EXTREMALS FOR THE HARDY-SOBOLEV INEQUALITIES ON CONES

INFORMAL NOTE

NASSIF GHOUSSOUB AND FRÉDÉRIC ROBERT

Let $\mathcal{C}$ be an open connected cone of $\mathbb{R}^n$, $n \geq 3$, centered at 0, that is

$$\left\{ \begin{array}{c} \mathcal{C} \text{ is a domain (that is open and connected)} \\ \forall x \in \mathcal{C},\, \forall r > 0,\; rx \in \mathcal{C}. \end{array} \right. \tag{1}$$

We fix $\gamma \in \mathbb{R}$ such that $-\Delta - \gamma |x|^{-2}$ is coercive, that is there exists $c > 0$ such that

$$\int_{\mathcal{C}} \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx \geq c \int_{\mathcal{C}} |\nabla u|^2 \, dx \tag{2}$$

for all $u \in D^{1,2}(\mathcal{C})$, where $D^{1,2}(\mathcal{C})$ is the completion of $C_c^\infty(\mathcal{C})$ for the norm $\|u\| := \|\nabla u\|_2$. We fix $s \in [0,2)$ and we define $2^\star(s) := \frac{2(n-s)}{n-2}$. It follows from the Hardy-Sobolev inequality (see for instance Ghoussoub-Robert [5] for general considerations on this inequality) that there exists $\mu_{\gamma,s}(\mathcal{C}) > 0$ such that

$$\mu_{\gamma,s}(\mathcal{C}) := \inf_{u \in D^{1,2}(\mathcal{C}) \setminus \{0\}} \frac{\int_{\mathcal{C}} \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx}{\left(\int_{\mathcal{C}} \frac{|u|^{2^\star(s)}}{|x|^s} \, dx \right)^{\frac{2}{2^\star(s)}}}. \tag{3}$$

We say that $u_0 \in D^{1,2}(\mathcal{C}) \setminus \{0\}$ is an extremal for $\mu_{\gamma,s}(\mathcal{C})$ if it achieves the infimum in (3). The question of the extremals on general cones has been tackled by Egnell [4] in the case $\{\gamma = 0 \text{ and } s > 0\}$. Theorem 0.1 below has been noted in several contexts by Bartsche-Peng-Zhang [2] and Lin-Wang [3]. In this note, we sketch an independent proof.

Theorem 0.1. *We let $\mathcal{C}$ be a cone of $\mathbb{R}^n$, $n \geq 3$, as in (1), $s \in [0,2)$ and $\gamma \in \mathbb{R}$ such that (2) holds. Then,*

1. *If $\{s > 0\}$ or $\{s = 0,\ \gamma > 0 \text{ and } n \geq 4\}$, then there are extremals for $\mu_{\gamma,s}(\mathcal{C})$.*
2. *If $\{s = 0 \text{ and } \gamma < 0\}$, there are no extremals for $\mu_{\gamma,0}(\mathcal{C})$.*
3. *If $\{s = 0 \text{ and } \gamma = 0\}$, there are extremals for $\mu_{0,0}(\mathcal{C})$ if and only if there exists $z \in \mathbb{R}^n$ such that $(1+|x-z|^2)^{1-n/2} \in D^{1,2}(\mathcal{C})$ (in particular $\overline{\mathcal{C}} = \mathbb{R}^n$).*

Moreover, if there are no extremals for $\mu_{\gamma,0}(\mathcal{C})$, then

$$\mu_{\gamma,0}(\mathcal{C}) = \frac{1}{K(n,2)^2} := \inf_{u \in D^{1,2}(\mathbb{R}^n) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2 \, dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\star} \, dx \right)^{\frac{2}{2^\star}}} \text{ where } 2^\star := 2^\star(0) = \frac{2n}{n-2}. \tag{4}$$

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As a consequence, the only unclear situation is when $\{s = 0,\ n = 3 \text{ and } \gamma > 0\}$.

Here are two corollaries. The first one covers most of the cones that are distinct from $\mathbb{R}^n$. The second one is essentially the case when the cone is $\mathbb{R}^n$.

Corollary 0.2. *We let $\mathcal{C}$ be a cone of $\mathbb{R}^n$, $n \geq 3$, as in (1) such that $\overline{\mathcal{C}} \neq \mathbb{R}^n$. We let $s \in [0,2)$ and $\gamma \in \mathbb{R}$ such that (2) holds. Then,*

(1) *If $\{s > 0\}$ or $\{s = 0,\ \gamma > 0 \text{ and } n \geq 4\}$, then there are extremals for $\mu_{\gamma,s}(\mathcal{C})$.*
(2) *If $\{s = 0 \text{ and } \gamma \leq 0\}$, there are no extremals for $\mu_{\gamma,0}(\mathcal{C})$.*

Here again, the case $\{s = 0,\ n = 3 \text{ and } \gamma > 0\}$ is unsettled.

Corollary 0.3. *We let $\mathcal{C}$ be a cone of $\mathbb{R}^n$, $n \geq 3$, as in (1). We assume that there exists $z \in \mathbb{R}^n$ such that $(1 + |x - z|^2)^{1-n/2} \in D^{1,2}(\mathcal{C})$ (in particular $\overline{\mathcal{C}} = \mathbb{R}^n$). We fix $s \in [0,2)$ and $\gamma \in \mathbb{R}$ such that (2) holds. Then,*

(1) *If $\{s > 0\}$ or $\{s = 0 \text{ and } \gamma \geq 0\}$, then there are extremals for $\mu_{\gamma,s}(\mathcal{C})$.*
(2) *If $\{s = 0 \text{ and } \gamma < 0\}$, there are no extremals for $\mu_{\gamma,0}(\mathcal{C})$.*

Note here that there is no specificity for dimension $n = 3$.

Proof of Theorem 0.1: This goes as the classical proof of the existence of extremals for the Sobolev inequalities using Lions's concentration-compactness Lemmae ([6,7], see also Struwe [8] for a classical exposition in book form).

We let $(\tilde{u}_k)_k \in D^{1,2}(\mathbb{R}^n_+)$ be a minimizing sequence for $\mu_{\gamma,s}(\mathcal{C})$ such that

$$\int_{\mathcal{C}} \frac{|\tilde{u}_k|^{2^\star(s)}}{|x|^s}\, dx = 1 \text{ and } \lim_{k \to +\infty} \int_{\mathcal{C}} \left(|\nabla \tilde{u}_k|^2 - \frac{\gamma}{|x|^2} \tilde{u}_k^2 \right) dx = \mu_{\gamma,s}(\mathcal{C}).$$

We use a concentration compactness argument in the spirit of Lions [6,7]. For any k, there exists $r_k > 0$ such that $\int_{B_{r_k}(0) \cap \mathcal{C}} \frac{|\tilde{u}_k|^{2^\star(s)}}{|x|^s}\, dx = 1/2$. We define $u_k(x) := r_k^{\frac{n-2}{2}} u_k(r_k x)$ for all $x \in \mathcal{C}$. Since $\mathcal{C}$ is a cone, we have that $u_k \in D^{1,2}(\mathcal{C})$. We then have that

$$\lim_{k \to +\infty} \int_{\mathcal{C}} \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx = \mu_{\gamma,s}(\mathcal{C}), \tag{5}$$

and

$$\int_{\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = 1 \ ,\ \int_{B_1(0) \cap \mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = \frac{1}{2}. \tag{6}$$

Step 1: We claim that, up to a subsequence,

$$\lim_{R \to +\infty} \lim_{k \to +\infty} \int_{B_R(0) \cap \mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = 1 \tag{7}$$

Proof of the claim: For $k \in \mathbb{N}$ and $r \geq 0$, we define

$$Q_k(r) := \int_{B_r(0) \cap \mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx.$$

Since $0 \leq Q_k \leq 1$ and $r \mapsto Q_k(r)$ is nondecreasing for all $k \in \mathbb{N}$, then, up to a subsequence, there exists $Q : [0, +\infty) \to \mathbb{R}$ nondecreasing such that $(Q_k(r)) \to Q(r)$ as $k \to +\infty$ for a.e. $r > 0$. We define

$$\alpha := \lim_{r \to +\infty} Q(r).$$

It follows from (5) and (6) that $\frac{1}{2} \leq \alpha \leq 1$. Up to taking another subsequence, there exists $(R_k)_k, (R'_k)_k \in (0, +\infty)$ such that

$$\left\{ \begin{array}{l} 2R_k \leq R'_k \leq 3R_k \text{ for all } k \in \mathbb{N}, \\ \lim_{k\to+\infty} R_k = \lim_{k\to+\infty} R'_k = +\infty, \\ \lim_{k\to+\infty} Q_k(R_k) = \lim_{k\to+\infty} Q_k(R'_k) = \alpha. \end{array} \right\}$$

In particular,

(8)

$$\lim_{k\to+\infty} \int_{B_{R_k}(0)\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = \alpha \text{ and } \lim_{k\to+\infty} \int_{(\mathbb{R}^n\setminus B_{R'_k}(0))\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = 1-\alpha$$

We claim that

$$\lim_{k\to+\infty} R_k^{-2} \int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} u_k^2\, dx = 0. \tag{9}$$

We prove the claim. A preliminary remark is that for all $x \in B_{R'_k}(0) \setminus B_{R_k}(0)$, we have that $R_k \leq |x| \leq 3R_k$. Therefore, Hölder's inequality yields

$$\begin{aligned} &\int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} u_k^2\, dx \\ &\leq \left(\int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} dx\right)^{1-\frac{2}{2^\star(s)}} \left(\int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} |u_k|^{2^\star(s)}\, dx\right)^{\frac{2}{2^\star(s)}} \\ &\leq CR_k^2 \left(\int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^\star(s)}} \end{aligned}$$

for all $k \in \mathbb{N}$. The conclusion (9) then follows from (8). This proves the claim.

We let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \varphi \leq 1$, $\varphi(x) = 1$ for $x \in B_1(0)$ and $\varphi(x) = 0$ for $x \in \mathbb{R}^n \setminus B_2(0)$. For $k \in \mathbb{N}$, we define

$$\varphi_k(x) := \varphi\left(\frac{|x|}{R'_k - R_k} + \frac{R'_k - 2R_k}{R'_k - R_k}\right) \text{ for all } x \in \mathbb{R}^n.$$

As one checks, $\varphi_k u_k, (1-\varphi_k)u_k \in D^{1,2}(\mathcal{C})$ for all $k \in \mathbb{N}$. Therefore, it follows from (8) that

$$\begin{aligned} \int_{\mathcal{C}} \frac{|\varphi_k u_k|^{2^\star(s)}}{|x|^s}\, dx &\geq \int_{B_{R_k}(0)\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = \alpha + o(1), \\ \int_{\mathcal{C}} \frac{|(1-\varphi_k)u_k|^{2^\star(s)}}{|x|^s}\, dx &\geq \int_{(\mathbb{R}^n\setminus B_{R'_k}(0))\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = 1-\alpha+o(1) \end{aligned}$$

as $k \to +\infty$. The Hardy-Sobolev inequality (3) and (9) yield

$$\begin{aligned} &\mu_{\gamma,s}(\mathcal{C}) \left(\int_{\mathcal{C}} \frac{|\varphi_k u_k|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^\star(s)}} \leq \int_{\mathcal{C}} \left(|\nabla(\varphi_k u_k)|^2 - \frac{\gamma}{|x|^2}\varphi_k^2 u_k^2\right) dx \\ &\leq \int_{\mathcal{C}} \varphi_k^2 \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2\right) dx + O\left(R_k^{-2} \int_{(B_{R'_k}(0)\setminus B_{R_k}(0))\cap\mathcal{C}} u_k^2\, dx\right) \\ &\leq \int_{\mathcal{C}} \varphi_k^2 \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2\right) dx + o(1) \end{aligned}$$

as $k \to +\infty$. Similarly,

$$\mu_{\gamma,s}(\mathcal{C})\left(\int_{\mathcal{C}}\frac{|(1-\varphi_k)u_k|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}\leq\int_{\mathcal{C}}(1-\varphi_k)^2\left(|\nabla u_k|^2-\frac{\gamma}{|x|^2}u_k^2\right)\,dx+o(1)$$

as $k \to +\infty$. Therefore, we have that

$$\begin{aligned}
&\mu_{\gamma,s}(\mathcal{C})\left(\alpha^{\frac{2}{2^\star(s)}}+(1-\alpha)^{\frac{2}{2^\star(s)}}+o(1)\right)\\
&\leq\mu_{\gamma,s}(\mathcal{C})\left(\left(\int_{\mathcal{C}}\frac{|\varphi_k u_k|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}+\left(\int_{\mathcal{C}}\frac{|(1-\varphi_k)u_k|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}\right)\\
&\leq\int_{\mathcal{C}}(\varphi_k^2+(1-\varphi_k)^2)\left(|\nabla u_k|^2-\frac{\gamma}{|x|^2}u_k^2\right)\,dx+o(1)\\
&\leq\int_{\mathcal{C}}(1-2\varphi_k(1-\varphi_k))\left(|\nabla u_k|^2-\frac{\gamma}{|x|^2}u_k^2\right)\,dx+o(1)\\
&\leq\mu_{\gamma,s}(\mathcal{C})+2\int_{\mathcal{C}}\varphi_k(1-\varphi_k)\frac{\gamma}{|x|^2}u_k^2\,dx+o(1)\\
&\leq\mu_{\gamma,s}(\mathcal{C})+O\left(R_k^{-2}\int_{(B_{R_k'}(0)\setminus B_{R_k}(0))\cap\mathcal{C}}u_k^2\,dx\right)+o(1)\leq\mu_{\gamma,s}(\mathcal{C})+o(1)
\end{aligned}$$

as $k \to +\infty$. Therefore, $\alpha^{\frac{2}{2^\star(s)}}+(1-\alpha)^{\frac{2}{2^\star(s)}}\leq 1$, which implies $\alpha = 1$ since $0 < \alpha \leq 1$. This proves the claim.

Step 2: We claim that there exists $u_\infty \in D^{1,2}(\mathcal{C})$ such that $u_k \rightharpoonup u_\infty$ weakly in $D^{1,2}(\mathcal{C})$ as $k \to +\infty$, there exists $x_0 \neq 0$ such that

$$\text{either } \lim_{k\to+\infty}\frac{|u_k|^{2^\star(s)}}{|x|^s}\mathbf{1}_{\mathcal{C}}\,dx=\frac{|u_\infty|^{2^\star(s)}}{|x|^s}\mathbf{1}_{\mathcal{C}}\,dx\quad\text{and}\quad\int_{\mathcal{C}}\frac{|u_\infty|^{2^\star(s)}}{|x|^s}\,dx=1\tag{10}$$

$$\text{or } \lim_{k\to+\infty}\frac{|u_k|^{2^\star(s)}}{|x|^s}\mathbf{1}_{\mathcal{C}}\,dx=\delta_{x_0}\quad\text{and } u_\infty\equiv 0.\tag{11}$$

Proof of the claim: Arguing as in Step 1, we get that for all $x \in \mathbb{R}^n$, we have that

$$\lim_{r\to 0}\lim_{k\to+\infty}\int_{B_r(0)\cap\mathcal{C}}\frac{|u_k|^{2^\star(s)}}{|x|^s}\,dx=\alpha_x\in\{0,1\}.$$

It then follows from the second identity of (6) that $\alpha_0 \leq 1/2$, and therefore $\alpha_0 = 0$. Moreover, it follows from the first identity of (6) that there exist as most one point $x_0 \in \mathbb{R}^n$ such that $\alpha_{x_0} = 1$. In particular $x_0 \neq 0$ since $\alpha_0 = 0$. Therefore, it follows from Lions's second concentration compactness lemma [6,7] (see also Struwe [8] for an exposition in book form) that, up to a subsequence, there exists $u_\infty \in D^{1,2}(\mathcal{C})$, $x_0 \in \mathbb{R}^n \setminus \{0\}$ and $C \in \{0,1\}$ such that $u_k \rightharpoonup u_\infty$ weakly in $D^{1,2}(\mathcal{C})$ and

$$\lim_{k\to+\infty}\frac{|u_k|^{2^\star(s)}}{|x|^s}\mathbf{1}_{\mathcal{C}}\,dx=\frac{|u_\infty|^{2^\star(s)}}{|x|^s}\mathbf{1}_{\mathcal{C}}\,dx+C\delta_{x_0}\text{ in the sense of measures}$$

In particular, due to (6) and the compactness (7), we have that

$$1=\lim_{k\to+\infty}\int_{\mathcal{C}}\frac{|u_k|^{2^\star(s)}}{|x|^s}\,dx=\int_{\mathcal{C}}\frac{|u_\infty|^{2^\star(s)}}{|x|^s}\,dx+C.$$

Since $C \in \{0,1\}$, the claim follows.

Step 3. We assume that $u_\infty \not\equiv 0$. We claim that $\lim_{k\to+\infty} u_k = u_\infty$ strongly in $D^{1,2}(\mathcal{C})$ and that u_∞ is an extremal for $\mu_{\gamma,s}(\mathcal{C})$.

Proof of the claim: It follows from (10)-(11) that $\int_{\mathcal{C}} \frac{|u_\infty|^{2^\star(s)}}{|x|^s}\, dx = 1$. It then follows from the Hardy-Sobolev inequality (3) that

$$\mu_{\gamma,s}(\mathcal{C}) \leq \int_{\mathcal{C}} \left(|\nabla u_\infty|^2 - \frac{\gamma}{|x|^2} u_\infty^2 \right) dx.$$

Moreover, since $u_k \rightharpoonup u_\infty$ weakly as $k \to +\infty$, we have that

$$\int_{\mathcal{C}} \left(|\nabla u_\infty|^2 - \frac{\gamma}{|x|^2} u_\infty^2 \right) dx \leq \liminf_{k\to+\infty} \int_{\mathcal{C}} \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx = \mu_{\gamma,s}(\mathcal{C}).$$

Therefore, equality holds in this latest inequality, u_∞ is an extremal for $\mu_{\gamma,s}(\mathcal{C})$ and reflexivity yields convergence of (u_k) to u_∞ in $D^{1,2}(\mathcal{C})$. This proves the claim.

Step 4: We assume that $u_\infty \equiv 0$. Then

$$s = 0\ ,\ \lim_{k\to+\infty} \int_{\mathcal{C}} \frac{u_k^2}{|x|^2}\, dx = 0 \text{ and } |\nabla u_k|^2\, dx \rightharpoonup \mu_{\gamma,s}(\mathcal{C}) \delta_{x_0}$$

as $k \to +\infty$ in the sense of measures.

Proof of the claim: Indeed, since $u_k \rightharpoonup u_\infty \equiv 0$ weakly in $D^{1,2}(\mathcal{C})$ as $k \to +\infty$, then for any $1 \leq q < 2^\star := \frac{2n}{n-2}$, $u_k \to 0$ strongly in $L^q_{loc}(\mathcal{C})$ when $k \to +\infty$. Assume by contradiction that $s > 0$: then $2^\star(s) < 2^\star$ and therefore, since $x_0 \neq 0$, we have that

$$\lim_{k\to+\infty} \int_{B_\delta(x_0)\cap\mathcal{C}} \frac{|u_k|^{2^\star(s)}}{|x|^s}\, dx = 0$$

for $\delta > 0$ small enough, contradicting (11). Therefore $s = 0$ and the first part of the claim is proved.

We prove the second part of the claim. We let $f \in C^\infty(\mathbb{R}^n)$ be such that $f(x) = 0$ for $x \in B_\delta(x_0)$, $f(x) = 1$ for $x \in \mathbb{R}^n \setminus B_{2\delta}(x_0)$ and $0 \leq f \leq 1$ ($0 < \delta < |x_0|/4$). We define $\varphi := 1 - f^2$ and $\psi := f\sqrt{2 - f^2}$. Clearly $\varphi, \psi \in C^\infty(\mathbb{R}^n)$ and $\varphi^2 + \psi^2 = 1$. Inequality (3) yields

$$\mu_{\gamma,s}(\mathcal{C}) \left(\int_{\mathcal{C}} |\varphi u_k|^{2^\star}\, dx \right)^{\frac{2}{2^\star}} \leq \int_{\mathcal{C}} \left(|\nabla(\varphi u_k)|^2 - \frac{\gamma}{|x|^2} (\varphi u_k)^2 \right) dx.$$

Integrating by parts, using (11), using that $u_k \to 0$ strongly in $L^2_{loc}(\mathbb{R}^n)$ as $k \to +\infty$, and that $\varphi^2 = 1 - \psi^2$, we get that

$$\mu_{\gamma,s}(\mathcal{C}) \left(|\varphi(x_0)|^{2^\star} + o(1) \right)^{\frac{2}{2^\star}} \leq \int_{\mathcal{C}} \varphi^2 \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx + O\left(\int_{\text{Supp } \varphi\Delta\varphi} u_k^2\, dx \right)$$

$$\mu_{\gamma,s}(\mathcal{C}) + o(1) \leq \int_{\mathcal{C}} \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx - \int_{\mathcal{C}} \psi^2 \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx + o(1)$$

as $k \to +\infty$. Using again (5), we then get that

$$\int_{\mathcal{C}} \psi^2 \left(|\nabla u_k|^2 - \frac{\gamma}{|x|^2} u_k^2 \right) dx \leq o(1)$$

as $k \to +\infty$. Integrating again by parts and using the strong local convergence to 0 in L^2_{loc}, we get that

$$\int_{\mathcal{C}} \left(|\nabla(\psi u_k)|^2 - \frac{\gamma}{|x|^2} (\psi u_k)^2 \right) dx \leq o(1)$$

as $k \to +\infty$. The coercivity (2) then yields $\lim_{k\to+\infty} \|\nabla(\psi u_k)\|_2 = 0$. Therefore, the Hardy inequality yields convergence of $|x|^{-1}(\psi u_k)_k$ to 0 in $L^2(\mathcal{C})$. Therefore,

$$\lim_{k\to+\infty} \int_{(B_{2\delta}(x_0))^c \cap \mathcal{C}} \frac{u_k^2}{|x|^2}\, dx = 0.$$

Taking $\delta > 0$ small enough and combining this result with the strong convergence of $(u_k)_k$ in L^2_{loc} around $x_0 \neq 0$ yields

$$\lim_{k\to+\infty} \int_{\mathcal{C}} \frac{u_k^2}{|x|^2}\, dx = 0.$$

Combining this equality, $\lim_{k\to+\infty} \|\nabla(\psi u_k)\|_2 = 0$ and (5) yields the third part of the claim. This proves the claim.

Step 5: We assume that $u_\infty \equiv 0$. Then $s = 0$ and

$$\mu_{\gamma,s}(\mathcal{C}) = \mu_{0,0}(\mathbb{R}^n) = \frac{1}{K(n,2)^2}.$$

Proof of the claim: Since $u_k \in D^{1,2}(\mathcal{C}) \subset D^{1,2}(\mathbb{R}^n)$, we have that

$$\mu_{0,0}(\mathbb{R}^n) \left(\int_{\mathbb{R}^n} |u_k|^{2^\star}\, dx \right)^{\frac{2}{2^\star}} \leq \int_{\mathbb{R}^n} |\nabla u_k|^2\, dx.$$

It then follows from Step 4, (5) and (6) that $\mu_{0,0}(\mathbb{R}^n) \leq \mu_{\gamma,s}(\mathcal{C})$. Conversely, the computations of Step 6 below yield $\mu_{\gamma,s}(\mathcal{C}) \leq \mu_{0,0}(\mathbb{R}^n) = K(n,2)^{-1}$. These two inequalities prove the claim.

Step 6: We assume that $\{s = 0 \text{ and } \gamma \leq 0\}$. Then

$$\mu_{\gamma,0}(\mathcal{C}) = \frac{1}{K(n,2)^2} = \mu_{0,0}(\mathbb{R}^n).$$

Moreover, there are extremals iff $\{\gamma = 0$ and there exists $z \in \mathbb{R}^n$ such that $(1 + |x-z|^2)^{1-n/2} \in D^{1,2}(\mathcal{C})\}$ (in particular $\overline{\mathcal{C}} = \mathbb{R}^n$).

Proof of the claim: Note that $2^\star(s) = 2^\star(0) = 2^\star$. Since $\gamma \leq 0$, we have for any $u \in C_c^\infty(\mathcal{C}) \setminus \{0\}$,

$$\frac{\int_{\mathcal{C}} \left(|\nabla u|^2 - \gamma \frac{u^2}{|x|^2} \right) dx}{\left(\int_{\mathcal{C}} |u|^{2^\star}\, dx \right)^{\frac{2}{2^\star}}} \geq \frac{\int_{\mathcal{C}} |\nabla u|^2\, dx}{\left(\int_{\mathcal{C}} |u|^{2^\star}\, dx \right)^{\frac{2}{2^\star}}} \geq \frac{1}{K(n,2)^2}, \tag{12}$$

and therefore $\mu_{\gamma,0}(\mathcal{C}) \geq \frac{1}{K(n,2)^2}$. Fix now $y_0 \in \Omega$ and let $\eta \in C_c^\infty(\mathcal{C})$ be such that $\eta(x) = 1$ around y_0. Set $u_\varepsilon(x) := \eta(x) \left(\frac{\varepsilon}{\varepsilon^2 + |x-y_0|^2} \right)^{\frac{n-2}{2}}$ for all $x \in \mathcal{C}$ and $\varepsilon > 0$. Since $y_0 \neq 0$, it is easy to check that $\lim_{\varepsilon\to 0} \int_{\mathcal{C}} \frac{u_\varepsilon^2}{|x|^2}\, dx = 0$. It is also classical (see for example Aubin [1]) that

$$\lim_{\varepsilon\to 0} \frac{\int_{\mathcal{C}} |\nabla u_\varepsilon|^2\, dx}{\left(\int_{\mathcal{C}} |u_\varepsilon|^{2^\star}\, dx \right)^{\frac{2}{2^\star}}} = \frac{1}{K(n,2)^2}.$$

It follows that $\mu_{\gamma,0}(\mathcal{C}) \leq \frac{1}{K(n,2)^2}$. This proves that $\mu_{\gamma,0}(\mathcal{C}) = \frac{1}{K(n,2)^2}$.

Assume now that there exists an extremal $u_0 \in D^{1,2}(\mathcal{C}) \setminus \{0\}$ for $\mu_{\gamma,0}(\mathcal{C})$. The inequalities in (12) and the fact that

$$\mu_{\gamma,0}(\mathcal{C}) = \frac{\int_{\mathcal{C}} \left(|\nabla u_0|^2 - \gamma \frac{u_0^2}{|x|^2}\right) dx}{\left(\int_{\mathcal{C}} |u_0|^{2^\star} dx\right)^{\frac{2}{2^\star}}} \geq \frac{\int_{\mathcal{C}} |\nabla u_0|^2 \, dx}{\left(\int_{\mathcal{C}} |u_0|^{2^\star} dx\right)^{\frac{2}{2^\star}}} = \frac{1}{K(n,2)^2},$$

yields $\gamma = 0$ and $u_0 \in D^{1,2}(\mathcal{C}) \subset D^{1,2}(\mathbb{R}^n)$ is an extremal for the classical Sobolev inequality on $\mathbb{R}^n$. Therefore, u_0 is of the form $x \mapsto a(b + |x - z_0|^2)^{1-n/2}$ for some $a \neq 0$ and $b > 0$ (see Aubin [1] or Talenti [9]). Using the homothetic invariance of the cone, we then get that there is an extremal of the form $x \mapsto (1 + |x - z|^2)^{1-n/2}$ for some $z \in \mathbb{R}^n$. Since an extremal has support in $\overline{\mathcal{C}}$, we then get that $\overline{\mathcal{C}} = \mathbb{R}^n$. This proves the claim. □

Step 7: We assume that $s = 0$, $\gamma > 0$ and $n \geq 4$. Then

$$\mu_{\gamma,s}(\mathcal{C}) < \mu_{0,0}(\mathbb{R}^n) = \frac{1}{K(n,2)^2}. \tag{13}$$

Proof of the claim: We consider the family u_ε as in Step 6. Well known computations by Aubin [1] yield

$$J_{\gamma,s}^{\mathcal{C}}(u_\varepsilon) = K(n,2)^{-2} - \gamma |x_0|^{-2} c\theta_\varepsilon + o(\theta_\varepsilon) \text{ as } \varepsilon \to 0,$$

where $c > 0$, $\theta_\varepsilon = \varepsilon^2$ if $n \geq 5$ and $\theta_\varepsilon = \varepsilon^2 \ln \varepsilon^{-1}$ if $n = 4$. It follows that if $\gamma > 0$ and $n \geq 4$, then $\mu_{\gamma,s}(\mathcal{C}) < K(n,2)^{-1}$. This proves the claim.

Step 8: We assume that $s = 0$ and that there exists $z \in \mathbb{R}^n$ such that $x \mapsto (1 + |x - z|^2)^{1-n/2} \in D^{1,2}(\mathcal{C})$. Then $\mu_{\gamma,0}(\mathcal{C}) < \frac{1}{K(n,2)^2}$ for all $\gamma > 0$.

Proof of the claim: We define $U(x) := (1 + |x - z|^2)^{1-n/2}$ for all $x \in \mathbb{R}^n$. We then have that $J_{\gamma,0}^{\mathcal{C}}(U) = J_{\gamma,0}^{\mathbb{R}^n}(U) < J_{0,0}^{\mathbb{R}^n}(U) = K(n,2)^{-1}$. This proves the claim.

Theorem 0.1 is a consequence of Steps 3 and 5 to 7. Corollary 0.2 is a direct consequence of Theorem 0.1. Corollary 0.3 is a direct consequence of Theorem 0.1 and Step 8. This ends the proof of Theorem 0.1.

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Nassif Ghoussoub, Department of Mathematics, 1984 Mathematics Road,The University of British Columbia, BC, Canada V6T 1Z2
E-mail address: nassif@math.ubc.ca

Frédéric Robert, Institut Élie Cartan, Université de Lorraine, BP 70239, F-54506 Vandœuvre-lès-Nancy, France
E-mail address: frederic.robert@univ-lorraine.fr