P_g^n in dimension $n \geq 5$ is a natural analogue of the conformal Laplacian $\Delta_g + \frac{n-2}{4(n-1)}S_g$ in dimension $n \geq 3$. Possible references on the subject are the survey articles [3] by Chang, and [4] by Chang and Yang.

We let here (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, and say that a fourth order operator P_g is a Paneitz type operator with constant coefficients if

$$P_g u = \Delta_g^2 u + \alpha\Delta_g u + au \tag{0.1}$$

where $\alpha, a \in \mathbb{R}$. When g is Einstein, $P_g^n = P_g$ for some α and a. Let $2^\sharp = 2n/(n-4)$ be the critical Sobolev exponent for the embedding of the Sobolev space H_2^2 in L^p-spaces. We are mainly concerned in this article with two questions. On the one hand to find necessary and sufficient conditions on α and a for P_g to be coercive. On the other hand to describe Palais-Smale sequences for the higher order analogue of Yamabe type equations

$$P_g u = |u|^{2^\sharp-2}u \tag{0.2}$$

By the mountain pass lemma of Ambrosetti and Rabinowitz [1], it easily follows that if P_g is coercive, then there exist Palais-Smale sequences for this equation. Minimizing positive solutions to (0.2) have been obtained in Djadli, Hebey and Ledoux [5]. Positivity for the 4-dimensional Paneitz operator P_g^4 is studied in the

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very nice Gursky [7]. The study of the analogue of (0.2) in dimension 4 is subject to an intensive literature. We refer to the survey articles [3] by Chang, and [4] by Chang and Yang, and to the references they contain, for more information.

1. Coercivity

Given (M, g) a smooth compact n-dimensional Riemannian manifold, $n \geq 5$, we let $H_2^2(M)$ be the Sobolev space defined as the completion of the space of smooth functions on M with respect to the norm

$$\|u\|_{H_2^2}^2 = \int_M (\Delta_g u)^2 \, dv_g + \int_M |\nabla u|^2 dv_g + \int_M u^2 dv_g$$

The Paneitz type operator P_g as given by (0.1) is said to be coercive if there exists $\lambda > 0$ such that for any $u \in H_2^2(M)$,

$$\int_M (P_g u)\, u dv_g \geq \lambda \|u\|_{H_2^2}^2$$

where the left hand side of this inequality has to be understood in the distributional sense. An equivalent definition is that there exists $\lambda > 0$ such that for all $u \in H_2^2(M)$,

$$\int_M (P_g u)\, u dv_g \geq \lambda \int_M u^2 dv_g$$

As already mentioned, we are concerned in this section with necessary and sufficient conditions on a and α for P_g to be coercive. By taking $u \equiv 1$ in the definition of the coercivity, one sees that a has to be positive. In what follows, we denote by

$$\lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots < +\infty$$

the ordered sequence of the eigenvalues of the Laplacian Δ_g, and let Λ_k be the eigenspace corresponding to the eigenvalue λ_k. Given $a > 0$, and $k \in \mathbb{N}$, $k \geq 1$, we also let

$$a_k = \lambda_k + \frac{a}{\lambda_k}$$

The answer to our question is given by the following result.

Theorem 1.1. *Given $a > 0$, let $k_a \in \mathbb{N}$, $k_a \geq 1$, be such that $\lambda_{k_a - 1} < \sqrt{a} \leq \lambda_{k_a}$. Let also $\alpha_0 = \alpha_0(a)$ be the largest α such that, for all $u \in H_2^2(M)$,*

$$\int_M (\Delta_g u)^2 \, dv_g + a \int_M u^2 dv_g \geq \alpha \int_M |\nabla u|^2 dv_g \tag{1.1}$$

Then, the following holds:

(1) $\alpha_0 = a_{k_a - 1}$ *if* $\lambda_{k_a-1}^2 < a < \lambda_{k_a-1}\lambda_{k_a}$;

(2) $\alpha_0 = \lambda_{k_a-1} + \lambda_{k_a}$ *if* $a = \lambda_{k_a-1}\lambda_{k_a}$;

(3) $\alpha_0 = a_{k_a}$ *if* $\lambda_{k_a-1}\lambda_{k_a} < a \leq \lambda_{k_a}^2$.

Moreover, u realizes the equality in the optimal inequality

$$\int_M (\Delta_g u)^2 \, dv_g + a \int_M u^2 dv_g \geq \alpha_0 \int_M |\nabla u|^2 dv_g \tag{1.2}$$

if and only if $u \in \Lambda_{k_a-1}$ in case (1), $u \in \Lambda_{k_a-1} \oplus \Lambda_{k_a}$ in case (2), and $u \in \Lambda_{k_a}$ in case (3). In particular, P_g as given by (0.1) is coercive if and only if $a > 0$ and $\alpha > -\alpha_0(a)$.

Proof. By definition,

$$\alpha_0 = \inf_{u\in\mathcal{H}} \int_M \left((\Delta_g u)^2 + au^2\right) dv_g$$

where

$$\mathcal{H} = \left\{u \in H_2^2(M)\ ,\ \int_M |\nabla u|^2 dv_g = 1\right\}$$

Given $k \in \mathbb{N}$, $k \geq 1$, and taking $u \in \Lambda_k$ in (1.2), one gets that $\alpha_0 \leq a_k$ for all $k \geq 1$. Independently, by standard variational technics, one gets that there exists $u_0 \in \mathcal{H}$ such that for all $\varphi \in H_2^2(M)$,

$$\int_M (\Delta_g u_0)(\Delta_g \varphi)\, dv_g + a\int_M u_0\varphi dv_g = \alpha_0 \int_M (\nabla u_0, \nabla\varphi)\, dv_g$$

Taking $\varphi \in \Lambda_k$, $k \geq 1$, in this relation gives that

$$\lambda_k\,(a_k - \alpha_0)\int_M u_0\varphi dv_g = 0 \tag{1.3}$$

In the same order of ideas, taking for φ a constant function, one gets that $u_0 \perp \Lambda_0$. Let $f : \mathbb{R} \to \mathbb{R}$ be the real valued function defined for $x > 0$ by

$$f(x) = x + \frac{a}{x}$$

Then f is decreasing for $x < \sqrt{a}$, and increasing for $x \geq \sqrt{a}$. Moreover, f goes from $+\infty$ to $2\sqrt{a}$ when x goes from 0^+ to $\sqrt{a}$, and f then goes from $2\sqrt{a}$ to $+\infty$ when x goes from $\sqrt{a}$ to $+\infty$. Set now

$$b_k = \min_{1\leq i\leq k} a_i$$

and let k_a be as in the theorem. As a first and main step, we claim that $\alpha_0 = b_{k_a}$. According to what we said above, $\alpha_0 \leq b_{k_a}$. Suppose that $\alpha_0 < b_{k_a}$. Then $\alpha_0 < a_k$ for any $k \geq 1$. By (1.3), it follows that $u_0 \perp \Lambda_k$ for all k. Since $L^2(M)$ possesses a basis of eigenfunctions, this implies that $u_0 \equiv 0$, a contradiction. Hence, $\alpha_0 = b_{k_a}$ and the claim is proved. Let now I_{k_a} be the set of the integers $i \geq 1$ for which $a_i = b_{k_a}$. If $i \notin I_{k_a}$, then, again by (1.3), $u_0 \perp \Lambda_i$. Hence, necessarily,

$$u_0 \in \oplus_{i\in I_{k_a}}\Lambda_i$$

Conversely, any function in this space realizes the equality in (1.2). As a consequence, u realizes the equality in (1.2) if and only if $u \in \oplus_{i\in I_{k_a}}\Lambda_i$. In order to end the proof of the first part of the theorem, note that, according to what we said on f,

$$b_{k_a} = \min\left(a_{k_a-1}, a_{k_a}\right)$$

It holds that $a_{k_a-1} < a_{k_a}$ if $a < \lambda_{k_a-1}\lambda_{k_a}$, $a_{k_a-1} = a_{k_a} = \lambda_{k_a-1} + \lambda_{k_a}$ if $a = \lambda_{k_a-1}\lambda_{k_a}$, and $a_{k_a-1} > a_{k_a}$ if $a > \lambda_{k_a-1}\lambda_{k_a}$. This ends the proof of the first part of the theorem.

Concerning the second part, it is clear that $a > 0$ and $\alpha > -\alpha_0(a)$ are necessary conditions for P_g to be coercive. Conversely, suppose that $a > 0$ and $\alpha > -\alpha_0(a)$. For $\varepsilon > 0$ sufficiently small, $\alpha > -\alpha_0(a-\varepsilon)$. Then, according to what is said above, and for all $u \in H_2^2(M)$,

$$\int_M (P_g u)\, u dv_g \geq \varepsilon \int_M u^2 dv_g$$

This proves the theorem. □

2. Struwe's compactness

As above, we let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 5$, and P_g be the fourth order operator given by (0.1). We let also I_g be the functional defined on $H_2^2(M)$ by

$$\begin{aligned} I_g(u) &= \frac{1}{2}\int_M (P_g u) u dv_g - \frac{1}{2^\sharp}\int_M |u|^{2^\sharp} dv_g \\ &= \frac{1}{2}\int_M (\Delta_g u)^2 dv_g + \frac{\alpha}{2}\int_M |\nabla u|^2 dv_g + \frac{a}{2}\int_M u^2 dv_g - \frac{1}{2^\sharp}\int_M |u|^{2^\sharp} dv_g \end{aligned}$$

and say that a sequence (u_m) in $H_2^2(M)$ is a Palais-Smale sequence for I_g if:

1. $I_g(u_m)$ is bounded in m, and
2. $DI_g(u_m) \to 0$ strongly as $m \to +\infty$.

When P_g is coercive, Palais-Smale sequences for I_g are easily produced by the Mountain-Pass lemma of Ambrosetti and Rabinowitz [1]. Indeed, it follows from the coercivity of P_g and the Sobolev inequality corresponding to the embedding $H_2^2 \subset L^{2^\sharp}$, that there exist $C_1, C_2 > 0$ such that for all $u \in H_2^2(M)$,

$$I_g(u) \geq C_1 \|u\|_{H_2^2}^2 - C_2 \|u\|_{H_2^2}^{2^\sharp}$$

Let B_r be the ball of center 0 and radius r in $H_2^2(M)$. Then, for $r > 0$ small, there exists $\rho = \rho(r)$, such that for $u \in \partial B_r$, $I_g(u) \geq \rho$. Independently, $I_g(0) = 0$, so that $I_g(0) < \rho$, while for $u_0 \in H_2^2(M)\backslash\{0\}$,

$$\lim_{t \to +\infty} I_g(tu_0) = -\infty$$

It follows that there exists an open neighbourhood B_r of 0 in $H_2^2(M)$, that there exists $\tilde{u} \in H_2^2(M)\backslash B_r$, and that there exists $\rho > 0$ such that

$$I_g(0) < \rho \ , \ \ I_g(\tilde{u}) < \rho \ , \text{ and } I_g(u) \geq \rho \text{ for all } u \in \partial B_r$$

The Mountain pass lemma of Ambrosetti and Rabinowitz then yields a Palais-Smale sequence (u_m) for I_g with the property that

$$\lim_{m \to \infty} I_g(u_m) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma} I_g(u)$$

where Γ stands for the class of continuous paths joining 0 to $\tilde{u}$.

Let $\mathcal{D}(\mathbb{R}^n)$ be the set of smooth functions in $\mathbb{R}^n$ with compact support. We let $D_2^2(\mathbb{R}^n)$ be the completion of $\mathcal{D}(\mathbb{R}^n)$ with respect to the norm

$$\|u\| = \sqrt{\int_{\mathbb{R}^n} |\nabla^2 u|^2 dx} = \sqrt{\int_{\mathbb{R}^n} (\Delta u)^2 dx}$$

For $u \in D_2^2(\mathbb{R}^n)$, we let also $E(u)$ be given by

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} (\Delta u)^2 dx - \frac{1}{2^\sharp}\int_{\mathbb{R}^n} |u|^{2^\sharp} dx$$

where Δ is the Euclidean Laplacian. Given $\delta > 0$, η_δ denotes a smooth cut-off function in $\mathbb{R}^n$ such that $\eta_\delta = 1$ in $B_0(\delta)$ and $\eta_\delta = 0$ in $\mathbb{R}^n\backslash B_0(2\delta)$. For $x \in M$, where (M, g) is a smooth compact Riemannian manifold, and $\delta < i_g/2$, where i_g is the injectivity radius, we let $\eta_{\delta,x}$ be the smooth cut-off function in M given by

$$\eta_{\delta,x}(y) = \eta_\delta \left(\exp_x^{-1}(y)\right)$$

where exp_x is the exponential map at x.

An important result of Struwe [12] describes the behavior of Palais-Smale sequences associated to second order equations of the type

$$\Delta_g u + au = |u|^{2^\star-2}u \tag{2.1}$$

where $2^\star = 2n/(n-2)$ is the critical exponent for the embedding of the Sobolev space H_1^2 in L^p-spaces. We prove here that the analogue of this result holds when passing from the above equations to the fourth order equations

$$\Delta_g^2 u + \alpha\Delta_g u + au = |u|^{2^\sharp-2}u \tag{2.2}$$

After blow-up, the limit equation of (2.2) is the equation in the Euclidean space

$$\Delta^2 u = |u|^{2^\sharp-2}u \tag{2.3}$$

The answer to the second question we asked in the introduction is then given by the following theorem. Remarks on the case where the Palais-Smale sequence consists of nonnegative functions, or when P_g is replaced by a more general operator, are in section 4.

Theorem 2.1. *Let (u_m) be a Palais-Smale sequence for I_g. There exists $k \in \mathbb{N}$, sequences (R_m^j), $R_m^j > 0$ and $R_m^j \to +\infty$ as $m \to \infty$, converging sequences (x_m^j) in M, a solution $u^0 \in H_2^2(M)$ of (2.2), and non-trivial solutions $u^j \in D_2^2(\mathbb{R}^n)$ of (2.3), $j = 1, \ldots, k$, such that, up to a subsequence,*

$$u_m = u^0 + \sum_{j=1}^k \eta_m^j u_m^j + o(1)$$

where

$$u_m^j(x) = \left(R_m^j\right)^{\frac{n-4}{2}} u^j\left(R_m^j \exp_{x_m^j}^{-1}(x)\right),$$

$\eta_m^j = \eta_{\delta,x_m^j}$, $\delta < i_g/2$, and $\|o(1)\|_{H_2^2} \to 0$ as $m \to +\infty$. Moreover,

$$I_g(u_m) = I_g(u^0) + \sum_{j=1}^k E(u^j) + o(1)$$

where $o(1) \to 0$ as $m \to \infty$.

In this paper we regard $\exp_x$ as defined in $\mathbb{R}^n$. An intrinsic definition is possible if M is parallelizable. If not we let Ω_i and $\tilde\Omega_i$, $i = 1, \ldots, N$, be open subsets of M such that for any i, $\tilde\Omega_i$ is parallelizable and $\overline{\Omega_i} \subset \tilde\Omega_i$, and such that $M = \cup\Omega_i$. The canonical exponential map gives N maps $\exp_x$ defined in $\Omega_i \times \mathbf{R}^n$, and $\exp_x$ is, depending on the situation, one of these maps. A property of $\exp_x$ that holds for any $x \in M$ should then be regarded as a property that holds for any i and any $x \in \overline{\Omega_i}$.

The proof of this theorem proceeds in several steps and follows for a large part the lines of the original proof by Struwe [12] where the behavior of Palais-Smale sequences associated to the second order equation (2.1) is described. First, we claim that the following result holds:

Step 1. Palais-Smale sequences for I_g are bounded in $H_2^2(M)$.

Proof of step 1. Let (u_m) be a Palais-Smale sequence for I_g. Then,

$$DI_g(u_m).u_m = \int_M (P_g u_m) u_m dv_g - \int_M |u_m|^{2^\sharp} dv_g = o\left(\|u_m\|_{H_2^2}\right)$$

so that

$$I_g(u_m) = \frac{2}{n}\int_M |u_m|^{2^\sharp} dv_g + o\left(\|u_m\|_{H_2^2}\right) \tag{2.4}$$

The embedding of $H_2^2(M)$ in $H_1^2(M)$ being compact, for any $\varepsilon > 0$ there exists $B_\varepsilon > 0$ such that for all $u \in H_2^2(M)$,

$$\|u\|_{H_1^2}^2 \le \varepsilon \|u\|_{H_2^2}^2 + B_\varepsilon \|u\|_{2^\sharp}^2 \tag{2.5}$$

where $\|u\|_{H_1^2}^2 = \|\nabla u\|_2^2 + \|u\|_2^2$. Clearly,

$$\|u_m\|_{H_2^2}^2 \le \int_M (P_g u_m)\, u_m dv_g + C(\alpha, a)\|u_m\|_{H_1^2}^2$$

where $C(\alpha, a) = \max(|\alpha - 1|, |a - 1|)$. Choosing ε in (2.5) sufficiently small such that $C(\alpha, a)\varepsilon \le 1/2$, and since $I_g(u_m) = O(1)$, we get with (2.4) and (2.5) that

$$\|u_m\|_{H_2^2}^2 \le O(1) + o\left(\|u_m\|_{H_2^2}\right)$$

This proves step 1. □

Now, we enter into a more specific study of Palais-Smale sequences, and claim that the following result holds:

Step 2. Let (u_m) be a Palais-Smale sequence for I_g such that $u_m \rightharpoonup u^0$ weakly in $H_2^2(M)$, $u_m \to u^0$ strongly in $H_1^2(M)$, and $u_m \to u^0$ almost everywhere. Let $v_m = u_m - u^0$, and J_g be the functional I_g when $\alpha = a = 0$. Then (v_m) is a Palais-Smale sequence for J_g and

$$J_g(v_m) = I_g(u_m) - I_g(u^0) + o(1)$$

where $o(1) \to 0$ as $m \to \infty$. Moreover, u^0 is a solution of (2.2).

Proof of step 2. We first observe that for any $\varphi \in C^\infty(M)$,

$$DI_g(u_m).\varphi = \int_M (P_g\varphi)\, u_m dv_g - \int_M |u_m|^{2^\sharp - 2} u_m \varphi dv_g = o(1)$$

By step 1, (u_m) is bounded in $H_2^2(M)$. Passing to the limit as $m \to +\infty$ in this relation, we get that u^0 is a solution of (2.2). Now, we compute the energy of v_m. Since $v_m \rightharpoonup 0$ weakly in $H_2^2(M)$, and $v_m \to 0$ strongly in $H_1^2(M)$,

$$I_g(u_m) = I_g(u^0) + J_g(v_m) - \frac{1}{2^\sharp}\int_M \left(|v_m + u^0|^{2^\sharp} - |v_m|^{2^\sharp} - |u^0|^{2^\sharp}\right) dv_g + o(1)$$

Let $C > 0$ be such that for any $x, y \in \mathbb{R}$,

$$\left||x+y|^{2^\sharp} - |x|^{2^\sharp} - |y|^{2^\sharp}\right| \le C\left(|x|^{2^\sharp - 1}|y| + |y|^{2^\sharp - 1}|x|\right)$$

Integration theory gives that

$$\int_M \left(|v_m + u^0|^{2^\sharp} - |v_m|^{2^\sharp} - |u^0|^{2^\sharp}\right) dv_g = o(1)$$

and we get that

$$J_g(v_m) = I_g(u_m) - I_g(u^0) + o(1)$$

Summarizing, we are left with the proof that (v_m) is a Palais-Smale sequence for J_g. Let $\varphi \in C^\infty(M)$. Then,

$$DI_g(u_m).\varphi = DJ_g(v_m).\varphi - \int_M \Phi_m \varphi dv_g + o\left(\|\varphi\|_{H_1^2}\right)$$

where

$$\Phi_m = |v_m + u^0|^{2^\sharp-2}(v_m + u^0) - |v_m|^{2^\sharp-2}v_m - |u^0|^{2^\sharp-2}u^0$$

We let $C > 0$ be such that for any $x, y \in \mathbb{R}$,

$$\left||x+y|^{2^\sharp-2}(x+y) - |x|^{2^\sharp-2}x - |y|^{2^\sharp-2}y\right| \le C\left(|x|^{2^\sharp-2}|y| + |y|^{2^\sharp-2}|x|\right)$$

By Hölder's inequality,

$$\left|\int_M \Phi_m \varphi dv_g\right| \le C\left(\left\||v_m|^{2^\sharp-2}u^0\right\|_{2^\sharp/(2^\sharp-1)} + \left\||u^0|^{2^\sharp-2}v_m\right\|_{2^\sharp/(2^\sharp-1)}\right)\|\varphi\|_{2^\sharp}$$

while,

$$\left\||v_m|^{2^\sharp-2}u^0\right\|_{2^\sharp/(2^\sharp-1)} + \left\||u^0|^{2^\sharp-2}v_m\right\|_{2^\sharp/(2^\sharp-1)} = o(1)$$

The Sobolev inequality corresponding to the embedding of $H_2^2(M)$ in $L^{2^\sharp}(M)$ then gives that

$$DI_g(u_m).\varphi = DJ_g(v_m).\varphi + o\left(\|\varphi\|_{H_2^2}\right)$$

This implies that (v_m) is a Palais-Smale sequence for J_g. Step 2 is proved. □

In what follows, we let $\beta^\sharp = \frac{2}{n}K_0^{-n/4}$, where K_0 is the best constant K in the Euclidean Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{2^\sharp} dx\right)^{2/2^\sharp} \le K \int_{\mathbb{R}^n} (\Delta u)^2 dx$$

By Edmunds, Fortunato and Janelli [6], Lieb [8], and Lions [10],

$$K_0^{-1} = \pi^2 n(n-4)(n^2-4)\Gamma\left(\frac{n}{2}\right)^{4/n}\Gamma(n)^{-4/n}$$

where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$, $x > 0$, is the Euler function. We claim that the following result holds:

Step 3. Let (v_m) be a Palais-Smale sequence for J_g such that $v_m \rightharpoonup 0$ weakly in $H_2^2(M)$, and such that $J_g(v_m) \to \beta$ where $\beta < \beta^\sharp$. Then $v_m \to 0$ strongly in $H_2^2(M)$.

Proof of step 3. By step 1, (v_m) is bounded in $H_2^2(M)$, and we have that

$$J_g(v_m) = \frac{2}{n}\|v_m\|_{2^\sharp}^{2^\sharp} + o(1) = \frac{2}{n}\|\Delta_g v_m\|_2^2 + o(1) = \beta + o(1) \tag{2.6}$$

As a consequence, $\beta \ge 0$. By Djadli, Hebey and Ledoux [5], for any $\varepsilon > 0$, there exists $B_\varepsilon > 0$ such that for all $u \in H_2^2(M)$,

$$\|u\|_{2^\sharp}^2 \le (K_0 + \varepsilon)\|\Delta_g u\|_2^2 + B_\varepsilon \|u\|_2^2$$

Since the embedding of $H_2^2(M)$ in $H_1^2(M)$ is compact, we may assume that $v_m \to 0$ strongly in $H_1^2(M)$, and in particular that $v_m \to 0$ strongly in $L^2(M)$. Then, applying the above sharp Sobolev inequality to v_m, and letting m go to $+\infty$, we get with (2.6) that for any $\varepsilon > 0$,

$$\left(\frac{n}{2}\beta\right)^{2/2^\sharp} \leq (K_0+\varepsilon)\frac{n}{2}\beta$$

Taking $\varepsilon > 0$ sufficiently small, this inequality is impossible if $\beta > 0$ and $\beta < \beta^\sharp$. Hence, $\beta = 0$, and by (2.6), $v_m \to 0$ strongly in $H_2^2(M)$. Step 3 is proved. □

As a remark, note that it follows from steps 2 and 3 that if (u_m) is a Palais-Smale sequence for I_g, and $I_g(u_m) \to \beta$, where $\beta < \beta^\sharp$, then, up to a subsequence, (u_m) converges strongly to some u^0 in $H_2^2(M)$. In other words, compactness holds for Palais-Smale sequences when the energy is (strictly) below the minimum energy. Another illustration of this fact is in Djadli, Hebey and Ledoux [5] when dealing with minimizing sub-critical sequences associated to (2.2).

The following lemma is the main ingredient in the proof of Theorem 2.1. We postpone its proof to section 3.

Lemma 2.1. *Let (v_m) be a Palais-Smale sequence for J_g such that $v_m \rightharpoonup 0$ weakly in $H_2^2(M)$ but not strongly. There exist a sequence (R_m), $R_m > 0$ and $R_m \to +\infty$ as $m \to \infty$, a converging sequence (x_m) in M, and a non-trivial solution $v \in D_2^2(\mathbb{R}^n)$ of* (2.3)*, such that, up to a subsequence, the following holds: if*

$$w_m = v_m - \eta_m \hat{v}_m ,$$

then (w_m) is a Palais-Smale sequence for J_g such that $w_m \rightharpoonup 0$ weakly in $H_2^2(M)$ and

$$J_g(w_m) = J_g(v_m) - E(v) + o(1) ,$$

where

$$\hat{v}_m(x) = (R_m)^{\frac{n-4}{2}} v\left(R_m \exp_{x_m}^{-1}(x)\right) ,$$

$\eta_m = \eta_{\delta,x_m}$, $\delta < i_g/2$, and $o(1) \to 0$ as $m \to \infty$.

By steps 1 to 3, and Lemma 2.1, we are now in position to prove the theorem. The proof proceeds as follows:

Proof of Theorem 2.1. First, we claim that non-trivial solutions to (2.3) have their energy bounded from below by $\beta^\sharp$. Indeed, if $u \in D_2^2(\mathbb{R}^n)$ is a non-trivial solution to (2.3), it follows from the sharp Euclidean Sobolev inequality that

$$\int_{\mathbb{R}^n} (\Delta u)^2\, dx = \int_{\mathbb{R}^n} |u|^{2^\sharp} dx \leq K_0^{2^\sharp/2}\left(\int_{\mathbb{R}^n} (\Delta u)^2\, dx\right)^{2^\sharp/2}$$

Then, $\|\Delta u\|_2^2 \geq K_0^{-n/4}$, and $E(u) \geq \beta^\sharp$. This proves the claim. In order to prove the theorem, we let (u_m) be a Palais-Smale sequence for I_g. According to step 1, (u_m) is bounded in $H_2^2(M)$. Up to a subsequence, we may therefore assume that for some $u^0 \in H_2^2(M)$, $u_m \rightharpoonup u^0$ weakly in $H_2^2(M)$, $u_m \to u^0$ strongly in $H_1^2(M)$, and $u_m \to u^0$ almost everywhere. We may also assume that $I_g(u_m) \to c$ as $m \to +\infty$. By step 2, u^0 is a solution of (2.2) and $v_m = u_m - u^0$ is a Palais-Smale sequence for J_g such that

$$J_g(v_m) = I_g(u_m) - I_g(u^0) + o(1)$$

If $v_m \to 0$ strongly in $H_2^2(M)$, note that by step 3 this holds if $c - I_g(u^0) < \beta^\sharp$, then $u_m = u^0 + o(1)$, and the theorem is proved. If not, according to the claim at the beginning of this proof, we apply Lemma 2.1 to get a new Palais-Smale sequence (v_m^1) of energy

$$J_g(v_m^1) \leq J_g(v_m) - \beta^\sharp + o(1)$$

Here again, either $v_m^1 \to 0$ strongly in $H_2^2(M)$, in which case the theorem is proved, or $v_m^1 \rightharpoonup 0$ weakly but not strongly in $H_2^2(M)$, in which case we apply again Lemma 2.1. By induction, we get at some point that the Palais-Smale sequence (v_m^k) obtained with this process has an energy which converges to some $\beta < \beta^\sharp$. Then, by step 3, $v_m^k \to 0$ strongly in $H_2^2(M)$, and the theorem is proved. □

3. Proof of Lemma 2.1

We prove Lemma 2.1 in this section. Special difficulties that occur in our context with respect to the original proof of Struwe [12] come from the Riemannian metric that we have to control (e.g. rescaling arguments change the metric), and from the fourth order operator we consider (the Laplacian of a function is more difficult to control than its gradient). If not, this lemma has its exact analogue in Struwe [12]. In essence, both reduce to the claim that substracting a suitable bubble to a Palais-Smale sequence, we are left with a Palais-Smale sequence of lower energy.

Up to a subsequence, we may assume that $J_g(v_m) \to \beta$ as $m \to +\infty$. We may also assume that v_m is smooth, since if not there always exists $\overline{v}_m$ smooth and such that $\|\overline{v}_m - v_m\|_{H_2^2} \to 0$. Then, $(\overline{v}_m)$ is a Palais-Smale sequence for J_g such that $\overline{v}_m \rightharpoonup 0$ weakly in $H_2^2(M)$ but not strongly, and, as easily checked, if the claim holds for $(\overline{v}_m)$, then it holds also for (v_m). Since $DJ_g(v_m) \to 0$, we get as in step 1 of section 2 that

$$\int_M (\Delta_g v_m)^2 \, dv_g = \frac{n}{2}\beta + o(1) \tag{3.1}$$

while, by step 3 of section 2, $\frac{n}{2}\beta \geq K_0^{-n/4}$. For $t > 0$, we let

$$\mu_m(t) = \max_{x \in M} \int_{B_x(t)} (\Delta_g v_m)^2 \, dv_g$$

Given $t_0 > 0$, it follows from (3.1) that there exist $x_0 \in M$ and $\lambda_0 > 0$ such that, up to a subsequence,

$$\int_{B_{x_0}(t_0)} (\Delta_g v_m)^2 \, dv_g \geq \lambda_0$$

for all m. Then, since $t \to \mu_m(t)$ is continuous, we get that for any $\lambda \in (0, \lambda_0)$, there exists $t_m \in (0, t_0)$ such that $\mu_m(t_m) = \lambda$. Clearly, there also exists $x_m \in M$ such that

$$\mu_m(t_m) = \int_{B_{x_m}(t_m)} (\Delta_g v_m)^2 \, dv_g$$

Up to a subsequence, (x_m) converges. We let $r_0 \in (0, i_g/2)$ be such that for all $x \in M$ and all $y, z \in \mathbb{R}^n$, if $|y| \leq r_0$ and $|z| \leq r_0$, then

$$d_g\left(exp_x(y), exp_x(z)\right) \leq C_0 \, |z - y|$$

for some $C_0 \in [1, 2]$ independent of x, y, and z. Given $R_m \geq 1$, and $x \in \mathbb{R}^n$ such that $|x| < i_g R_m$, we let

$$\tilde{v}_m(x) = R_m^{\frac{4-n}{2}} v_m\left(exp_{x_m}(R_m^{-1}x)\right) \text{ and } \tilde{g}_m(x) = \left(exp_{x_m}^\star g\right)(R_m^{-1}x)$$

Then,

$$(\Delta_g v_m)\left(\exp_{x_m}(R_m^{-1}x)\right) = R_m^{n/2}\left(\Delta_{\tilde g_m}\tilde v_m\right)(x)$$

and if $|z| + r < i_g R_m$,

$$\int_{B_z(r)} \left(\Delta_{\tilde g_m}\tilde v_m\right)^2 dv_{\tilde g_m} = \int_{exp_{x_m}\left(R_m^{-1}B_z(r)\right)} \left(\Delta_g v_m\right)^2 dv_g \tag{3.2}$$

Moreover, when $|z| + r < r_0 R_m$,

$$exp_{x_m}\left(R_m^{-1}B_z(r)\right) \subset B_{exp_{x_m}(R_m^{-1}z)}(C_0 r R_m^{-1}) \tag{3.3}$$

while

$$exp_{x_m}\left(R_m^{-1}B_0(C_0 r)\right) = B_{x_m}(C_0 r R_m^{-1}) \tag{3.4}$$

Given $r \in (0, r_0)$, we fix t_0 such that $C_0 r t_0^{-1} \ge 1$. Then, for any $\lambda \in (0, \lambda_0)$, we let $R_m \ge 1$ be such that $C_0 r R_m^{-1} = t_m$. By (3.2) to (3.4), for any $z \in \mathbb{R}^n$ such that $|z| < r_0 R_m - r$,

$$\int_{B_z(r)} \left(\Delta_{\tilde g_m}\tilde v_m\right)^2 dv_{\tilde g_m} \le \lambda \text{ and } \int_{B_0(C_0 r)} \left(\Delta_{\tilde g_m}\tilde v_m\right)^2 dv_{\tilde g_m} = \lambda \tag{3.5}$$

As a technical point we will use in the sequel, we claim that there exist $\delta \in (0, i_g)$ and $C_1 > 1$ such that for any $x \in M$, and any $R \ge 1$, if $\tilde g_{x,R}(y) = exp_x^\star g(R^{-1}y)$, then

$$\frac{1}{C_1}\int_{\mathbb{R}^n} (\Delta u)^2 dx \le \int_{\mathbb{R}^n} \left(\Delta_{\tilde g_{x,R}} u\right)^2 dv_{\tilde g_{x,R}} \le C_1 \int_{\mathbb{R}^n} (\Delta u)^2 dx \tag{3.6}$$

for all $u \in D_2^2(\mathbb{R}^n)$ such that $\mathrm{Supp} u \subset B_0(\delta R)$. Indeed, given $\varepsilon > 0$, we choose $\delta > 0$ sufficiently small such that for any $x \in M$, $exp_x^\star g$ and the Euclidean metric ξ, when restricted to $B_0(\delta)$, are ε-close in the C^1-topology. Then,

$$\Delta_{\tilde g_{x,R}} u = \Delta u + O\left(\varepsilon|\nabla^2 u| + \frac{\varepsilon}{R}|\nabla u|\right)$$

for all $u \in D_2^2(\mathbb{R}^n)$ such that $\mathrm{Supp} u \subset B_0(\delta R)$, while, according to the Hölder and Sobolev inequalities,

$$\begin{aligned}\int_{B_0(\delta R)} |\nabla u|^2 dx &\le |B_0(\delta R)|^{2/n}\left(\int_{B_0(\delta R)} |\nabla u|^{2n/(n-2)} dx\right)^{(n-2)/n}\\ &\le AR^2 \int_{\mathbb{R}^n} |\nabla^2 u|^2 dx\end{aligned}$$

where $|B_0(\delta R)|$ is the Euclidean volume of $B_0(\delta R)$. Taking ε sufficiently small, we then get the existence of $\delta > 0$ and $C_1 > 1$ as in the above claim. Clearly, we may also ask that for all $u \in L^1(\mathbb{R}^n)$ such that $\mathrm{Supp} u \subset B_0(\delta R)$,

$$\frac{1}{C_1}\int_{\mathbb{R}^n} |u| dx \le \int_{\mathbb{R}^n} |u| dv_{\tilde g_{x,R}} \le C_1 \int_{\mathbb{R}^n} |u| dx \tag{3.7}$$

Now, we let $\tilde\eta \in \mathcal{D}(\mathbb{R}^n)$ be a cut-off function such that $0 \le \tilde\eta \le 1$, $\tilde\eta = 1$ in $B_0(1/4)$ and $\tilde\eta = 0$ in $\mathbb{R}^n \backslash B_0(3/4)$. We set $\tilde\eta_m(x) = \tilde\eta(\delta^{-1}R_m^{-1}x)$, where δ is as above. Then,

$$\int_{\mathbb{R}^n} \left(\Delta_{\tilde g_m}\tilde\eta_m\tilde v_m\right)^2 dv_{\tilde g_m} = O(1)$$

and it follows from the above claim that $\tilde\eta_m\tilde v_m$ is bounded in $D_2^2(\mathbb{R}^n)$. In particular, up to a subsequence, there exists $v \in D_2^2(\mathbb{R}^n)$ such that $\tilde\eta_m\tilde v_m \rightharpoonup v$ weakly in

$D_2^2(\mathbb{R}^n)$. As a first step in the proof of Lemma 2.1, we claim that the following holds:

Step 1. We have that

$$\tilde{\eta}_m\tilde{v}_m \to v \quad \text{strongly in } \; H_2^2\left(B_0(C_0r)\right) \tag{3.8}$$

for r and λ sufficiently small.

Proof of step 1. In order to prove this claim, we let $x_0 \in \mathbb{R}^n$, and for $\rho > 0$, we denote by h_ρ the standard metric on $\partial B_{x_0}(\rho)$. By Fatou's lemma,

$$\int_r^{2r}\left(\liminf_{m\to+\infty}\int_{\partial B_{x_0}(\rho)} N_\xi(\tilde{\eta}_m\tilde{v}_m)dv_{h_\rho}\right)d\rho \le \liminf_{m\to+\infty}\int_{B_{x_0}(2r)} N_\xi(\tilde{\eta}_m\tilde{v}_m)dx \le C$$

where $N_h(u) = |\nabla_h^2 u|_h^2 + |\nabla u|_h^2 + u^2$, and ξ is the Euclidean metric. It follows that there exists $\rho \in [r, 2r]$ such that, up to a subsequence, and for all m,

$$\int_{\partial B_{x_0}(\rho)} N_\xi(\tilde{\eta}_m\tilde{v}_m)dv_{h_\rho} \le C$$

We let $C = C(\rho) > 0$ be such that for any $\varphi \in C^\infty(\mathbb{R}^n)$, $N_{h_\rho}(\varphi_{|\partial B_{x_0}(\rho)}) \le CN_\xi(\varphi)$ on $\partial B_{x_0}(\rho)$. By the above inequality,

$$\|\tilde{\eta}_m\tilde{v}_m\|_{H_2^2\left(\partial B_{x_0}(\rho)\right)} \le C \quad \text{and} \quad \|\partial_n(\tilde{\eta}_m\tilde{v}_m)\|_{H_1^2\left(\partial B_{x_0}(\rho)\right)} \le C$$

where $\partial_n u$ stands for the derivative in the direction of the inward normal to $\partial B_{x_0}(\rho)$. By compactness of the embeddings $H_2^2\left(\partial B_{x_0}(\rho)\right) \subset H_{3/2}^2\left(\partial B_{x_0}(\rho)\right)$ and $H_1^2\left(\partial B_{x_0}(\rho)\right) \subset H_{1/2}^2\left(\partial B_{x_0}(\rho)\right)$, and continuity of the trace operators $u \to u_{|\partial B}$ and $u \to (\partial_n u)_{|\partial B}$, we get that, up to a subsequence,

$$\tilde{\eta}_m\tilde{v}_m \to v \;\text{ in }\; H_{3/2}^2\left(\partial B_{x_0}(\rho)\right) \quad \text{and} \quad \partial_n(\tilde{\eta}_m\tilde{v}_m) \to \partial_n v \;\text{ in }\; H_{1/2}^2\left(\partial B_{x_0}(\rho)\right)$$

Let $A = B_{x_0}(3r)\backslash B_{x_0}(\rho)$, and $\varphi_m \in D_2^2(\mathbb{R}^n)$ be such that $\varphi_m = \tilde{\eta}_m\tilde{v}_m - v$ on $B_{x_0}(\rho+\varepsilon)$ and $\varphi_m = 0$ on $\mathbb{R}^n\backslash B_{x_0}(3r-\varepsilon)$, $\varepsilon << 1$. Let also $D_2^2(A)$ be the closure in $H_2^2(A)$ of $\mathcal{D}(A)$, the space of smooth functions with compact support in A. Then,

$$\|\tilde{\eta}_m\tilde{v}_m - v\|_{H_{3/2}^2(\partial B_{x_0}(\rho))} = \|\varphi_m\|_{H_{3/2}^2(\partial A)}$$

and

$$\|\partial_n(\tilde{\eta}_m\tilde{v}_m - v)\|_{H_{1/2}^2(\partial B_{x_0}(\rho))} = \|\partial_n\varphi_m\|_{H_{1/2}^2(\partial A)}$$

while there exists $\varphi_m^0 \in D_2^2(A)$ such that

$$\|\varphi_m + \varphi_m^0\|_{H_2^2(A)} \le C_1\|\varphi_m\|_{H_{3/2}^2(\partial A)} + C_2\|\partial_n\varphi_m\|_{H_{1/2}^2(\partial A)}$$

Minimization arguments give that there exists $z_m \in H_2^2(A)$ such that

$$\Delta^2 z_m = 0 \;\text{ in } A\;, \quad z_m - \varphi_m - \varphi_m^0 \in D_2^2(A)$$

and $\|z_m\|_{H_2^2(A)} \le C\|\varphi_m + \varphi_m^0\|_{H_2^2(A)}$. Hence, $z_m \to 0$ strongly in $H_2^2(A)$. We let

$$\psi_m = \tilde{\eta}_m\tilde{v}_m - v \;\text{ in } \overline{B}_{x_0}(\rho)\;, \quad \psi_m = z_m \;\text{ in } \overline{B}_{x_0}(3r)\backslash B_{x_0}(\rho)\;, \quad \psi_m = 0 \;\text{ otherwise}$$

Clearly, $\psi_m \in D_2^2(\mathbb{R}^n)$. Choosing r such that $r < \min(i_g/6, \delta/24)$, we set

$$\tilde{\psi}_m(x) = R_m^{\frac{n-4}{2}}\psi_m\left(R_m exp_{x_m}^{-1}(x)\right) \text{ if } d_g(x_m, x) < 6r \;\;,\;\; \tilde{\psi}_m = 0 \;\text{ otherwise}$$

Then, $\tilde{\eta}\left(\delta^{-1}exp_{x_m}^{-1}(x)\right) = 1$ if $d_g(x_m, x) < 6r$, and if in addition $|x_0| < 3r$, then

$$\begin{aligned} DJ_g(v_m).\tilde{\psi}_m &= DJ_g(\hat{\eta}_m v_m).\tilde{\psi}_m \\ &= \int_{B_{x_0}(3r)} \left(\Delta_{\tilde{g}_m}(\tilde{\eta}_m \tilde{v}_m)\right)\left(\Delta_{\tilde{g}_m}\psi_m\right) dv_{\tilde{g}_m} \\ &\quad - \int_{B_{x_0}(3r)} |\tilde{\eta}_m \tilde{v}_m|^{2^\sharp - 2}\left(\tilde{\eta}_m \tilde{v}_m\right)\psi_m dv_{\tilde{g}_m} \end{aligned}$$

where $\hat{\eta}_m(x) = \tilde{\eta}\left(\delta^{-1}exp_{x_m}^{-1}(x)\right)$. We have that $\|\tilde{\psi}_m\|_{H_2^2(M)} \leq C\|\psi_m\|_{D_2^2(\mathbb{R}^n)}$. Hence, the $\tilde{\psi}_m$'s are bounded in $H_2^2(M)$, and it follows that $DJ_g(v_m).\tilde{\psi}_m = o(1)$. Since $\psi_m \to 0$ strongly in $H_2^2(A)$, and $\psi_m \rightharpoonup 0$ weakly in $D_2^2(\mathbb{R}^n)$,

$$\begin{aligned} &\int_{B_{x_0}(3r)} \left(\Delta_{\tilde{g}_m}(\tilde{\eta}_m \tilde{v}_m)\right)\left(\Delta_{\tilde{g}_m}\psi_m\right) dv_{\tilde{g}_m} \\ &= \int_{B_{x_0}(\rho)} \Delta_{\tilde{g}_m}(\psi_m + v)\Delta_{\tilde{g}_m}\psi_m dv_{\tilde{g}_m} + o(1) \\ &= \int_{\mathbb{R}^n} \left(\Delta_{\tilde{g}_m}\psi_m\right)^2 dv_{\tilde{g}_m} + o(1) \end{aligned}$$

Similarly, one easily gets that

$$\int_{B_{x_0}(3r)} |\tilde{\eta}_m \tilde{v}_m|^{2^\sharp - 2}\left(\tilde{\eta}_m \tilde{v}_m\right)\psi_m dv_{\tilde{g}_m} = \int_{\mathbb{R}^n} |\psi_m|^{2^\sharp} dv_{\tilde{g}_m} + o(1)$$

and since $DJ_g(v_m).\tilde{\psi}_m = o(1)$, it follows that

$$\int_{\mathbb{R}^n} \left(\Delta_{\tilde{g}_m}\psi_m\right)^2 dv_{\tilde{g}_m} - \int_{\mathbb{R}^n} |\psi_m|^{2^\sharp} dv_{\tilde{g}_m} = o(1) \tag{3.9}$$

By the strong convergence $\psi_m \to 0$ in $H_2^2(A)$, and the weak convergence $\psi_m \rightharpoonup 0$ in $D_2^2(\mathbb{R}^n)$,

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\Delta_{\tilde{g}_m}\psi_m\right)^2 dv_{\tilde{g}_m} &= \int_{B_{x_0}(\rho)} \left(\Delta_{\tilde{g}_m}(\tilde{\eta}_m \tilde{v}_m - v)\right)^2 dv_{\tilde{g}_m} + o(1) \\ &= \int_{B_{x_0}(\rho)} \left(\Delta_{\tilde{g}_m}(\tilde{\eta}_m \tilde{v}_m)\right)^2 dv_{\tilde{g}_m} - \int_{B_{x_0}(\rho)} \left(\Delta_{\tilde{g}_m} v\right)^2 dv_{\tilde{g}_m} + o(1) \end{aligned}$$

It follows that

$$\int_{\mathbb{R}^n} \left(\Delta_{\tilde{g}_m}\psi_m\right)^2 dv_{\tilde{g}_m} \leq \int_{B_{x_0}(\rho)} \left(\Delta_{\tilde{g}_m}(\tilde{\eta}_m \tilde{v}_m)\right)^2 dv_{\tilde{g}_m} + o(1)$$

Let N be an integer such that $B_0(2)$ is covered by N balls of radius 1 and center in $B_0(2)$. Then there exist N points $x_1, \dots, x_N$ in $B_{x_0}(2r)$ such that

$$B_{x_0}(\rho) \subset B_{x_0}(2r) \subset \bigcup_{i=1}^{N} B_{x_i}(r)$$

and we get with (3.5) that for x_0 and r such that $|x_0| + 3r < r_0$,

$$\int_{\mathbb{R}^n} \left(\Delta_{\tilde{g}_m}\psi_m\right)^2 dv_{\tilde{g}_m} \leq N\lambda + o(1) \tag{3.10}$$

For C_1 as in (3.6) and (3.7), and x_0 and r such that $|x_0| + 3r < \delta$,

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |\psi_m|^{2^\sharp} dv_{\tilde{g}_m}\right)^{2/2^\sharp} &\leq C_1^{2/2^\sharp} \left(\int_{\mathbb{R}^n} |\psi_m|^{2^\sharp} dx\right)^{2/2^\sharp} \\ &\leq C_1^{2/2^\sharp} K_0 \int_{\mathbb{R}^n} (\Delta\psi_m)^2 dx \\ &\leq C_1^{1+(2/2^\sharp)} K_0 \int_{\mathbb{R}^n} (\Delta_{\tilde{g}_m}\psi_m)^2 dv_{\tilde{g}_m}\end{aligned}$$

By (3.9) and (3.10) we then get that

$$\int_{\mathbb{R}^n} (\Delta_{\tilde{g}_m}\psi_m)^2 dv_{\tilde{g}_m} \leq K^{2^\sharp/2} \int_{\mathbb{R}^n} (\Delta_{\tilde{g}_m}\psi_m)^2 dv_{\tilde{g}_m} + o(1)$$

where $K = C_1^{1+(2/2^\sharp)} K_0 (N\lambda + o(1))^{1-(2/2^\sharp)}$. Choosing $\lambda > 0$ sufficiently small such that $NC_1^{(2^\sharp+2)/(2^\sharp-2)} K_0^{2/(2^\sharp-2)}\lambda < 1$, it follows that

$$\int_{\mathbb{R}^n} (\Delta_{\tilde{g}_m}\psi_m)^2 dv_{\tilde{g}_m} = o(1)$$

and hence that $\psi_m \to 0$ strongly in $D_2^2(\mathbb{R}^n)$. Since $r \leq \rho$, it follows that

$$\tilde{\eta}_m\tilde{v}_m \to v \text{ strongly in } H_2^2\left(B_{x_0}(r)\right) \tag{3.11}$$

and the convergence holds as soon as $NC_1^{(2^\sharp+2)/(2^\sharp-2)} K_0^{2/(2^\sharp-2)}\lambda < 1$, $|x_0| < 3r$, $|x_0| + 3r < r_0$, $|x_0| + 3r < \delta$, and $r < \min(i_g/6, \delta/24)$. We choose $\lambda > 0$ such that the above inequality is satisfied, and $r > 0$ such that $r < \min(i_g/6, \delta/24, r_0/6)$. Then (3.11) holds for any x_0 such that $|x_0| < 2r$. Since $C_0 \leq 2$, $B_0(C_0 r)$ is covered by N balls of radius r and center in $B_0(2r)$. It follows that $\tilde{\eta}_m\tilde{v}_m \to v$ strongly in $H_2^2\left(B_0(C_0 r)\right)$, and this proves (3.8). □

In particular, we get from (3.8) that $v \not\equiv 0$. Indeed,

$$\begin{aligned}\lambda &= \int_{B_0(C_0 r)} (\Delta_{\tilde{g}_m}\tilde{v}_m)^2 dv_{\tilde{g}_m} \\ &= \int_{B_0(C_0 r)} (\Delta_{\tilde{g}_m}(\tilde{\eta}_m\tilde{v}_m))^2 dv_{\tilde{g}_m} \\ &\leq C_1 \int_{B_0(C_0 r)} (\Delta v)^2 dx + o(1)\end{aligned}$$

and it follows that $v \not\equiv 0$. Another consequence of (3.8) is that $R_m \to +\infty$ as $m \to +\infty$. Indeed, if $R_m \to R$ as $m \to +\infty$, $R \geq 1$, then $\tilde{v}_m \rightharpoonup 0$ weakly in $H_2^2\left(B_0(C_0 r)\right)$ since $v_m \rightharpoonup 0$ weakly in $H_2^2(M)$, and this is in contradiction with (3.8) and the fact that $v \not\equiv 0$. Hence,

$$\lim_{m\to+\infty} R_m = +\infty \tag{3.12}$$

Now, let $R \geq 1$ be given. By (3.12), for m large, $R_m > R$. Then, coming back to the beginning of the proof of the lemma, (3.5) holds for z such that $|z| < r_0 R - r$. Thus, as easily checked, it follows from the proof of (3.8) that (3.11) holds if $|x_0| < 3r(2R-1)$, $|x_0| + 3r < r_0 R$ and $|x_0| + 3r < \delta R$, where r is as above. In particular, (3.11) holds if $|x_0| < 2rR$. Hence, $\tilde{\eta}_m\tilde{v}_m \to 0$ strongly in $H_2^2\left(B_0(2rR)\right)$. Since $R \geq 1$ is arbitrary, and $\tilde{\eta}_m(x) = 1$ for m large if $|x| \leq R$, we get that for any $R > 0$,

$$\tilde{v}_m \to v \text{ strongly in } H_2^2\left(B_0(R)\right) \tag{3.13}$$

It also follows from (3.12) that the following holds:

Step 2. v is a solution of (2.3).

Proof of step 2. Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and let $R_0 > 0$ be such that $\text{Supp}\varphi \subset B_0(R_0)$. Let also $\hat{\varphi}_m$ be given by

$$\hat{\varphi}_m(x) = R_m^{\frac{n-4}{2}} \varphi(R_m x)$$

Then $\text{Supp}\hat{\varphi}_m \subset B_0(R_0 R_m^{-1})$. For m large, we let φ_m be the smooth function on M given by the relation $\hat{\varphi}_m = \varphi_m \circ exp_{x_m}$. Then, for m large,

$$\int_M \Delta_g v_m \Delta_g \varphi_m dv_g = \int_{\mathbb{R}^n} \Delta_{\tilde{g}_m}(\tilde{\eta}_m \tilde{v}_m) \Delta_{\tilde{g}_m} \varphi dv_{\tilde{g}_m}$$

and

$$\int_M |v_m|^{2^\sharp - 2} v_m \varphi_m dv_g = \int_{\mathbb{R}^n} |\tilde{\eta}_m \tilde{v}_m|^{2^\sharp - 2} \tilde{\eta}_m \tilde{v}_m \varphi dv_{\tilde{g}_m}$$

Since $R_m \to +\infty$, $\tilde{g}_m \to \xi$ in $C^1(B_0(R))$ for any $R > 0$. Moreover, (φ_m) is bounded in $H_2^2(M)$. Since (v_m) is a Palais-Smale sequence for J_g, and $\tilde{\eta}_m \tilde{v}_m \rightharpoonup v$ in $D_2^2(\mathbb{R}^n)$, we get by passing to the limit as $m \to +\infty$ in the above two relations that

$$\int_{\mathbb{R}^n} \Delta v \Delta \varphi dx = \int_{\mathbb{R}^n} |v|^{2^\sharp - 2} v \varphi dx$$

In other words, $v \in D_2^2(\mathbb{R}^n)$ is a solution of (2.3). □

Now, for $x \in M$ and $\hat{\delta} \in (0, \delta/8)$, we let

$$V_m(x) = \eta_m(x) R_m^{\frac{n-4}{2}} v\left(R_m exp_{x_m}^{-1}(x)\right) \tag{3.14}$$

where $\eta_m = \eta_{\hat{\delta}, x_m}$, and set $w_m = v_m - V_m$.

Step 3. The following relations hold. On the one hand,

$$w_m \rightharpoonup 0 \text{ weakly in } H_2^2(M) \tag{3.15}$$

On the other hand,

$$DJ_g(V_m) \to 0 \text{ and } DJ_g(w_m) \to 0 \text{ strongly} \tag{3.16}$$

At last,

$$J_g(w_m) = J_g(v_m) - E(v) + o(1) \tag{3.17}$$

where $o(1) \to 0$ as $m \to +\infty$.

Proof of step 3. We start with the proof of (3.15). There, it suffices to prove that $V_m \rightharpoonup 0$ weakly in $H_2^2(M)$. Given $R > 0$, we let $\Omega_m(R) = B_{x_m}(R_m^{-1} R)$. For φ a smooth function on M, and m large,

$$\int_{\Omega_m(R)} V_m \varphi dv_g = R_m^{\frac{n-4}{2}} \int_{B_0(R_m^{-1} R)} \eta_{\hat{\delta}}(x) v(R_m x) \varphi(exp_{x_m}(x)) dv_{g_m}$$

where $g_m = exp_{x_m}^\star g$. It follows that for $C > 0$ such that $dv_{g_m} \le C dx$,

$$\left| \int_{\Omega_m(R)} V_m \varphi dv_g \right| \le C \|\varphi\|_\infty R_m^{-(n+4)/2} \int_{B_0(R)} |v| dx$$

Similarly, by Hölder's inequality,

$$\left|\int_{M\backslash\Omega_m(R)} V_m\varphi dv_g\right| \leq C\|\varphi\|_\infty R_m^{-(n+4)/2}\int_{B_0(\delta R_m)\backslash B_0(R)}|v|dx$$
$$\leq C\|\varphi\|_\infty\left(\int_{B_0(\delta R_m)\backslash B_0(R)}|v|^{2^\sharp}dx\right)^{1/2^\sharp}$$

Taking $R>0$ sufficiently large, and since $R_m\to+\infty$ as $m\to+\infty$, it follows that $\int_M V_m\varphi dv_g\to 0$ as $m\to+\infty$. With similar estimates, one gets that

$$\int_M(\nabla V_m,\nabla\varphi)_g\,dv_g\to 0\quad\text{and}\quad\int_M\Delta_g V_m\Delta_g\varphi dv_g\to 0$$

as $m\to+\infty$. We also do have that (V_m) is bounded in $H_2^2(M)$. This proves (3.15). Now we prove (3.16). Here again, we let φ be a smooth function on M. Then,

$$DJ_g(V_m).\varphi=\int_M\Delta_g V_m\Delta_g\varphi dv_g-\int_M|V_m|^{2^\sharp-2}V_m\varphi dv_g$$

Given $R>0$, we write that

$$\int_M\Delta_g V_m\Delta_g\varphi dv_g=\int_{B_{x_m}(R_m^{-1}R)}\Delta_g V_m\Delta_g\varphi dv_g+\int_{B_{x_m}(\delta)\backslash B_{x_m}(R_m^{-1}R)}\Delta_g V_m\Delta_g\varphi dv_g$$

Easy computations give that

$$\int_{B_{x_m}(\delta)\backslash B_{x_m}(R_m^{-1}R)}\Delta_g V_m\Delta_g\varphi dv_g=O\left(\|\varphi\|_{H_2^2}\right)\varepsilon_R$$

where $\varepsilon_R\to 0$ as $R\to+\infty$. Independently, let $\overline{\varphi}_m$ be the function of $D_2^2(\mathbb{R}^n)$ given by

$$\overline{\varphi}_m(x)=R_m^{\frac{4-n}{2}}\eta_{m,\hat\delta}(x)\left(\varphi\circ exp_{x_m}\right)\left(R_m^{-1}x\right)$$

where $\eta_{m,\hat\delta}(x)=\eta_{\hat\delta}(R_m^{-1}x)$. Then, for m large,

$$\int_{B_{x_m}(R_m^{-1}R)}\Delta_g V_m\Delta_g\varphi dv_g=\int_{B_0(R)}\Delta_{\tilde g_m}v\Delta_{\tilde g_m}\overline{\varphi}_m dv_{\tilde g_m}$$

Noting that $\tilde g_m\to\xi$ in $C^1\left(B_0(\tilde R)\right)$, $\tilde R>R$, and that

$$\int_{B_{x_m}(R_m^{-1}R)}(\Delta_g\varphi)^2\,dv_g=\int_{B_0(R)}\left(\Delta_{\tilde g_m}\overline{\varphi}_m\right)^2dv_{\tilde g_m}$$

we get that

$$\int_{B_0(R)}\Delta_{\tilde g_m}v\Delta_{\tilde g_m}\overline{\varphi}_m dv_{\tilde g_m}=\int_{B_0(R)}\Delta v\Delta\overline{\varphi}_m dx+o\left(\|\varphi\|_{H_2^2}\right)$$

We also do have that

$$\int_{B_0(R)}\Delta v\Delta\overline{\varphi}_m dx=\int_{\mathbb{R}^n}\Delta v\Delta\overline{\varphi}_m dx+O\left(\|\varphi\|_{H_2^2}\right)\varepsilon_R$$

where ε_R is as above. Hence,

$$\int_M\Delta_g V_m\Delta_g\varphi dv_g=\int_{\mathbb{R}^n}\Delta v\Delta\overline{\varphi}_m dx+o\left(\|\varphi\|_{H_2^2}\right)+O\left(\|\varphi\|_{H_2^2}\right)\varepsilon_R\qquad(3.18)$$

In a similar way, we get that

$$\int_M |V_m|^{2^\sharp-2}V_m\varphi dv_g = \int_{\mathbb{R}^n} |v|^{2^\sharp-2}v\overline{\varphi}_m dx + o\left(\|\varphi\|_{H_2^2}\right) + O\left(\|\varphi\|_{H_2^2}\right)\varepsilon_R \tag{3.19}$$

Since v is a solution of (2.3), it follows from (3.18) and (3.19) that

$$DJ_g(V_m).\varphi = o\left(\|\varphi\|_{H_2^2}\right) + O\left(\|\varphi\|_{H_2^2}\right)\varepsilon_R$$

and since $R > 0$ is arbitrary, we get that $DJ_g(V_m) \to 0$ strongly. Now, we write that

$$DJ_g(w_m).\varphi = DJ_g(v_m).\varphi - DJ_g(V_m).\varphi - A(m) \tag{3.20}$$

where

$$A(m) = \int_M \Phi_m\varphi dv_g = \int_{B_{x_m}(2\hat{\delta})} \Phi_m\varphi dv_g$$

and $\Phi_m = |w_m|^{2^\sharp-2}w_m - |v_m|^{2^\sharp-2}v_m + |V_m|^{2^\sharp-2}V_m$. By the Hölder and Sobolev inequalities,

$$|A(m)| \le \|\Phi_m\|_{2^\sharp/(2^\sharp-1)}\|\varphi\|_{H_2^2}$$

Given $R > 0$, we set $B_m = B_{x_m}(R_m^{-1}R)$ and $B_m^c = B_{x_m}(2\hat{\delta})\backslash B_{x_m}(R_m^{-1}R)$. Then, for m large,

$$\|\Phi_m\|_{2^\sharp/(2^\sharp-1)} \le \|\Phi_m\|_{L^{2^\sharp/(2^\sharp-1)}(B_m)} + \|\Phi_m\|_{L^{2^\sharp/(2^\sharp-1)}(B_m^c)}$$

and as in step 2 of section 2,

$$\|\Phi_m\|_{L^{2^\sharp/(2^\sharp-1)}(B_m^c)} \le C\left(\|\Phi_m^1\|_{L^{2^\sharp/(2^\sharp-1)}(B_m^c)} + \|\Phi_m^2\|_{L^{2^\sharp/(2^\sharp-1)}(B_m^c)}\right)$$

where $\Phi_m^1 = |v_m|^{2^\sharp-2}V_m$ and $\Phi_m^2 = |V_m|^{2^\sharp-2}v_m$. We have that

$$\int_{B_m} |\Phi_m|^{\frac{2^\sharp}{2^\sharp-1}} dv_g = \int_{B_0(R)} |\tilde{\Phi}_m|^{\frac{2^\sharp}{2^\sharp-1}} dv_{\tilde{g}_m}$$

where $\tilde{\Phi}_m = |\tilde{v}_m - v|^{2^\sharp-2}(\tilde{v}_m - v) - |\tilde{v}_m|^{2^\sharp-2}\tilde{v}_m + |v|^{2^\sharp-2}v$. Then, by (3.13), we get that

$$\int_{B_m} |\Phi_m|^{\frac{2^\sharp}{2^\sharp-1}} dv_g = o(1)$$

Independently,

$$\begin{aligned}\int_{B_m^c} |\Phi_m^1|^{\frac{2^\sharp}{2^\sharp-1}} dv_g &= \int_{B_0(2\hat{\delta}R_m)\backslash B_0(R)} |\tilde{\eta}_m\tilde{v}_m|^{\frac{2^\sharp(2^\sharp-2)}{2^\sharp-1}} |v|^{\frac{2^\sharp}{2^\sharp-1}} \hat{\eta}_m^{\frac{2^\sharp}{2^\sharp-1}} dv_{\tilde{g}_m}\\ &\le C\int_{\mathbb{R}^n\backslash B_0(R)} |\tilde{\eta}_m\tilde{v}_m|^{\frac{2^\sharp(2^\sharp-2)}{2^\sharp-1}} |v|^{\frac{2^\sharp}{2^\sharp-1}} dx\end{aligned}$$

where $\hat{\eta}_m = \eta_{\hat{\delta},x_m}\left(exp_{x_m}(R_m^{-1}x)\right)$, and $C > 0$ is such that $dv_{\tilde{g}_m} \le Cdx$. Without loss of generality, we may assume that $\tilde{\eta}_m\tilde{v}_m \to v$ almost everywhere in $\mathbb{R}^n$. Set

$$f_m = |\tilde{\eta}_m\tilde{v}_m|^{\frac{2^\sharp(2^\sharp-2)}{2^\sharp-1}} \quad\text{and}\quad f = |v|^{\frac{2^\sharp(2^\sharp-2)}{2^\sharp-1}}$$

Then (f_m) is bounded in $L^{(2^\sharp-1)/(2^\sharp-2)}(\mathbb{R}^n)$ and (f_m) converges almost everywhere to f, so that, by classical integration theory, (f_m) converges weakly to f in $L^{(2^\sharp-1)/(2^\sharp-2)}(\mathbb{R}^n)$. It follows that

$$\lim_{m\to+\infty}\int_{\mathbb{R}^n\backslash B_0(R)} |\tilde{\eta}_m\tilde{v}_m|^{\frac{2^\sharp(2^\sharp-2)}{2^\sharp-1}} |v|^{\frac{2^\sharp}{2^\sharp-1}} dx = \int_{\mathbb{R}^n\backslash B_0(R)} |v|^{2^\sharp} dx$$

and we get that

$$\lim_{R\to+\infty}\limsup_{m\to+\infty}\int_{B_m^c}|\Phi_m^1|^{\frac{2^\sharp}{2^\sharp-1}}dv_g=0$$

Similarly,

$$\lim_{R\to+\infty}\limsup_{m\to+\infty}\int_{B_m^c}|\Phi_m^2|^{\frac{2^\sharp}{2^\sharp-1}}dv_g=0$$

Coming back to (3.20), and since $R>0$ is arbitrary, we get that $DJ_g(w_m)\to 0$ strongly. In particular, (3.16) is proved, and we are left with the proof of (3.17). We have here that

$$J_g(w_m)=\frac{1}{2}\int_M(\Delta_g w_m)^2\,dv_g-\frac{1}{2^\sharp}\int_M|w_m|^{2^\sharp}dv_g \tag{3.21}$$

Concerning the first term, we write that

$$\int_M(\Delta_g w_m)^2\,dv_g=\int_{B_{x_m}(2\hat\delta)}(\Delta_g w_m)^2\,dv_g+\int_{M\setminus B_{x_m}(2\hat\delta)}(\Delta_g v_m)^2\,dv_g$$

and for B_m and B_m^c as above, we write that

$$\int_{B_{x_m}(2\hat\delta)}(\Delta_g w_m)^2\,dv_g=\int_{B_m}(\Delta_g w_m)^2\,dv_g+\int_{B_m^c}(\Delta_g w_m)^2\,dv_g$$

We have that

$$\int_{B_m}(\Delta_g w_m)^2\,dv_g=\int_{B_0(R)}(\Delta_{\tilde g_m}(\tilde v_m-v))^2\,dv_{\tilde g_m}$$

and it follows from (3.13) that

$$\int_{B_m}(\Delta_g w_m)^2\,dv_g=o(1)$$

Moreover, it follows from rough estimates that

$$\lim_{R\to+\infty}\limsup_{m\to+\infty}\int_{B_m^c}(\Delta_g V_m)^2\,dv_g=0$$

Since $w_m=v_m-V_m$ and (v_m) is bounded in $H_2^2(M)$, it follows that

$$\int_{B_m^c}(\Delta_g w_m)^2\,dv_g=\int_{B_m^c}(\Delta_g v_m)^2\,dv_g+B_R(m)$$

and

$$\int_M(\Delta_g w_m)^2\,dv_g=\int_M(\Delta_g v_m)^2\,dv_g-\int_{B_m}(\Delta_g v_m)^2\,dv_g+B_R(m)+o(1)$$

where

$$\lim_{R\to+\infty}\limsup_{m\to+\infty}B_R(m)=0 \tag{3.22}$$

Here again,

$$\int_{B_m}(\Delta_g v_m)^2\,dv_g=\int_{B_0(R)}(\Delta_{\tilde g_m}\tilde v_m)^2\,dv_{\tilde g_m}$$

and since $\tilde g_m\to\xi$ in $C^1(B_0(R))$, it follows from (3.13) that

$$\int_{B_m}(\Delta_g v_m)^2\,dv_g=\int_{B_0(R)}(\Delta v)^2\,dx+o(1)=\int_{\mathbb{R}^n}(\Delta v)^2\,dx+B_R(m)+o(1)$$

where $B_R(m)$ satisfies (3.22). Summarizing, we have that

$$\int_M (\Delta_g w_m)^2 \, dv_g = \int_M (\Delta_g v_m)^2 \, dv_g - \int_{\mathbb{R}^n} (\Delta v)^2 \, dx + B_R(m) + o(1) \tag{3.23}$$

where $B_R(m)$ satisfies (3.22). It follows from similar arguments that

$$\int_M |w_m|^{2^\sharp} dv_g = \int_M |v_m|^{2^\sharp} dv_g - \int_{\mathbb{R}^n} |v|^{2^\sharp} dx + B_R(m) + o(1) \tag{3.24}$$

where $B_R(m)$ satisfies (3.22). Then, combining (3.21), (3.23) and (3.24),

$$J_g(w_m) = J_g(v_m) - E(v) + B_R(m) + o(1)$$

and since $R > 0$ is arbitrary, we actually do have that

$$J_g(w_m) = J_g(v_m) - E(v) + o(1)$$

This proves (3.17), and step 3. □

According to what we said up to now, and to steps 1 to 3, Lemma 2.1 holds for some $\delta \in (0, i_g/2)$ small. Given $\delta_1 < \delta_2$ in $(0, i_g/2)$,

$$\|(\eta_{\delta_2,x_m} - \eta_{\delta_1,x_m})\, \hat{v}_m\|_{H_2^2} = o(1)$$

It follows that Lemma 2.1 holds for any $\delta \in (0, i_g/2)$. This ends the proof of Lemma 2.1.

4. Miscellaneous on Theorem 2.1

We briefly comment on Theorem 2.1 when the u_m's in this theorem are nonnegative. Let us consider equation (2.3) for nonnegative functions,

$$\Delta^2 u = u^{2^\sharp - 1} \ , \ u \geq 0 \tag{4.1}$$

As a first result, we claim that the following holds:

Lemma 4.1. *If $u \in D_2^2(\mathbb{R}^n)$ is a nontrivial nonnegative solution to* (4.1)*, then*

$$u(x) = \alpha_n \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{n-4}{2}} \tag{4.2}$$

for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, where $\alpha_n = \big(n(n-4)(n^2-4)\big)^{(n-4)/8}$.

The functions given by (4.2) are extremal functions for the sharp Euclidean Sobolev inequality

$$\left(\int_{\mathbb{R}^n} |u|^{2^\sharp} dx \right)^{2/2^\sharp} \leq K_0 \int_{\mathbb{R}^n} (\Delta u)^2 \, dx \tag{4.3}$$

in the sense that they realize the equality in (4.3). By the works of Lions [10], Lieb [8], and Edmunds, Fortunato and Janelli [6], the functions given by (4.2) are the only extremal functions for (4.3), and the only nontrivial and nonnegative spherically symmetric solutions of (4.1) which are decreasing in $|x|$. More recently, it has been proved by Lin [9] that smooth positive solutions to (4.1) are also given by (4.2). In order to prove our claim, it thus suffices to prove that if $u \in D_2^2(\mathbb{R}^n)$ is a nontrivial nonnegative solution to (4.1), then u is smooth and positive. The proof of the lemma then proceeds as follows:

Proof. Let (S^n, h) be the unit sphere, and P be some point in S^n. We let also $\Phi_P : S^n \backslash \{P\} \to \mathbb{R}^n$ be the stereographic projection of pole P. Then,

$$\left(\Phi_P^{-1}\right)^\star h = \varphi^{4/(n-4)}\xi$$

where ξ is the Euclidean metric and

$$\varphi(x) = 4^{\frac{n}{4}-1}\left(1+|x|^2\right)^{-\frac{n-4}{2}}$$

By conformal invariance properties, if $u \in \mathcal{D}(\mathbb{R}^n)$, then $\varphi^{2^\sharp-1}\left(P_h^n \hat{u}\right) \circ \Phi_P^{-1} = \Delta^2 u$ and

$$\int_{\mathbb{R}^n} \left(\Delta^2 u\right) u dx = \int_{S^n} \left(P_h^n \hat{u}\right) \hat{u} dv_h \tag{4.4}$$

where $\hat{u} = \left(u\varphi^{-1}\right) \circ \Phi_P$ and P_h^n is the Branson-Paneitz operator on the sphere. Namely,

$$P_h^n u = \Delta_h^2 u + c_n \Delta_h u + d_n u$$

where

$$c_n = \frac{n^2-2n-4}{2} \quad \text{and} \quad d_n = \frac{n(n-4)(n^2-4)}{16}$$

Let now (u_k) be a sequence of smooth functions with compact support in $\mathbb{R}^n$ which converges to u in $D_2^2(\mathbb{R}^n)$. Clearly, $\|u\|^2 = \int_{S^n} \left(P_h^n u\right) u dv_h$ is a norm on $H_2^2(S^n)$. It follows from (4.4) that $(\hat{u}_k)$ is a Cauchy sequence in $H_2^2(S^n)$, where $\hat{u}_k$ is given by $\hat{u}_k = \left(u_k\varphi^{-1}\right) \circ \Phi_P$. Hence, $(\hat{u}_k)$ converges to some $\hat{u}$ in $H_2^2(S^n)$. Moreover, $\hat{u} = \left(u\varphi^{-1}\right) \circ \Phi_P$ almost everywhere. Let $(\eta_s)_{s\geq 0}$ be a family of smooth functions on S^n such that $0 \leq \eta_s \leq 1$, $\eta_s = 0$ in $B_P(s)$, $\eta_s = 1$ in $S^n \backslash B_P(2s)$, and

$$|\nabla \eta_s| \leq \frac{C_1}{s} \quad \text{and} \quad |\Delta_h \eta_s| \leq \frac{C_2}{s^2}$$

where C_1, C_2 are positive constants which do not depend on s. For any $v \in C^\infty(S^n)$, $(\eta_s v)$ converges to v in $H_2^2(S^n)$ as $s \to 0$. On such an assertion, note that

$$\lim_{s\to 0} \frac{1}{s^2} \mathrm{Vol}_h\left(B_P(2s)\right) = 0 \quad \text{and} \quad \lim_{s\to 0} \frac{1}{s^4} \mathrm{Vol}_h\left(B_P(2s)\right) = 0$$

since $n \geq 5$. It follows that

$$\lim_{s\to 0} \int_{S^n} \left(P_h^n \hat{u}\right) \eta_s v dv_h = \int_{S^n} \left(P_h^n \hat{u}\right) v dv_h$$

where the integrals have to be understood in the distributional sense. It also follows that

$$\lim_{s\to 0} \int_{S^n} \hat{u}^{2^\sharp-1} \eta_s v dv_h = \int_{S^n} \hat{u}^{2^\sharp-1} v dv_h$$

Noting that

$$\int_{S^n} \left(P_h^n \hat{u}\right) \eta_s v dv_h = \int_{S^n} \hat{u}^{2^\sharp-1} \eta_s v dv_h$$

we get that $\hat{u} \in H_2^2(S^n)$ is a nontrivial nonnegative solution of the equation

$$P_h^n \hat{u} = \hat{u}^{2^\sharp-1} \tag{4.5}$$

There, we can apply Lemma 2.1 of Djadli, Hebey and Ledoux [5]. It follows from this lemma that $\hat{u} \in L^s(S^n)$ for all $s \geq 1$. Let L_h be the second order operator given by

$$L_h u = \Delta_h u + \frac{c_n}{2} u$$

Equation (4.5) can be rewritten as

$$L_h\left(L_h\hat{u}\right) = \hat{u}^{2^\sharp-1} + \beta_n\hat{u} \tag{4.6}$$

where $\beta_n = \frac{c_n^2}{4} - d_n$ is positive. By standard regularity results, since $\hat{u} \in L^s(S^n)$ for all $s \geq 1$, we get that $\hat{u} \in H_4^s(S^n)$ for all $s \geq 1$. In particular, $\hat{u}$ is C^3, and we obtain by coming back to (4.6) that $\hat{u}$ is actually at least C^4. The right hand side in (4.6) being nonnegative, it follows from elementary considerations and the maximum principle that $\hat{u}$ is positive. Then $\hat{u}$ is smooth, and coming back to our original solution u of (4.1), we get that u is smooth and positive. By the work of Lin [9], this proves the lemma. □

As another result on Theorem 2.1, we claim that if the u_m's in this theorem are nonnegative, then u^0 and the u^i's of Theorem 2.1 are also nonnegative. According to Lemma 4.1, the u^i's are then given by (4.2). That u^0 is nonnegative is straightforward. On the other hand, the u^i's, $i \geq 1$, are obtained by rescaling $u_m - u^0 - \mathcal{S}$, where $\mathcal{S}$ is a sum of bubbles, and it is not anymore straightforward that $u_m \geq 0$ implies that $u^i \geq 0$. The following proposition holds:

Proposition 4.1. *Let (u_m) be a Palais-Smale sequence for I_g. We suppose that $u_m \geq 0$ for all m. Then the u^i's of Theorem 2.1 are also nonnegative. In particular, u^i is given by* (4.2) *and, up to the assimilation through the exponential map at x_m^i,*

$$u_m^i(y) = \alpha_n\left(\frac{\lambda_m^i}{(\lambda_m^i)^2 + |y - \frac{x^i}{R_m^i}|^2}\right)^{\frac{n-4}{2}} \tag{4.7}$$

where $x^i \in \mathbb{R}^n$, $\lambda_m^i = \lambda^i/R_m^i$ for some $\lambda^i > 0$, and α_n is as in Lemma 2.1. Moreover,

$$E(u^i) = \beta^\sharp = \frac{2}{n}K_0^{-n/4}$$

so that the Palais-Smale property holds for I_g at all levels which are not of the form $\beta_0 + k\beta^\sharp$ where $k \geq 1$ and β_0 is the energy of some nonnegative solution u^0 of (2.2).

Proof. Let $v_m = u_m - u^0$ and $\mu_m^i = 1/R_m^i$. First we prove the following: for any N integer in $[1,k]$, and for any s integer in $[0, N-1]$, there exists an integer p, there exist sequences (y_m^j) and (λ_m^j), $j = 1,\dots,p$, $y_m^j \in M$ and $\lambda_m^j > 0$, such that for any j, $d_g(x_m^N, y_m^j)/\mu_m^N$ is bounded and $\lambda_m^j/\mu_m^N \to 0$, and such that for any $R, R' > 0$,

$$\int_{B_{x_m^N}(R\mu_m^N)\setminus\bigcup_{j=1}^p B_{y_m^j}(R'\lambda_m^j)}\left|v_m - \sum_{i=1}^s u_m^i - u_m^N\right|^{2^\sharp} dv_g = o(1) + \varepsilon(R') \tag{4.8}$$

where $\varepsilon(R') \to 0$ as $R' \to 0$, and the (u_m^i)'s and (x_m^i)'s are the ordered sequences in i that come from the proof of Theorem 2.1. We proceed here by inverse induction on s. If $s = N-1$, then, by (3.13),

$$\int_{B_{x_m^N}(R\mu_m^N)}\left|v_m - \sum_{i=1}^{N-1} u_m^i - u_m^N\right|^{2^\sharp} dv_g = o(1)$$

so that (4.8) holds with $p = 0$. Now, we suppose that (4.8) holds for some s, $s \leq N-1$. If the $d_g(x_m^s, x_m^N)$'s do not converge to 0, then, up to a subsequence,

$B_{x_m^N}(R\mu_m^N)\bigcap B_{x_m^s}(\tilde{R}\mu_m^s)=\emptyset$ for $\tilde{R}>0$. As a consequence,

$$\int_{B_{x_m^N}(R\mu_m^N)\setminus\bigcup_{j=1}^p B_{y_m^j}(R'\lambda_m^j)}|u_m^s|^{2^\sharp}\,dv_g\le\int_{M\setminus B_{x_m^s}(\tilde{R}\mu_m^s)}|u_m^s|^{2^\sharp}\,dv_g$$

and it follows, see the proof of Lemma 2.1 in section 3, that

$$\int_{B_{x_m^N}(R\mu_m^N)\setminus\bigcup_{j=1}^p B_{y_m^j}(R'\lambda_m^j)}|u_m^s|^{2^\sharp}\,dv_g\le\int_{\mathbb{R}^n\setminus B_0(\tilde{R})}|u^s|^{2^\sharp}\,dx$$

Since $\tilde{R}>0$ is arbitrary, and $u^s\in L^{2^\sharp}(\mathbb{R}^n)$, we get that

$$\int_{B_{x_m^N}(R\mu_m^N)\setminus\bigcup_{j=1}^p B_{y_m^j}(R'\lambda_m^j)}|u_m^s|^{2^\sharp}\,dv_g=o(1)$$

and then that

$$\int_{B_{x_m^N}(R\mu_m^N)\setminus\bigcup_{j=1}^p B_{y_m^j}(R'\lambda_m^j)}\left|v_m-\sum_{i=1}^{s-1}u_m^i-u_m^N\right|^{2^\sharp}dv_g=o(1)+\varepsilon(R')$$

In particular, (4.8) holds for $s-1$. Now, we deal with the case $d_g(x_m^s,x_m^N)\to 0$. We let $r_0>0$ and $C\ge 1$ be such that for all $x\in M$, and all $y,z\in\mathbb{R}^n$, if $|y|\le r_0$ and $|z|\le r_0$, then

$$\frac{1}{C}|z-y|\le d_g\left(exp_x(y),exp_x(z)\right)\le C|z-y|$$

If $\tilde{x}_m^s$ and $\tilde{y}_m^j$ are such that $x_m^s=exp_{x_m^N}(\mu_m^N\tilde{x}_m^s)$ and $y_m^j=exp_{x_m^N}(\mu_m^N\tilde{y}_m^j)$, then

$$B_{\tilde{y}_m^j}\left(\frac{R'}{C}\frac{\lambda_m^j}{\mu_m^N}\right)\subset\frac{1}{\mu_m^N}exp_{x_m^N}^{-1}\left(B_{y_m^j}(R'\lambda_m^j)\right)\subset B_{\tilde{y}_m^j}\left(R'C\frac{\lambda_m^j}{\mu_m^N}\right)\tag{4.9}$$

and

$$B_{\tilde{x}_m^s}\left(\frac{R'}{C}\frac{\mu_m^s}{\mu_m^N}\right)\subset\frac{1}{\mu_m^N}exp_{x_m^N}^{-1}\left(B_{x_m^s}(R'\mu_m^s)\right)\subset B_{\tilde{x}_m^s}\left(R'C\frac{\mu_m^s}{\mu_m^N}\right)\tag{4.10}$$

Given $\tilde{R}>0$, we have by (3.13) that

$$\int_{B_{x_m^s}(\tilde{R}\mu_m^s)}\left|v_m-\sum_{i=1}^{s}u_m^i\right|^{2^\sharp}dv_g=o(1)$$

Hence, by (4.8),

$$\int_{\left(B_{x_m^N}(R\mu_m^N)\setminus\bigcup_{j=1}^p B_{y_m^j}(R'\lambda_m^j)\right)\bigcap B_{x_m^s}(\tilde{R}\mu_m^s)}\left|u_m^N\right|^{2^\sharp}dv_g=o(1)+\varepsilon(R')$$

and it follows from (4.9) and (4.10) that

$$\int_{\left(B_0(R)\setminus\bigcup_{j=1}^p B_{\tilde{y}_m^j}(R'C\frac{\lambda_m^j}{\mu_m^N})\right)\bigcap B_{\tilde{x}_m^s}(\frac{\tilde{R}}{C}\frac{\mu_m^s}{\mu_m^N})}\left|u^N\right|^{2^\sharp}dx=o(1)+\varepsilon(R')\tag{4.11}$$

Now, we distinguish two cases. In the first case we assume that as $m\to+\infty$, $d_g(x_m^s,x_m^N)/\mu_m^N\to+\infty$. Then we also do have that $d_g(x_m^s,x_m^N)/\mu_m^s\to+\infty$, since if not, we get by (4.11) with $\tilde{R}$ large enough that $\mu_m^s/\mu_m^N\to 0$, while

$$\frac{d_g(x_m^s,x_m^N)}{\mu_m^s}=\frac{d_g(x_m^s,x_m^N)}{\mu_m^N}\times\frac{\mu_m^N}{\mu_m^s}$$

Then it follows that $B_{x_m^N}(R\mu_m^N)\bigcap B_{x_m^s}(\tilde{R}\mu_m^s)=\emptyset$ for $\tilde{R}>0$, and we may proceed as in the case where the $d_g(x_m^s,x_m^N)$'s do not converge to 0 to get that (4.8) holds for $s-1$. In the second case we assume that as $m\to+\infty$, the $d_g(x_m^s,x_m^N)/\mu_m^N$'s converge. By (4.11), we must have that $\mu_m^s/\mu_m^N\to 0$. We set $y_m^{p+1}=x_m^s$ and $\lambda_m^{p+1}=\mu_m^s$. Clearly,

$$\int_{B_{x_m^N}(R\mu_m^N)\backslash\bigcup_{j=1}^{p+1}B_{y_m^j}(R'\lambda_m^j)}\left|v_m-\sum_{i=1}^{s}u_m^i-u_m^N\right|^{2^\sharp}dv_g=o(1)+\varepsilon(R')$$

while

$$\int_{B_{x_m^N}(R\mu_m^N)\backslash\bigcup_{j=1}^{p+1}B_{y_m^j}(R'\lambda_m^j)}|u_m^s|^{2^\sharp}dv_g\le\int_{M\backslash B_{x_m^s}(R'\mu_m^s)}|u_m^s|^{2^\sharp}dv_g\le\varepsilon(R')$$

It follows that

$$\int_{B_{x_m^N}(R\mu_m^N)\backslash\bigcup_{j=1}^{p+1}B_{y_m^j}(R'\lambda_m^j)}\left|v_m-\sum_{i=1}^{s-1}u_m^i-u_m^N\right|^{2^\sharp}dv_g=o(1)+\varepsilon(R')$$

and (4.8) holds for $s-1$. Therefore, we proved that (4.8) always holds. Let us now prove the original claim that if the u_m's in Theorem 2.1 are nonnegative, then u^0 and the u^i's of Theorem 2.1 are also nonnegative. By the construction of u^0, it is clear that u^0 is nonnegative. We let $\tilde{v}_m^N$ be given by

$$\tilde{v}_m^N(x)=(\mu_m^N)^{\frac{n-4}{2}}v_m\left(exp_{x_m^N}(\mu_m^Nx)\right)$$

We apply (4.8) with $s=0$. Then,

$$\int_{B_{x_m^N}(R\mu_m^N)\backslash\bigcup_{j=1}^{p}B_{y_m^j}(R'\lambda_m^j)}\left|v_m-u_m^N\right|^{2^\sharp}dv_g=o(1)+\varepsilon(R')$$

and it follows that

$$\int_{B_0(R)\backslash\bigcup_{j=1}^{p}B_{\tilde{y}_m^j}(R'C\frac{\lambda_m^j}{\mu_m^N})}\left|\tilde{v}_m^N-u^N\right|^{2^\sharp}dx=o(1)+\varepsilon(R')\qquad(4.12)$$

where the $\tilde{y}_m^j$'s are as above. In particular, the $\tilde{y}_m^j$'s are bounded. Up to a subsequence we may assume that $\tilde{y}_m^j\to\tilde{y}^j$ as $m\to+\infty$. Then we get from (4.12) that

$$\tilde{v}_m^N\to u^N\quad\text{in }L^{2^\sharp}_{loc}\left(B_0(R)\backslash\{\tilde{y}^j,j=1,\dots,p\}\right)$$

and thus we may assume that $\tilde{v}_m^N\to u^N$ almost everywhere in $\mathbb{R}^n$. Independently, let

$$\tilde{u}_m^{0,N}(x)=(\mu_m^N)^{\frac{n-4}{2}}u^0\left(exp_{x_m^N}(\mu_m^Nx)\right)$$

Then,

$$\int_{B_{x_m^N}(R\mu_m^N)}\left|u^0\right|^{2^\sharp}dv_g=\int_{B_0(R)}\left|\tilde{u}_m^{0,N}\right|^{2^\sharp}dv_{\tilde{g}_m}$$

where $\tilde{g}_m=\left(exp_{x_m^N}^\star g\right)(\mu_m^Nx)$, and we get that $\tilde{u}_m^{0,N}\to 0$ in $L^{2^\sharp}(B_0(R))$. Thus, $\tilde{u}_m^{0,N}\to 0$ almost everywhere in $\mathbb{R}^n$. It follows that the $\tilde{u}_m^N$'s given by

$$\tilde{u}_m^N(x)=(\mu_m^N)^{\frac{n-4}{2}}u_m\left(exp_{x_m^N}(\mu_m^Nx)\right)$$

converge almsot everywhere to u^N. In particular, u^N is nonnegative and, thanks to Lemma 3.1, the proposition is proved. □

As a remark, note that it follows from the above proof that for any $i \neq j$,

$$\frac{R_m^j}{R_m^i} + \frac{R_m^i}{R_m^j} + R_m^i R_m^j d_g(x_m^i, x_m^j)^2 \to +\infty$$

as $m \to +\infty$. There, we recover well-known relations that hold when dealing with the Laplace operator instead of the Paneitz operator. At last, note that Theorem 2.1 and the above remarks do hold if instead of a Paneitz operator P_g with constant coefficients, one deals with the Paneitz-Branson operator P_g^n of the introduction, or more generally with operators of the form

$$\mathcal{P}_g u = \Delta_g^2 u - div_g\left(A \nabla u\right) + au$$

where A is a smooth section of the space of smooth symmetric $(0,2)$ tensors on M, and a is a smooth function.

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