HARDY-SINGULAR BOUNDARY MASS AND SOBOLEV-CRITICAL VARIATIONAL PROBLEMS

NASSIF GHOUSSOUB AND FRÉDÉRIC ROBERT

Abstract. We investigate the Hardy-Schrödinger operator $L_\gamma = -\Delta - \frac{\gamma}{|x|^2}$ on smooth domains $\Omega \subset \mathbb{R}^n$ whose boundaries contain the singularity 0. We prove a Hopf-type result and optimal regularity for variational solutions of corresponding linear and nonlinear Dirichlet boundary value problems, including the equation $L_\gamma u = \frac{u^{2^\star(s)-1}}{|x|^s}$, where $\gamma < \frac{n^2}{4}$, $s \in [0,2)$ and $2^\star(s) := \frac{2(n-s)}{n-2}$ is the critical Hardy-Sobolev exponent. We also give a complete description of the profile of all positive solutions –variational or not– of the corresponding linear equation on the punctured domain. The value $\gamma = \frac{n^2-1}{4}$ turned out to be a critical threshold for the operator L_γ. When $\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$, a notion of *"Hardy singular boundary mass"* $m_\gamma(\Omega)$ associated to the operator L_γ, can be assigned to any conformally bounded domain Ω such that $0 \in \partial\Omega$. As a byproduct, we give a complete answer to problems of existence of extremals for Hardy-Sobolev inequalities, and consequently for those of Caffarelli-Kohn-Nirenberg. These results extend previous contributions by the authors in the case $\gamma = 0$, and by Chern-Lin for the case $\gamma < \frac{(n-2)^2}{4}$. More specifically, we show that extremals exist when $0 \le \gamma \le \frac{n^2-1}{4}$ if the mean curvature of $\partial\Omega$ at 0 is negative. On the other hand, if $\frac{n^2-1}{4} < \gamma < \frac{n^2}{4}$, extremals then exist whenever the Hardy singular boundary mass $m_\gamma(\Omega)$ of the domain is positive.

Contents

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1. Introduction

The borderline Dirichlet boundary value problem

$$\left\{\begin{array}{rcll} -\Delta u-\gamma\frac{u}{|x|^2} &=& u^{\frac{n+2}{n-2}} & \text{on } \Omega\\ u &>& 0 & \text{on } \Omega\\ u &=& 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1.1}$$

on a smooth bounded domain Ω of $\mathbb{R}^n$ $(n\geq 3)$ has no energy minimizing solutions if the singularity 0 belongs to the interior of the domain Ω (See discussion after inequality (1.15)). The situation changes dramatically however, if 0 is situated on the boundary $\partial\Omega$. Indeed, C. S. Lin and his collaborators [5, 6] showed that solutions exist in this case provided the mean curvature of $\partial\Omega$ at 0 is negative, $n\geq 4$, and $0<\gamma<\frac{(n-2)^2}{4}$. The condition on γ insures that the Hardy-Schrödinger operator $L_\gamma:=-\Delta-\frac{\gamma}{|x|^2}$ is positive on $H_0^1(\Omega)$. This is the case as long as $\gamma<\gamma_H(\Omega)$, the latter being the best constant in the corresponding Hardy inequality, i.e.,

$$\gamma_H(\Omega):=\inf\left\{\frac{\int_\Omega|\nabla u|^2\,dx}{\int_\Omega\frac{u^2}{|x|^2}\,dx};\ u\in D^{1,2}(\Omega)\setminus\{0\}\right\}. \tag{1.2}$$

Here $D^{1,2}(\Omega)$ – or $H_0^1(\Omega)$ if the domain is bounded – is the completion of $C_c^\infty(\Omega)$ with respect to the norm given by $||u||^2=\int_\Omega|\nabla u|^2dx$, and it is well known that for any domain Ω having 0 in its interior, we have

$$\gamma(\Omega)=\gamma_H(\mathbb{R}^n)=\frac{(n-2)^2}{4}. \tag{1.3}$$

On the other hand, $\gamma_H(\mathbb{R}^n_+)=\frac{n^2}{4}$ when $\mathbb{R}^n_+:=\{x\in\mathbb{R}^n;\,x_1>0\}$ is the half-space, and if Ω is any domain having 0 on its boundary, then necessarily

$$\frac{(n-2)^2}{4}<\gamma_H(\Omega)\leq\frac{n^2}{4}. \tag{1.4}$$

The question of what happens when $\frac{(n-2)^2}{4}<\gamma<\gamma_H(\Omega)$ provided the initial motivation for this paper. To start with, we shall show that the negative mean curvature condition at 0 is still sufficient for the existence of solutions for (1.1) as long as γ remains below a new (higher) threshold, namely when $n\geq 4$ and

$$0<\gamma\leq\frac{n^2-1}{4}. \tag{1.5}$$

However, the situation changes dramatically for the remaining interval, i.e., when

$$\frac{n^2-1}{4}<\gamma<\gamma_H(\Omega). \tag{1.6}$$

In this case, we show that local geometric conditions at 0 become irrelevant for solving (1.1) and more global properties of the domain must come into play. This will be illustrated by the notion of *Hardy singular boundary mass* of the domain Ω that we introduce as follows.

We first consider the Hardy-Schrödinger operator $L_\gamma:=-\Delta-\frac{\gamma}{|x|^2}$ on $\mathbb{R}^n_+$, and notice that the most basic solutions for $L_\gamma u=0$ satisfying $u=0$ on $\partial\mathbb{R}^n_+$ are of the form $u_\alpha(x)=x_1|x|^{-\alpha}$, and that $L_\gamma u_\alpha=0$ on $\mathbb{R}^n_+$ if and only if α is either $\alpha_-(\gamma)$ or $\alpha_+(\gamma)$, where

$$\alpha_\pm(\gamma):=\tfrac{n}{2}\pm\sqrt{\tfrac{n^2}{4}-\gamma}. \tag{1.7}$$

Actually, a byproduct of our analysis below gives that any non-negative solution of $L_\gamma u=0$ on $\mathbb{R}^n_+$ with $u=0$ on $\partial\mathbb{R}^n_+$ is a linear combination of these two solutions. Note that $\alpha_-(\gamma)<\frac{n}{2}<\alpha_+(\gamma)$, which points to the difference –in terms of behaviour around 0– between the "small" solution $x\mapsto$

$x_1|x|^{-\alpha_-(\gamma)}$, and the "large" one $x \mapsto x_1|x|^{-\alpha_+(\gamma)}$. Indeed, the small solution is "variational", i.e. is locally in $D^{1,2}(\mathbb{R}^n_+)$, while the large one is not.
This turned out to hold in more general settings, as we show that any variational solution of $L_\gamma u = a(x)u$ behaves like $x \mapsto d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}$ around 0, while any positive non-variational solution is necessarily like $x \mapsto d(x,\partial\Omega)|x|^{-\alpha_+(\gamma)}$ around 0. The profile can be made more explicit when $\gamma > \frac{n^2-1}{4}$, as it is the only situation in which one can write a solution of $L_\gamma u = 0$ as the sum of the two above described profiles (plus lower-order terms), while if $\gamma \leq \frac{n^2-1}{4}$, there might be some intermediate terms between the two profiles. This led us to define the following notion of mass, which is reminiscent of the positive mass theorem of Schoen-Yau [27] that was used to complete the solution of the Yamabe problem. This will allow us to settle the remaining cases left by Chern-Lin, since we establish that the positivity of such a boundary singular mass is sufficient to guarantee the existence of solutions for (1.1) in low dimensions.

Theorem 1.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. Assume that $\frac{n^2-1}{4} < \gamma < \gamma_H(\Omega)$. Then, up to multiplication by a positive constant, there exists a unique function $H \in C^2(\overline{\Omega} \setminus \{0\})$ such that*

$$-\Delta H - \frac{\gamma}{|x|^2}H = 0 \text{ in } \Omega\ ,\ H > 0 \text{ in } \Omega\ ,\ H = 0 \text{ on } \partial\Omega \setminus \{0\}. \tag{1.8}$$

Moreover, there exists a constant $c \in \mathbb{R}$ and H satisfying (1.8) such that

$$H(x) = \frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}} + c\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \qquad \text{as } x \to 0.$$

Due to the uniqueness of solutions to (1.8) up to multiplication by a constant, the coefficient c is uniquely defined. It will be denoted by $m_\gamma(\Omega) := c \in \mathbb{R}$, and will be referred to as the "Hardy singular boundary mass" of Ω.

It will be shown in section 7 that this notion of mass is conformally invariant in the following sense: if two sets are diffeomorphic via an inversion fixing 0 (see Definition 7.3 and (7.16)), then they have the same mass. As a consequence, we shall be able to define a notion of *Hardy singular boundary mass* for unbounded domains that are conformally bounded (that is, those that are smooth and bounded up to an inversion that fixes 0). We shall show that $\Omega \to m_\gamma(\Omega)$ is a monotone set-function and that $m_\gamma(\mathbb{R}^n_+) = 0$. These properties will allow us to construct in section 9, examples of bounded domains Ω in $\mathbb{R}^n$ with $0 \in \partial\Omega$ with either positive or negative boundary mass, while satisfying any local behavior at 0 one wishes. In other words, the sign of the Hardy-singular boundary mass is totally independent of the local properties of $\partial\Omega$ around 0.
One motivation for considering equation (1.1) came from the problem of existence of extremals for the Caffarelli-Kohn-Nirenberg (CKN) inequalities [4]. These state that in dimension $n \geq 3$, there is a constant $C := C(a,b,n) > 0$ such that for all $u \in C_c^\infty(\mathbb{R}^n)$, the following inequality holds:

$$\left(\int_{\mathbb{R}^n} |x|^{-bq}|u|^q\right)^{\frac{2}{q}} \leq C\int_{\mathbb{R}^n} |x|^{-2a}|\nabla u|^2 dx, \tag{1.9}$$

where

$$-\infty < a < \frac{n-2}{2},\quad 0 \leq b-a \leq 1 \text{ and } q = \frac{2n}{n-2+2(b-a)}. \tag{1.10}$$

If we let $D_a^{1,2}(\Omega)$ be the completion of $C_c^\infty(\Omega)$ with respect to the norm $||u||_a^2 = \int_\Omega |x|^{-2a}|\nabla u|^2 dx$, then the best constant in (1.9) is given by

$$S(a,b,\Omega) = \inf\left\{\frac{\int_\Omega |x|^{-2a}|\nabla u|^2 dx}{\left(\int_\Omega |x|^{-bq}|u|^q\right)^{\frac{2}{q}}\ dx}; u \in D_a^{1,2}(\Omega)\backslash\{0\}\right\}. \tag{1.11}$$

The extremal functions for $S(a,b,\Omega)$ –whenever they exist– are then the least-energy solutions of the corresponding Euler-Lagrange equations:

$$\left\{\begin{array}{rcll} -\mathrm{div}(|x|^{-2a}\nabla u) & = & |x|^{-bq}u^{q-1} & \text{on } \Omega\\ u & > & 0 & \text{on } \Omega\\ u & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{1.12}$$

To make the connection with Hardy-Schrödinger operator, note that the substitution $v(x)=|x|^{-a}u(x)$ with $a<\frac{n-2}{2}$, gives –via the Hardy inequality– that $u\in D^{1,2}_a(\Omega)$ if and only if $v\in D^{1,2}(\Omega)$ and that u is a variational solution of (1.12) if and only if w is a solution of equation

$$\left\{\begin{array}{rcll} -\Delta v-\gamma\frac{v}{|x|^2} & = & \frac{v^{2^\star(s)-1}}{|x|^s} & \text{on } \Omega\\ v & > & 0 & \text{on } \Omega\\ v & = & 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1.13}$$

where

$$\gamma=a(n-2-a),\ s=(b-a)q\ \text{ and }\ 2^\star=\frac{2n}{n-2+2(b-a)}. \tag{1.14}$$

The Caffarelli-Kohn-Nirenberg inequalities are then equivalent to the Hardy-Sobolev inequality

$$C\left(\int_\Omega\frac{u^{2^\star(s)}}{|x|^s}dx\right)^{\frac{2}{2^\star(s)}}\leq\int_\Omega|\nabla u|^2dx-\gamma\int_\Omega\frac{u^2}{|x|^2}dx\quad\text{for all } u\in D^{1,2}(\Omega), \tag{1.15}$$

at least in the case when $\gamma<\frac{(n-2)^2}{4}$, which is optimal for domains Ω having 0 in their interior. If Ω is also bounded, then the best constant in (1.15) is never attained, that is (1.13) has no energy minimizing solution.
However, when $0\in\partial\Omega$, inequality (1.15) holds for γ all the way to $\frac{n^2}{4}$, and we shall work thereafter towards solving (1.13) by finding extremals for the variational problem

$$\mu_{\gamma,s}(\Omega):=\inf\left\{J^\Omega_{\gamma,s}(u);u\in D^{1,2}(\Omega)\setminus\{0\}\right\}, \tag{1.16}$$

where $J^\Omega_{\gamma,s}$ is the functional on $D^{1,2}(\Omega)$ defined by

$$J^\Omega_{\gamma,s}(u):=\frac{\int_\Omega|\nabla u|^2-\gamma\int_\Omega\frac{u^2}{|x|^2}dx}{\left(\int_\Omega\frac{u^{2^\star(s)}}{|x|^s}dx\right)^{\frac{2}{2^\star(s)}}}. \tag{1.17}$$

We shall therefore consider thereafter the more general equation (1.13). The study of this type of nonlinear singular problems when $0\in\partial\Omega$ was initiated by Ghoussoub-Kang [13] and studied extensively by Ghoussoub-Robert [15,16] in the case $\gamma=0$. C. S. Lin and his collaborators [5,6,22] dealt with the case $\gamma<\frac{(n-2)^2}{4}$. For more contributions, we refer to Attar-Merchán-Peral [1], Dávila-Peral [8], and Gmira-Véron [19].

Theorem 1.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ $(n\geq 3)$ such that $0\in\partial\Omega$. Assume $\gamma\leq\frac{n^2-1}{4}$ and $0\leq s<2$. If either $\{s>0\}$ or $\{s=0, n\geq 4$ and $\gamma>0\}$, then there are extremals for $\mu_{\gamma,s}(\Omega)$ provided the mean curvature of $\partial\Omega$ at 0 is negative.*

As mentioned above, our main contribution here to this problem is however to consider the cases when $\frac{n^2-1}{4}\leq\gamma<\frac{n^2}{4}$, as well as the case when $n=3$, $s=0$ and $\gamma>0$, which were left open by Chern-Lin [6]. We now discuss the new ingredients that we bring to the discussion.
We first note that standard compactness arguments [6,13] yield that for $\mu_{\gamma,s}(\Omega)$ to be attained it is sufficient to have that

$$\mu_{\gamma,s}(\Omega)<\mu_{\gamma,s}(\mathbb{R}^n_+), \tag{1.18}$$

and in order to prove the existence of such a gap, one tries to construct test functions for $\mu_{\gamma,s}(\Omega)$ that are based on the extremals of $\mu_{\gamma,s}(\mathbb{R}^n_+)$ provided the latter exist. The cases where this is known are given by the following standard proposition. See for instance Bartsch-Peng-Zhang [3] and Chern-Lin [6]. A complete proof is given in [17].

Proposition 1.3. *Assume $\gamma < \frac{n^2}{4}$, $n \geq 3$ and $0 \leq s < 2$. Then,*

1. *$\mu_{\gamma,s}(\mathbb{R}^n_+)$ is attained provided either $\{s > 0\}$ or $\{s = 0, n \geq 4$ and $\gamma > 0\}$.*
2. *On the other hand, there are no extremals for $\mu_{\gamma,s}(\mathbb{R}^n_+)$ for any $n \geq 3$, if $\{s = 0$ and $\gamma \leq 0\}$.*
3. *Furthermore, whenever $\mu_{\gamma,0}(\mathbb{R}^n_+)$ has no extremals, then necessarily*

$$\mu_{\gamma,0}(\mathbb{R}^n_+) = \inf_{u \in D^{1,2}(\mathbb{R}^n)\setminus\{0\}} \frac{\int_{\mathbb{R}^n} |\nabla u|^2\, dx}{\left(\int_{\mathbb{R}^n} |u|^{2^\star}\, dx\right)^{\frac{2}{2^\star}}} = \frac{1}{K(n,2)^2}, \tag{1.19}$$

where $2^\star := \frac{2n}{n-2}$ and $\frac{1}{K(n,2)^2}$ is the best constant in the Sobolev inequality.

The only unknown situation on $\mathbb{R}^n_+$ is again when $s = 0$, $n = 3$ and $\gamma > 0$, which we address in Section 10.

Assuming first that an extremal for $\mu_{\gamma,s}(\mathbb{R}^n_+)$ exists and that one knows its profile at infinity and at 0, then this information can be used to construct test functions for $\mu_{\gamma,s}(\Omega)$. This classical method has been used by Kang-Ghoussoub [13], Ghoussoub-Robert [15, 16] when $\gamma = 0$, and by Chern-Lin [6] for $0 < \gamma < \frac{(n-2)^2}{4}$ in order to establish (1.18) under the assumption that $\partial\Omega$ has a negative mean curvature at 0. Actually, the estimates of Chern-Lin [6] extend directly to establish Theorem 1.2 for all $\gamma < \frac{n^2-1}{4}$ under the same negative mean curvature condition. However, the case where $\gamma = \frac{n^2-1}{4}$ already requires estimates on the profile of variational solutions of (1.13) on $\mathbb{R}^n_+$ that are finer than those used by Chern-Lin [6]. The following description of such a profile will allow us to construct sharper test functions and to prove existence of solutions for (1.13) when $\gamma = \frac{n^2-1}{4}$.

Theorem 1.4. *Assume $\gamma < \frac{n^2}{4}$, $0 \leq s < 2$, and let $u \in D^{1,2}(\mathbb{R}^n_+)$, $u \geq 0$, $u \not\equiv 0$ be a weak solution to*

$$-\Delta u - \frac{\gamma}{|x|^2} u = \frac{u^{2^\star(s)-1}}{|x|^s} \text{ in } \mathbb{R}^n_+. \tag{1.20}$$

Then, there exist $K_1, K_2 > 0$ such that

$$u(x) \sim_{x\to 0} K_1 \frac{x_1}{|x|^{\alpha_-(\gamma)}} \quad \text{and} \quad u(x) \sim_{|x|\to+\infty} K_2 \frac{x_1}{|x|^{\alpha_+(\gamma)}}.$$

The solution of the problem on $\mathbb{R}^n_+$ also enjoys the following natural symmetry that will be crucial for the sequel. This was carried out by Ghoussoub-Robert [16] when $\gamma = 0$, and their proof extends immediately to the case $0 \leq \gamma < n^2/4$. Chern-Lin [6] gave another proof which also includes the case where $\gamma < 0$.

Theorem 1.5. (Chern-Lin [6]) *If u is a non-negative solution to (1.20) in $D^{1,2}(\mathbb{R}^n_+)$, then $u \circ \sigma = u$ for all isometries of $\mathbb{R}^n$ such that $\sigma(\mathbb{R}^n_+) = \mathbb{R}^n_+$. In particular, there exists $v \in C^\infty((0,+\infty) \times \mathbb{R})$ such that for all $x_1 > 0$ and all $x' \in \mathbb{R}^{n-1}$, we have that $u(x_1, x') = v(x_1, |x'|)$.*

The following theorem summarizes the situation for low dimensions.

Theorem 1.6. *Let Ω be a bounded smooth domain of $\mathbb{R}^n$ $(n \geq 3)$ such that $0 \in \partial\Omega$, hence $\frac{(n-2)^2}{4} < \gamma_H(\Omega) \leq \frac{n^2}{4}$. Let $0 \leq s < 2$.*

1. *If $\gamma_H(\Omega) \leq \gamma < \frac{n^2}{4}$, then there are extremals for $\mu_{\gamma,s}(\Omega)$ for all $n \geq 3$.*
2. *If $\frac{n^2-1}{4} < \gamma < \gamma_H(\Omega)$ and either $\{s > 0\}$ or $\{s = 0,\ n \geq 4$ and $\gamma > 0\}$, then there are extremals for $\mu_{\gamma,s}(\Omega)$ provided the Hardy singular boundary mass $m_\gamma(\Omega)$ is positive.*
3. *If $\{s = 0$ and $\gamma \leq 0\}$, then there are no extremals for $\mu_{\gamma,0}(\Omega)$ for any $n \geq 3$.*

Finally, we address in section 10 the only remaining case, i.e., $n=3$, $s=0$ and $\gamma\in(0,\frac{9}{4})$. In this situation, there may or may not be extremals for $\mu_{\gamma,0}(\mathbb{R}^3_+)$. If they do exist, we can then argue as before –using the same test functions– to conclude existence of extremals under the same conditions, that is either $\gamma\leq 2$ and the mean curvature of $\partial\Omega$ at 0 is negative, or $\gamma>2$ and the mass $m_\gamma(\Omega)$ is positive. However, if no extremal exist for $\mu_{\gamma,0}(\mathbb{R}^3_+)$, then as noted in (1.19), we have that

$$\mu_{\gamma,0}(\mathbb{R}^3_+)=\inf_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}}\frac{\int_{\mathbb{R}^3}|\nabla u|^2\,dx}{\left(\int_{\mathbb{R}^3}|u|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}}=\frac{1}{K(3,2)^2},$$

and we are back to the case of the Yamabe problem with no boundary singularity. This means that one needs to resort to a more standard notion of mass $R_\gamma(\Omega,x_0)$ associated to L_γ and an interior point $x_0\in\Omega$, in order to construct suitable test-functions in the spirit of Schoen [26]. Such an *"interior mass"* will be introduced in section 10. We get the following (note that the boundary mass $m_\gamma(\Omega)$ was defined in Theorem 1.1).

Theorem 1.7. *Let Ω be a bounded smooth domain of $\mathbb{R}^3$ such that $0\in\partial\Omega$. In particular $\frac{1}{4}<\gamma_H(\Omega)\leq\frac{9}{4}$.*

(1) *If $\gamma_H(\Omega)\leq\gamma<\frac{9}{4}$, then there are extremals for $\mu_{\gamma,0}(\Omega)$.*
(2) *If $0<\gamma<\gamma_H(\Omega)$ and if there exists $x_0\in\Omega$ such that $R_\gamma(\Omega,x_0)>0$, then there are extremals for $\mu_{\gamma,0}(\Omega)$, under either one of the following conditions:*
   (a) *$\gamma\leq 2$ and the mean curvature of $\partial\Omega$ at 0 is negative.*
   (b) *$\gamma>2$ and the boundary mass $m_\gamma(\Omega)$ is positive.*

More precisely, if there are extremals for $\mu_{\gamma,0}(\mathbb{R}^3)$, then conditions (a) and (b) are sufficient to get extremals for $\mu_{\gamma,0}(\Omega)$. If there are no extremals for $\mu_{\gamma,0}(\mathbb{R}^3)$, then the positivity of the internal mass $R_\gamma(\Omega,x_0)$ is sufficient to get extremals for $\mu_{\gamma,0}(\Omega)$. The following table summarizes our findings.

Table 1. Singular Sobolev-Critical term: $s>0$

Hardy term	**Dimension**	**Geometric condition**	**Extremal**
$-\infty<\gamma\leq\frac{n^2-1}{4}$	$n\geq 3$	Negative mean curvature at 0	Yes
$\frac{n^2-1}{4}<\gamma<\frac{n^2}{4}$	$n\geq 3$	Positive boundary-mass	Yes

Table 2. Non-singular Sobolev-Critical term: $s=0$

Hardy term	**Dim.**	**Geometric condition**	**Extr.**
$0<\gamma\leq\frac{n^2-1}{4}$	$n=3$	Negative mean curvature at 0 & Positive internal mass	Yes
	$n\geq 4$	Negative mean curvature at 0	Yes
$\frac{n^2-1}{4}<\gamma<\frac{n^2}{4}$	$n=3$	Positive boundary-mass & Positive internal mass	Yes
	$n\geq 4$	Positive boundary mass	Yes
$\gamma\leq 0$	$n\geq 3$	–	No

Notations: in the sequel, $C_i(a,b,...)$ $(i=1,2,...)$ will denote constants depending on $a,b,....$ The same notation can be used for different constants, even in the same line. We will always refer to the monograph [18] by Gilbarg and Trudinger for the standard results on elliptic PDEs.

2. Old and new inequalities involving singular weights

The following general form of the Hardy inequality is well known. See for example Cowan [7] or the book of Ghoussoub-Moradifam [14].

Theorem 2.1. *Let Ω be a connected open subset of $\mathbb{R}^n$ and consider $\rho \in C^\infty(\Omega)$ such that $\rho > 0$ and $-\Delta\rho > 0$. Then, for any $u \in D^{1,2}(\Omega)$ we have*

$$\int_\Omega \frac{-\Delta\rho}{\rho} u^2\, dx \leq \int_\Omega |\nabla u|^2\, dx. \tag{2.1}$$

Moreover, the case of equality is achieved exactly on $\mathbb{R}\rho \cap D^{1,2}(\Omega)$. In particular, if $\rho \notin D^{1,2}(\Omega)$, there are no nontrivial extremals for (2.1).

The above theorem applies to various weight functions ρ. See for example [7] or [14]. For this paper, we use it to derive the following inequality.

Corollary 2.2. *Fix $1 \leq k \leq n$, we then have the following inequality.*

$$\left(\frac{n+2k-2}{2}\right)^2 = \inf_u \frac{\int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k}} |\nabla u|^2\, dx}{\int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k}} \frac{u^2}{|x|^2}\, dx},$$

where the infimum is taken over all u in $D^{1,2}(\mathbb{R}^k_+ \times \mathbb{R}^{n-k}) \setminus \{0\}$. Moreover, the infimum is never achieved.

Proof of Corollary 2.2: Take $\rho(x) := x_1 ... x_k |x|^{-\alpha}$ for all $x \in \Omega := \mathbb{R}^k_+ \times \mathbb{R}^{n-k} \setminus \{0\}$. Then $\frac{-\Delta\rho}{\rho} = \frac{\alpha(n+2k-2-\alpha)}{|x|^2}$. We then maximize the constant by taking $\alpha := (n+2k-2)/2$. Since $\rho \notin D^{1,2}(\mathbb{R}^k_+ \times \mathbb{R}^{n-k})$, Theorem 2.1 applies and we obtain that

$$\left(\frac{n+2k-2}{2}\right)^2 \int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k}} \frac{u^2}{|x|^2}\, dx \leq \int_{\mathbb{R}^k_+ \times \mathbb{R}^{n-k}} |\nabla u|^2\, dx \tag{2.2}$$

for all $u \in D^{1,2}(\mathbb{R}^k_+ \times \mathbb{R}^{n-k})$, and that the extremals are trivial.

It remains to prove that the constant in (2.2) is optimal. This will be achieved via the following test-function estimates. Construct a sequence $(\rho_\epsilon)_{\epsilon>0} \in D^{1,2}(\mathbb{R}^k_+ \times \mathbb{R}^{n-k})$ as follows. Starting with $\rho(x) = x_1 ... x_k |x|^{-\alpha}$, we fix $\beta > 0$ and define

$$\rho_\epsilon(x) := \begin{cases} \left|\frac{x}{\epsilon}\right|^\beta \rho(x) & \text{if } |x| < \epsilon \\ \rho(x) & \text{if } \epsilon \leq |x| \leq \frac{1}{\epsilon} \\ |\epsilon \cdot x|^{-\beta} \rho(x) & \text{if } |x| > \frac{1}{\epsilon} \end{cases} \tag{2.3}$$

with $\alpha := (n+2k-2)/2$. As one checks, $\rho_\epsilon \in D^{1,2}(\mathbb{R}^k_+ \times \mathbb{R}^{n-k})$ for all $\epsilon > 0$. The changes of variables $x = \epsilon y$ and $x = \epsilon^{-1} z$ yield

$$\begin{array}{cc} \int_{B_\epsilon(0)} \frac{\rho_\epsilon^2}{|x|^2}\, dx = O(1), & \int_{B_\epsilon(0)} |\nabla \rho_\epsilon|^2\, dx = O(1), \\ \int_{\mathbb{R}^n \setminus \overline{B}_{\epsilon^{-1}}(0)} \frac{\rho_\epsilon^2}{|x|^2}\, dx = O(1), & \int_{\mathbb{R}^n \setminus \overline{B}_{\epsilon^{-1}}(0)} |\nabla \rho_\epsilon|^2\, dx = O(1) \end{array} \tag{2.4}$$

when $\epsilon \to 0$. By integrating by parts, we get

$$\begin{aligned} \int_{B_{\epsilon^{-1}}(0) \setminus \overline{B}_\epsilon(0)} |\nabla \rho_\epsilon|^2\, dx &= \int_{B_{\epsilon^{-1}}(0) \setminus \overline{B}_\epsilon(0)} \frac{-\Delta\rho}{\rho} \rho^2\, dx + O(1) \\ &= \left(\frac{n+2k-2}{2}\right)^2 \int_{B_{\epsilon^{-1}}(0) \setminus \overline{B}_\epsilon(0)} \frac{\rho^2}{|x|^2}\, dx + O(1), \end{aligned} \tag{2.5}$$

when $\epsilon \to 0$. Using polar coordinates, we obtain

$$\int_{B_{\epsilon^{-1}}(0)\setminus\overline{B}_\epsilon(0)} \frac{\rho^2}{|x|^2}\,dx = C(2)\ln\frac{1}{\epsilon} \text{ where } C(2) := 2\int_{\mathbb{S}^{n-1}}\left|\prod_{i=1}^k x_i\right|^2\,d\sigma. \tag{2.6}$$

Therefore, by using (2.4), (2.5) and (2.6),

$$\frac{\int_{\mathbb{R}^k_+\times\mathbb{R}^{n-k}}|\nabla\rho_\epsilon|^2\,dx}{\int_{\mathbb{R}^k_+\times\mathbb{R}^{n-k}}\frac{\rho_\epsilon^2}{|x|^2}\,dx} = \left(\frac{n+2k-2}{2}\right)^2 + o(1)$$

as $\epsilon \to 0$, and we are done. Note that the infimum is never achieved since $\rho \notin D^{1,2}(\mathbb{R}^k_+\times\mathbb{R}^{n-k})$. $\Box$

Another approach to prove Corollary 2.2 is to see $\mathbb{R}^k_+\times\mathbb{R}^{n-k}$ as a cone generated by a domain of the unit sphere. Then the Hardy constant is given by the Hardy constant of $\mathbb{R}^n$ plus the first eigenvalue of the Laplacian of the Dirichlet of the above domain of the unit sphere endowed with its canonical metric. This point of view is developed in Pinchover-Tintarev [24] (see also Fall-Musina [12] and Ghoussoub-Moradifam [14] for an exposition in book form).

We also have the following generalized Caffarelli-Kohn-Nirenberg inequality.

Proposition 2.3. *Let Ω be an open subset of $\mathbb{R}^n$. Let $\rho,\rho'\in C^\infty(\Omega)$ be such that $\rho,\rho'>0$ and $-\Delta\rho,-\Delta\rho'>0$. Fix $s\in[0,2]$ and assume that there exists $\varepsilon\in(0,1)$ and $\rho_\varepsilon\in C^\infty(\Omega)$ such that*

$$\frac{-\Delta\rho}{\rho}\le(1-\varepsilon)\frac{-\Delta\rho_\varepsilon}{\rho_\varepsilon} \text{ in } \Omega \text{ with } \rho_\varepsilon,-\Delta\rho_\varepsilon>0.$$

Then, for all $u\in C^\infty_c(\Omega)$,

$$\left(\int_\Omega\left(\frac{-\Delta\rho'}{\rho'}\right)^{s/2}\rho^{2^\star(s)}|u|^{2^\star(s)}\,dx\right)^{\frac{2}{2^\star(s)}} \le C\int_\Omega\rho^2|\nabla u|^2\,dx. \tag{2.7}$$

Proof: The Sobolev inequality yields the existence of $C(n)>0$ such that

$$\left(\int_\Omega|u|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}\le C(n)\int_\Omega|\nabla u|^2\,dx$$

for all $u\in C^\infty_c(\Omega)$, where $2^\star=2^\star(0)=\frac{2n}{n-2}$. A Hölder inequality interpolating between this Sobolev inequality and the Hardy inequality (2.1) for ρ' yields the existence of $C>0$ such that for all $u\in C^\infty_c(\Omega)$,

$$\left(\int_\Omega\left(\frac{-\Delta\rho'}{\rho'}\right)^{s/2}|u|^{2^\star(s)}\,dx\right)^{\frac{2}{2^\star(s)}}\le C\int_\Omega|\nabla u|^2\,dx \tag{2.8}$$

By applying (2.1) to ρ_ε, we get for $v\in C^\infty_c(\Omega)$,

$$\begin{aligned}\int_\Omega\rho^2|\nabla v|^2\,dx &= \int_\Omega|\nabla(\rho v)|^2\,dx-\int_\Omega\frac{-\Delta\rho}{\rho}(\rho v)^2\,dx\\ &\ge \int_\Omega|\nabla(\rho v)|^2\,dx-(1-\varepsilon)\int_\Omega\frac{-\Delta\rho_\varepsilon}{\rho_\varepsilon}(\rho v)^2\,dx\\ &\ge \varepsilon\int_\Omega|\nabla(\rho v)|^2.\end{aligned}$$

Taking $u:=\rho v$ in (2.8) and using this latest inequality yield (2.7). $\Box$

Corollary 2.4. *Fix $k \in \{1, \dots, n-1\}$. There exists then a constant $C := C(a,b,n) > 0$ such that for all $u \in C_c^\infty(\mathbb{R}_+^k \times \mathbb{R}^{n-k})$,*

$$\left(\int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k}} |x|^{-bq} \left(\Pi_{i=1}^k x_i\right)^q |u|^q \right)^{\frac{2}{q}} \le C \int_{\mathbb{R}_+^k \times \mathbb{R}^{n-k}} \left(\Pi_{i=1}^k x_i\right)^2 |x|^{-2a} |\nabla u|^2 dx, \tag{2.9}$$

where

$$-\infty < a < \frac{n-2+2k}{2}, \quad 0 \le b-a \le 1, \quad q = \frac{2n}{n-2+2(b-a)}. \tag{2.10}$$

Proof: Apply Proposition 2.3 with $\rho(x) = \rho'(x) = \left(\Pi_{i=1}^k x_i\right) |x|^{-a}$ and $\rho_\varepsilon(x) = \left(\Pi_{i=1}^k x_i\right) |x|^{-\frac{n-2+2k}{2}}$ for all $x \in \mathbb{R}_+^k \times \mathbb{R}^{n-k}$. Corollary 2.4 then follows for suitable a, b, q. $\square$

Remark: Observe that by taking $k = 0$, we recover the classical Caffarelli-Kohn-Nirenberg inequalities (1.9). However, one does not see any improvement in the integrability of the weight functions since $\left(\Pi_{i=1}^k x_i\right) |x|^{-a}$ is of order $k - a > -(n-2)/2$, hence as close as we wish to $(n-2)/2$ with the right choice of a. The relevance here appears when one considers the Hardy inequality of Corollary 2.2.

3. On the best constants in the Hardy and Hardy-Sobolev inequalities

As mentioned in the introduction, the best constant in the Hardy inequality $\gamma_H(\Omega)$ does not depend on the domain $\Omega \subset \mathbb{R}^n$ if the singularity 0 belongs to the interior of Ω, and it is always equal to $\frac{(n-2)^2}{4}$. We have seen, however, in the last section that the situation changes whenever $0 \in \partial\Omega$, since $\gamma_H(\mathbb{R}_+^n) = \frac{n^2}{4}$. Some properties of the best Hardy constants have been studied by Fall-Musina [12] and Fall [11]. In this section, we shall collect whatever information we shall need later on about γ_H.

Proposition 3.1. *The best Hardy constant γ_H satisfies the following properties:*

1. $\gamma_H(\Omega) = \frac{(n-2)^2}{4}$ *for any smooth domain Ω such that $0 \in \Omega$.*
2. *If $0 \in \partial\Omega$, then $\frac{(n-2)^2}{4} < \gamma_H(\Omega) \le \frac{n^2}{4}$.*
3. $\gamma_H(\Omega) = \frac{n^2}{4}$ *for every Ω such that $0 \in \partial\Omega$ and $\Omega \subset \mathbb{R}_+^n$.*
4. *If $\gamma_H(\Omega) < \frac{n^2}{4}$, then it is attained in $D^{1,2}(\Omega)$.*
5. *We have $\inf\{\gamma_H(\Omega);\, 0 \in \partial\Omega\} = \frac{(n-2)^2}{4}$ for $n \ge 3$.*
6. *For every $\epsilon > 0$, there exists a smooth domain $\mathbb{R}_+^n \subsetneq \Omega_\epsilon \subsetneq \mathbb{R}^n$ such that $0 \in \partial\Omega_\epsilon$ and $\frac{n^2}{4} - \epsilon \le \gamma_H(\Omega_\epsilon) < \frac{n^2}{4}$.*

Proof of Proposition 3.1: Properties (1)-(2)-(3)-(4) are well known (See [12] and [11]). We sketch proofs since we will make frequent use of the test functions involved. Note first that Corollary 2.2 already yields that $\gamma_H(\mathbb{R}_+^n) = \frac{n^2}{4}$.

(2) Since $\Omega \subset \mathbb{R}^n$, we have that $\gamma_H(\Omega) \ge \gamma_H(\mathbb{R}^n) = \frac{(n-2)^2}{4}$. Assume by contradiction that $\gamma_H(\Omega) = \frac{(n-2)^2}{4}$. It then follows from Theorem 3.6 below (applied with $s = 2$) that $\gamma_H(\Omega)$ is achieved by a function in $u_0 \in D^{1,2}(\Omega) \setminus \{0\}$ (note that $\mu_{0,\gamma}(\Omega) = \gamma_H(\Omega) - \gamma$). Therefore, $\gamma_H(\mathbb{R}^n)$ is achieved in $D^{1,2}(\mathbb{R}^n)$. Up to taking $|u_0|$, we can assume that $u_0 \ge 0$. Therefore, the Euler-Lagrange equation and the maximum principle yield $u_0 > 0$ in $\mathbb{R}^n$: this is impossible since $u_0 \in D^{1,2}(\Omega)$. Therefore $\gamma_H(\Omega) > \frac{(n-2)^2}{4}$.

For the other inequality, the standard proof normally uses the fact that the domain contains an interior sphere that is tangent to the boundary at 0. We choose here to perform another proof based on test-functions, which will be used again to prove Proposition 3.3. It goes as follows: since Ω is a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$, there exist U, V open subsets of $\mathbb{R}^n$ such that $0 \in U$, $0 \in V$ and there exists $\varphi \in C^\infty(U, V)$ a diffeomophism such that $\varphi(0) = 0$ and

$$\varphi(U \cap \{x_1 > 0\}) = \varphi(U) \cap \Omega \text{ and } \varphi(U \cap \{x_1 = 0\}) = \varphi(U) \cap \partial\Omega.$$

Moreover, we can and shall assume that $d\varphi_0$ is an isometry. Let $\eta \in C_c^\infty(U)$ such that $\eta(x) = 1$ for $x \in B_\delta(0)$ for some $\delta > 0$ small enough, and consider $(\alpha_\epsilon)_{\epsilon>0} \in (0, +\infty)$ such that $\alpha_\epsilon = o(\epsilon)$ as $\epsilon \to 0$. For $\epsilon > 0$, define

$$u_\varepsilon(x) := \begin{cases} \eta(y)\alpha_\epsilon^{-\frac{n-2}{2}}\rho_\epsilon\left(\frac{y}{\alpha_\epsilon}\right) & \text{for all } x \in \varphi(U) \cap \Omega,\ x = \varphi(y), \\ 0 & \text{elsewhere.} \end{cases} \tag{3.1}$$

Here ρ_ϵ is constructed as in (2.3) with $k = 1$. Now fix $\sigma \in [0, 2]$, and note that only the case $\sigma = 2$ is needed for the above proposition. Immediate computations yield

$$\int_\Omega \frac{|u_\varepsilon(y)|^{2^\star(\sigma)}}{|y|^\sigma}\, dy = C(\sigma) \ln \frac{1}{\epsilon} + O(1) \quad \text{as } \epsilon \to 0, \tag{3.2}$$

where $C(\sigma) := 2\int_{\mathbb{S}^{n-1}} \left|\prod_{i=1}^k x_i\right|^{2^\star(\sigma)} d\sigma$. Similar arguments yield

$$\int_\Omega |\nabla u_\varepsilon|^2\, dy = \frac{n^2}{4} C(2) \ln \frac{1}{\epsilon} + O(1) \quad \text{as } \epsilon \to 0. \tag{3.3}$$

As a consequence, we get that

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2\, dx}{\int_\Omega \frac{u_\varepsilon^2}{|x|^2}\, dx} = \frac{n^2}{4} + o(1) \quad \text{as } \epsilon \to 0.$$

In particular, we get that $\gamma_H(\Omega) \leq \frac{n^2}{4}$, which proves the upper bound in item 2) of the proposition.

(3) Assume that $\Omega \subset \mathbb{R}^n_+$, then $D^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n_+)$, and therefore $\gamma_H(\Omega) \geq \gamma_H(\mathbb{R}^n_+) = n^2/4$. With the reverse inequality already given by Point (2), we get that $\gamma_H(\Omega) = n^2/4$ for all $\Omega \subset \mathbb{R}^n_+$ such that $0 \in \partial\Omega$.

(4) This will be a particular case of Theorem 3.6 when $s = 2$.

(5) Let Ω_0 be a bounded domain of $\mathbb{R}^n$ such that $0 \in \Omega_0$ (i.e., it is not on the boundary). Given $\delta > 0$, we chop out a ball of radius $\delta/4$ with 0 on its boundary to define $\Omega_\delta := \Omega_0 \setminus \overline{B}_{\frac{\delta}{4}}\left((\frac{-\delta}{4}, 0, \ldots, 0)\right)$. Note that for $\delta > 0$ small enough, Ω is smooth and $0 \in \partial\Omega$. We now prove that

$$\lim_{\delta \to 0} \gamma_H(\Omega_\delta) = \frac{(n-2)^2}{4}. \tag{3.4}$$

Define $\eta_1 \in C^\infty(\mathbb{R}^n)$ such that $\eta_1(x) = 0$ if $|x| < 1$ $\eta_1(x) = 1$ if $|x| > 2$. Let $\eta_\delta(x) := \eta_1(\delta^{-1}x)$ for all $\delta > 0$ and $x \in \mathbb{R}^n$. Fix $U \in C_c^\infty(\mathbb{R}^n)$ and consider for any $\delta > 0$, an $\varepsilon_\delta > 0$ such that $\lim_{\delta \to 0} \frac{\delta}{\varepsilon_\delta} = \lim_{\delta \to 0} \varepsilon_\delta = 0$. For $\delta > 0$, we define

$$u_\delta(x) := \eta_\delta(x)\varepsilon_\delta^{-\frac{n-2}{2}} U(\varepsilon_\delta^{-1}x) \text{ for all } x \in \Omega_\delta.$$

For $\delta > 0$ small enough, we have that $u_\delta \in C_c^\infty(\Omega_\delta)$. Since $\delta = o(\varepsilon_\delta)$ as $\delta \to 0$, a change of variable yields $\lim_{\delta \to 0} \int_{\Omega_\delta} \frac{u_\delta^2}{|x|^2}\, dx = \int_{\mathbb{R}^n} \frac{U^2}{|x|^2}\, dx$. We also have for δ small,

$$\begin{aligned} \int_{\Omega_\delta} |\nabla u_\delta|^2\, dx &= \int_{\mathbb{R}^n} |\nabla u_\delta|^2\, dx = \int_{\mathbb{R}^n} |\nabla \left(U \cdot \eta_{\frac{\delta}{\varepsilon_\delta}}\right)|^2\, dx \\ &= \int_{\mathbb{R}^n} |\nabla U|^2 \eta_{\frac{\delta}{\varepsilon_\delta}}^2\, dx + \int_{\mathbb{R}^n} \eta_{\frac{\delta}{\varepsilon_\delta}} \left(-\Delta \eta_{\frac{\delta}{\varepsilon_\delta}}\right) U^2\, dx. \end{aligned} \tag{3.5}$$

Let $R>0$ be such that U has support in $B_R(0)$. Since $n\geq 3$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\eta_{\frac{\delta}{\varepsilon_\delta}}\left(-\Delta\eta_{\frac{\delta}{\varepsilon_\delta}}\right)U^2\,dx &= O\left(\left(\frac{\varepsilon_\delta}{\delta}\right)^2\mathrm{Vol}(B_R(0)\cap\mathrm{Supp}\,\left(-\Delta\eta_{\frac{\delta}{\varepsilon_\delta}}\right))\right)\\ &= O\left(\left(\frac{\delta}{\varepsilon_\delta}\right)^{n-2}\right)=o(1)\end{aligned}$$

as $\delta\to 0$. This latest identity, (3.5) and the dominated convergence theorem yield

$$\lim_{\delta\to 0}\int_{\Omega_\delta}|\nabla u_\delta|^2\,dx=\int_{\mathbb{R}^n}|\nabla U|^2\,dx.$$

Therefore, for $U\in C_c^\infty(\mathbb{R}^n)$, we have

$$\limsup_{\delta\to 0}\gamma_H(\Omega_\delta)\leq\lim_{\delta\to 0}\frac{\int_{\Omega_\delta}|\nabla u_\delta|^2\,dx}{\int_{\Omega_\delta}\frac{u_\delta^2}{|x|^2}\,dx}=\frac{\int_{\mathbb{R}^n}|\nabla U|^2\,dx}{\int_{\mathbb{R}^n}\frac{U^2}{|x|^2}\,dx}.$$

Taking the infimum over all $U\in C_c^\infty(\mathbb{R}^n)$, we get that

$$\limsup_{\delta\to 0}\gamma_H(\Omega_\delta)\leq\inf_{U\in D^{1,2}(\mathbb{R}^n)\setminus\{0\}}\frac{\int_{\mathbb{R}^n}|\nabla U|^2\,dx}{\int_{\mathbb{R}^n}\frac{U^2}{|x|^2}\,dx}=\gamma_H(\mathbb{R}^n)=\frac{(n-2)^2}{4}.$$

Since $\gamma_H(\Omega_\delta)\geq\frac{(n-2)^2}{4}$ for all $\delta>0$, this completes the proof of (3.4), yielding (5).

For (6) we use the following observation.

Lemma 3.2. *Let $(\Phi_k)_{k\in\mathbb{N}}\in C^1(\mathbb{R}^n,\mathbb{R}^n)$ be such that*

$$\lim_{k\to+\infty}(\|\Phi_k-Id_{\mathbb{R}^n}\|_\infty+\|\nabla(\Phi_k-Id_{\mathbb{R}^n})\|_\infty)=0\text{ and }\Phi_k(0)=0.\tag{3.6}$$

Let $D\subset\mathbb{R}^n$ be an open domain such that $0\in\partial D$ (not necessarily bounded nor regular), and set $D_k:=\Phi_k(D)$ for all $k\in\mathbb{N}$. Then $0\in\partial D_k$ for all $k\in\mathbb{N}$ and

$$\lim_{k\to+\infty}\gamma_H(D_k)=\gamma_H(D).\tag{3.7}$$

Proof of Lemma 3.2: If $u\in C_c^\infty(D_k)$, then $u\circ\Phi_k\in C_c^\infty(D)$ and

$$\int_{D_k}|\nabla u|^2\,dx = \int_{\mathbb{R}^n_+}|\nabla(u\circ\Phi_k)|^2_{\Phi_k^\star\mathrm{Eucl}}|\mathrm{Jac}(\Phi_k)|\,dx,\tag{3.8}$$

$$\int_{D_k}\frac{u^2}{|x|^2}\,dx = \int_{\mathbb{R}^n_+}\frac{(u\circ\Phi_k(x))^2}{|\Phi_k(x)|^2}|\mathrm{Jac}(\Phi_k)|\,dx,\tag{3.9}$$

where here and in the sequel $\Phi_k^\star\mathrm{Eucl}$ is the pull-back of the Euclidean metric via the diffeomorphism Φ_k. Assumption (3.6) yields

$$\lim_{k\to+\infty}\sup_{x\in D}\left(\left|\frac{|\Phi_k(x)|}{|x|}-1\right|+\sup_{i,j}|(\partial_i\Phi_k(x),\partial_j\Phi_k(x))-\delta_{ij})|+|\mathrm{Jac}(\Phi_k)-1|\right)=0,$$

where $\delta_{ij}=1$ if $i=j$ and 0 otherwise. This limit, (3.8), (3.9) and a density argument yield (3.7). □

We now prove (6) of Proposition 3.1. Let $\varphi\in C^\infty(\mathbb{R}^{n-1})$ such that $0\leq\varphi\leq 1$, $\varphi(0)=0$, and $\varphi(x')=1$ for all $x'\in\mathbb{R}^{n-1}$ be such that $|x'|\geq 1$. For $t\geq 0$, define $\Phi_t(x_1,x'):=(x_1-t\varphi(x'),x')$ for all $(x_1,x')\in\mathbb{R}^n$. Set $\tilde\Omega_t:=\Phi_t(\mathbb{R}^n_+)$ and apply Lemma 3.2 to note that $\lim_{\varepsilon\to 0}\gamma_H(\tilde\Omega_t)=\gamma_H(\mathbb{R}^n_+)=\frac{n^2}{4}$. Since $\varphi\geq 0$, $\varphi\not\equiv 0$, we have that $\mathbb{R}^n_+\subsetneq\tilde\Omega_t$ for all $t>0$. To get (6) it suffices to take $\Omega_\varepsilon:=\tilde\Omega_t$ for $t>0$ small enough. □

As in the case of $\gamma_H(\Omega)$, the best Hardy-Sobolev constant

$$\mu_{\gamma,s}(\Omega) := \inf\left\{\frac{\int_\Omega |\nabla u|^2\,dx - \gamma\int_\Omega \frac{u^2}{|x|^2}dx}{(\int_\Omega \frac{u^{2^\star(s)}}{|x|^s}dx)^{\frac{2}{2^\star(s)}}};\ u\in D^{1,2}(\Omega)\setminus\{0\}\right\}$$

will depend on the geometry of Ω whenever $0\in\partial\Omega$.

Proposition 3.3. *Let Ω be a bounded smooth domain such that $0\in\partial\Omega$.*

(1) *If $\gamma<\frac{n^2}{4}$, then $\mu_{\gamma,s}(\Omega)>-\infty$.*
(2) *If $\gamma>\frac{n^2}{4}$, then $\mu_{\gamma,s}(\Omega)=-\infty$.*
Moreover,
(3) *If $\gamma<\gamma_H(\Omega)$, then $\mu_{\gamma,s}(\Omega)>0$.*
(4) *If $\gamma_H(\Omega)<\gamma<\frac{n^2}{4}$, then $0>\mu_{\gamma,s}(\Omega)>-\infty$.*
(5) *If $\gamma=\gamma_H(\Omega)<\frac{n^2}{4}$, then $\mu_{\gamma,s}(\Omega)=0$.*

Proof: Assume that $\gamma<\frac{n^2}{4}$ and let $\epsilon>0$ be such that $(1+\epsilon)\gamma\le\frac{n^2}{4}$. It follows from Proposition 3.5 that there exists $C_\epsilon>0$ such that for $u\in D^{1,2}(\Omega)$,

$$\frac{n^2}{4}\int_\Omega\frac{u^2}{|x|^2}\,dx\le(1+\epsilon)\int_\Omega|\nabla u|^2\,dx+C_\epsilon\int_\Omega u^2\,dx.$$

For any $u\in D^{1,2}(\Omega)\setminus\{0\}$, we have

$$\begin{aligned}J^\Omega_{\gamma,s}(u) &\ge \frac{\left(1-\frac{4\gamma}{n^2}(1+\epsilon)\right)\int_\Omega|\nabla u|^2\,dx-\frac{4\gamma}{n^2}C_\epsilon\int_\Omega u^2\,dx}{\left(\int_\Omega\frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}}\\ &\ge -\frac{4\gamma}{n^2}C_\epsilon\frac{\int_\Omega u^2\,dx}{\left(\int_\Omega\frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}}.\end{aligned}$$

It follows from Hölder's inequality that there exists $C>0$ independent of u such that $\int_\Omega u^2\,dx\le C\left(\int_\Omega\frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}$. It then follows that $J^\Omega_{\gamma,s}(u)\ge-\frac{4\gamma}{n^2}C_\epsilon C$ for all $u\in D^{1,2}(\Omega)\setminus\{0\}$. Therefore $\mu_{\gamma,s}(\Omega)>-\infty$ whenever $\gamma<\frac{n^2}{4}$.

Assume now that $\gamma>\frac{n^2}{4}$ and define for every $\varepsilon>0$ a function $u_\varepsilon\in D^{1,2}(\Omega)$ as in (3.1). It then follows from (3.2) and (3.3) that as $\varepsilon\to0$,

$$J^\Omega_{\gamma,s}(u_\varepsilon)=\frac{\left(\frac{n^2}{4}-\gamma\right)C(2)\ln\frac{1}{\varepsilon}+O(1)}{\left(C(s)\ln\frac{1}{\varepsilon}+O(1)\right)^{\frac{2}{2^\star(s)}}}=\left(\left(\frac{n^2}{4}-\gamma\right)\frac{C(2)}{C(s)^{\frac{2}{2^\star(s)}}}+o(1)\right)\left(\ln\frac{1}{\varepsilon}\right)^{\frac{2-s}{n-s}}.$$

Since $s<2$ and $\gamma>\frac{n^2}{4}$, we have $\lim_{\varepsilon\to0}J^\Omega_{\gamma,s}(u_\varepsilon)=-\infty$, therefore $\mu_{\gamma,s}(\Omega)=-\infty$.

If $\gamma<\gamma_H(\Omega)$, Sobolev's embedding theorem yields $\mu_{0,s}(\Omega)>0$, hence the result is clear for all $\gamma\le0$ since then $\mu_{\gamma,s}(\Omega)\ge\mu_{0,s}(\Omega)$. If now $0\le\gamma<\gamma_H(\Omega)$, it follows from the definition of $\gamma_H(\Omega)$ that for all $u\in D^{1,2}(\Omega)\setminus\{0\}$,

$$\begin{aligned}J^\Omega_{\gamma,s}(u)=\frac{\int_\Omega|\nabla u|^2-\gamma\int_\Omega\frac{u^2}{|x|^2}dx}{(\int_\Omega\frac{u^{2^\star(s)}}{|x|^s}dx)^{\frac{2}{2^\star(s)}}} &\ge \left(1-\frac{\gamma}{\gamma_H(\Omega)}\right)\frac{\int_\Omega|\nabla u|^2\,dx}{\left(\int_\Omega\frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}}\\ &\ge \left(1-\frac{\gamma}{\gamma_H(\Omega)}\right)\mu_{0,s}(\Omega).\end{aligned}$$

Therefore $\mu_{\gamma,s}(\Omega) \geq \left(1-\frac{\gamma}{\gamma_H(\Omega)}\right)\mu_{0,s}(\Omega) > 0$ when $\gamma < \gamma_H(\Omega)$.

If $\gamma_H(\Omega) < \gamma < \frac{n^2}{4}$, then Proposition 3.1 (4) yields that $\gamma_H(\Omega)$ is attained. We let u_0 be such an extremal. In particular $J^{\Omega}_{\gamma_H(\Omega),s}(u) \geq 0 = J^{\Omega}_{\gamma_H(\Omega),s}(u_0)$, and therefore $\mu_{\gamma_H(\Omega),s}(\Omega) = 0$. Since $\gamma_H(\Omega) < \gamma < \frac{n^2}{4}$, we have that $J^{\Omega}_{\gamma,s}(u_0) < 0$, and therefore $\mu_{\gamma,s}(\Omega) < 0$ when $\gamma_H(\Omega) < \gamma < \frac{n^2}{4}$. □

Remark 3.4. The case $\gamma = \frac{n^2}{4}$ is unclear and anything can happen at that value of γ. For example, if $\gamma_H(\Omega) < \frac{n^2}{4}$ then $\mu_{\frac{n^2}{4},s}(\Omega) < 0$, while if $\gamma_H(\Omega) = \frac{n^2}{4}$ then $\mu_{\frac{n^2}{4},s}(\Omega) \geq 0$. It is our guess that many examples reflecting different regimes can be constructed.

We shall need the following standard result.

Proposition 3.5. *Assume $\gamma < \frac{n^2}{4}$ and $s \in [0,2]$. Then, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all $u \in D^{1,2}(\Omega)$,*

$$\left(\int_\Omega \frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}} \leq \left(\frac{1}{\mu_{\gamma,s}(\mathbb{R}^n_+)} + \epsilon\right)\int_\Omega \left(|\nabla u|^2 - \gamma\frac{u^2}{|x|^2}\right)dx + C_\varepsilon \int_\Omega u^2\,dx. \tag{3.10}$$

This result says that, up to adding an L^2-term (indeed, any subcritical term fits), the best constant in the Hardy-Sobolev embedding can be chosen to be as close as one wishes to the best constant in the model space $\mathbb{R}^n_+$. One can see this by noting that for functions that are supported in a small neighborhood of 0, the domain Ω looks like $\mathbb{R}^n_+$, and the distortion is determined by the radius of the neighborhood. The case of general functions in $D^{1,2}(\Omega)$ is dealt with by using a cut-off, which induces the L^2-norm. A detailed proof is given in [17].

The following result is central for the sequel. The proof is standard, ever since T. Aubin's proof of the Yamabe conjecture in high dimensions, where he noted that the compactness of minimizing sequences is restored if the infimum is strictly below the energy of a "bubble". In our case below, this translates to $\mu_{\gamma,s}(\Omega) < \mu_{\gamma,s}(\mathbb{R}^n_+)$. We omit the proof, which can be found in [17].

Theorem 3.6. *Assume that $\gamma < \frac{n^2}{4}$, $0 \leq s \leq 2$ and that $\mu_{\gamma,s}(\Omega) < \mu_{\gamma,s}(\mathbb{R}^n_+)$. Then there are extremals for $\mu_{\gamma,s}(\Omega)$. In particular, there exists a minimizer u in $D^{1,2}(\Omega)\setminus\{0\}$ that is a positive solution to the equation*

$$\left\{\begin{array}{rcll} -\Delta u - \gamma\frac{u}{|x|^2} & = & \mu_{\gamma,s}(\Omega)\frac{u^{2^\star(s)-1}}{|x|^s} & \text{in } \Omega\\ u & > & 0 & \text{in } \partial\Omega\\ u & = & 0 & \text{on } \partial\Omega. \end{array}\right. \tag{3.11}$$

4. Profile at 0 of the variational solutions of $L_\gamma u = a(x)u$

Here and in the sequel, we shall assume that $0 \in \partial\Omega$, where Ω is a smooth domain. Recall from the introduction that two solutions for $L_\gamma u = 0$, with $u = 0$ on $\partial\mathbb{R}^n_+$ are of the form $u_\alpha(x) = x_1|x|^{-\alpha}$, where $\alpha \in \{\alpha_-(\gamma), \alpha_+(\gamma)\}$ with

$$\alpha_-(\gamma) := \frac{n}{2} - \sqrt{\frac{n^2}{4}-\gamma} \quad \text{and} \quad \alpha_+(\gamma) := \frac{n}{2} + \sqrt{\frac{n^2}{4}-\gamma}. \tag{4.1}$$

These solutions will be the building blocks for sub- and super-solutions of more general linear equations involving L_γ on other domains. This section is devoted to the proof of the following result. To state the theorem, we use the following terminology:
We say that $u \in D^{1,2}(\Omega)_{loc,0}$ if there exists $\eta \in C^\infty_c(\mathbb{R}^n)$ such that $\eta \equiv 1$ around 0 and $\eta u \in D^{1,2}(\Omega)$. Say that $u \in D^{1,2}(\Omega)_{loc,0}$ is *a weak solution* to the equation

$$-\Delta u = F \in \left(D^{1,2}(\Omega)_{loc,0}\right)',$$

if for any $\varphi \in D^{1,2}(\Omega)$ and $\eta \in C_c^\infty(\mathbb{R}^n)$ with sufficiently small support around 0, we have $\int_\Omega (\nabla u, \nabla(\eta\varphi))\, dx = \langle F, \eta\varphi\rangle$.

Theorem 4.1. *Fix $\gamma < \frac{n^2}{4}$, $\tau > 0$, and let $u \in D^{1,2}(\Omega)_{loc,0}$ be a weak solution of*

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = 0 \text{ in } D^{1,2}(\Omega)_{loc,0}. \tag{4.2}$$

Then, there exists $K \in \mathbb{R}$ such that

$$\lim_{x\to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} = K.$$

Moreover, if $u \geq 0$ and $u \not\equiv 0$, we have that $K > 0$.

By a slight abuse of notation, $u \mapsto -\Delta u - \frac{\gamma+O(|x|^\tau)}{|x|^2}u$ will denote an operator $u \mapsto -\Delta u - \frac{\gamma+a(x)}{|x|^2}u$ where $a \in C^0(\overline{\Omega})$ such that $a(x) = O(|x|^\tau)$ as $\tau \to 0$. In section 6, we will make a full description of solutions to (4.2) that are not necessarily variational (we also refer to Pinchover [23] for related problems).

We need the following lemmas, which will be used frequently throughout the paper. The first is only a first step towards proving rigidity for the solutions of $L_\gamma u = 0$ on $\mathbb{R}^n_+$. Indeed, the pointwise assumption $u(x) \leq C|x|^{1-\alpha}$ will not be necessary as it will be eventually removed in Proposition 6.4, which will be a consequence of the classification Theorem 6.1. We omit the proof as it can be inferred from the work of Pinchover-Tintarev [24].

Lemma 4.2. (Rigidity) *Let $u \in C^2(\overline{\mathbb{R}^n_+} \setminus \{0\})$ be a nonnegative solution of*

$$-\Delta u - \frac{\gamma}{|x|^2} u = 0 \textit{ in } \mathbb{R}^n_+ \;;\; u = 0 \textit{ on } \partial\mathbb{R}^n_+. \tag{4.3}$$

Suppose $u(x) \leq C|x|^{1-\alpha}$ on $\mathbb{R}^n_+$ for $\alpha \in \{\alpha_-(\gamma), \alpha_+(\gamma)\}$, then there exists $\lambda \geq 0$ such that $u(x) = \lambda x_1 |x|^{-\alpha}$ for all $x \in \mathbb{R}^n_+$.

We now construct basic sub- and super-solutions for the equation $L_\gamma u = a(x)u$, where $a(x) = O(|x|^{\tau-2})$ for some $\tau > 0$.

Proposition 4.3. *Let $\gamma < \frac{n^2}{4}$ and $\alpha \in \{\alpha_-(\gamma), \alpha_+(\gamma)\}$. Let $0 < \tau \leq 1$ and $\beta \in \mathbb{R}$ such that $\alpha - \tau < \beta < \alpha$ and $\beta \notin \{\alpha_-(\gamma), \alpha_+(\gamma)\}$. Then, there exist $r > 0$, and $u_{\alpha,+}, u_{\alpha,-} \in C^\infty(\overline{\Omega} \setminus \{0\})$ such that*

$$\left\{\begin{array}{rl} u_{\alpha,+}, u_{\alpha,-} > 0 & \textit{in } \Omega \cap B_r(0)\\ u_{\alpha,+}, u_{\alpha,-} = 0 & \textit{on } \partial\Omega \cap B_r(0)\\ -\Delta u_{\alpha,+} - \frac{\gamma+O(|x|^\tau)}{|x|^2} u_{\alpha,+} > 0 & \textit{in } \Omega \cap B_r(0)\\ -\Delta u_{\alpha,-} - \frac{\gamma+O(|x|^\tau)}{|x|^2} u_{\alpha,-} < 0 & \textit{in } \Omega \cap B_r(0). \end{array}\right. \tag{4.4}$$

Moreover, we have as $x \to 0$, $x \in \Omega$, that

$$u_{\alpha,+}(x) = \frac{d(x,\partial\Omega)}{|x|^\alpha}(1 + O(|x|^{\alpha-\beta})) \;\;\&\;\; u_{\alpha,-}(x) = \frac{d(x,\partial\Omega)}{|x|^\alpha}(1 + O(|x|^{\alpha-\beta})). \tag{4.5}$$

Proof of Proposition 4.3: We first choose an adapted chart to lift the basic solutions from $\mathbb{R}^n_+$. Since $0 \in \partial\Omega$ and Ω is smooth, there exist $\tilde{U}, \tilde{V}$ two bounded domains of $\mathbb{R}^n$ such that $0 \in \tilde{U}$, $0 \in \tilde{V}$, and there exists $c \in C^\infty(\tilde{U}, \tilde{V})$ a $C^\infty-$diffeomorphism such that $c(0) = 0$,

$$c(\tilde{U} \cap \{x_1 > 0\}) = c(\tilde{U}) \cap \Omega \text{ and } c(\tilde{U} \cap \{x_1 = 0\}) = c(\tilde{U}) \cap \partial\Omega.$$

The orientation of $\partial\Omega$ is chosen in such a way that for any $x' \in \tilde{U} \cap \{x_1 = 0\}$,

$$\{\partial_1 c(0,x'), \partial_2 c(0,x'), \ldots, \partial_n c(0,x')\}$$

is a direct basis of $\mathbb{R}^n$ (canonically oriented). For $x' \in \tilde{U} \cap \{x_1 = 0\}$, we define $\nu(x')$ as the unique orthonormal inner vector at the tangent space $T_{c(0,x')}\partial\Omega$ (it is chosen such that $\{\nu(x'), \partial_2 c(0,x'), \ldots, \partial_n c(0,x')\}$ is a direct basis of $\mathbb{R}^n$). In particular, on $\mathbb{R}^n_+ := \{x_1 > 0\}$, $\nu(x') := (1, 0, \ldots, 0)$.

Here and in the sequel, we write for any $r > 0$

$$\tilde{B}_r := (-r, r) \times B_r^{(n-1)}(0) \tag{4.6}$$

where $B_r^{(n-1)}(0)$ denotes the ball of center 0 and radius r in $\mathbb{R}^{n-1}$. It is standard that there exists $\delta > 0$ such that

$$\begin{array}{cccc} \varphi : & \tilde{B}_{2\delta} & \to & \mathbb{R}^n \\ & (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} & \mapsto & c(0, x') + x_1 \nu(x') \end{array} \tag{4.7}$$

is a $C^\infty-$diffeomorphism onto its open image $\varphi(\tilde{B}_{2\delta})$, and

$$\varphi(\tilde{B}_{2\delta} \cap \{x_1 > 0\}) = \varphi(\tilde{B}_{2\delta}) \cap \Omega \text{ and } \varphi(\tilde{B}_{2\delta} \cap \{x_1 = 0\}) = \varphi(\tilde{B}_{2\delta}) \cap \partial\Omega. \tag{4.8}$$

We also have for all $x' \in B_\delta(0)^{(n-1)}$,

$$\nu(x') \text{ is the inner orthonormal unit vector at the tangent space } T_{\varphi(0,x')}\partial\Omega. \tag{4.9}$$

An important remark is that

$$d(\varphi(x_1, x'), \partial\Omega) = |x_1| \text{ for all } (x_1, x') \in \tilde{B}_{2\delta} \text{ close to } 0. \tag{4.10}$$

Consider the metric $g := \varphi^\star \mathrm{Eucl}$ on $\tilde{B}_{2\delta}$, that is the pull-back of the Euclidean metric Eucl via the diffeomorphism φ. Following classical notations, we define

$$g_{ij}(x) := (\partial_i \varphi(x), \partial_j \varphi(x))_{\mathrm{Eucl}} \text{ for all } x \in \tilde{B}_{2\delta} \text{ and } i, j = 1, ..., n. \tag{4.11}$$

Up to a change of coordinates, we can assume that $(\partial_2\varphi(0), ..., \partial_n\varphi(0))$ is an orthogonal basis of $T_0\partial\Omega$. In other words, we then have that

$$g_{ij}(0) = \delta_{ij} \text{ for all } i, j = 1, ..., n. \tag{4.12}$$

As one checks,

$$g_{i1}(x) = \delta_{i1} \text{ for all } x \in \tilde{B}_{2\delta} \text{ and } i = 1, ..., n. \tag{4.13}$$

Fix now $\alpha \in \mathbb{R}$ and consider $\Theta \in C^\infty(\tilde{B}_{2\delta})$ such that $\Theta(0) = 0$ and which will be constructed later (independently of α) with additional needed properties. Fix $\eta \in C_c^\infty(\tilde{B}_{2\delta})$ such that $\eta(x) = 1$ for all $x \in \tilde{B}_\delta$. Define $u_\alpha \in C^\infty(\overline{\Omega} \setminus \{0\})$ as

$$u_\alpha \circ \varphi(x_1, x') := \eta(x) x_1 |x|^{-\alpha} (1 + \Theta(x)) \text{ for all } (x_1, x') \in \tilde{B}_{2\delta} \setminus \{0\}. \tag{4.14}$$

In particular, $u_\alpha(x) > 0$ for all $x \in \varphi(\tilde{B}_{2\delta}) \cap \Omega$ and $u_\alpha(x) = 0$ on $\Omega \setminus \varphi(\tilde{B}_{2\delta})$.

We claim that with a good choice of Θ, we have that

$$-\Delta u_\alpha = \frac{\alpha(n-\alpha)}{|x|^2} u_\alpha + O\left(\frac{u_\alpha(x)}{|x|}\right) \quad \text{as } x \to 0. \tag{4.15}$$

Indeed, using the chart φ, we have that

$$(-\Delta u_\alpha) \circ \varphi(x_1, x') = -\Delta_g (u_\alpha \circ \varphi)(x_1, x')$$

for all $(x_1, x') \in \tilde{B}_\delta \setminus \{0\}$. Here, $-\Delta_g$ is the Laplace operator associated to the metric g, that is

$$-\Delta_g := -g^{ij} \left(\partial_{ij} - \Gamma_{ij}^k \partial_k\right),$$

where

$$\Gamma_{ij}^k := \frac{1}{2} g^{km} \left(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right),$$

and (g^{ij}) is the inverse of the matrix (g_{ij}). Here and in the sequel, we have adopted Einstein's convention of summation. It follows from (4.13) that

$$\begin{aligned}(-\Delta u_\alpha)\circ\varphi &= -\Delta_{\text{Eucl}}(u_\alpha\circ\varphi)-\sum_{i,j\geq 2}\left(g^{ij}-\delta^{ij}\right)\partial_{ij}(u_\alpha\circ\varphi)\\ &\quad +g^{ij}\Gamma_{ij}^1\partial_1(u_\alpha\circ\varphi)+\sum_{k\geq 2}g^{ij}\Gamma_{ij}^k\partial_k(u_\alpha\circ\varphi).\end{aligned}\tag{4.16}$$

It follows from the definition (4.14) that there exists $C>0$ such that for any $i,j,k\geq 2$, we have that

$$|\partial_{ij}(u_\alpha\circ\varphi)(x_1,x')|\leq C|x_1|\cdot|x|^{-\alpha-2}\text{ and }|\partial_k(u_\alpha\circ\varphi)(x_1,x')|\leq C|x_1|\cdot|x|^{-\alpha-1},$$

for all $(x_1,x')\in\tilde{B}_\delta\setminus\{0\}$. It follows from (4.12) that $g^{ij}-\delta^{ij}=O(|x|)$ as $x\to 0$. Therefore, (4.16) yields that as $x\to 0$,

$$(-\Delta u_\alpha)\circ\varphi = -\Delta_{\text{Eucl}}(u_\alpha\circ\varphi)+g^{ij}\Gamma_{ij}^1\partial_1(u_\alpha\circ\varphi)+O(x_1|x|^{-\alpha-1})\tag{4.17}$$

The definition of g_{ij} and the expression of $\varphi(x_1,x')$ then yield that as $x\to 0$,

$$\begin{aligned}g^{ij}\Gamma_{ij}^1 &= -\frac{1}{2}\sum_{i,j\geq 2}g^{ij}\partial_1 g_{ij}\\ &= -\sum_{i,j\geq 2}g^{ij}(x_1,x')\left((\partial_i\varphi(0,x'),\partial_j\nu(x'))+x_1(\partial_i(x'),\partial_j\nu(x'))\right)\\ &= -\sum_{i,j\geq 2}g^{ij}(0,x')\left(\partial_i\varphi(0,x'),\partial_j\nu(x')\right)+O(|x_1|)\\ &= H(x')+O(|x_1|),\end{aligned}$$

where $H(x')$ is the mean curvature of the $(n-1)-$manifold $\partial\Omega$ at $\varphi(0,x')$ oriented by the outer normal vector $-\nu(x')$. Using the expression (4.14) and using the smoothness of Θ, (4.17) yields

$$\begin{aligned}(-\Delta u_\alpha)\circ\varphi &= \left(-\Delta_{\text{Eucl}}(x_1|x|^{-\alpha})\right)\cdot(1+\Theta)+|x|^{-\alpha}\left(H(x')(1+\Theta)-2\partial_1\Theta\right)\\ &\quad +O(x_1|x|^{-\alpha-1})\qquad\text{as }x\to 0.\end{aligned}$$

We now define

$$\Theta(x_1,x') := e^{-\frac{1}{2}x_1H(x')}-1\text{ for all }x=(x_1,x')\in\tilde{B}_{2\delta}.$$

Clearly $\Theta(0)=0$ and $\Theta\in C^\infty(\tilde{B}_{2\delta})$. We then get that as $x\to 0$,

$$(-\Delta u_\alpha)\circ\varphi = \frac{\alpha(n-\alpha)}{|x|^2}x_1|x|^{-\alpha}\cdot(1+\Theta)+O(x_1|x|^{-\alpha-1}).\tag{4.18}$$

With the choice that $g_{ij}(0)=\delta_{ij}$, we have that $(\partial_i\varphi(0))_{i=1,\dots,n}$ is an orthonormal basis of $\mathbb{R}^n$, and therefore $|\varphi(x)|=|x|(1+O(|x|))$ as $x\to 0$. It then follows from (4.18) and (4.14) that

$$-\Delta u_\alpha=\frac{\alpha(n-\alpha)}{|x|^2}u_\alpha+O(|x|^{-1}u_\alpha)\qquad\text{as }x\to 0.\tag{4.19}$$

This proves (4.15). We now proceed with the construction of the sub- and super-solutions. Let $\alpha\in\{\alpha_-(\gamma),\alpha_+(\gamma)\}$ in such a way that $\alpha(n-\alpha)=\gamma$ and consider $\beta,\lambda\in\mathbb{R}$ to be chosen later. It follows from (4.15) that

$$\begin{aligned}\left(-\Delta-\frac{\gamma+O(|x|^{\tau})}{|x|^2}\right)(u_\alpha+\lambda u_\beta) &= \frac{\lambda(\beta(n-\beta)-\gamma)}{|x|^2}u_\beta\\ &\quad+\frac{O(|x|^{\tau})}{|x|^2}u_\alpha+O(|x|^{-1}u_\alpha)+O(|x|^{\tau-2}u_\beta)\\ &= \frac{u_\beta}{|x|^2}\big(\lambda(\beta(n-\beta)-\gamma)\\ &\quad+O(|x|^{\tau})+O(|x|^{\tau+\beta-\alpha})+O(|x|^{1+\beta-\alpha})\big)\end{aligned}$$

as $x\to 0$. Choose β such that $\alpha-\tau<\beta<\alpha$ in such a way that $\beta\neq\alpha_-(\gamma)$ and $\beta\neq\alpha_+(\gamma)$. In particular, $\beta>\alpha-1$ and $\beta(n-\beta)-\gamma\neq 0$. We then have

$$\left(-\Delta-\frac{\gamma+O(|x|^{\tau})}{|x|^2}\right)(u_\alpha+\lambda u_\beta)=\frac{u_\beta}{|x|^2}\left(\lambda(\beta(n-\beta)-\gamma)+O(|x|^{\tau+\beta-\alpha})\right)\tag{4.20}$$

as $x\to 0$. Choose $\lambda\in\mathbb{R}$ such that $\lambda(\beta(n-\beta)-\gamma)>0$. Finally, let $u_{\alpha,+}:=u_\alpha+\lambda u_\beta$ and $u_{\alpha,-}:=u_\alpha-\lambda u_\beta$. They clearly satisfy (4.4) and (4.5), which completes the proof of Proposition 4.3. □

Lemma 4.4. *Assume that $u\in D^{1,2}(\Omega)_{loc,0}$ is a weak solution of*

$$\begin{cases} -\Delta u-\frac{\gamma+O(|x|^{\tau})}{|x|^2}u=0 & \text{in } D^{1,2}(\Omega)_{loc,0}\\ \qquad\qquad\qquad u=0 & \text{on } B_{2\delta}(0)\cap\partial\Omega,\end{cases}\tag{4.21}$$

for some $\tau>0$ and $\delta>0$. Then, there exists $C_1>0$ such that

$$|u(x)|\leq C_1 d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)} \text{ for } x\in\Omega\cap B_\delta(0).\tag{4.22}$$

Moreover, if $u>0$ in Ω, then there exists $C_2>0$ such that

$$u(x)\geq C_2 d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)} \text{ for } x\in\Omega\cap B_\delta(0).\tag{4.23}$$

Proof of Lemma 4.4: Assume first that $u\in D^{1,2}(\Omega)_{loc,0}$ and $u>0$ on $B_\delta(0)\cap\Omega$. We claim that there exists $C_0>0$ such that

$$\frac{1}{C_0}\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\leq u(x)\leq C_0\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} \text{ for all } x\in\Omega\cap B_\delta(0).\tag{4.24}$$

Indeed, since u is smooth outside 0, it follows from Hopf's Maximum principle that there exists $C_1,C_2>0$ such that

$$C_1 d(x,\partial\Omega)\leq u(x)\leq C_2 d(x,\partial\Omega) \text{ for all } x\in\Omega\cap\partial B_\delta(0).\tag{4.25}$$

Let $u_{\alpha_-(\gamma),+}$ be the super-solution constructed in Proposition 4.3. It follows from (4.25) and the asymptotics (4.5) of $u_{\alpha_-(\gamma),+}$ that there exists $C_3>0$ such that

$$u(x)\leq C_3 u_{\alpha_-(\gamma),+}(x) \text{ for all } x\in\partial(B_\delta(0)\cap\Omega).$$

Since u is a solution and $u_{\alpha_-(\gamma),+}$ is a supersolution, both being in $D^{1,2}(\Omega)_{loc,0}$, it follows from the maximum principle (by choosing $\delta>0$ small enough so that $-\Delta-(\gamma+O(|x|^{\tau}))|x|^{-2}$ is coercive on $B_\delta(0)\cap\Omega$) that $u(x)\leq C_3 u_{\alpha_-(\gamma),+}(x)$ for all $x\in B_\delta(0)\cap\Omega$. In particular, it follows from the asymptotics (4.5) of $u_{\alpha_-(\gamma),+}$ that there exists $C_4>0$ such that $u(x)\leq C_4 d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}$ for all $x\in\Omega\cap B_\delta(0)$. Arguing similarly with the lower-bound in (4.25) and the subsolution $u_{\alpha_-(\gamma),-}$, we get the existence of $C_0>0$ such that (4.24) holds. This yields Lemma 4.4 for $u>0$.

Now we deal with the case when u is a sign-changing solution for (4.21). We then define $u_1, u_2 : B_\delta(0)\cap\Omega \to \mathbb{R}$ be such that

$$\left\{\begin{array}{ll} -\Delta u_1 - \frac{\gamma+O(|x|^\tau)}{|x|^2}u_1 = 0 & \hbox{in } B_\delta(0)\cap\Omega\\ u_1(x)=\max\{u(x),0\} & \hbox{on } \partial(B_\delta(0)\cap\Omega).\end{array}\right.$$

$$\left\{\begin{array}{ll} -\Delta u_2 - \frac{\gamma+O(|x|^\tau)}{|x|^2}u_2 = 0 & \hbox{in } B_\delta(0)\cap\Omega\\ u_2(x)=\max\{-u(x),0\} & \hbox{on } \partial(B_\delta(0)\cap\Omega).\end{array}\right.$$

The existence of such solutions is ensured by choosing $\delta>0$ small enough so that the operator $-\Delta-(\gamma+O(|x|^\tau))|x|^{-2}$ is coercive on $B_\delta(0)\cap\Omega$. In particular, $u_1, u_2 \in D^{1,2}(\Omega)_{loc,0}$, $u_1, u_2 \geq 0$ and $u = u_1 - u_2$. It follows from the maximum principle that for all i, either $u_i \equiv 0$ or $u_i > 0$. The first part of the proof yields the upper bound for u_1, u_2. Since $u = u_1 - u_2$, we then get (4.22). □
The following lemma allows to construct sub- and super solutions with Dirichlet boundary conditions on any small smooth domain.

Proposition 4.5. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, and let W be a smooth domain of $\mathbb{R}^n$ such that for some $r>0$ small enough, we have*

$$B_r(0)\cap\Omega \subset W \subset B_{2r}(0)\cap\Omega \hbox{ and } B_r(0)\cap\partial W = B_r(0)\cap\partial\Omega. \tag{4.26}$$

Fix $\gamma < \frac{n^2}{4}$, $0<\tau\leq 1$ and $\beta\in\mathbb{R}$ such that $\alpha_+(\gamma)-\tau < \beta < \alpha_+(\gamma)$ and $\beta \neq \alpha_-(\gamma)$. Then, for r small enough, there exists $u^{(d)}_{\alpha_+(\gamma),+}, u^{(d)}_{\alpha_+(\gamma),-} \in C^\infty(\overline{W}\setminus\{0\})$ such that

$$\left\{\begin{array}{ll} u^{(d)}_{\alpha_+(\gamma),+}, u^{(d)}_{\alpha_+(\gamma),+} = 0 & \hbox{in } \partial W\setminus\{0\}\\ -\Delta u^{(d)}_{\alpha_+(\gamma),+} - \frac{\gamma+O(|x|^\tau)}{|x|^2}u^{(d)}_{\alpha_+(\gamma),+} > 0 & \hbox{in } W\\ -\Delta u^{(d)}_{\alpha_+(\gamma),-} - \frac{\gamma+O(|x|^\tau)}{|x|^2}u^{(d)}_{\alpha_+(\gamma),-} < 0 & \hbox{in } W.\end{array}\right. \tag{4.27}$$

Moreover, we have as $x\to 0$, $x\in\Omega$ that

$$u^{(d)}_{\alpha_+(\gamma),+}(x) = \frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}(1+O(|x|^{\alpha-\beta})), \tag{4.28}$$

and

$$u^{(d)}_{\alpha_+(\gamma),-}(x) = \frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}(1+O(|x|^{\alpha-\beta})). \tag{4.29}$$

Proof of Proposition 4.5: Take $\eta\in C^\infty(\mathbb{R}^n)$ such that $\eta(x)=0$ for $x\in B_{\delta/4}(0)$ and $\eta(x)=1$ for $x\in\mathbb{R}^n\setminus B_{\delta/3}(0)$. Define on W the function

$$f(x) := \left(-\Delta - \frac{\gamma+O(|x|^\tau)}{|x|^2}\right)(\eta u_{\alpha_+(\gamma),+}),$$

where $u_{\alpha_+(\gamma),+}$ is given by Proposition 4.3. Note that f vanishes around 0 and that it is in $C^\infty(\overline{W})$. Let $v\in D^{1,2}(W)$ be such that

$$\left\{\begin{array}{ll} -\Delta v - \frac{\gamma+O(|x|^\tau)}{|x|^2}v = f & \hbox{in } W\\ v = 0 & \hbox{on } \partial W.\end{array}\right.$$

Note that for $r>0$ small enough, $-\Delta-(\gamma+O(|x|^\tau))|x|^{-2}$ is coercive on W, and therefore, the existence of v is ensured for small r. Define

$$u^{(d)}_{\alpha_+(\gamma),+} := u_{\alpha_+(\gamma),+} - \eta u_{\alpha_+(\gamma),+} + v.$$

The properties of W and the definition of η and v yield

$$\left\{\begin{array}{rl} u_{\alpha_+(\gamma),+}^{(d)}=0 & \text{in } \partial W\setminus\{0\}\\ -\Delta u_{\alpha_+(\gamma),+}^{(d)}-\frac{\gamma+O(|x|^\tau)}{|x|^2}u_{\alpha_+(\gamma),+}^{(d)}>0 & \text{in } W.\end{array}\right.$$

Since $-\Delta v-(\gamma+O(|x|^\tau))|x|^{-2}v=0$ around 0 and $v\in D^{1,2}(W)$, it follows from Lemma 4.4 that there exists $C>0$ such that $|v(x)|\leq Cd(x,W)|x|^{-\alpha_-(\gamma)}$ for all $x\in W$. Then (4.28) follows from the asymptotics (4.5) of $u_{\alpha_+(\gamma),+}$ and the fact that $\alpha_-(\gamma)<\alpha_+(\gamma)$. We argue similarly for $u_{\alpha_+(\gamma),-}^{(d)}$. □

Lemma 4.6. *Let $u\in D^{1,2}(\Omega)_{loc,0}$ such that (4.2) holds. Assume there exists $C>0$ and $\alpha\in\{\alpha_+(\gamma),\alpha_-(\gamma)\}$ such that*

$$|u(x)|\leq C|x|^{1-\alpha}\text{ for } x\to 0,\ x\in\Omega. \tag{4.30}$$

(1) *Then, there exists $C_1>0$ such that*

$$|\nabla u(x)|\leq C_1|x|^{-\alpha}\text{ as } x\to 0,\ x\in\Omega. \tag{4.31}$$

(2) *If $\lim_{x\to 0}|x|^{\alpha-1}u(x)=0$, then $\lim_{x\to 0}|x|^\alpha|\nabla u(x)|=0$. Moreover, if $u>0$, then there exists $l\geq 0$ such that*

$$\lim_{x\to 0}\frac{|x|^\alpha u(x)}{d(x,\partial\Omega)}=l\text{ and }\lim_{x\to 0,\,x\in\partial\Omega}|x|^\alpha|\nabla u(x)|=l. \tag{4.32}$$

Proof of Lemma 4.6: Assume that (4.30) holds. Set $\omega(x):=\frac{|x|^\alpha u(x)}{d(x,\partial\Omega)}$ for $x\in\Omega$. Let $(x_i)_i\in\Omega$ be such that

$$\lim_{i\to+\infty}x_i=0\text{ and }\lim_{i\to+\infty}\omega(x_i)=l. \tag{4.33}$$

Choose a chart φ as in (4.7) such that $d\varphi_0=Id_{\mathbb{R}^n}$. For any i, define $X_i\in\mathbb{R}^n_+$ such that $x_i=\varphi(X_i)$, $r_i:=|X_i|$ and $\theta_i:=\frac{X_i}{|X_i|}$. In particular, $\lim_{i\to+\infty}r_i=0$ and $|\theta_i|$=1 for all i. Set

$$\tilde{u}_i(x):=r_i^{\alpha-1}u(\varphi(r_ix))\text{ for all } i\text{ and } x\in B_R(0)\cap\mathbb{R}^n_+\,;\,x\neq 0.$$

Equation (4.2) then rewrites

$$\left\{\begin{array}{rl} -\Delta_{g_i}\tilde{u}_i-\frac{\gamma+o(1)}{|x|^2}\tilde{u}_i=0 & \text{in } B_R(0)\cap\mathbb{R}^n_+\\ \tilde{u}_i=0 & \text{in } B_R(0)\cap\partial\mathbb{R}^n_+,\end{array}\right. \tag{4.34}$$

where $g_i(x):=(\varphi^\star\text{Eucl})(r_ix)$ is a metric that goes to Eucl on every compact subset of $\mathbb{R}^n$ as $i\to\infty$. Here, $o(1)\to 0$ in $C^0_{loc}(\overline{\mathbb{R}^n_+}\setminus\{0\})$. It follows from (4.30) and (4.33) that

$$|\tilde{u}_i(x)|\leq C|x|^{1-\alpha}\text{ for all } i\text{ and all } x\in B_R(0)\cap\mathbb{R}^n_+, \tag{4.35}$$

It follows from elliptic theory, that there exists $\tilde{u}\in C^2(\overline{\mathbb{R}^n_+}\setminus\{0\})$ such that $\tilde{u}_i\to\tilde{u}$ in $C^1_{loc}(\overline{\mathbb{R}^n_+}\setminus\{0\})$. By letting $\theta:=\lim_{i\to+\infty}\theta_i$ ($|\theta|=1$), we then have that for any $j=1,...,n$, $\partial_j\tilde{u}_i(\theta_i)\to\partial_j\tilde{u}(\theta)$ as $i\to+\infty$, which rewrites

$$\lim_{i\to+\infty}|x_i|^\alpha\partial_ju(x_i)=\partial_j\tilde{u}(\theta)\text{ for all } j=1,...,n. \tag{4.36}$$

We now prove (4.31). For that, we argue by contradiction and assume that there exists a sequence $(x_i)_i\in\Omega$ that goes to 0 as $i\to+\infty$ and such that $|x_i|^\alpha|\nabla u(x_i)|\to+\infty$ as $i\to+\infty$. It then follows from (4.36) that $|x_i|^\alpha|\nabla u(x_i)|=O(1)$ as $i\to+\infty$. This is a contradiction to our assumption, which proves (4.31). The case when $|x|^\alpha u(x)\to 0$ as $x\to 0$ goes similarly.

Now we consider the case when $u>0$, which implies that $\tilde{u}_i\geq 0$ and $\tilde{u}\geq 0$. We let $l\in[0,+\infty]$ and $(x_i)_i\in\Omega$ be such that

$$\lim_{i\to+\infty}x_i=0\text{ and }\lim_{i\to+\infty}\omega(x_i)=l. \tag{4.37}$$

We claim that

$$0 \leq l < +\infty \text{ and } \lim_{x\to 0} \omega(x) = l \in [0,+\infty). \tag{4.38}$$

Indeed, using the notations above, we get that

$$\lim_{i\to+\infty} \frac{\tilde{u}_i(\theta_i)}{(\theta_i)_1} = l.$$

The convergence of $\tilde{u}_i$ in $C^1_{loc}(\overline{\mathbb{R}^n_+} \setminus \{0\})$ then yields $l < +\infty$. Passing to the limit as $i \to +\infty$ in (4.34), we get

$$\begin{cases} -\Delta_{\text{Eucl}}\tilde{u} - \frac{\gamma}{|x|^2}\tilde{u} = 0 & \text{in } \mathbb{R}^n_+ \\ \tilde{u} \geq 0 & \text{in } \mathbb{R}^n_+ \\ \tilde{u} = 0 & \text{in } \partial\mathbb{R}^n_+. \end{cases}$$

The limit (4.37) can be rewritten as $\tilde{u}(\theta) = l\theta_1$ if $\theta \in \mathbb{R}^n_+$ and $\partial_1\tilde{u}(\theta) = l$ if $\theta \in \partial\mathbb{R}^n_+$. The rigidity Lemma 4.2 then yields

$$\tilde{u}(x) = lx_1|x|^{-\alpha} \text{ for all } x \in \mathbb{R}^n_+.$$

In particular, since the differential of φ at 0 is the identity map, it follows from the convergence of $\tilde{u}_i$ to $\tilde{u}$ locally in C^1 that

$$\lim_{i\to+\infty} \sup_{x\in\Omega\cap\partial B_{r_i}(0)} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha}} = \sup_{x\in\mathbb{R}^n_+\cap\partial B_1(0)} \frac{\tilde{u}(x)}{x_1|x|^{-\alpha}} = l \tag{4.39}$$

and

$$\lim_{i\to+\infty} \inf_{x\in\Omega\cap\partial B_{r_i}(0)} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha}} = \inf_{x\in\mathbb{R}^n_+\cap\partial B_1(0)} \frac{\tilde{u}(x)}{x_1|x|^{-\alpha}} = l. \tag{4.40}$$

We distinguish two cases:

Case 1: $\alpha = \alpha_+(\gamma)$**.** Let W and $u^{(d)}_{\alpha_+(\gamma),-}$ be as in Proposition 4.5, and fix $\epsilon > 0$. Note that the existence and properties of $u^{(d)}_{\alpha_+(\gamma),-}$ do not use the Lemma that is currently proved. It follows from (4.40) that there exists i_0 such that for $i \geq i_0$, we have that

$$u(x) \geq (l-\epsilon)u^{(d)}_{\alpha_+(\gamma),-}(x) \text{ for all } x \in W \cap \partial B_{r_i}(0).$$

Since $(-\Delta - (\gamma + O(|x|^\tau))|x|^{-2})(u - (l-\epsilon)u^{(d)}_{\alpha_+(\gamma),-}) \geq 0$ in $W \setminus B_{r_i}(0)$ and since $u_{\alpha_+(\gamma),-}$ vanishes on $\partial W \setminus \{0\}$, it follows from the comparison principle that

$$u(x) \geq (l-\epsilon)u^{(d)}_{\alpha_+(\gamma),-}(x) \text{ for all } x \in W \setminus \partial B_{r_i}(0).$$

Letting $i \to +\infty$ yields

$$u(x) \geq (l-\epsilon)u^{(d)}_{\alpha_+(\gamma),-}(x) \text{ for all } x \in W \setminus \{0\}.$$

It follows from this inequality and the asymptotics for $u^{(d)}_{\alpha_+(\gamma),-}$ that

$$\liminf_{x\to 0} \omega(x) \geq l.$$

Note that this is valid for any $l \in \mathbb{R}$ satisfying (4.37). By taking $l := \limsup_{x\to 0} \omega(x)$, we then get that $\lim_{x\to 0} \omega(x) = l$.

Case 2: $\alpha = \alpha_-(\gamma)$**.** Consider the super- and sub-solutions $u_{\alpha_-(\gamma),+}, u_{\alpha_-(\gamma),-}$ constructed in Proposition 4.3. It follows from (4.39) and (4.40) that for $\epsilon > 0$, there exists i_0 such that for $i \geq i_0$, we have

$$(l-\epsilon)u_{\alpha_-(\gamma),-}(x) \leq u(x) \leq (l+\epsilon)u_{\alpha_-(\gamma),+}(x) \text{ for all } x \in \Omega \cap \partial B_{r_i}(0).$$

Since the operator $-\Delta - (\gamma + O(|x|^\tau))|x|^{-2}$ is coercive on $\Omega \cap B_{r_i}(0)$ and that the functions we consider are in $D^{1,2}_{loc,0}(\Omega \cap B_{r_i}(0))$ (i.e., they are variational), it follows from the maximum principle that

$$(l-\epsilon)u_{\alpha_-(\gamma),-}(x) \leq u(x) \leq (l+\epsilon)u_{\alpha_-(\gamma),+}(x) \text{ for all } x \in \Omega \cap B_{r_i}(0).$$

Using the asymptotics (4.5) of the sub- and super-solution, we get that

$$(l-\epsilon) \leq \liminf_{x\to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} \leq \limsup_{x\to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} \leq (l+\epsilon).$$

Letting $\epsilon \to 0$ yields $\lim_{x\to 0} \omega(x) = l \geq 0$. This ends Case 2 and completes the proof of (4.38).

The case $u > 0$ is a consequence of (4.38) and (4.36) (note that for the second limit, $x_i \in \partial\Omega$ rewrites as $\theta_i \in \partial\mathbb{R}^n_+$ and therefore $(\theta_i)_1 = 0$). This ends the proof of Lemma 4.6. □

Proof of Theorem 4.1: First, assume that $u \in D^{1,2}(\Omega)_{loc,0}$ satisfies (4.2) and $u > 0$ on $B_\delta(0) \cap \Omega$. It then follows from Lemma 4.4 that there exists $C_0 > 0$ such that

$$\frac{1}{C_0}\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} \leq u(x) \leq C_0 \frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} \text{ for all } x \in \Omega \cap B_\delta(0).$$

Since $u > 0$, this estimate coupled with Lemma 4.6 yields the theorem for $u > 0$.

If now u is a sign-changing solution for (4.2), we define $u_1, u_2 : B_\delta(0) \cap \Omega \to \mathbb{R}_{\geq 0}$ as in the proof of Lemma 4.4. The first part of the proof yields that there exist $l_1, l_2 \geq 0$ such that

$$\lim_{x\to 0} \frac{u_1(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} = l_1 \text{ and } \lim_{x\to 0} \frac{u_2(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} = l_2.$$

Since $u = u_1 - u_2$, we get Theorem 4.1 by taking $l := l_1 - l_2$. □

Here is an immediate consequence.

Corollary 4.7. *Suppose $\gamma < \gamma_H(\Omega)$ and consider the first eigenvalue of L_γ, i.e.,*

$$\lambda_1(\Omega,\gamma) := \inf_{u \in D^{1,2}(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(|\nabla u|^2 - \frac{\gamma}{|x|^2}u^2\right) dx}{\int_\Omega u^2\, dx} > 0.$$

If $u_0 \in D^{1,2}(\Omega) \setminus \{0\}$ is a minimizer, then there exists $A \neq 0$ such that

$$u_0(x) \sim_{x\to 0} A\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}.$$

Proof: The existence of a minimizer u_0 that doesn't change sign is standard. The Euler-Lagrange equation is $-\Delta u - \frac{\gamma}{|x|^2}u = ku$ for some $k \in \mathbb{R}$. We then apply Theorem 4.1. □

5. Regularity of solutions for related nonlinear variational problems

This section is devoted to the proof of the following key result.

Theorem 5.1 (Optimal regularity and Generalized Hopf's Lemma)**.** *Fix $\gamma < \frac{n^2}{4}$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that*

$$|f(x,v)| \leq C|v|\left(1 + \frac{|v|^{2^\star(s)-2}}{|x|^s}\right) \text{ for all } x \in \Omega \text{ and } v \in \mathbb{R}.$$

Let $u \in D^{1,2}(\Omega)_{loc,0}$ be a weak solution of

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2}u = f(x,u) \text{ in } D^{1,2}(\Omega)_{loc,0} \tag{5.1}$$

for some $\tau > 0$. Then, there exists $K \in \mathbb{R}$ such that

$$\lim_{x\to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_-(\gamma)}} = K. \tag{5.2}$$

Moreover, if $u \geq 0$ and $u \not\equiv 0$, we have that $K > 0$.

Note that when $f \equiv 0$, this is nothing but Theorem 4.1. The result can be viewed as a generalization of Hopf's Lemma in the following sense: when $\gamma = 0$ (and then $\alpha_-(\gamma) = 0$), the classical Nash-Moser regularity scheme yields $u \in C^1_{loc}$, and when $u \geq 0$, $u \not\equiv 0$, Hopf's comparison principle yields $\partial_\nu u(0) < 0$, which is a reformulation of (5.2) when $\alpha_-(\gamma) = 0$.
The following lemma will be of frequent use in the sequel.

Lemma 5.2. *Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be as in the statement of Theorem 5.1, and consider $u \in D^{1,2}(\Omega)_{loc,0}$ such that (5.1) holds. Assume that for some $C > 0$,*

$$|u(x)| \leq C|x|^{1-\alpha_-(\gamma)} \text{ for } x \to 0,\ x \in \Omega. \tag{5.3}$$

Then, u satisfies the conclusion of Lemma 4.6.

Proof of Lemma 5.2: Assume that (5.3) holds. We claim that we can assume that for some $\tau > 0$,

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = 0 \text{ in } D^{1,2}(\Omega)_{loc,0}. \tag{5.4}$$

Indeed, we have as $x \to 0$,

$$\begin{aligned} |f(x,u)| &\leq C|u| \left(1 + |x|^{-s}|x|^{-(2^\star(s)-2)(\alpha_-(\gamma)-1)}\right) \\ &\leq C\frac{|u|}{|x|^2}\left(|x|^2 + |x|^{(2^\star(s)-2)(\frac{n}{2}-\alpha_-(\gamma))}\right) = O\left(|x|^{\tau'}\frac{u}{|x|^2}\right) \end{aligned}$$

for some $\tau' > 0$. Plugging this inequality into (5.1) and replacing τ by $\min\{\tau, \tau'\}$ yields (5.4). The lemma now follows from Lemma 4.6. □

Proof of Theorem 5.1: We let here $u \in D^{1,2}(\Omega)_{loc,0}$ be a solution to (5.1), that is

$$-\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u = f(x,u) \text{ weakly in } D^{1,2}(\Omega)_{loc,0} \tag{5.5}$$

for some $\tau > 0$. We shall first use the classical DeGiorgi-Nash-Moser iterative scheme (see Gilbarg-Trudinger [18], and Hebey [20] for expositions in book form). We skip most of the computations and refer to Ghoussoub-Robert (Proposition A.1 of [16]) for the details. We fix $\delta_0 > 0$ such that
(i) there exists $\tilde\eta \in C^\infty(B_{4\delta_0}(0))$ such that $\tilde\eta(x) = 1$ for $x \in B_{2\delta_0}(0)$.
(ii) $\tilde\eta u \in D^{1,2}(\Omega)$ and,
(iii) u is a weak solution to (5.5) when tested on $\tilde\eta\varphi$ with $\varphi \in D^{1,2}(\Omega)$ (see the definition of weak solution given in the preceding section).
The proof goes through four steps.

Step 1: Let $\beta \geq 1$ be such that $\frac{4\beta}{(\beta+1)^2} > \frac{4}{n^2}\gamma$. Assume that $u \in L^{\beta+1}(\Omega \cap B_{\delta_0}(0))$. We claim that

$$u \in L^{\frac{n}{n-2}(\beta+1)}(\Omega \cap B_{\delta_0}(0)). \tag{5.6}$$

Indeed, fix $\beta \geq 1$, $L > 0$, and define $G_L, H_L : \mathbb{R} \to \mathbb{R}$ as

$$G_L(t) := \left\{\begin{array}{ll} |t|^{\beta-1}t & \text{if } |t| \leq L \\ \beta L^{\beta-1}(t-L) + L^\beta & \text{if } t \geq L \\ \beta L^{\beta-1}(t+L) - L^\beta & \text{if } t \leq -L \end{array}\right. \tag{5.7}$$

and

$$H_L(t) := \begin{cases} |t|^{\frac{\beta-1}{2}} t & \text{if } |t| \le L \\ \frac{\beta+1}{2} L^{\frac{\beta-1}{2}}(t-L) + L^{\frac{\beta+1}{2}} & \text{if } t \ge L \\ \frac{\beta+1}{2} L^{\frac{\beta-1}{2}}(t+L) - L^{\frac{\beta+1}{2}} & \text{if } t \le -L \end{cases} \tag{5.8}$$

As it is easily checked,

$$0 \le tG_L(t) \le H_L(t)^2 \text{ and } G_L'(t) = \frac{4\beta}{(\beta+1)^2}(H_L'(t))^2 \tag{5.9}$$

for all $t \in \mathbb{R}$ and all $L > 0$. We fix $\delta > 0$ small that will be chosen later. We let $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 1$ for $x \in B_{\delta/2}(0)$ and $\eta(x) = 0$ for $x \in \mathbb{R}^n \setminus B_\delta(0)$. Multiplying equation (5.5) with $\eta^2 G_L(u) \in D^{1,2}(\Omega)$, we get that

$$\begin{aligned} \int_\Omega (\nabla u, \nabla(\eta^2 G_L(u)))\, dx \quad &- \quad \int_\Omega \frac{\gamma + O(|x|^\tau)}{|x|^2}\eta^2 u G_L(u)\, dx \\ &= \quad \int_\Omega f(x,u)\eta^2 G_L(u)\, dx. \end{aligned} \tag{5.10}$$

Integrating by parts, and using formulae (5.7) to (5.9) (see [16] for details) yields

$$\begin{aligned} \int_\Omega (\nabla u, \nabla(\eta^2 G_L(u)))\, dx \quad &= \quad \frac{4\beta}{(\beta+1)^2}\int_\Omega \left(|\nabla(\eta H_L(u))|^2 - \eta(-\Delta)\eta H_L(u)^2\right)\, dx \\ &\quad + \int_\Omega -\Delta(\eta^2) J_L(u)\, dx \end{aligned} \tag{5.11}$$

where $J_L(t) := \int_0^t G_L(\tau)\, d\tau$. This identity and (5.10) yield

$$\begin{aligned} \frac{4\beta}{(\beta+1)^2}\int_\Omega |\nabla(\eta H_L(u))|^2\, dx \quad &- \quad \int_\Omega \frac{\gamma + O(|x|^\tau)}{|x|^2}\eta^2 u G_L(u)\, dx \\ &\le \quad \int_\Omega |-\Delta(\eta^2)| \cdot |J_L(u)|\, dx \\ &\quad + C(\beta,\delta)\int_{\Omega\cap B_\delta(0)} |H_L(u)|^2\, dx \\ &\quad + C\int_\Omega \frac{|u|^{2^\star(s)-2}}{|x|^s}(\eta H_L(u))^2\, dx. \end{aligned} \tag{5.12}$$

Hölder's inequality and the Sobolev constant given in (1.16) yield

$$\begin{aligned} &\int_\Omega \frac{|u|^{2^\star(s)-2}}{|x|^s}(\eta H_L(u))^2\, dx \\ &\le \left(\int_{\Omega\cap B_\delta(0)} \frac{|u|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2^\star(s)-2}{2^\star(s)}} \left(\int_\Omega \frac{|\eta H_L(u)|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^\star(s)}} \\ &\le \left(\int_{\Omega\cap B_\delta(0)} \frac{|u|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2^\star(s)-2}{2^\star(s)}} \cdot \frac{1}{\mu_{0,s}(\Omega)}\int_\Omega |\nabla(\eta H_L(u))|^2\, dx. \end{aligned}$$

Plugging this estimate into (5.12) and defining $\gamma_+ := \max\{\gamma, 0\}$ yields

$$\begin{aligned}\frac{4\beta}{(\beta+1)^2}\int_\Omega |\nabla(\eta H_L(u))|^2\,dx \quad - \quad & (\gamma_+ + C\delta^\tau)\int_\Omega \frac{(\eta H_L(u))^2}{|x|^2}\,dx\\ & \leq C(\beta,\delta)\int_{\Omega\cap B_\delta(0)}\left(|H_L(u)|^2 + |J_L(u)|\right)\,dx\\ & +\alpha(\delta)\int_\Omega |\nabla(\eta H_L(u))|^2\,dx,\end{aligned}$$

where

$$\alpha(\delta) := C\left(\int_{\Omega\cap B_\delta(0)}\frac{|u|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2^\star(s)-2}{2^\star(s)}}\cdot\frac{1}{\mu_{0,s}(\Omega)},$$

so that

$$\lim_{\delta\to 0}\alpha(\delta) = 0.$$

It follows from Hardy's inequality that

$$\frac{n^2}{4}\int_\Omega \frac{(\eta H_L(u))^2}{|x|^2}\,dx \leq (1+\epsilon(\delta))\int_\Omega |\nabla(\eta H_L(u))|^2\,dx,$$

where $\lim_{\delta\to 0}\epsilon(\delta) = 0$. Therefore, we get that

$$\begin{aligned}&\left(\frac{4\beta}{(\beta+1)^2} - \alpha(\delta) - (\gamma_+ + C\delta^\tau)\,\frac{4}{n^2}(1+\epsilon(\delta))\right)\int_\Omega |\nabla(\eta H_L(u))|^2\,dx\\ &\leq C(\beta,\delta)\int_{\Omega\cap B_\delta(0)}\left(|H_L(u)|^2 + |J_L(u)|\right)\,dx \leq C(\beta,\delta)\int_{B_\delta(0)\cap\Omega}|u|^{\beta+1}\,dx.\end{aligned}$$

Let $\delta \in (0,\delta_0)$ be such that

$$\frac{4\beta}{(\beta+1)^2} - \alpha(\delta) - (\gamma_+ + C\delta^\tau)\,\frac{4}{n^2}(1+\epsilon(\delta)) > 0.$$

This is possible since $\frac{4\beta}{(\beta+1)^2} > \frac{4}{n^2}\gamma$. Using Sobolev's embedding, we then get that

$$\begin{aligned}\left(\int_{B_{\delta/2}(0)\cap\Omega}|H_L(u)|^{2^\star}\,dx\right)^{\frac{2}{2^\star}} \quad & \leq \quad \left(\int_{\mathbb{R}^n}|\eta H_L(u)|^{2^\star}\,dx\right)^{\frac{2}{2^\star}}\\ & \leq \quad \mu_{0,0}(\Omega)^{-1}\int_\Omega |\nabla(\eta H_L(u))|^2\,dx\\ & \leq \quad C(\beta,\delta,\gamma)\int_{B_\delta(0)\cap\Omega}|u|^{\beta+1}\,dx.\end{aligned}$$

Since $u \in L^{\beta+1}(B_{\delta_0}(0)\cap\Omega)$, let $L \to +\infty$ and use Fatou's Lemma to obtain that $u \in L^{\frac{2^\star}{2}(\beta+1)}(B_{\delta/2}(0)\cap\Omega)$. The standard iterative scheme then yields that $u \in C^1(\overline{\Omega}\cap B_{\delta_0}(0)\setminus\{0\})$. Therefore $u \in L^{\frac{2^\star}{2}(\beta+1)}(B_{\delta_0}(0)\cap\Omega)$.

Step 2: We now show that

$$\text{if } \gamma \leq 0, \text{ then } u \in L^p(\Omega\cap B_\delta(0)) \text{ for all } p \geq 1, \tag{5.13}$$

$$\text{if } \gamma > 0, \text{ then } u \in L^p(\Omega\cap B_\delta(0)) \text{ for all } p \in \left(1, \frac{n}{n-2}\frac{n}{\alpha_-(\gamma)}\right). \tag{5.14}$$

The case $\gamma \leq 0$ is standard, so we only consider the case where $\gamma > 0$. Fix $p \geq 2$ and set $\beta := p-1$. we have

$$\frac{4\beta}{(\beta+1)^2} > \frac{4}{n^2}\gamma \;\Leftrightarrow\; \frac{n}{\alpha_+(\gamma)} < p < \frac{n}{\alpha_-(\gamma)}.$$

Since $\alpha_+(\gamma) > n/2$ and $p \geq 2$, then

$$\frac{4\beta}{(\beta+1)^2} > \frac{4}{n^2}\gamma \;\Leftrightarrow\; p < \frac{n}{\alpha_-(\gamma)}.$$

Therefore, it follows from Step 1 that if $u \in L^p(\Omega \cap B_{\delta_0})$, with $p < n/\alpha_-(\gamma)$, then $u \in L^{\frac{n}{n-2}p}(\Omega \cap B_{\delta_0})$. Since $u \in L^2(\Omega \cap B_{\delta_0})$, (5.14) follows.

Step 3: We claim that for any $\lambda > 0$, then

$$|x|^{\frac{n-2}{2}}|u(x)| = O(|x|^{\frac{n-2}{n}\left(\frac{n}{2}-\max\{\alpha_-(\gamma),0\}-\lambda\right)} \quad \text{as } x \to 0. \tag{5.15}$$

Indeed, take $p \in \left(2^\star, \frac{n^2}{(n-2)\alpha_-(\gamma)}\right)$ if $\gamma > 0$, and $p > 2^\star$ if $\gamma \leq 0$. This is possible since $2^\star = 2n/(n-2)$ and $\alpha_-(\gamma) < n/2$. We fix a sequence $(\varepsilon_i)_i \in (0,+\infty)$ such that $\lim_{i\to+\infty} \varepsilon_i = 0$ and we fix a chart φ as in (4.7) to (4.12). For any $i \in \mathbb{N}$, we define

$$u_i(x) := \varepsilon_i^{\frac{n}{p}} u(\varphi(\varepsilon_i x)) \text{ for all } x \in \tilde{B}_{\delta/\varepsilon_i}.$$

Equation (5.5) then rewrites

$$-\Delta_{g_i} u_i - \frac{\epsilon_i^2(\gamma + O(\epsilon_i^\tau |x|^\tau))}{|\varphi(\epsilon_i x)|^2} u_i = f_i(x, u_i)\ ;\ u_i = 0 \text{ on } \partial\mathbb{R}^n_+ \cap \tilde{B}_{\delta/\varepsilon_i} \tag{5.16}$$

where $g_i(x) := \varphi^\star \mathrm{Eucl}(\epsilon_i x)$ and

$$|f_i(x,u_i)| \leq C\epsilon_i^2|u_i| + C\varepsilon_i^{(2^\star(s)-2)\left(\frac{n-2}{2}-\frac{n}{p}\right)}|x|^{-s}|u_i|^{2^\star(s)-1} \qquad \text{in } \tilde{B}_{\delta/\varepsilon_i}.$$

We fix $R > 0$ and define $\omega_R := \left(\tilde{B}_R \setminus \tilde{B}_{R^{-1}}\right) \cap \mathbb{R}^n_+$. With our choice of p above and using (5.14), we get that

$$\|u_i\|_{L^p(\omega_R)} \leq C, \tag{5.17}$$

and

$$|f_i(x,u_i)| \leq C_R|u_i| + C_R|u_i|^{2^\star(s)-1} \quad \text{for all } x \in \omega_R. \tag{5.18}$$

Fix $q \geq p > 2^\star$. It follows from elliptic regularity that

$$\|u_i\|_{L^q(\omega_R)} \leq C \;\Rightarrow\; \begin{cases} \|u_i\|_{L^{q'}(\omega_{R/2})} \leq C' & \text{if } q < \frac{n}{2}(2^\star(s)-1) \\ \|u_i\|_{L^r(\omega_{R/2})} \leq C' & \text{for all } r \geq 1 \text{ if } q = \frac{n}{2}(2^\star(s)-1) \\ \|u_i\|_{L^\infty(\omega_{R/2})} \leq C' & \text{if } q > \frac{n}{2}(2^\star(s)-1) \end{cases}$$

where $\frac{1}{q'} = \frac{2^\star(s)-1}{q} - \frac{2}{n}$ and the constants C, C' are uniform with respect to i. It then follows from the standard bootstrap iterative argument and the initial bound (5.17) that $\|u_i\|_{L^\infty(\omega_{R/4})} \leq C'$. Taking $R > 0$ large enough and going back to the definition of u_i, we get that for all $i \in \mathbb{N}$,

$$|x|^{\frac{n}{p}}|u(x)| \leq C \text{ for all } x \in \Omega \cap B_{2\varepsilon_i}(0) \setminus B_{\varepsilon_i/2}(0).$$

Since this holds for any sequence $(\varepsilon_i)_i$, we get that $|x|^{\frac{n}{p}}|u(x)| \leq C$ around 0 for any $2^\star < p < \frac{n^2}{(n-2)\alpha_-(\gamma)}$ when $\gamma > 0$. Letting p go to $\frac{n^2}{(n-2)\alpha_-(\gamma)}$ yields (5.15) when $\gamma > 0$. For $\gamma \leq 0$, we let $p \to +\infty$.

To finish the proof of Theorem 5.1, we rewrite equation (5.5) as

$$-\Delta u - \frac{a(x)}{|x|^2}u = 0,$$

where for $x \in \Omega$,

$$\begin{aligned} a(x) &= \gamma + O(|x|^\tau) + O(|x|^2) + O\left(|x|^{2-s}|u|^{2^\star(s)-2}\right) \\ &= \gamma + O(|x|^\tau) + O(|x|^2) + O\left(|x|^{\frac{n-2}{2}}|u(x)|\right)^{2^\star(s)-2} \end{aligned}$$

Since $\alpha_-(\gamma) < \frac{n}{2}$, it then follows from (5.15) that there exists $\tau' > 0$ such that $a(x) = \gamma + O(|x|^{\tau'})$ as $x \to 0$. We are therefore back to the linear case, hence we can apply Theorem 4.1 and deduce Theorem 5.1. ☐

As a consequence we get the following result that will be crucial for the sequel.

Corollary 5.3. *Suppose $u \in D^{1,2}(\mathbb{R}^n_+)$, $u \geq 0$, $u \not\equiv 0$ is a weak solution of*

$$-\Delta u - \frac{\gamma}{|x|^2}u = \frac{u^{2^\star-1}}{|x|^s} \text{ in } \mathbb{R}^n_+.$$

Then, there exist $K_1, K_2 > 0$ such that

$$u(x) \sim_{x\to 0} K_1 \frac{x_1}{|x|^{\alpha_-(\gamma)}} \text{ and } u(x) \sim_{|x|\to+\infty} K_2 \frac{x_1}{|x|^{\alpha_+(\gamma)}}. \tag{5.19}$$

Proof: Theorem 5.1 yields the behavior when $x \to 0$. The Kelvin transform $\hat{u}(x) := |x|^{2-n}u(x/|x|^2)$ is a solution to the same equation in $D^{1,2}(\mathbb{R}^n_+)$, and its behavior at 0 is given by Theorem 5.1. Going back to u yields the behavior at ∞. ☐

6. Profile around 0 of positive singular solutions of $L_\gamma u = a(x)u$

In this section we describe the profile of any positive solution –variational or not– of linear equations involving L_γ. Here is the main result of this section.

Theorem 6.1. *Let $u \in C^2(B_\delta(0) \cap (\overline{\Omega} \setminus \{0\}))$ be such that*

$$\left\{\begin{array}{rl} -\Delta u - \frac{\gamma+O(|x|^\tau)}{|x|^2}u = 0 & \text{in } \Omega \cap B_\delta(0) \\ u > 0 & \text{in } \Omega \cap B_\delta(0) \\ u = 0 & \text{on } (\partial\Omega \cap B_\delta(0)) \setminus \{0\}. \end{array}\right. \tag{6.1}$$

Then, there exists $K > 0$ such that

$$\text{either } u(x) \sim_{x\to 0} K\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} \quad \text{or} \quad u(x) \sim_{x\to 0} K\frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}.$$

In the first case, the solution $u \in D^{1,2}(\Omega)_{loc,0}$ is a variational solution to (6.1).

It is worth noting that Pinchover [23] tackled similar issues. The proof of Theorem 6.1 will require the following two lemmas. The first is a Harnack-type result.

Proposition 6.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, and let $a \in L^\infty(\Omega)$ be such that $\|a\|_\infty \leq M$ for some $M > 0$. Assume U is an open subset of $\mathbb{R}^n$ and consider $u \in C^2(U \cap \overline{\Omega})$ to be a solution of*

$$\left\{\begin{array}{rl} -\Delta_g u + au = 0 & \text{in } U \cap \Omega \\ u \geq 0 & \text{in } U \cap \Omega \\ u = 0 & \text{on } U \cap \partial\Omega. \end{array}\right.$$

Here g is a smooth metric on U. If $U' \subset\subset U$ is such that $U' \cap \Omega$ is connected, then there exists $C > 0$ depending only on Ω, U', M and g such that

$$\frac{u(x)}{d(x,\partial\Omega)} \leq C\frac{u(y)}{d(y,\partial\Omega)} \text{ for all } x, y \in U' \cap \Omega. \tag{6.2}$$

Proof of Proposition 6.2: We first prove a local result. The global result will be the consequence of a covering of U'. Fix $x_0 \in \partial\Omega$. For $\delta > 0$ small enough, there exists a smooth open domain W such that

$$B_\delta(x_0) \cap \Omega \subset W \subset B_{2\delta}(x_0) \cap \Omega \text{ and } B_\delta(x_0) \cap \partial W = B_\delta(x_0) \cap \partial\Omega. \tag{6.3}$$

Let G be the Green's function of $-\Delta_g + a$ with Dirichlet boundary condition on W, then its representation formula reads as

$$u(x) = \int_{\partial W} u(\sigma)\,(-\partial_{\nu,\sigma} G(x,\sigma))\,d\sigma = \int_{\partial W \setminus \partial\Omega} u(\sigma)\,(-\partial_{\nu,\sigma} G(x,\sigma))\,d\sigma, \tag{6.4}$$

for all $x \in W$, where $\partial_{\nu,\sigma} G(x,\sigma)$ is the normal derivative of $y \mapsto G(x,y)$ at $\sigma \in \partial W$. Estimates of the Green's function (see Robert [25] and Ghoussoub-Robert [16]) yield the existence of $C > 0$ such that for all $x \in W$ and $\sigma \in \partial W$,

$$\frac{1}{C} \frac{d(x,\partial W)}{|x-\sigma|^n} \leq -\partial_{\nu,\sigma} G(x,\sigma) \leq C \frac{d(x,\partial W)}{|x-\sigma|^n}.$$

It follows from (6.3) that there exists $C(\delta) > 0$ such that for all $x \in B_{\delta/2}(x_0) \cap \Omega \subset W$ and $\sigma \in \partial W \setminus \partial\Omega$,

$$\frac{1}{C(\delta)} d(x,\partial W) \leq -\partial_{\nu,\sigma} G(x,\sigma) \leq C(\delta) d(x,\partial W)$$

Since u vanishes on $\partial\Omega$, it then follows from (6.4) that for all $x \in B_{\delta/2}(x_0) \cap \Omega$,

$$\frac{1}{C(\delta)} d(x,\partial W) \int_{\partial W} u(\sigma)\,d\sigma \leq u(x) \leq C(\delta) d(x,\partial W) \int_{\partial W} u(\sigma)\,d\sigma.$$

It is easy to check, that under the assumption (6.3), we have that $d(x,\partial\Omega) = d(x,\partial W)$. Therefore, we have for all $x \in B_{\delta/2}(x_0) \cap \Omega$,

$$\frac{1}{C(\delta)} \int_{\partial W} u(\sigma)\,d\sigma \leq \frac{u(x)}{d(x,\partial\Omega)} \leq C(\delta) \int_{\partial W} u(\sigma)\,d\sigma.$$

These lower and upper bounds being independent of x, we get inequality (6.2) for any $x, y \in B_{\delta/2}(x_0) \cap \Omega$.

The general case is a consequence of a covering of $U' \cap \Omega$ by finitely many balls. Note that for balls intersecting $\partial\Omega$, we apply the preceding result, while for balls not intersecting $\partial\Omega$, we apply the classical Harnack inequality. This completes the proof of Proposition 6.2. □

Proof of Theorem 6.1: Let u be a solution of (6.1) as in the statement of Theorem 6.1. We claim that

$$u(x) = O(d(x,\partial\Omega)|x|^{-\alpha_+(\gamma)}) \text{ for } x \to 0,\ x \in \Omega. \tag{6.5}$$

Indeed, otherwise we can assume that

$$\limsup_{x \to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_+(\gamma)}} = +\infty. \tag{6.6}$$

In particular, there exists $(x_k)_k \in \Omega$ such that for all $k \in \mathbb{N}$,

$$\lim_{k \to +\infty} x_k = 0 \text{ and } \frac{u(x_k)}{d(x_k,\partial\Omega)|x_k|^{-\alpha_+(\gamma)}} \geq k. \tag{6.7}$$

We claim that there exists $C > 0$ such that

$$\frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_+(\gamma)}} \geq Ck \text{ for all } x \in \Omega \cap \partial B_{r_k}(0), \text{ with } r_k := |x_k| \to 0. \tag{6.8}$$

We prove the claim by using the Harnack inequality (6.2): first take the chart φ at 0 as in (4.7), and define

$$u_k(x) := u \circ \varphi(r_k x) \text{ for } x \in \mathbb{R}^n_+ \cap B_3(0) \setminus \{0\}.$$

Equation (6.1) rewrites

$$-\Delta_{g_k} u_k + a_k u_k = 0 \text{ in } \mathbb{R}^n_+ \cap B_3(0) \setminus \{0\}, \tag{6.9}$$

with $a_k(x) := -r_k^2 \frac{\gamma + O(r_k^\tau |x|^\tau)}{|\varphi(r_k x)|^2}$. In particular, there exists $M > 0$ such that $|a_k(x)| \leq M$ for all $x \in \mathbb{R}^n_+ \cap B_3(0) \setminus \overline{B}_{1/3}(0)$. Since $u_k \geq 0$, the Harnack inequality (6.2) yields the existence of $C > 0$ such that

$$\frac{u_k(y)}{y_1} \geq C \frac{u_k(x)}{x_1} \quad \text{for all } x, y \in \mathbb{R}^n_+ \cap B_2(0) \setminus \overline{B}_{1/2}(0). \tag{6.10}$$

Let $\tilde{x}_k \in \mathbb{R}^n_+$ be such that $x_k = \varphi(r_k \tilde{x}_k)$. In particular, $|\tilde{x}_k| = 1 + o(1)$ as $k \to +\infty$. It then follows from (6.7), (6.9) and (6.10) that

$$\frac{u \circ \varphi(r_k y)}{d(\varphi(r_k y), \partial\Omega)} \geq C \cdot k \quad \text{for all } y \in \mathbb{R}^n_+ \cap B_2(0) \setminus \overline{B}_{1/2}(0).$$

In particular, (6.8) holds.

We let now W be a smooth domain such that (4.26) holds for $r > 0$ small enough. Take the super-solution $u^{(d)}_{\alpha_+(\gamma),-}$ defined in Proposition 4.5. We have that

$$u(x) \geq \frac{C \cdot k}{2} u^{(d)}_{\alpha_+(\gamma),-}(x) \text{ for all } x \in W \cap \partial B_{r_k}(0).$$

Since $u^{(d)}_{\alpha_+(\gamma),-}$ vanishes on ∂W, we have $u(x) \geq \frac{C \cdot k}{2} u^{(d)}_{\alpha_+(\gamma),-}(x)$ for all $x \in \partial(W \cap B_{r_k}(0))$. Moreover, we have that

$$-\Delta u^{(d)}_{\alpha_+(\gamma),-} - \frac{\gamma + O(|x|^\tau)}{|x|^2} u^{(d)}_{\alpha_+(\gamma),-} < 0 = -\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2} u \quad \text{on } W.$$

Up to taking r even smaller, it follows from the coercivity of the operator and the maximum principle that

$$u(x) \geq \frac{C \cdot k}{2} u^{(d)}_{\alpha_+(\gamma),-}(x) \text{ for all } x \in W \cap B_{r_k}(0). \tag{6.11}$$

For any $x \in W$, we let $k_0 \in \mathbb{N}$ such that $r_k < |x|$ for all $k \geq k_0$. It then follows from (6.11) that $u(x) \geq \frac{C \cdot k}{2} u^{(d)}_{\alpha_+(\gamma),-}(x)$ for all $k \geq k_0$. Letting $k \to +\infty$ yields that $u^{(d)}_{\alpha_+(\gamma),-}(x)$ goes to zero for all $x \in W$. This is in contradiction with (4.29). Hence (6.6) does not hold, and therefore (6.5) holds. A straightforward consequence of (6.5) and Lemma 5.2 is that there exists $l \in \mathbb{R}$ such that

$$\lim_{x \to 0} \frac{u(x)}{d(x, \partial\Omega)|x|^{-\alpha_+(\gamma)}} = l. \tag{6.12}$$

We now show the following lemma:

Lemma 6.3. *If* $\lim_{x \to 0} \frac{u(x)}{d(x,\partial\Omega)|x|^{-\alpha_+(\gamma)}} = 0$*, then* $u \in D^{1,2}(\Omega)_{loc,0}$ *and there exists* $K > 0$ *such that* $u(x) \sim_{x \to 0} K \frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}$.

Proof of Lemma 6.3: We shall use Theorem 4.1. Take W as in (4.26) and let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 0$ for $x \in B_{\delta/4}(0)$ and $\eta(x) = 1$ for $x \in \mathbb{R}^n \setminus B_{\delta/3}(0)$. Define

$$f(x) := \left(-\Delta - \frac{\gamma + O(|x|^\tau)}{|x|^2}\right)(\eta u) \text{ for } x \in W.$$

The function $f \in C^\infty(\overline{W})$ vanishes around 0. Let $v \in D^{1,2}(\Omega)$ be such that

$$\left\{\begin{array}{rl} -\Delta v - \frac{\gamma+O(|x|^\tau)}{|x|^2} v = f & \text{in } W \\ v = 0 & \text{on } \partial W. \end{array}\right.$$

Note again that for $r > 0$ small enough, $-\Delta - (\gamma + O(|x|^\tau))|x|^{-2}$ is coercive on W, and therefore, the existence of v is ensured for small r. Define

$$\tilde{u} := u - \eta u + v.$$

The properties of W and the definition of η and v yield

$$\left\{\begin{array}{rl} -\Delta \tilde{u} - \frac{\gamma+O(|x|^\tau)}{|x|^2} \tilde{u} = 0 & \text{in } W \\ \tilde{u} = 0 & \text{in } \partial W \setminus \{0\}. \end{array}\right.$$

Moreover, since $-\Delta v - (\gamma + O(|x|^\tau))|x|^{-2} v = 0$ around 0 and $v \in D^{1,2}(W)$, it follows from Theorem 4.1 that there exists $C > 0$ such that $|v(x)| \leq C d(x, W)|x|^{-\alpha_-(\gamma)}$ for all $x \in W$. Therefore, we have that

$$\lim_{x\to 0} \frac{\tilde{u}(x)}{d(x,\partial\Omega)|x|^{-\alpha_+(\gamma)}} = 0. \tag{6.13}$$

It then follows from Lemma 5.2 that

$$\lim_{x\to 0} |x|^{\alpha_+(\gamma)} |\nabla \tilde{u}(x)| = 0. \tag{6.14}$$

Let $\psi \in C_c^\infty(W)$ and $w \in D^{1,2}(W)$ be such that

$$\left\{\begin{array}{rl} -\Delta w - \frac{\gamma+O(|x|^\tau)}{|x|^2} w = \psi & \text{in } W \\ w = 0 & \text{on } \partial W. \end{array}\right.$$

Since ψ vanishes around 0, it follows from Theorem 4.1 and Lemma 5.2 that

$$w(x) = O(d(x,\partial W)|x|^{-\alpha_-(\gamma)}) \quad \text{and} \quad |\nabla w(x)| = O(|x|^{-\alpha_-(\gamma)}) \quad \text{as } x \to 0. \tag{6.15}$$

Fix $\epsilon > 0$ small and integrate by parts using that both $\tilde{u}$ and w vanish on ∂W, to get

$$\begin{aligned} 0 &= \int_{W\setminus B_\epsilon(0)} \left(-\Delta \tilde{u} - \frac{\gamma + O(|x|^\tau)}{|x|^2} \tilde{u}\right) w \, dx \\ &= \int_{W\setminus B_\epsilon(0)} \left(-\Delta w - \frac{\gamma + O(|x|^\tau)}{|x|^2} w\right) \tilde{u} \, dx + \int_{\partial(W\setminus B_\epsilon(0))} (-w\partial_\nu \tilde{u} + \tilde{u}\partial_\nu w) \, d\sigma \\ &= \int_{W\setminus B_\epsilon(0)} \psi \tilde{u} \, dx - \int_{\Omega\cap\partial B_\epsilon(0)} (-w\partial_\nu \tilde{u} + \tilde{u}\partial_\nu w) \, d\sigma. \end{aligned}$$

Using the limits and estimates (6.13), (6.14) and (6.15), and that ψ vanishes around 0, we get

$$\begin{aligned} 0 &= \int_{W\setminus B_\epsilon(0)} \psi \tilde{u} \, dx + o\left(\epsilon^{n-1}(\epsilon^{1-\alpha_-(\gamma)}\epsilon^{-\alpha_+(\gamma)} + \epsilon^{1-\alpha_+(\gamma)}\epsilon^{-\alpha_-(\gamma)})\right) \\ &= \int_{W\setminus B_\epsilon(0)} \psi \tilde{u} \, dx + o(1), \quad \text{as } \epsilon \to 0. \end{aligned}$$

Therefore, we have $\int_W \psi \tilde{u} \, dx = 0$ for all $\psi \in C_c^\infty(W)$. Since $\tilde{u} \in L^p$ is smooth outside 0, we then get that $\tilde{u} \equiv 0$, and therefore $u = \eta u + v$. In particular, $u \in D^{1,2}(\Omega)_{loc,0}$ is a distributional positive solution to $-\Delta u - \frac{\gamma+O(|x|^\tau)}{|x|^2} u = 0$ on W. It then follows from Theorem 4.1 that there exists $K > 0$ such that $u(x) \sim_{x\to 0} K \frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}$. This proves Lemma 6.3. □

Combining Lemma 6.3 with (6.12) completes the proof of Theorem 6.1. □

As a consequence of Theorem 6.1, we improve Lemma 4.2 as follows.

Proposition 6.4. *Let $u \in C^2(\overline{\mathbb{R}^n_+} \setminus \{0\})$ be a nonnegative function such that*

$$-\Delta u - \frac{\gamma}{|x|^2} u = 0 \text{ in } \mathbb{R}^n_+ \; ; \; u = 0 \text{ on } \partial\mathbb{R}^n_+. \tag{6.16}$$

Then there exist $\lambda_-, \lambda_+ \geq 0$ such that

$$u(x) = \lambda_- x_1 |x|^{-\alpha_-(\gamma)} + \lambda_+ x_1 |x|^{-\alpha_+(\gamma)} \text{ for all } x \in \mathbb{R}^n_+.$$

Proof of Proposition 6.4: Without loss of generality, we assume that $u \not\equiv 0$, so that $u > 0$. We consider the Kelvin transform of u defined by $\hat{u}(x) := |x|^{2-n} u(x/|x|^2)$ for all $x \in \mathbb{R}^n_+$. Both u and $\hat{u}$ are then nonnegative solutions of (6.16). It follows from Theorem 6.1 that, after performing back the Kelvin transform, there exist $\alpha_1, \alpha_2 \in \{\alpha_+(\gamma), \alpha_-(\gamma)\}$ such that

$$\lim_{x\to 0} \frac{u(x)}{x_1|x|^{-\alpha_1}} = l_1 > 0 \text{ and } \lim_{|x|\to\infty} \frac{u(x)}{x_1|x|^{-\alpha_2}} = l_2 > 0.$$

If $\alpha_1 \leq \alpha_2$, then $u(x) \leq C x_1 |x|^{-\alpha_1}$ for all $x \in \mathbb{R}^n_+$. The result then follows from Lemma 4.2. If $\alpha_1 > \alpha_2$, then $\alpha_1 = \alpha_+(\gamma)$ and $\alpha_2 = \alpha_-(\gamma)$. We define

$$\tilde{u}(x) := u(x) - l_1 x_1 |x|^{-\alpha_+(\gamma)} \text{ for all } x \in \mathbb{R}^n_+.$$

to obtain that $-\Delta\tilde{u} - \frac{\gamma}{|x|^2}\tilde{u} = 0$ in $\mathbb{R}^n_+$, $\tilde{u} = 0$ on $\partial\mathbb{R}^n_+$, and $\tilde{u}(x) = o(x_1|x|^{-\alpha_+(\gamma)})$ as $x \to 0$. Arguing as in the proof of Lemma 6.3, we get that $\tilde{u} \in D^{1,2}(\mathbb{R}^n_+)_{loc,0}$ and $\tilde{u}(x) = O(x_1|x|^{-\alpha_-(\gamma)})$ as $x \to 0$. Moreover, we have that $\tilde{u}(x) = (l_2 + o(1)) x_1 |x|^{-\alpha_-(\gamma)}$ as $|x| \to +\infty$, therefore $\tilde{u}(x) > 0$ for $|x| >> 1$. Since $\tilde{u} \in D^{1,2}(\mathbb{R}^n_+)_{loc,0}$, the comparison principle then yields $\tilde{u} > 0$ everywhere. We also have that $\tilde{u}(x) \leq C x_1 |x|^{-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_+$. It then follows from Lemma 4.2 that there exists $\lambda_- \geq 0$ such that $\tilde{u}(x) = \lambda_- x_1 |x|^{-\alpha_-(\gamma)}$ for all $x \in \mathbb{R}^n_+$, from which Proposition 6.4 follows. $\square$

7. The Hardy singular boundary mass of a domain Ω when $0 \in \partial\Omega$

We shall proceed in the following theorem to define the mass of a smooth bounded domain Ω of $\mathbb{R}^n$ such as $0 \in \partial\Omega$. It will involve the expansion of positive singular solutions of the Dirichlet boundary problem $L_\gamma u = 0$.

Theorem 7.1. *Let Ω be a smooth bounded domain Ω of $\mathbb{R}^n$ such as $0 \in \partial\Omega$, and assume that $\frac{n^2-1}{4} < \gamma < \gamma_H(\Omega)$. Then, up to multiplication by a positive constant, there exists a unique function $H \in C^2(\overline{\Omega} \setminus \{0\})$ such that*

$$-\Delta H - \frac{\gamma}{|x|^2} H = 0 \text{ in } \Omega \;,\; H > 0 \text{ in } \Omega \;,\; H = 0 \text{ on } \partial\Omega \setminus \{0\}. \tag{7.1}$$

Moreover, there exists $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$H(x) = c_1 \frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}} + c_2 \frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \text{ as } x \to 0. \tag{7.2}$$

The quantity $m_\gamma(\Omega) := \frac{c_2}{c_1} \in \mathbb{R}$, which is independent of the choice of H satisfying (7.1), will be called the Hardy b-mass of Ω associated to L_γ.

Proof of Theorem 7.1. First, we start by constructing a singular solution H_0 for (7.1). For that, consider $u_{\alpha_+(\gamma)}$ as in (4.14) and let $\eta \in C^\infty_c(\mathbb{R}^n)$ be such that $\eta(x) = 1$ for $x \in B_{\delta/2}(0)$ and $\eta(x) = 0$ for $x \in \mathbb{R}^n \setminus B_\delta(0)$. Set

$$f := -\Delta(\eta u_{\alpha_+(\gamma)}) - \frac{\gamma}{|x|^2}(\eta u_{\alpha_+(\gamma)}) \text{ in } \overline{\Omega} \setminus \{0\}.$$

It follows from (4.19) and (4.5) that f is smooth outside 0 and that

$$f(x) = O\left(d(x,\partial\Omega)|x|^{-\alpha_+(\gamma)-1}\right) = O\left(|x|^{-\alpha_+(\gamma)}\right) \text{ in } \Omega \cap B_{\delta/2}(0).$$

Since $\gamma > \frac{n^2-1}{4}$, we have that $\alpha_+(\gamma) < \frac{n+1}{2}$, and therefore $f \in L^{\frac{2n}{n+2}}(\Omega) = \left(L^{2^\star}(\Omega)\right)' \subset \left(D^{1,2}(\Omega)\right)'$. It then follows from the coercivity assumption $\gamma < \gamma_H(\Omega)$ that there exists $v \in D^{1,2}(\Omega)$ such that

$$-\Delta v - \frac{\gamma}{|x|^2} v = f \text{ in } \left(D^{1,2}(\Omega)\right)'.$$

Let $v_1, v_2 \in D^{1,2}(\Omega)$ be such that

$$-\Delta v_1 - \frac{\gamma}{|x|^2} v_1 = f_+ \text{ and } -\Delta v_2 - \frac{\gamma}{|x|^2} v_2 = f_- \text{ in } \left(D^{1,2}(\Omega)\right)'. \tag{7.3}$$

In particular, $v = v_1 - v_2$ and $v_1, v_2 \in C^1(\overline{\Omega} \setminus \{0\})$, and they vanish on $\partial\Omega \setminus \{0\}$. Assume that $f_+ \not\equiv 0$. Since $f_+ \geq 0$, the comparison principle yields $v_1 > 0$ on $\Omega \setminus \{0\}$ and $\partial_\nu v_1 < 0$ on $\partial\Omega \setminus \{0\}$. Therefore, for any $\delta > 0$ small enough, there exists $C(\delta) > 0$ such that $v_1(x) \geq C(\delta) d(x, \partial\Omega)$ for all $x \in \partial B_\delta(0) \cap \Omega$. Let $u_{\alpha_-(\gamma),-}$ be the sub-solution defined in (4.4). It follows from the asymptotic (4.5) that there exists $C'(\delta) > 0$ such that $v_1 \geq C'(\delta) u_{\alpha_-(\gamma),-}$ in $\partial B_\delta(0) \cap \Omega$. Since this inequality also holds on $\partial(B_\delta(0) \cap \Omega)$ and that

$$(-\Delta - \tfrac{\gamma}{|x|^2})(v_1 - C'(\delta) u_{\alpha_-(\gamma),-}) \geq 0 \quad \text{in } B_\delta(0) \cap \Omega,$$

coercivity and the maximum principle yield $v_1 \geq C'(\delta) u_{\alpha_-(\gamma),-}$ in $B_\delta(0) \cap \Omega$. It then follows from (4.5) that there exists $c > 0$ such that

$$v_1(x) \geq c \cdot d(x, \partial\Omega) |x|^{-\alpha_-(\gamma)} \text{ in } B_\delta(0) \cap \Omega.$$

Therefore, we have for $x \in B_\delta(0) \cap \Omega$,

$$\begin{aligned} f_+(x) &\leq C d(x, \partial\Omega) |x|^{-\alpha_+(\gamma) - 1} \\ &\leq \frac{C}{c} |x|^{\alpha_-(\gamma) - \alpha_+(\gamma) - 1} v_1(x) \\ &\leq \frac{C}{c} |x|^{\alpha_-(\gamma) - \alpha_+(\gamma) + 1} \frac{v_1(x)}{|x|^2} \end{aligned}$$

Therefore, (7.3) yields

$$-\Delta v_1 + \frac{\gamma + O(|x|^{\alpha_-(\gamma) - \alpha_+(\gamma) + 1})}{|x|^2} v_1 = 0 \text{ in } B_\delta(0) \cap \Omega.$$

Since $\gamma > \frac{n^2-1}{4}$, we have that $\alpha_-(\gamma) - \alpha_+(\gamma) + 1 > 0$. Since $v_1 \in D^{1,2}(\Omega)$, $v_1 \geq 0$ and $v_1 \not\equiv 0$, it follows from Theorem 4.1 that there exists $K_1 > 0$ such that

$$v_1(x) = K_1 \frac{d(x, \partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x, \partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \quad \text{as } x \to 0. \tag{7.4}$$

If $f_+ \equiv 0$, then $v_1 \equiv 0$ and (7.4) holds with $K_1 = 0$. Arguing similarly for f_-, and using that $v = v_1 - v_2$, we then get that there exists $K \in \mathbb{R}$ such that

$$v(x) = -K \frac{d(x, \partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x, \partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \quad \text{as } x \to 0. \tag{7.5}$$

Set

$$H_0(x) := \eta(x) u_{\alpha_+(\gamma)}(x) - v(x) \text{ for all } x \in \overline{\Omega} \setminus \{0\}. \tag{7.6}$$

It follows from the definition of v and the regularity outside 0 that

$$-\Delta H_0 - \frac{\gamma}{|x|^2} H_0 = 0 \text{ in } \Omega\,;\ H_0(x) = 0 \text{ in } \partial\Omega \setminus \{0\}.$$

Moreover, the asymptotics (4.5) and (7.5) yield $H_0(x) > 0$ on $\Omega \cap B_{\delta'}(0)$ for some $\delta' > 0$ small enough. It follows from the comparison principle that $H_0 > 0$ in Ω.

We now perform an expansion of H_0. First note that from the definition (4.14) of $u_{\alpha_+(\gamma)}$, the asymptotic (7.5) of v and the fact that $\alpha_+(\gamma)-\alpha_-(\gamma)<1$, we have

$$\begin{aligned} H_0(x) &= \frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}(1+O(|x|))+K\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}+o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right)\\ &= \frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}+K\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}+o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right)\end{aligned}$$

as $x\to 0$. In particular, since in addition $H_0>0$ in Ω, there exists $c>1$ such that

$$\frac{1}{c}\frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}\leq H_0(x)\leq c\frac{d(x,\partial\Omega)}{|x|^{\alpha_+(\gamma)}}\quad \text{for all } x\in\Omega. \tag{7.7}$$

Finally, we establish the uniqueness. For that, we let $H\in C^2(\overline{\Omega}\setminus\{0\})$ be as in (7.1) and set

$$\lambda_0:=\max\{\lambda\geq 0/\ H\geq \lambda H_0\}.$$

The number λ_0 is clearly defined, and so we set $\tilde{H}:=H-\lambda_0 H_0\geq 0$. Assume that $\tilde{H}\not\equiv 0$. Since $-\Delta\tilde{H}-\gamma|x|^{-2}\tilde{H}=0$, it follows from Theorem 6.1 that there exists $\alpha\in\{\alpha_+(\gamma),\alpha_-(\gamma)\}$ and $K>0$ such that

$$H(x)\sim_{x\to 0} K\frac{d(x,\partial\Omega)}{|x|^{\alpha}}. \tag{7.8}$$

If $\alpha=\alpha_-(\gamma)$, then $\tilde{H}\in D^{1,2}(\Omega)$ is a variational solution to $-\Delta\tilde{H}-\frac{\gamma}{|x|^2}\tilde{H}=0$ in Ω. The coercivity then yields that $\tilde{H}\equiv 0$, contradicting the initial hypothesis.
Therefore $\alpha=\alpha_+(\gamma)$. Since $\tilde{H}>0$ vanishes on $\partial\Omega\setminus\{0\}$, then for any $\delta>0$, there exists $c(\delta)>0$ such that

$$\tilde{H}(x)\geq c(\delta)d(x,\partial\Omega) \text{ for } x\in\Omega\setminus B_\delta(0). \tag{7.9}$$

Therefore, (7.8), (7.9) and (7.7) yield the existence of $c>0$ such that $\tilde{H}\geq cH_0$, and then $H\geq(\lambda_0+c)H_0$, contradicting the definition of λ_0. It follows that $\tilde{H}\equiv 0$, which means that $H=\lambda_0 H_0$ for some $\lambda_0>0$. This proves uniqueness and completes the proof of Theorem 7.1. □
Now we establish the monotonicity of the mass with respect to set inclusion.

Proposition 7.2. *The mass m_γ is a strictly increasing set-function in the following sense: Assume Ω_1,Ω_2 are two smooth bounded domains such that* $0\in\partial\Omega_1\cap\partial\Omega_2$, *and* $\frac{n^2-1}{4}<\gamma<\min\{\gamma_H(\Omega_1),\gamma_H(\Omega_2)\}$, *then*

$$\Omega_1\subsetneq\Omega_2\ \Rightarrow\ m_\gamma(\Omega_1)<m_\gamma(\Omega_2). \tag{7.10}$$

Moreover, if $\Omega\subsetneq\mathbb{R}^n_+$ and $\frac{n^2-1}{4}<\gamma<\frac{n^2}{4}$, then $m_\gamma(\Omega)<0$.

Proof of Proposition 7.2: It follows from the definition of the mass that for $i=1,2$, there exists $H_i\in C^2(\overline{\Omega_i}\setminus\{0\})$ such that

$$-\Delta H_i-\frac{\gamma}{|x|^2}H_i=0 \text{ in } \Omega_i\ ,\ H_i>0 \text{ in } \Omega_i\ ,\ H_i=0 \text{ on } \partial\Omega_i, \tag{7.11}$$

with

$$H_i(x)=\frac{d(x,\partial\Omega_i)}{|x|^{\alpha_+(\gamma)}}+m_\gamma(\Omega_i)\frac{d(x,\partial\Omega_i)}{|x|^{\alpha_-(\gamma)}}+o\left(\frac{d(x,\partial\Omega_i)}{|x|^{\alpha_-(\gamma)}}\right) \tag{7.12}$$

as $x\to 0$, $x\in\Omega_i$. Set $h:=H_2-H_1$ on Ω_1. Since $\Omega_1\subsetneq\Omega_2$, we have that

$$\left\{\begin{array}{ll} -\Delta h-\frac{\gamma}{|x|^2}h=0 & \text{in } \Omega_1\\ h\geq 0,\ h\not\equiv 0 & \text{on } \partial\Omega_1\end{array}\right. \tag{7.13}$$

First, we claim that $h \in H^{1,2}(\Omega_1)$. Indeed, it follows from the construction of the singular function (see (7.6)), that there exists $w \in H^{1,2}(\Omega_1)$ such that

$$h(x) = \frac{d(x, \partial\Omega_2) - d(x, \partial\Omega_1)}{|x|^{\alpha_+(\gamma)}} + w(x) \text{ for all } x \in \Omega_1. \tag{7.14}$$

Since $\Omega_1 \subset \Omega_2$ and 0 is on the boundary of both domains, then the tangent spaces at 0 of Ω_1 and Ω_2 are equal, and one gets that $d(x, \partial\Omega_1) - d(x, \partial\Omega_2) = O(|x|^2)$ as $x \to 0$. Since $\alpha_+(\gamma) - \alpha_-(\gamma) < 1$, we then get that

$$\tilde{h}(x) := \frac{d(x, \partial\Omega_2) - d(x, \partial\Omega_1)}{|x|^{\alpha_+(\gamma)}} = O(|x|^{1-\alpha_-(\gamma)}) \text{ as } x \to 0.$$

Similarly, $|\nabla \tilde{h}(x)| = O(|x|^{-\alpha_-(\gamma)})$ as $x \to 0$. Therefore, we deduce that $\tilde{h} \in H^{1,2}(\Omega_1)$. It then follows from (7.14) that $h \in H^{1,2}(\Omega_1)$.
To prove the monotonicity, note first that since $\gamma < \gamma_H(\Omega_1)$ and $h \in H^{1,2}(\Omega_1)$, it follows from (7.13) and the comparison principle that $h \geq 0$ in Ω_1 (indeed, this is obtained by multiplying (7.13) by $h_- \in D_1^2(\Omega)$ and integrating: therefore, coercivity yields $h_- \equiv 0$). Since $h \not\equiv 0$, it follows from Hopf's maximum principle that for any $\delta > 0$ small, there exists $C(\delta) > 0$ such that $h(x) \geq C(\delta) d(x, \partial\Omega_1)$ for all $x \in \partial B_\delta(0) \cap \Omega_1$. We define the sub-solution $u_{\alpha_-(\gamma),-}$ as in Proposition 4.3. It then follows from the inequality above and the asymptotics in (4.5) that there exists $\epsilon_0 > 0$ such that $h(x) \geq 2\epsilon_0 u_{\alpha_-(\gamma),-}(x)$ for all $x \in \partial B_\delta(0) \cap \Omega_1$. This inequality also holds on $B_\delta(0) \cap \partial\Omega_1$ since $u_{\alpha_-(\gamma),-}$ vanishes on $\partial\Omega_1$. It then follows from the maximum principle that $h(x) \geq 2\epsilon_0 u_{\alpha_-(\gamma),-}(x)$ for all $x \in B_\delta(0) \cap \Omega_1$. With the definition of h and the asymptotic (4.5), we then have that for $\delta' > 0$ small enough

$$H_2(x) - H_1(x) \geq \epsilon_0 \frac{d(x, \partial\Omega_1)}{|x|^{\alpha_-(\gamma)}} \text{ for all } x \in B_{\delta'}(0) \cap \Omega_1. \tag{7.15}$$

We let $\vec{\nu}$ be the inner unit normal vector of $\partial\Omega_1$ at 0. This is also the inner unit normal vector of $\partial\Omega_2$ at 0. Therefore, for any $t > 0$ small enough, we have that $d(t\vec{\nu}, \partial\Omega_i) = t$ for $i = 1, 2$. It then follows from the expressions (7.12) and (7.15) that

$$(m_\gamma(\Omega_2) - m_\gamma(\Omega_1)) \tfrac{t}{t^{\alpha_-(\gamma)}} + o\left(\tfrac{t}{t^{\alpha_-(\gamma)}}\right) \geq \epsilon_0 \tfrac{t}{t^{\alpha_-(\gamma)}} \quad \text{as } t \downarrow 0.$$

We then get that $m_\gamma(\Omega_2) - m_\gamma(\Omega_1) \geq \epsilon_0$, and therefore $m_\gamma(\Omega_2) > m_\gamma(\Omega_1)$. This proves (7.10) and ends the first part of Proposition 7.2.

The proof of the second part is similar. Indeed, we take $\Omega_2 := \mathbb{R}^n_+$ and we define $H_2(x) := \frac{x_1}{|x|^{\alpha_+(\gamma)}}$. Arguing as above, we get that $0 > m_\gamma(\Omega)$, which completes the proof of Proposition 7.2. □

The proof of the second part is similar. Indeed, we take $\Omega_2 := \mathbb{R}^n_+$ and we define $H_2(x) := \frac{x_1}{|x|^{\alpha_+(\gamma)}}$. Arguing as above, we get that $0 > m_\gamma(\Omega)$, which completes the proof of Proposition 7.2. □

Note that we have used above that the mass $m_\gamma(\mathbb{R}^n_+) = 0$ even though we had only defined the mass for bounded sets. In the rest of the section, we shall extend the notion of mass to certain unbounded sets that include $\mathbb{R}^n_+$. For that, we shall use the Kelvin transformation, defined as follows: For any $x_0 \in \mathbb{R}^n$, let

$$i_{x_0}(x) := x_0 + |x_0|^2 \frac{x - x_0}{|x - x_0|^2} \quad \text{for all } x \in \mathbb{R}^n \setminus \{x_0\}. \tag{7.16}$$

The inversion i_{x_0} is clearly the identity map on $\partial B_{|x_0|}(x_0)$ (the ball of center x_0 and of radius $|x_0|$), and in particular $i_{x_0}(0) = 0$.

Definition 7.3. *We say that a domain $\Omega \subset \mathbb{R}^n$ ($0 \in \partial\Omega$) is conformally bounded if there exists $x_0 \notin \overline{\Omega}$ such that $i_{x_0}(\Omega)$ is a smooth bounded domain of $\mathbb{R}^n$ having both 0 and x_0 on its boundary $\partial(i_{x_0}(\Omega))$.*

One can easily check that $\mathbb{R}^n_+$ is a smooth domain at infinity (take $x_0 := (-1, 0, \dots, 0)$). The following proposition shows that the notion of mass extends to unbounded domains that are conformally bounded.

Proposition 7.4. *Let Ω be a conformally bounded domain in $\mathbb{R}^n$ such that $0 \in \partial\Omega$. Assume that $\gamma_H(\Omega) > \frac{n^2-1}{4}$ and that $\gamma \in \left(\frac{n^2-1}{4}, \gamma_H(\Omega)\right)$. Then, up to a multiplicative constant, there exists a unique function $H \in C^2(\overline{\Omega} \setminus \{0\})$ such that*

$$\left\{\begin{array}{rl} -\Delta H - \frac{\gamma}{|x|^2} H = 0 & \text{in } \Omega \\ H > 0 & \text{in } \Omega \\ H = 0 & \text{on } \partial\Omega \setminus \{0\} \\ H(x) \leq C|x|^{1-\alpha_+(\gamma)} & \text{for } x \in \Omega. \end{array}\right. \tag{7.17}$$

Moreover, there exists $c_1 > 0$ and $c_2 \in \mathbb{R}$ such that

$$H(x) = c_1 \frac{d(x, \partial\Omega)}{|x|^{\alpha_+(\gamma)}} + c_2 \frac{d(x, \partial\Omega)}{|x|^{\alpha_-(\gamma)}} + o\left(\frac{d(x, \partial\Omega)}{|x|^{\alpha_-(\gamma)}}\right) \quad \text{as } x \to 0.$$

We define the mass $b_\gamma(\Omega) := \frac{c_2}{c_1}$, which is independent of the choice of H in (7.17).

Proof: For convenience, up to a rotation and a dilation, we can assume that $x_0 := (-1, 0, \dots, 0) \in \mathbb{R}^n$ so that the inversion becomes

$$i(x) := x_0 + \frac{x - x_0}{|x - x_0|^2} \quad \text{for all } x \in \mathbb{R}^n \setminus \{x_0\}.$$

For any $u \in C^2(U)$, with $U \subset \mathbb{R}^n$, we define its Kelvin transform $\hat{u} : \hat{U} \to \mathbb{R}$ by

$$\hat{u}(x) := |x - x_0|^{2-n} u(i(x)) \text{ for all } x \in \hat{U} := i^{-1}(U \setminus \{x_0\}).$$

This transform leaves the Laplacian invariant in the following sense:

$$-\Delta\hat{u}(x) = |x - x_0|^{-(n+2)}(-\Delta u)(i(x)) \text{ for all } x \in \hat{U}. \tag{7.18}$$

Define $\tilde{\Omega} := i(\Omega)$ and suppose $u \in C^2(\overline{\Omega} \setminus \{0\})$ is such that

$$-\Delta u - \frac{\gamma}{|x|^2} u = 0 \text{ in } \Omega\ ,\ u > 0 \text{ in } \Omega\ ,\ u = 0 \text{ on } \partial\Omega.$$

The Kelvin transform $\tilde{u}$ of u then satisfies

$$-\Delta\tilde{u} - V\tilde{u} = 0 \text{ in } \tilde{\Omega},$$

where

$$V(x) := \frac{\gamma}{|x|^2|x - x_0|^2} \text{ for } x \in \mathbb{R}^n \setminus \{0, x_0\}. \tag{7.19}$$

It is easy to check that

$$V(x) = \frac{\gamma + O(|x|)}{|x|^2} \text{ as } x \to 0 \quad \text{ and } \quad V(x) = \frac{\gamma + O(|x - x_0|)}{|x - x_0|^2} \text{ as } x \to x_0.$$

In other words, the Kelvin transform allows us to reduce the study of the Hardy-singular boundary mass of a conformally bounded domain Ω into defining a notion of mass for the Schrödinger operator $-\Delta + V$ on $\tilde{\Omega}$.

Note that the coercivity of $-\Delta-\gamma|x|^{-2}$ on Ω (since $\gamma<\gamma_H(\Omega)$) yields the coercivity of $-\Delta-V$ on $\tilde{\Omega}$, that is there exists $c_0>0$ such that

$$\int_{\tilde{\Omega}}\left(|\nabla u|^2-V(x)u^2\right)\,dx\geq c_0\int_{\tilde{\Omega}}|\nabla u|^2\,dx \text{ for all } u\in D^{1,2}(\tilde{\Omega}).$$

Arguing as is Section 4, we get for $\delta>0$ small enough, a function u_{α_+}

$$\left\{\begin{array}{lll}(-\Delta-V)u_{\alpha_+} & =O(d(x,\partial\tilde{\Omega})|x|^{-\alpha_+(\gamma)-1}) & \text{in } \tilde{\Omega}\cap\tilde{B}_\delta\\ u_{\alpha_+} & >0 & \text{in } \tilde{\Omega}\cap\tilde{B}_\delta\\ u_{\alpha_+} & =0 & \text{on } \partial\tilde{\Omega}\setminus\{0\},\end{array}\right.$$

and

$$u_{\alpha_+}(x)=\frac{d(x,\partial\tilde{\Omega})}{|x|^{\alpha_+(\gamma)}}(1+O(|x|)\text{ as } x\to 0.$$

The function $f_0:=-\Delta u_{\alpha_+}-Vu_{\alpha_+}$, then satisfies for all $x\in\tilde{\Omega}\cap\tilde{B}_\delta$,

$$|f_0(x)|\leq Cd(x,\partial\tilde{\Omega})|x|^{-\alpha_+(\gamma)-1}\leq C|x|^{-\alpha_+(\gamma)}$$

where C is a positive constant. Since $\gamma>\frac{n^2-1}{4}$, it follows that $f_0\in L^{\frac{2n}{n+2}}(\tilde{\Omega})$. Let now $v_0\in D^{1,2}(\tilde{\Omega})$ be such that

$$-\Delta v_0-Vv_0=f_0\text{ weakly in } D^{1,2}(\tilde{\Omega}). \tag{7.20}$$

The existence follows from the coercivity of $-\Delta-V$ on $\tilde{\Omega}$, and the proof of Theorem 7.1) yields that around 0, $|v_0(x)|$ is bounded by $|x|^{1-\alpha_-(\gamma)}$. Note that around x_0, we have $-\Delta v_0-Vv_0=0$ and the regularity Theorem 5.1 yields a control by $|x-x_0|^{1-\alpha_-(\gamma)}$, which means that there exists $C>0$ such that

$$|v_0(x)|\leq Cd(x,\partial\tilde{\Omega})\left(|x|^{-\alpha_-(\gamma)}+|x-x_0|^{-\alpha_-(\gamma)}\right)\quad\text{for all } x\in\tilde{\Omega}.$$

The construction of the mass (Theorem 7.1) and the regularity Theorem 5.1 then yield that there exists $K_0\in\mathbb{R}$ such that

$$v_0(x)=K_0\frac{d(x,\partial\tilde{\Omega})}{|x|^{\alpha_-(\gamma)}}+o\left(\frac{d(x,\partial\tilde{\Omega})}{|x|^{\alpha_-(\gamma)}}\right). \tag{7.21}$$

Define now $\tilde{H}_0(x):=u_{\alpha_+(\gamma)}(x)-v_0(x)$ for all $x\in\overline{\tilde{\Omega}}\setminus\{0,x_0\}$, and consider its Kelvin transform

$$H_0(x):=|x-x_0|^{2-n}\tilde{H}_0(i(x))=|x-x_0|^{2-n}\left(u_{\alpha_+(\gamma)}-v_0\right)(i(x)),x\in\Omega. \tag{7.22}$$

It follows from (7.18), the definitions of $u_{\alpha_+(\gamma)}$ and v_0 that H_0 satisfies the following properties:

$$\left\{\begin{array}{rll}-\Delta H_0-\frac{\gamma}{|x|^2}H_0 & =0 & \text{in } \Omega\\ H_0 & >0 & \text{in } \Omega\\ H_0 & =0 & \text{in } \partial\Omega\setminus\{0\}.\end{array}\right. \tag{7.23}$$

Concerning the pointwise behavior, we have that

$$H_0(x)=\frac{d(x,\partial\Omega)}{|x|^{\alpha_+}}-K_0\frac{d(x,\partial\Omega)}{|x|^{\alpha_-}}+o\left(\frac{d(x,\partial\Omega)}{|x|^{\alpha_-}}\right)\quad\text{as } x\to 0,\ x\in\Omega, \tag{7.24}$$

and

$$H_0(x)\leq C|x|^{1-\alpha_+}\text{ for all } x\in\Omega,\ |x|>1. \tag{7.25}$$

This proves the existence part in Proposition 7.4. In order to show uniqueness, we let $H \in C^2(\overline{\Omega}\setminus\{0\})$ be as in Proposition 7.4, and consider its Kelvin transform $\tilde{H}(x) := |x-x_0|^{2-n}H(i(x))$ for all $x \in \overline{\tilde{\Omega}} \setminus \{0, x_0\}$. The transformation law (7.18) yields

$$\begin{cases} -\Delta\tilde{H} - V\tilde{H} &= 0 \quad \text{in } \tilde{\Omega} \\ \tilde{H} &> 0 \quad \text{in } \tilde{\Omega} \\ \tilde{H} &= 0 \quad \text{in } \partial\tilde{\Omega} \setminus \{0, x_0\}. \end{cases} \tag{7.26}$$

Moreover, we have that $\tilde{H}(x) \leq C|x|^{1-\alpha_+(\gamma)} + C|x-x_0|^{1-\alpha_-(\gamma)}$ for all $x \in \tilde{\Omega}$. It then follows from Theorem 6.1 that there exist $C_1, C_2 > 0$ such that

$$\tilde{H}(x) \sim_{x\to 0} C_1 \frac{d(x,\partial\tilde{\Omega})}{|x|^\alpha} \text{ and } \tilde{H}(x) \sim_{x\to x_0} C_2 \frac{d(x,\partial\tilde{\Omega})}{|x-x_0|^{\alpha_-(\gamma)}}, \tag{7.27}$$

where $\alpha \in \{\alpha_-(\gamma), \alpha_+(\gamma)\}$. We claim that $\alpha = \alpha_+(\gamma)$. Indeed, otherwise, we would have $\tilde{H} \in D^{1,2}(\tilde{\Omega})$ (see Theorem 6.1) and then (7.26) and coercivity would yield $\tilde{H} \equiv 0$, which is a contradiction. Therefore $\alpha = \alpha_+(\gamma)$. By the same reasoning, the estimates (7.27) hold for $\tilde{H}_0$ (with different constants C_1, C_2). Arguing as in the proof of Theorem 7.1, we get that there exists $\lambda > 0$ such that $\tilde{H} = \lambda\tilde{H}_0$, and therefore $H = \lambda H_0$. This proves uniqueness and completes the proof of Proposition 7.4. Note that as a consequence of (7.24), the mass $m_\gamma(\Omega)$ is well-defined and is equal to $-K_0$.

8. Test functions and the existence of extremals

Let Ω be a domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$. For $\gamma \in \mathbb{R}$ and $s \in [0,2)$, recall that

$$\mu_{\gamma,s}(\Omega) := \inf_{u \in D^{1,2}(\Omega)\setminus\{0\}} J^\Omega_{\gamma,s}(u), \tag{8.1}$$

where

$$J^\Omega_{\gamma,s}(u) := \frac{\int_\Omega \left(|\nabla u|^2 - \frac{\gamma}{|x|^2}u^2\right)\, dx}{\left(\int_\Omega \frac{|u|^{2^\star}}{|x|^s}\, dx\right)^{\frac{2}{2^\star}}}.$$

Note that critical points $u \in D^{1,2}(\Omega)$ of $J^\Omega_{\gamma,s}$ are weak solutions to the pde

$$-\Delta u - \frac{\gamma}{|x|^2} = \lambda \frac{|u|^{2^\star-2}u}{|x|^s} \text{for some } \lambda \in \mathbb{R}, \tag{8.2}$$

which can be rescaled to be equal to 1 if $\lambda > 0$ and to be -1 if $\lambda < 0$. In this section, we investigate the existence of minimizers for $J^\Omega_{\gamma,s}$. We start with the following easy case, where we do not have extremals.

Proposition 8.1. *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain such that $0 \in \partial\Omega$ (no boundedness is assumed). When $s = 0$ and $\gamma \leq 0$, we have that $\mu_{\gamma,0}(\Omega) = \frac{1}{K(n,2)^2}$ (where $K(n,2)^{-2} = \mu_{0,0}(\mathbb{R}^n)$ is the best constant in the Sobolev inequality* (1.19)*) and there is no extremal.*

Proof of Proposition 8.1: Note that $2^\star(s) = 2^\star(0) = 2^\star$. Since $\gamma \leq 0$, we have for any $u \in C^\infty_c(\Omega) \setminus \{0\}$,

$$\frac{\int_\Omega \left(|\nabla u|^2 - \gamma\frac{u^2}{|x|^2}\right)\, dx}{\left(\int_\Omega |u|^{2^\star}\, dx\right)^{\frac{2}{2^\star}}} \geq \frac{\int_\Omega |\nabla u|^2\, dx}{\left(\int_\Omega |u|^{2^\star}\, dx\right)^{\frac{2}{2^\star}}} \geq \frac{1}{K(n,2)^2}, \tag{8.3}$$

and therefore $\mu_{\gamma,0}(\Omega) \geq \frac{1}{K(n,2)^2}$. Fix now $x_0 \in \Omega$ and let $\eta \in C^\infty_c(\Omega)$ be such that $\eta(x) = 1$ around x_0. Set $u_\varepsilon(x) := \eta(x)\left(\frac{\varepsilon}{\varepsilon^2 + |x-x_0|^2}\right)^{\frac{n-2}{2}}$ for all $x \in \Omega$ and $\varepsilon > 0$. Since $x_0 \neq 0$, it is classical (see for example Aubin [2]) that $\lim_{\varepsilon\to 0} J_{0,0}^{\Omega}(u_\varepsilon) = K(n,2)^{-2}$. It follows that $\mu_{\gamma,0}(\Omega) \leq \frac{1}{K(n,2)^2}$. This proves that $\mu_{\gamma,0}(\Omega) = \frac{1}{K(n,2)^2}$.

Assume now that there exists an extremal u_0 for $\mu_{\gamma,0}(\Omega)$ in $D^{1,2}(\Omega)\setminus\{0\}$. It then follows from (8.3) that $u_0 \in D^{1,2}(\Omega) \subset D^{1,2}(\mathbb{R}^n)$ is an extremal for the classical Sobolev inequality on $\mathbb{R}^n$. But these extremals are known (see Aubin [2]) and their support is the whole of $\mathbb{R}^n$, which is a contradiction since u_0 has bounded support in Ω. It follows that there is no extremal for $\mu_{\gamma,0}(\Omega)$. □
The remainder of the section is devoted to the proof of the following.

Theorem 8.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ ($n \geq 3$) such that $0 \in \partial\Omega$ and let $0 \leq s < 2$ and $\gamma < \frac{n^2}{4}$. Assume that either $s > 0$, or that $\{s = 0,\ n \geq 4$ and $\gamma > 0\}$. There are then extremals for $\mu_{\gamma,s}(\Omega)$ under one of the following two conditions:*

(1) *$\gamma \leq \frac{n^2-1}{4}$ and the mean curvature of $\partial\Omega$ at 0 is negative.*
(2) *$\gamma > \frac{n^2-1}{4}$ and the mass $m_\gamma(\Omega)$ of Ω is positive.*

Moreover, if $\gamma < \gamma_H(\Omega)$ (resp., $\gamma \geq \gamma_H(\Omega)$), then such extremals are positive solutions for (8.2) with $\lambda > 0$ (resp., $\lambda \leq 0$).

The remaining case $n = 3$, $s = 0$ and $\gamma > 0$ will be dealt with in Section 11.

According to Theorem 3.6, in order to establish existence of extremals, it suffices to show that $\mu_{\gamma,s}(\Omega) < \mu_{\gamma,s}(\mathbb{R}^n_+)$. The rest of the section consists in showing that the above mentioned geometric conditions lead to such gap. The existence of extremals on $\mathbb{R}^n_+$ as described in Theorem 1.3 is essential here.

In the sequel, $h_\Omega(0)$ will denote the mean curvature of $\partial\Omega$ at 0. The orientation is chosen such that the mean curvature of the canonical sphere (as the boundary of the ball) is positive. Since $\{s > 0\}$, or that $\{s = 0,\ n \geq 4$ and $\gamma > 0\}$, it follows from Theorem 1.3 that there are extremals for $\mu_{\gamma,s}(\mathbb{R}^n_+)$. The following proposition combined with Theorem 3.6 clearly yield the claims in Theorem 8.2.

Proposition 8.3. *We fix $\gamma < \frac{n^2}{4}$. Assume that there are extremals for $\mu_{\gamma,s}(\mathbb{R}^n_+)$. There exist then two families $(u^1_\varepsilon)_{\varepsilon>0}$ and $(u^2_\varepsilon)_{\varepsilon>0}$ in $D^{1,2}(\Omega)$, and two positive constants $c^1_{\gamma,s}$ and $c^2_{\gamma,s}$ such that:*

(1) *For $\gamma < \frac{n^2-1}{4}$, we have that*

$$J(u^1_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+)\left(1 + c^1_{\gamma,s}\cdot h_\Omega(0)\cdot\varepsilon + o(\varepsilon)\right) \text{ when } \varepsilon \to 0. \tag{8.4}$$

(2) *For $\gamma = \frac{n^2-1}{4}$, we have that*

$$J(u^1_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+)\left(1 + c^1_{\gamma,s}\cdot h_\Omega(0)\cdot\varepsilon\ln\frac{1}{\varepsilon} + o\left(\varepsilon\ln\frac{1}{\varepsilon}\right)\right) \text{ when } \varepsilon \to 0. \tag{8.5}$$

(3) *For $\gamma > \frac{n^2-1}{4}$, we have as $\epsilon \to 0$, that*

$$J(u^2_\epsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+)\left(1 - c^2_{\gamma,s}\cdot m_\gamma(\Omega)\cdot\varepsilon^{\alpha_+(\gamma)-\alpha_-(\gamma)} + o(\varepsilon^{\alpha_+(\gamma)-\alpha_-(\gamma)})\right). \tag{8.6}$$

Remark: When $\gamma < \frac{n^2-1}{4}$, this result is due to Chern-Lin [6]. Actually, they stated the result for $\gamma < \frac{(n-2)^2}{4}$, but their proof works for $\gamma < \frac{n^2-1}{4}$. However, when $\gamma \geq \frac{n^2-1}{4}$, we need the exact asymptotic profile of U that was described by Corollary 5.3.

Proof of Proposition 8.3: By assumption, there exists $U \in D^{1,2}(\mathbb{R}^n_+)\setminus\{0\}$, $U \geq 0$, that is a minimizer for $\mu_{\gamma,s}(\mathbb{R}^n_+)$. In other words,

$$J^{\mathbb{R}^n_+}_{\gamma,s}(U) = \frac{\int_{\mathbb{R}^n_+}\left(|\nabla U|^2 - \frac{\gamma}{|x|^2}U^2\right)dx}{\left(\int_{\mathbb{R}^n_+}\frac{|U|^{2^\star(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^\star(s)}}} = \mu_{\gamma,s}(\mathbb{R}^n_+).$$

Therefore, there exists $\lambda > 0$ such that

$$\left\{\begin{array}{rl} -\Delta U - \frac{\gamma}{|x|^2}U = \lambda\frac{U^{2^\star(s)-1}}{|x|^s} & \text{in } \mathbb{R}^n_+ \\ U > 0 & \text{in } \mathbb{R}^n_+ \\ U = 0 & \text{in } \partial\mathbb{R}^n_+ \end{array}\right\} \tag{8.7}$$

and there exist $K_1, K_2 > 0$ such that

$$U(x) \sim_{x\to 0} K_1\frac{x_1}{|x|^{\alpha_-}} \text{ and } U(x) \sim_{|x|\to+\infty} K_2\frac{x_1}{|x|^{\alpha_+}}, \tag{8.8}$$

where here and in the sequel, we write for convenience

$$\alpha_+ := \alpha_+(\gamma) \text{ and } \alpha_- := \alpha_-(\gamma).$$

In particular, it follows from Lemma 5.2 (after reducing all limits to happen at 0 via the Kelvin transform) that there exists $C > 0$ such that

$$U(x) \leq Cx_1|x|^{-\alpha_+} \text{ and } |\nabla U(x)| \leq C|x|^{-\alpha_+} \text{ for all } x \in \mathbb{R}^n_+. \tag{8.9}$$

We shall now construct a suitable test-function for each range of γ. First note that

$$\gamma < \frac{n^2-1}{4} \quad \Leftrightarrow \quad \alpha_+ - \alpha_- > 1$$
$$\gamma = \frac{n^2-1}{4} \quad \Leftrightarrow \quad \alpha_+ - \alpha_- = 1.$$

Concerning terminology, here and in the sequel, we define as in (4.6)

$$\tilde{B}_r := (-r, r) \times B_r^{(n-1)}(0) \subset \mathbb{R} \times \mathbb{R}^{n-1},$$

for all $r > 0$ and

$$V_+ := V \cap \mathbb{R}^n_+$$

for all $V \subset \mathbb{R}^n$. Since Ω is smooth, up to a rotation, there exists $\delta > 0$ and $\varphi_0 : B_\delta^{(n-1)}(0) \to \mathbb{R}$ such that $\varphi_0(0) = |\nabla\varphi_0(0)| = 0$ and

$$\left\{\begin{array}{llll} \varphi : & \tilde{B}_{3\delta} & \to & \mathbb{R}^n \\ & (x_1, x') & \mapsto & (x_1 + \varphi_0(x'), x'), \end{array}\right. \tag{8.10}$$

that realizes a diffeomorphism onto its image and such that

$$\varphi(\tilde{B}_{3\delta} \cap \mathbb{R}^n_+) = \varphi(\tilde{B}_{3\delta}) \cap \Omega \text{ and } \varphi(\tilde{B}_{3\delta} \cap \partial\mathbb{R}^n_+) = \varphi(\tilde{B}_{3\delta}) \cap \partial\Omega.$$

Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be such that $\eta(x) = 1$ for all $x \in \tilde{B}_\delta$, $\eta(x) = 0$ for all $x \notin \tilde{B}_{2\delta}$.

Case 1: $\gamma \leq \frac{n^2-1}{4}$. As in Chern-Lin [6], for any $\epsilon > 0$, we define

$$u_\epsilon(x) := \left(\eta\epsilon^{-\frac{n-2}{2}}U(\epsilon^{-1}x)\right) \circ \varphi^{-1}(x) \text{ for } x \in \varphi(\tilde{B}_{2\delta}) \cap \Omega \text{ and 0 elsewhere.}$$

This subsection is devoted to give a Taylor expansion of $J^\Omega_{\gamma,s}(u_\varepsilon)$ as $\epsilon \to 0$. In the sequel, we adopt the following notation: given $(a_\epsilon)_{\epsilon>0} \in \mathbb{R}$, $\Theta_\gamma(a_\epsilon)$ denotes a quantity such that, as $\epsilon \to 0$.

$$\Theta_\gamma(a_\epsilon) := \left\{\begin{array}{ll} o(a_\epsilon) & \text{if } \gamma < \frac{n^2-1}{4} \\ O(a_\epsilon) & \text{if } \gamma = \frac{n^2-1}{4}. \end{array}\right.$$

A. Estimate of $\int_\Omega |\nabla u_\varepsilon|^2\,dx$

It follows from (8.9) that

$$|\nabla u_\varepsilon(x)| \leq C\varepsilon^{\alpha_+ - \frac{n}{2}}|x|^{-\alpha_+} \text{ for all } x \in \Omega \text{ and } \varepsilon > 0. \tag{8.11}$$

Therefore, $\int_{\varphi((\tilde{B}_{3\delta}\setminus\tilde{B}_\delta)\cap\mathbb{R}^n_+)}|\nabla u_\varepsilon|^2\,dx=\Theta_\gamma(\epsilon)$ as $\varepsilon\to 0$. It follows that

$$\int_\Omega|\nabla u_\varepsilon|^2\,dx=\int_{\tilde{B}_{\delta,+}}|\nabla(u_\varepsilon\circ\varphi)|^2_{\varphi^\star\mathrm{Eucl}}|\mathrm{Jac}(\varphi)|\,dx+\Theta_\gamma(\epsilon)\quad\text{as }\varepsilon\to 0,$$

where $\tilde{B}_{\delta,+}:=\tilde{B}_\delta\cap\mathbb{R}^n_+$. The definition (8.10) of φ yields $\mathrm{Jac}(\varphi)=1$. Moreover, for any $\theta\in(0,1)$, we have as $x\to 0$,

$$\varphi^\star\mathrm{Eucl}:=\left(\begin{array}{cc}1 & \partial_j\varphi_0\\ \partial_i\varphi_0 & \delta_{ij}+\partial_i\varphi_0\partial_j\varphi_0\end{array}\right)=Id+H+O(|x|^{1+\theta})$$

where

$$H:=\left(\begin{array}{cc}0 & \partial_j\varphi_0\\ \partial_i\varphi_0 & 0\end{array}\right).$$

It follows that

$$\begin{aligned}\int_\Omega|\nabla u_\varepsilon|^2\,dx &= \int_{\tilde{B}_{\delta,+}}|\nabla(u_\varepsilon\circ\varphi)|^2_{\mathrm{Eucl}}\,dx-\int_{\tilde{B}_{\delta,+}}H^{ij}\partial_i(u_\varepsilon\circ\varphi)\partial_j(u_\varepsilon\circ\varphi)\,dx\\ &\quad +O\left(\int_{\tilde{B}_{\delta,+}}|x|^{1+\theta}|\nabla(u_\varepsilon\circ\varphi)|^2\,dx\right)+\Theta_\gamma(\epsilon)\quad\text{as }\varepsilon\to 0.\end{aligned}\tag{8.12}$$

We have that

$$\begin{aligned}&\int_{\tilde{B}_{\delta,+}}H^{ij}\partial_i(u_\varepsilon\circ\varphi)\partial_j(u_\varepsilon\circ\varphi)\,dx\\ &=2\sum_{i\geq 2}\int_{\tilde{B}_{\delta,+}}H^{1i}\partial_1(u_\varepsilon\circ\varphi)\partial_i(u_\varepsilon\circ\varphi)\,dx\\ &=2\sum_{i\geq 2}\int_{\tilde{B}_{\delta,+}}\partial_i\varphi_0(x')\partial_1(u_\varepsilon\circ\varphi)\partial_i(u_\varepsilon\circ\varphi)\,dx\\ &=2\sum_{i,j\geq 2}\int_{\tilde{B}_{\delta,+}}\partial_{ij}\varphi_0(0)(x')^j\partial_1(u_\varepsilon\circ\varphi)\partial_i(u_\varepsilon\circ\varphi)\,dx\\ &\quad+O\left(\int_{\tilde{B}_{\delta,+}}|x|^2|\nabla(u_\varepsilon\circ\varphi)|^2\,dx\right)\quad\text{as }\varepsilon\to 0.\end{aligned}\tag{8.13}$$

We let II be the second fundamental form at 0 of the oriented boundary $\partial\Omega$. By definition, for any $X,Y\in T_0\partial\Omega$, we have that

$$II(X,Y):=(d\vec{\nu}_0(X),Y)_{\mathrm{Eucl}}$$

where $\vec{\nu}:\partial\Omega\to\mathbb{R}^n$ is the outer unit normal vector of $\partial\Omega$. In particular, we have that $\vec{\nu}(0)=(-1,0,\cdot,0)$. For any $i,j\geq 2$, we have that

$$II_{ij}:=II(\partial_i\varphi(0),\partial_j\varphi(0))=(\partial_i(\vec{\nu}\circ\varphi)(0),\partial_j\varphi(0))=-(\vec{\nu}(0),\partial_{ij}\varphi(0))=\partial_{ij}\varphi_0(0).$$

Plugging (8.13) in (8.12), and using a change of variables, we get that

$$\begin{aligned}\int_\Omega|\nabla u_\varepsilon|^2\,dx &= \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}|\nabla U|^2\,dx-2II_{ij}\sum_{i,j\geq 2}\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}(x')^j\partial_1U\partial_iU\,dx\\ &\quad+O\left(\int_{\tilde{B}_{\delta,+}}|x|^{1+\theta}|\nabla(u_\varepsilon\circ\varphi)|^2\,dx\right)+\Theta_\gamma(\epsilon)\quad\text{as }\varepsilon\to 0.\end{aligned}\tag{8.14}$$

We now choose θ in the following way:

(i) If $\gamma<\frac{n^2-1}{4}$, then take θ in $(0,\alpha_+-\alpha_--1)$,

(ii) If $\gamma = \frac{n^2-1}{4}$, take $\theta \in (0,1)$.
In both cases, we get by using (8.11), that

$$\int_{\tilde{B}_{\delta,+}} |x|^{1+\theta} |\nabla(u_\varepsilon \circ \varphi)|^2\, dx = \Theta_\gamma(\epsilon) \quad \text{as } \varepsilon \to 0. \tag{8.15}$$

Moreover, using (8.9), we have that

$$\textstyle\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} |\nabla U|^2\, dx = \int_{\mathbb{R}^n_+} |\nabla U|^2\, dx + \Theta_\gamma(\varepsilon) \text{ as } \varepsilon \to 0. \tag{8.16}$$

Plugging together (8.14), (8.15), (8.16) yields

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^2\, dx &= \int_{\mathbb{R}^n_+} |\nabla U|^2\, dx \\ &\quad -2II_{ij} \sum_{i,j\geq 2} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} (x')^j \partial_1 U \partial_i U\, dx + \Theta_\gamma(\epsilon). \end{aligned} \tag{8.17}$$

B. Estimate of $\int_\Omega \frac{|u_\varepsilon|^{2^\star(s)}}{|x|^s}\, dx$

Fix $\sigma \in [0,2]$. We will apply the estimates below to $\sigma = s \in [0,2)$ or to $\sigma := 2$. The first estimate in (8.9) yields

$$|u_\varepsilon(x)| \leq C\varepsilon^{\alpha_+ - \frac{n}{2}} d(x,\partial\Omega)|x|^{-\alpha_+} \leq C\varepsilon^{\alpha_+ - \frac{n}{2}} |x|^{1-\alpha_+} \tag{8.18}$$

for all $\varepsilon > 0$ and all $x \in \Omega$. Since Jac $\varphi = 1$, this estimate then yields

$$\begin{aligned} \int_\Omega \frac{|u_\varepsilon|^{2^\star(\sigma)}}{|x|^\sigma}\, dx &= \int_{\varphi(\tilde{B}_{\delta,+})} \frac{|u_\varepsilon|^{2^\star(\sigma)}}{|x|^\sigma}\, dx + \Theta_\gamma(\varepsilon) \\ &= \textstyle\int_{\tilde{B}_{\delta,+}} \frac{|u_\varepsilon \circ \varphi|^{2^\star(\sigma)}}{|\varphi(x)|^\sigma}\, dx + \Theta_\gamma(\varepsilon) \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{8.19}$$

If $\gamma < \frac{n^2-1}{4}$ or if $\gamma = \frac{n^2-1}{4}$ and $\sigma < 2$, we choose $\theta \in (0, (\alpha_+ - \alpha_-)\frac{2^\star(\sigma)}{2} - 1) \cap (0,1)$. If $\gamma = \frac{n^2-1}{4}$ and $\sigma = 2$, we choose any $\theta \in (0,1)$. Using the expression of $\varphi(x_1, x')$, a Taylor expansion yields

$$|\varphi(x)|^{-\sigma} = |x|^{-\sigma} \left(1 - \frac{\sigma}{2} \frac{x_1}{|x|^2} \sum_{i,j\geq 2} \partial_{ij}\varphi_0(0)(x')^i(x')^j + O(|x|^{1+\theta})\right) \quad \text{as } \varepsilon \to 0. \tag{8.20}$$

The choice of θ yields

$$\textstyle\int_{\tilde{B}_{\delta,+}} \frac{|u_\varepsilon \circ \varphi|^{2^\star(\sigma)}}{|\varphi(x)|^\sigma} |x|^{1+\theta}\, dx = \Theta_\gamma(\varepsilon) \quad \text{as } \varepsilon \to 0. \tag{8.21}$$

Plugging together (8.19), (8.20), (8.21), using a change of variable and (8.9), we get as $\varepsilon \to 0$ that

$$\begin{aligned} \int_\Omega \frac{|u_\varepsilon|^{2^\star(\sigma)}}{|x|^\sigma}\, dx &= \int_{\mathbb{R}^n_+} \frac{|U|^{2^\star(\sigma)}}{|x|^\sigma}\, dx \\ &\quad -\frac{\sigma}{2} \sum_{i,j\geq 2} \varepsilon II_{ij} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|U|^{2^\star(\sigma)}}{|x|^\sigma} \frac{x_1}{|x|^2} (x')^i(x')^j\, dx + \Theta_\gamma(\varepsilon). \end{aligned} \tag{8.22}$$

We now compute the terms in U by using its symmetry property established in Chern-Lin [6]. Indeed, there exists $\tilde{U} : (0,+\infty) \times \mathbb{R}$ such that $U(x_1, x') = \tilde{U}(x_1, |x'|)$ for all $(x_1, x') \in \mathbb{R}^n_+$. Therefore, for any $i,j \geq 2$, we get that

$$\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|U|^{2^\star(\sigma)}}{|x|^\sigma} \frac{x_1}{|x|^2} (x')^i(x')^j\, dx = \frac{\delta_{ij}}{n-1} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|U|^{2^\star(\sigma)}}{|x|^\sigma} \frac{x_1}{|x|^2} |x'|^2\, dx$$

and that

$$\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} (x')^j \partial_1 U \partial_i U\, dx = \frac{\delta_{ij}}{n-1} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U (x', \nabla U)\, dx$$

where $x = (x_1, x') \in \mathbb{R}^n_+$. Therefore, the identities (8.17) and (8.22) rewrite as

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx \quad = \quad \int_{\mathbb{R}^n_+} |\nabla U|^2\, dx - \frac{2h_\Omega(0)}{n-1}\varepsilon \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U(x', \nabla U)\, dx + \Theta_\gamma(\epsilon) \tag{8.23}$$

and

$$\begin{aligned} \int_\Omega \frac{|u_\varepsilon|^{2^\star(\sigma)}}{|x|^\sigma}\, dx \quad &= \quad \int_{\mathbb{R}^n_+} \frac{|U|^{2^\star(\sigma)}}{|x|^\sigma}\, dx \\ &\quad - \frac{\sigma h_\Omega(0)}{2(n-1)}\varepsilon \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|U|^{2^\star(\sigma)}}{|x|^\sigma} \frac{x_1}{|x|^2} |x'|^2\, dx + \Theta_\gamma(\varepsilon) \end{aligned} \tag{8.24}$$

as $\varepsilon \to 0$, where $h_\Omega(0) = \sum_i II_{ii}$ is the mean curvature at 0.

C. An intermediate identity. We now claim that as $\varepsilon \to 0$,

$$\begin{aligned} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U(x', \nabla U)\, dx \quad &= \quad \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|x'|^2 x_1}{2|x|^2} \left(\lambda \frac{s}{2^\star(s)} \frac{U^{2^\star(s)}}{|x|^s} + \gamma \frac{U^2}{|x|^2}\right) dx \\ &\quad - \int_{\partial\mathbb{R}^n_+ \cap \tilde{B}_{\varepsilon^{-1}\delta}} \frac{|x'|^2 (\partial_1 U)^2}{4}\, dx + \Theta_\gamma(1) \end{aligned} \tag{8.25}$$

where $\lambda > 0$ is as in (8.7). This was shown by Chern-Lin [6], and we include it for the sake of completeness. Here and in the sequel, ν_i denotes the i^{th} coordinate of the direct outward normal vector on the boundary of the relevant domain (for instance, on $\partial\mathbb{R}^n_+$, we have that $\nu_i = -\delta_{1i}$). We write

$$\begin{aligned} &\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U(x', \nabla U)\, dx = \sum_{j\geq 2} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U (x')^j \partial_j U\, dx \\ &= \sum_{j\geq 2} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U \partial_j \left(\frac{|x'|^2}{2}\right) \partial_j U\, dx \\ &= \sum_{j\geq 2} \int_{\partial(\tilde{B}_{\varepsilon^{-1}\delta,+})} \partial_1 U \frac{|x'|^2}{2} \partial_j U \nu_j\, d\sigma - \sum_{j\geq 2} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|x'|^2}{2} \partial_j \left(\partial_1 U \partial_j U\right)\, dx \\ &= \sum_{j\geq 2} \int_{\partial\mathbb{R}^n_+ \cap \tilde{B}_{\varepsilon^{-1}\delta}} \partial_1 U \frac{|x'|^2}{2} \partial_j U \nu_j\, d\sigma + O\left(\int_{\mathbb{R}^n_+ \cap \partial\tilde{B}_{\varepsilon^{-1}\delta}} |x'|^2 |\nabla U|^2(x)\, d\sigma\right) \\ &- \sum_{j\geq 2} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|x'|^2}{2} \left(\partial_{1j} U \partial_j U + \partial_1 U \partial_{jj} U\right)\, dx. \end{aligned} \tag{8.26}$$

Since $U(0,x')=0$ for all $x'\in\mathbb{R}^{n-1}$, using the upper-bound (8.9) and writing $\nabla'=(\partial_2,\dots,\partial_n)$, we get that

$$
\begin{aligned}
\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\partial_1 U(x',\nabla U)\,dx &= -\sum_{j\geq 2}\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\left(\partial_{1j}U\partial_j U+\partial_1 U\partial_{jj}U\right)\,dx+\Theta_\gamma(1)\\
&= -\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{4}\partial_1\left(|\nabla' U|^2\right)\,dx\\
&\quad+\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U\left(-\Delta U+\partial_{11}U\right)\,dx+\Theta_\gamma(1)\\
&= -\int_{\partial(\tilde{B}_{\varepsilon^{-1}\delta,+})}\frac{|x'|^2|\nabla' U|^2}{4}\nu_1\,dx+\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U(-\Delta U)\,dx\\
&\quad+\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\partial_1\left(\frac{|x'|^2(\partial_1 U)^2}{4}\right)\,dx+\Theta_\gamma(1).
\end{aligned}\tag{8.27}
$$

Using again that U vanishes on $\partial\mathbb{R}^n_+$ and the bound (8.9), we get as $\varepsilon\to 0$,

$$
\begin{aligned}
\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\partial_1 U(x',\nabla U)\,dx &= \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U(-\Delta U)\,dx\\
&\quad+\int_{\partial\mathbb{R}^n_+\cap\tilde{B}_{\varepsilon^{-1}\delta}}\frac{|x'|^2(\partial_1 U)^2}{4}\nu_1\,dx\\
&\quad+O\left(\int_{\partial(\tilde{B}_{\varepsilon^{-1}\delta})\cap\mathbb{R}^n_+}|x'|^2|\nabla U|^2\,dx\right)+\Theta_\gamma(1)\\
&= \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U(-\Delta U)\,dx\\
&\quad-\int_{\partial\mathbb{R}^n_+\cap\tilde{B}_{\varepsilon^{-1}\delta}}\frac{|x'|^2(\partial_1 U)^2}{4}\,dx+\Theta_\gamma(1).
\end{aligned}\tag{8.28}
$$

Now use equation (8.7) to get that

$$
\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U(-\Delta U)\,dx=\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}}\frac{|x'|^2}{2}\partial_1 U\left(\lambda\frac{U^{2^\star(s)-1}}{|x|^s}+\gamma\frac{U}{|x|^2}\right)\,dx.\tag{8.29}
$$

Integrating by parts, using that U vanishes on $\partial\mathbb{R}^n_+$ and the upper-bound (8.9), for $\sigma \in [0,2]$, we get that

$$\begin{aligned}
\int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} |x'|^2 \partial_1 U \frac{U^{2^\star(\sigma)-1}}{|x|^\sigma}\, dx &= \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} |x'|^2 |x|^{-\sigma} \partial_1 \left(\frac{U^{2^\star(\sigma)}}{2^\star(\sigma)}\right) dx \\
&= \int_{\partial(\tilde{B}_{\varepsilon^{-1}\delta,+})} |x'|^2 |x|^{-\sigma} \frac{U^{2^\star(\sigma)}}{2^\star(\sigma)} \nu_1 \, dx \\
&\quad - \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 (|x'|^2 |x|^{-\sigma}) \left(\frac{U^{2^\star(\sigma)}}{2^\star(\sigma)}\right) dx \\
&= O\left(\int_{\mathbb{R}^n_+ \cap \partial \tilde{B}_{\varepsilon^{-1}\delta,+}} |x|^{2-\sigma} U^{2^\star(\sigma)}\, d\sigma\right) \\
&\quad + \frac{\sigma}{2^\star(s)} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|x'|^2 x_1}{|x|^{\sigma+2}} U^{2^\star(\sigma)}\, dx \\
&= \frac{\sigma}{2^\star(s)} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|x'|^2 x_1}{|x|^{\sigma+2}} U^{2^\star(\sigma)}\, dx + \Theta_\gamma(1) \quad \text{as } \varepsilon \to 0. \qquad (8.30)
\end{aligned}$$

Putting together (8.28) to (8.30) yields (8.25).

D. Estimate for $J^\Omega_{\gamma,s}(u_\varepsilon)$

Since $U \in D^{1,2}(\mathbb{R}^n)$, it follows from (8.7) that

$$\int_{\mathbb{R}^n_+} \left(|\nabla U|^2 - \frac{\gamma}{|x|^2} U^2\right) dx = \lambda \int_{\mathbb{R}^n_+} \frac{U^{2^\star(s)}}{|x|^s}\, dx. \qquad (8.31)$$

This equality, combined with (8.23) and (8.24) gives

$$\begin{aligned}
J^\Omega_{\gamma,s}(u_\varepsilon) &= \frac{\int_\Omega \left(|\nabla u_\varepsilon|^2 - \frac{\gamma}{|x|^2} u_\varepsilon^2\right) dx}{\left(\int_\Omega \frac{|u_\varepsilon|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^\star(s)}}} \\
&= \frac{\int_{\mathbb{R}^n_+} \left(|\nabla U|^2 - \frac{\gamma}{|x|^2} U^2\right) dx}{\left(\int_{\mathbb{R}^n_+} \frac{|U|^{2^\star(s)}}{|x|^s}\, dx\right)^{\frac{2}{2^\star(s)}}} \left(1 + \epsilon \frac{h_\Omega(0)}{(n-1)\lambda \int_{\mathbb{R}^n_+} \frac{|U|^{2^\star(s)}}{|x|^s}\, dx} C_\varepsilon + \Theta_\gamma(\varepsilon)\right) \qquad (8.32)
\end{aligned}$$

where for all $\varepsilon > 0$,

$$\begin{aligned}
C_\varepsilon &:= -2 \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \partial_1 U (x', \nabla U)\, dx + \gamma \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|x'|^2 x_1}{|x|^2} \frac{U^2}{|x|^2}\, dx \\
&\quad + \lambda \frac{s}{2^\star(s)} \int_{\tilde{B}_{\varepsilon^{-1}\delta,+}} \frac{|x'|^2 x_1}{|x|^2} \frac{U^{2^\star(s)}}{|x|^s}\, dx.
\end{aligned}$$

The identity (8.25) then yields as $\varepsilon \to 0$,

$$C_\varepsilon = \int_{\partial\mathbb{R}^n_+ \cap \tilde{B}_{\varepsilon^{-1}\delta}} \frac{|x'|^2 (\partial_1 U)^2}{2}\, dx + \Theta_\gamma(1)$$

Therefore, (8.32) yields that as $\varepsilon \to 0$,

$$J^\Omega_{\gamma,s}(u_\varepsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+) \left(1 + \epsilon \frac{h_\Omega(0) \int_{\partial\mathbb{R}^n_+ \cap \tilde{B}_{\varepsilon^{-1}\delta}} |x'|^2 (\partial_1 U)^2\, dx'}{2(n-1)\lambda \int_{\mathbb{R}^n_+} \frac{|U|^{2^\star(s)}}{|x|^s}\, dx} + \Theta_\gamma(\varepsilon)\right). \qquad (8.33)$$

We now distinguish two cases:

Case i) $\gamma < \frac{n^2-1}{4}$: The bound (8.9) then yields $x' \mapsto |x'|^2|\partial_1 U(x')|^2$ is in $L^1(\partial\mathbb{R}^n_+)$ and so we get from (8.33) that

$$J^{\Omega}_{\gamma,s}(u_\varepsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+)\left(1 + C_0 \cdot h_\Omega(0)\cdot \epsilon + o(\varepsilon)\right) \text{ as } \varepsilon \to 0, \tag{8.34}$$

with

$$C_0 := \frac{\int_{\partial\mathbb{R}^n_+} |x'|^2(\partial_1 U)^2\, dx'}{2(n-1)\lambda \int_{\mathbb{R}^n_+} \frac{|U|^{2^\star(s)}}{|x|^s}\, dx} > 0.$$

Case ii) $\gamma = \frac{n^2-1}{4}$: From (8.8), Lemma 5.2 and the Kelvin transform, we have that $\lim_{|x'|\to+\infty} |x'|^{\alpha_+}|\partial_1 U(0,x')| = K_2 > 0$. Since $2\alpha_+ - 2 = n-1$, we get that

$$\int_{\partial\mathbb{R}^n_+ \cap \tilde{B}_{\varepsilon^{-1}\delta}} |x'|^2(\partial_1 U)^2\, dx' = \omega_{n-1}K_2^2 \ln\frac{1}{\varepsilon} + o\left(\ln\frac{1}{\varepsilon}\right)$$

as $\varepsilon \to 0$. Therefore, (8.33) yields

$$J^{\Omega}_{\gamma,s}(u_\varepsilon) = \mu_{\gamma,s}(\mathbb{R}^n_+)\left(1 + C_0' h_\Omega(0)\varepsilon \ln\tfrac{1}{\varepsilon} + o\left(\ln\tfrac{1}{\varepsilon}\right)\right) \quad \text{as } \varepsilon \to 0, \tag{8.35}$$

where

$$C_0' := \frac{\omega_{n-1}K_2^2}{2(n-1)\lambda \int_{\mathbb{R}^n_+} \frac{|U|^{2^\star(s)}}{|x|^s}\, dx} > 0.$$

Cases i) and ii) prove Proposition 8.3 when $\gamma \le \frac{n^2-1}{4}$.

Case 2: $\gamma > \frac{n^2-1}{4}$. In this case, the construction of test-functions is more subtle. First, use Theorem 7.1 to obtain $H \in C^2(\overline{\Omega}\setminus\{0\})$ such that (7.1) holds and

$$H(x) = \tfrac{d(x,\partial\Omega)}{|x|^{\alpha_+}} + m_\gamma(\Omega)\tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-}} + o\left(\tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-}}\right) \quad \text{when } x \to 0. \tag{8.36}$$

As above, we fix $\eta \in C_c^\infty(\mathbb{R}^n)$ such that $\eta(x) = 1$ for all $x \in \tilde{B}_\delta$, $\eta(x) = 0$ for all $x \notin \tilde{B}_{2\delta}$. We then define β such that

$$H(x) = \left(\eta \frac{x_1}{|x|^{\alpha_+}}\right) \circ \varphi^{-1}(x) + \beta(x) \quad \text{for all } x \in \Omega.$$

Here φ is as in (4.7) to (4.12). Note that $\beta \in D^{1,2}(\Omega)$ and

$$\beta(x) = m_\gamma(\Omega)\tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-}} + o\left(\tfrac{d(x,\partial\Omega)}{|x|^{\alpha_-}}\right) \quad \text{as } x \to 0. \tag{8.37}$$

Indeed, since $\alpha_+ - \alpha_- < 1$, an essential point underlying all this subsection is that

$$|x| = o\left(|x|^{\alpha_+ - \alpha_-}\right) \quad \text{as } x \to 0.$$

We choose U as in (8.7). By multiplying by a constant if necessary, we assume that $K_2 = 1$, that is

$$U(x) \sim_{x\to 0} K_1 \frac{x_1}{|x|^{\alpha_-}} \text{ and } U(x) \sim_{|x|\to+\infty} \frac{x_1}{|x|^{\alpha_+}}. \tag{8.38}$$

Now define

$$u_\epsilon(x) := \left(\eta \epsilon^{-\frac{n-2}{2}} U(\epsilon^{-1}\cdot)\right) \circ \varphi^{-1}(x) + \epsilon^{\frac{\alpha_+ - \alpha_-}{2}}\beta(x) \text{ for } x \in \Omega \text{ and } \epsilon > 0. \tag{8.39}$$

We start by showing that for any $k \ge 0$

$$\lim_{\varepsilon\to 0} \frac{u_\varepsilon}{\varepsilon^{\frac{\alpha_+ - \alpha_-}{2}}} = H \text{ in } C^k_{loc}(\overline{\Omega}\setminus\{0\}). \tag{8.40}$$

Indeed, the convergence in $C^0_{loc}(\overline{\Omega}\backslash\{0\})$ is a consequence of the definition of u_ε, the choice $K_2=1$ and the asymptotic behavior (8.38). For convergence in C^k, we need in addition that $\nabla^i(U-x_1|x|^{-\alpha_+}) = o\left(|x|^{1-\alpha_+-i}\right)$ as $x\to+\infty$ for all $i\geq 0$. This estimate follows from (8.38) and Lemma 5.2.

In the sequel, we adopt the following notation: θ_c^ε will denote any quantity such that there exists $\theta:\mathbb{R}\to\mathbb{R}$ such that $\lim_{c\to 0}\lim_{\varepsilon\to 0}\theta_c^\varepsilon=0$. We first claim that for any $c>0$, we have that

$$\begin{aligned}&\int_{\Omega\backslash\varphi(B_c(0)_+)}\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)dx\\ &=\varepsilon^{\alpha_+-\alpha_-}\left((\alpha_+-1)c^{n-2\alpha_+}\frac{\omega_{n-1}}{2n}+m_\gamma(\Omega)\frac{(n-2)\omega_{n-1}}{2n}\right)+\theta_c^\varepsilon\varepsilon^{\alpha_+-\alpha_-}.\end{aligned}\tag{8.41}$$

Indeed, it follows from (8.40) that

$$\lim_{\varepsilon\to 0}\frac{\int_{\Omega\backslash\varphi(B_c(0)_+)}\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)dx}{\varepsilon^{\alpha_+-\alpha_-}}=\int_{\Omega\backslash\varphi(B_c(0)_+)}\left(|\nabla H|^2-\frac{\gamma}{|x|^2}H^2\right)dx.\tag{8.42}$$

Since H vanishes on $\partial\Omega\setminus\{0\}$ and satisfies $-\Delta H-\frac{\gamma}{|x|^2}H=0$, integrating by parts yields

$$\begin{aligned}\int_{\Omega\backslash\varphi(B_c(0)_+)}\left(|\nabla H|^2-\frac{\gamma}{|x|^2}H^2\right)dx &= -\int_{\varphi(\mathbb{R}^n_+\cap\partial B_c(0))}H\partial_\nu H\,d\sigma\\ &= -\int_{\mathbb{R}^n_+\cap\partial B_c(0)}H\circ\varphi\,\partial_{\varphi_\star\nu}(H\circ\varphi)\,d(\varphi^\star\sigma),\end{aligned}\tag{8.43}$$

where in the two last equalities, $\nu(x)$ is the outer normal vector of $B_c(0)$ at $x\in\partial B_c(0)$.

We now estimate $H\circ\varphi\,\partial_{\varphi_\star\nu}H\circ\varphi$. Since $\varphi_\star\nu(x)=\frac{x}{|x|}+O(|x|)$ as $x\to 0$, it follows from from (8.36) that

$$-H\circ\varphi\partial_{\varphi_\star\nu}(H\circ\varphi)=\tfrac{(\alpha_+-1)x_1^2}{|x|^{2\alpha_++1}}+(n-2)m_\gamma(\Omega)\tfrac{x_1^2}{|x|^{n+1}}+o\left(|x|^{1-n}\right)\quad\text{as } x\to 0.$$

Integrating this expression on $B_c(0)_+=\mathbb{R}^n_+\cap\partial B_c(0)$ and plugging into (8.43) yields

$$\begin{aligned}\int_{\Omega\backslash\varphi(B_c(0)_+)}\left(|\nabla H|^2-\frac{\gamma}{|x|^2}H^2\right)dx &= \frac{(\alpha_+-1)c^{n-2\alpha_+}\omega_{n-1}}{2n}\\ &\quad+(n-2)m_\gamma(\Omega)\frac{\omega_{n-1}}{2n}+\theta_c,\end{aligned}$$

where $\lim_{c\to 0}\theta_c=0$. Here, we have used that

$$\int_{\mathbb{S}^{n-1}_+}x_1^2\,d\sigma=\frac{1}{2}\int_{\mathbb{S}^{n-1}}x_1^2\,d\sigma=\frac{1}{2n}\int_{\mathbb{S}^{n-1}}|x|^2\,d\sigma=\frac{\omega_{n-1}}{2n},\ \omega_{n-1}:=\int_{\mathbb{S}^{n-1}}d\sigma.$$

This equality and (8.42) prove (8.41).

We now claim that

$$\begin{aligned}\int_\Omega\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)dx &= \lambda\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)}}{|x|^s}\,dx\\ &\quad+m_\gamma(\Omega)\tfrac{(n-2)\omega_{n-1}}{2n}\varepsilon^{\alpha_+-\alpha_-}+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\quad\text{as }\varepsilon\to 0.\end{aligned}\tag{8.44}$$

Indeed, define $U_\varepsilon(x):=\varepsilon^{-\frac{n-2}{2}}U(\varepsilon^{-1}x)$ for all $x\in\mathbb{R}^n_+$. The definition (8.39) of u_ε rewrites as:

$$u_\varepsilon\circ\varphi(x)=U_\epsilon(x)+\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\beta\circ\varphi(x)\quad\text{for all } x\in\mathbb{R}^n_+\cap\tilde{B}_\delta.$$

Fix $c\in(0,\delta)$ that we will eventually let go to 0. Since $d\varphi_0$ is an isometry, we get that

$$
\begin{aligned}
&\int_{\varphi(B_c(0)_+)}\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)\,dx \\
&=\int_{B_c(0)_+}\left(|\nabla(u_\varepsilon\circ\varphi)|^2_{\varphi^\star\mathrm{Eucl}}-\frac{\gamma}{|\varphi(x)|^2}(u_\varepsilon\circ\varphi)^2\right)|\mathrm{Jac}(\varphi)|\,dx \\
&=\int_{B_c(0)_+}\left(|\nabla U_\varepsilon|^2_{\varphi^\star\mathrm{Eucl}}-\frac{\gamma}{|\varphi(x)|^2}U_\varepsilon^2\right)|\mathrm{Jac}(\varphi)|\,dx \\
&\quad+2\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_c(0)_+}\left((\nabla U_\varepsilon,\nabla(\beta\circ\varphi))_{\varphi^\star\mathrm{Eucl}}-\frac{\gamma}{|\varphi(x)|^2}U_\varepsilon(u_\varepsilon\circ\varphi)\right)|\mathrm{Jac}(\varphi)|\,dx \\
&\quad+\varepsilon^{\alpha_+-\alpha_-}\int_{B_c(0)_+}\left(|\nabla(\beta\circ\varphi)|^2_{\varphi^\star\mathrm{Eucl}}-\frac{\gamma}{|\varphi(x)|^2}(\beta\circ\varphi)^2\right)|\mathrm{Jac}(\varphi)|\,dx
\end{aligned}\tag{8.45}
$$

Since $\varphi^\star\mathrm{Eucl}=\mathrm{Eucl}+O(|x|)$, $|\varphi(x)|=|x|+O(|x|^2)$ and $\beta\in D^{1,2}(\Omega)$, we get that

$$
\begin{aligned}
&\int_{\varphi(B_c(0)_+)}\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)\,dx=\int_{B_c(0)_+}\left(|\nabla U_\varepsilon|^2_{\mathrm{Eucl}}-\frac{\gamma}{|x|^2}U_\varepsilon^2\right)|\,dx \\
&+O\left(\int_{B_c(0)_+}|x|\left(|\nabla U_\varepsilon|^2_{\mathrm{Eucl}}+\frac{U_\varepsilon^2}{|x|^2}\right)|\,dx\right) \\
&+2\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_c(0)_+}\left((\nabla U_\varepsilon,\nabla(\beta\circ\varphi))_{\mathrm{Eucl}}-\frac{\gamma}{|x|^2}U_\varepsilon(\beta\circ\varphi)\right)\,dx \\
&+O\left(\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_c(0)_+}|x|\left(|\nabla U_\varepsilon|\cdot|\nabla(\beta\circ\varphi)|+\frac{U_\varepsilon|\beta\circ\varphi|}{|x|^2}\right)\,dx\right)+\varepsilon^{\alpha_+-\alpha_-}\theta_c^\varepsilon
\end{aligned}\tag{8.46}
$$

as $\varepsilon\to 0$. The pointwise estimates (8.38) yield

$$
\begin{aligned}
&\int_{\varphi(B_c(0)_+)}\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)\,dx=\int_{B_c(0)_+}\left(|\nabla U_\varepsilon|^2_{\mathrm{Eucl}}-\frac{\gamma}{|x|^2}U_\varepsilon^2\right)\,dx \\
&+2\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_c(0)_+}\left((\nabla U_\varepsilon,\nabla(\beta\circ\varphi))_{\mathrm{Eucl}}-\frac{\gamma}{|x|^2}U_\varepsilon(\beta\circ\varphi)\right)\,dx \\
&+\varepsilon^{\alpha_+-\alpha_-}\theta_c^\varepsilon
\end{aligned}
$$

as $\varepsilon\to 0$. Integrating by parts yields

$$
\begin{aligned}
&\int_{\varphi(B_c(0)_+)}\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)\,dx \\
&=\int_{B_c(0)_+}\left(-\Delta U_\varepsilon-\frac{\gamma}{|x|^2}U_\varepsilon\right)U_\varepsilon\,dx+\int_{\partial(B_c(0)_+)}U_\varepsilon\partial_\nu U_\varepsilon\,d\sigma \\
&+2\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\left(\int_{B_c(0)_+}\left(-\Delta U_\varepsilon-\frac{\gamma}{|x|^2}U_\varepsilon\right)\beta\circ\varphi\,dx+\int_{\partial(B_c(0)_+)}\beta\circ\varphi\partial_\nu U_\varepsilon\,d\sigma\right) \\
&+\varepsilon^{\alpha_+-\alpha_-}\theta_c^\varepsilon
\end{aligned}
$$

as $\varepsilon\to 0$. Since both U and $\beta\circ\varphi$ vanish on $\partial\mathbb{R}^n_+\setminus\{0\}$, we get that

$$\begin{aligned}(8.47)\qquad &\int_{\varphi(B_c(0)_+)}\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)dx\\ &=\int_{B_c(0)_+}\left(-\Delta U_\varepsilon-\frac{\gamma}{|x|^2}U_\varepsilon\right)U_\varepsilon\,dx+\int_{\mathbb{R}^n_+\cap\partial B_c(0)}U_\varepsilon\partial_\nu U_\varepsilon\,d\sigma\\ &+2\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\left(\int_{B_c(0)_+}\left(-\Delta U_\varepsilon-\frac{\gamma}{|x|^2}U_\varepsilon\right)\beta\circ\varphi\,dx+\int_{\mathbb{R}^n_+\cap\partial B_c(0)}\beta\circ\varphi\partial_\nu U_\varepsilon\,d\sigma\right)\\ &+\varepsilon^{\alpha_+-\alpha_-}\theta_c^\varepsilon,\end{aligned}$$

as $\varepsilon\to 0$. The asymptotic estimate (8.38) of U and Lemma 5.2 yield (after a Kelvin transform)

$$\partial_\nu U_\varepsilon=-(\alpha_+-1)\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}x_1|x|^{-\alpha_+-1}+o\left(\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}|x|^{-\alpha_+}\right)$$

as $\varepsilon\to 0$ uniformly on compact subsets of $\overline{\mathbb{R}^n_+}\setminus\{0\}$. We then get that

$$\beta\circ\varphi\partial_\nu U_\varepsilon=\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\left(-m_\gamma(\Omega)(\alpha_+-1)x_1^2|x|^{-n-1}+o\left(|x|^{1-n}\right)\right)$$

and

$$U_\varepsilon\partial_\nu U_\varepsilon=\varepsilon^{\alpha_+-\alpha_-}\left(-(\alpha_+-1)x_1^2|x|^{-2\alpha_+-1}+o\left(|x|^{1-2\alpha_+}\right)\right)$$

as $\varepsilon\to 0$ uniformly on compact subsets of $\overline{\mathbb{R}^n_+}\setminus\{0\}$. Plugging these identities in (8.47) and using equation (8.7) yield, as $\varepsilon\to 0$,

$$\begin{aligned}\int_{\varphi(B_c(0)_+)}\left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)dx &= \int_{B_c(0)_+}\lambda\frac{U_\varepsilon^{2^\star(s)}}{|x|^s}\,dx-(\alpha_+-1)\frac{\omega_{n-1}}{2n}c^{n-2\alpha_+}\varepsilon^{\alpha_+-\alpha_-}\\ &+2\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_c(0)_+}\lambda\frac{U_\varepsilon^{2^\star(s)-1}}{|x|^s}\beta\circ\varphi\,dx\\ (8.48)\qquad &-(\alpha_+-1)\frac{\omega_{n-1}}{n}m_\gamma(\Omega)\varepsilon^{\alpha_+-\alpha_-}+\varepsilon^{\alpha_+-\alpha_-}\theta_c^\varepsilon.\end{aligned}$$

As $\varepsilon\to 0$, we have that

$$(8.49)\qquad \int_{B_c(0)_+}\lambda\frac{U_\varepsilon^{2^\star(s)}}{|x|^s}\,dx=\int_{\mathbb{R}^n_+}\lambda\frac{U_\varepsilon^{2^\star(s)}}{|x|^s}\,dx+o\left(\varepsilon^{\alpha_+-\alpha_-}\right).$$

The expansion (8.37) and the change of variable $x:=\varepsilon y$ yield as $\varepsilon\to 0$,

$$(8.50)\qquad \int_{B_c(0)_+}\lambda\frac{U_\varepsilon^{2^\star(s)-1}}{|x|^s}\beta\circ\varphi\,dx=\lambda m_\gamma(\Omega)\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)-1}}{|y|^s}\frac{y_1}{|y|^{\alpha_-}}\,dy+\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\theta_\varepsilon^c$$

Integrating by parts, and using the asymptotics (8.38) for U yield

$$\begin{aligned}\lambda\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)-1}}{|y|^s}\frac{y_1}{|y|^{\alpha_-}}\,dy &= \lim_{R\to+\infty}\int_{B_R(0)_+}\lambda\frac{U^{2^\star(s)-1}}{|y|^s}\frac{y_1}{|y|^{\alpha_-}}\,dy\\ &= \lim_{R\to+\infty}\int_{B_R(0)_+}\left(-\Delta U-\frac{\gamma}{|y|^2}U\right)\frac{y_1}{|y|^{\alpha_-}}\,dy\\ &= \lim_{R\to+\infty}\int_{B_R(0)_+}U\left(-\Delta-\frac{\gamma}{|y|^2}\right)\left(\frac{y_1}{|y|^{\alpha_-}}\right)dy\\ (8.51)\qquad &-\int_{\partial B_R(0)_+}\partial_\nu U\frac{y_1}{|y|^{\alpha_-}}\,d\sigma=(\alpha_+-1)\frac{\omega_{n-1}}{2n}.\end{aligned}$$

Putting together (8.49), (8.50) and (8.51) yield

$$\begin{aligned}\int_\Omega \left(|\nabla u_\varepsilon|^2-\frac{\gamma}{|x|^2}u_\varepsilon^2\right)\,dx &= \lambda\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)}}{|x|^s}\,dx\\ &\quad +m_\gamma(\Omega)\frac{(n-2)\omega_{n-1}}{2n}\varepsilon^{\alpha_+-\alpha_-}+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\end{aligned}$$

as $\varepsilon\to 0$. This finally yields (8.44).

We finally claim that

$$\begin{aligned}\int_\Omega \frac{u_\varepsilon^{2^\star(s)}}{|x|^s}\,dx &= \int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)}}{|x|^s}\,dx+\frac{2^\star(s)}{\lambda}m_\gamma(\Omega)\frac{(\alpha_+-1)\omega_{n-1}}{2n}\varepsilon^{\alpha_+-\alpha_-}\\ &\quad +o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\qquad \text{as }\varepsilon\to 0.\end{aligned}\tag{8.52}$$

Indeed, fix $c>0$. Due to estimates (8.37) and (8.38), we have that

$$\begin{aligned}\int_\Omega \frac{u_\varepsilon^{2^\star(s)}}{|x|^s}\,dx &= \int_{\varphi(B_c(0)_+)}\frac{u_\varepsilon^{2^\star(s)}}{|x|^s}\,dx+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\\ &= \int_{B_c(0)_+}\frac{|U_\varepsilon+\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\beta\circ\varphi|^{2^\star(s)}}{|\varphi(x)|^s}|\mathrm{Jac}(\varphi)|\,dx+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\\ &= \int_{B_c(0)_+}\frac{|U_\varepsilon+\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\beta\circ\varphi|^{2^\star(s)}}{|x|^s}|(1+O(|x|))\,dx+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\end{aligned}$$

as $\varepsilon\to 0$. As one checks, there exists $C>0$ such that for all $X,Y\in\mathbb{R}$,

$$||X+Y|^{2^\star(s)}-|X|^{2^\star(s)}-2^\star(s)|X|^{2^\star(s)-2}XY|\le C\left(|X|^{2^\star(s)-2}|Y|^2+|Y|^{2^\star(s)}\right)\tag{8.53}$$

Therefore, using the asymptotics (8.37) and (8.38) of U and β, we get that

$$\begin{aligned}\int_\Omega \frac{u_\varepsilon^{2^\star(s)}}{|x|^s}\,dx &= \int_{B_c(0)_+}\frac{U_\varepsilon^{2^\star(s)}}{|x|^s}|(1+O(|x|))\,dx\\ &\quad +2^\star(s)\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_c(0)_+}\frac{U_\varepsilon^{2^\star(s)-1}}{|x|^s}\beta\circ\varphi(1+O(|x|))\,dx\\ &\quad +\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\theta_\varepsilon^c\\ &= \int_{B_c(0)_+}\frac{U_\varepsilon^{2^\star(s)}}{|x|^s}\,dx+2^\star(s)\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\int_{B_c(0)_+}\frac{U_\varepsilon^{2^\star(s)-1}}{|x|^s}\beta\circ\varphi\,dx\\ &\quad +\varepsilon^{\frac{\alpha_+-\alpha_-}{2}}\theta_\varepsilon^c\quad \text{as }\varepsilon\to 0.\end{aligned}$$

Then (8.52) follows from this latest identity, combined with (8.49), (8.50), and (8.51).

We finally use (8.31), (8.44) and (8.52) to get

$$J_{\gamma,s}^\Omega(u_\varepsilon)=J_{\gamma,s}^{\mathbb{R}^n_+}(U)\left(1-\frac{\left(\alpha_+-\frac{n}{2}\right)\omega_{n-1}}{n\lambda\int_{\mathbb{R}^n_+}\frac{U^{2^\star(s)}}{|x|^s}\,dx}m_\gamma(\Omega)\varepsilon^{\alpha_+-\alpha_-}+o\left(\varepsilon^{\alpha_+-\alpha_-}\right)\right)\quad \text{as }\varepsilon\to 0,$$

which proves (8.6). This completes Proposition 8.3 and therefore Theorem 8.2.

9. Domains with positive mass and an arbitrary geometry at 0

In this section, we construct smooth bounded domains in $\mathbb{R}^n$ with positive or negative mass, regardless of the local geometry of $\partial\Omega$ at 0. This is illustrated by the following result.

Theorem 9.1. *Let ω be a smooth open set of $\mathbb{R}^n$. Then, there exist $r_0 > 0$ and two smooth bounded domains Ω_+, Ω_- of $\mathbb{R}^n$ such that*

$$\Omega_+ \cap B_{r_0}(0) = \Omega_- \cap B_{r_0}(0) = \omega \cap B_{r_0}(0), \tag{9.1}$$

$$\min\{\gamma_H(\Omega_+), \gamma_H(\Omega_-)\} > \frac{n^2-1}{4}, \tag{9.2}$$

and

$$m_\gamma(\Omega_+) > 0 > m_\gamma(\Omega_-), \tag{9.3}$$

whenever $\frac{n^2-1}{4} < \gamma < \min\{\gamma_H(\Omega_+), \gamma_H(\Omega_-)\}$.

We shall need the following stability result for the mass under continuous deformations and truncations.

Proposition 9.2. *Let $\Omega \subset \mathbb{R}^n$ be a conformally bounded domain such that $0 \in \partial\Omega$. Assume that $\gamma_H(\Omega) > \frac{n^2-1}{4}$ and fix $\gamma \in \left(\frac{n^2-1}{4}, \gamma_H(\Omega)\right)$. For any $R > 0$, let D_R be a smooth domain of $\mathbb{R}^n$ such that*

- *$B_R(x_0) \subset D_R \subset B_{2R}(x_0)$,*
- *$\Omega \cap D_R$ is a smooth domain of $\mathbb{R}^n$.*

Let $\Phi \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ be such that

- *$\Phi_t := \Phi(t, \cdot)$ is a smooth diffeomorphism of $\mathbb{R}^n$,*
- *$\Phi_t(x) = x$ for all $|x| > 1/2$ and all $t \in \mathbb{R}$,*
- *$\Phi_t(0) = 0$ for all $t \in \mathbb{R}$,*
- *$\Phi_0 = Id_{\mathbb{R}^n}$.*

Set $\Omega_{t,R} := \Phi_t(\Omega) \cap D_R$. Then as $t \to 0$, $R \to +\infty$, we have that $\gamma_H(\Omega_{t,R}) > \frac{n^2-1}{4}$ and $m_\gamma(\Omega_{t,R})$ is well defined. In addition,

$$\lim_{t\to 0,\, R\to+\infty} m_\gamma(\Omega_{t,R}) = m_\gamma(\Omega).$$

As a preliminary remark, we claim that if Ω is a conformally bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$, then

$$\liminf_{t\to 0, R\to\infty} \gamma_H(\Omega_{t,R}) \geq \gamma_H(\Omega), \tag{9.4}$$

where $\Omega_{t,R}$ are defined as in Proposition 9.2. Indeed, by definition, $\gamma_H(\Omega_{t,R}) \geq \gamma_H(\Omega_t) = \gamma_H(\Phi_t(\Omega))$. Inequality (9.4) then follows from (3.7) of Lemma 3.2.

We shall use the same approach as in the proof of Proposition 7.4. Assuming that $x_0 := (-1, 0, \dots, 0) \in \mathbb{R}^n$, and denoting the corresponding Kelvin inversion by i, this transformation allows to map the operator $-\Delta - \frac{\gamma}{|x|^2}$ on a conformally bounded domain Ω into the Schrödinger operator $-\Delta + V$ on the bounded domain $\tilde{\Omega}$, where V is the potential defined in (7.19).

Set now $\tilde{\Omega} := i(\Omega)$, $\tilde{\Phi}(t,x) := i \circ \Phi(t, i(x))$ for $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, and the complement $\tilde{D}_r := \mathbb{R}^n \setminus i(D_{r^{-1}})$ in $\mathbb{R}^n$. Observe that $R \to +\infty$ in Proposition 9.2 is equivalent to $r \to 0$ in here. Note that $\tilde{\Phi} \in C^\infty(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$ is such that

- For any $t \in (-2, 2)$, $\tilde{\Phi}_t := \tilde{\Phi}(t, \cdot)$ is a $C^\infty-$diffeomorphism onto its open image $\tilde{\Phi}_t(\mathbb{R}^n)$.
- $\tilde{\Phi}_0 = \mathrm{Id}$,
- $\tilde{\Phi}_t(0) = 0$ for all $t \in (-2, 2)$,
- $\tilde{\Phi}_t(x) = x$ for all $t \in (-2, 2)$ and all $x \in B_{2\delta}(x_0)$ with $\delta < 1/4$.

Set $\tilde{\Omega}_t := \tilde{\Phi}_t(\tilde{\Omega})$ and note that the sets $\tilde{D}_r$ satisfy the following properties:

- $B_{r/2}(x_0) \subset \tilde{D}_r \subset B_r(x_0)$,
- $\tilde{\Omega}_{t,r} := \tilde{\Omega}_t \setminus \tilde{D}_r$ is a smooth domain of $\mathbb{R}^n$.

In particular, we have that $\tilde{\Omega}_{t,r} = i(\Omega_{t,r^{-1}})$. Let $u \in C^2(\overline{\Omega_{t,r}} \setminus \{0\})$ be such that

$$-\Delta u - \frac{\gamma}{|x|^2} u = 0 \text{ in } \Omega_{t,r}\ ,\ u > 0 \text{ in } \Omega_{t,r}\ ,\ u = 0 \text{ on } \partial\Omega_{t,r}.$$

We shall need the following.

Lemma 9.3. *For any $t \in (-1,1)$, there exists $u_t \in C^2(\overline{\tilde{\Omega}_t} \setminus \{0, x_0\})$ such that*

$$\left\{\begin{array}{lll} -\Delta u_t - V u_t & = 0 & \text{in } \tilde{\Omega}_t \\ u_t & > 0 & \text{in } \tilde{\Omega}_t \\ u_t & = 0 & \text{on } \partial\tilde{\Omega}_t \setminus \{0, x_0\} \\ u_t(x) & \leq C|x|^{1-\alpha_+(\gamma)} + C|x - x_0|^{1-\alpha_-(\gamma)} & \text{for } x \in \tilde{\Omega}_t. \end{array}\right. \tag{9.5}$$

Moreover, we have that

$$u_t(x) = \frac{d(x, \partial\tilde{\Omega}_t)}{|x|^{\alpha_+(\gamma)}}(1 + O(|x|^{\alpha_+(\gamma) - \alpha_-(\gamma)})) \tag{9.6}$$

as $x \to 0$, uniformly with respect to $t \in (-1,1)$.

Proof of Lemma 9.3. We construct approximate singular solutions as in Section 4. For all $t \in (-2,2)$, there exists a chart φ_t that satisfies (4.7) to (4.12) for $\tilde{\Omega}_t$. Without restriction, we assume that $\lim_{t\to 0} \varphi_t = \varphi_0$ in $C^k(\tilde{B}_{2\delta}, \mathbb{R}^n)$. We define a cut-off function η_δ such that $\eta_\delta(x) = 1$ for $x \in \tilde{B}_\delta$ and $\eta_\delta(x) = 0$ for $x \notin \tilde{B}_{2\delta}$. As in (4.14), we define $u_{\alpha_+(\gamma),t} \in C^2(\overline{\tilde{\Omega}_t} \setminus \{0\})$ with compact support in $\varphi_t(\tilde{B}_{2\delta})$ such that

$$u_{\alpha_+,t} \circ \varphi_t(x_1, x') := \eta_\delta(x_1, x') x_1 |x|^{-\alpha_+}(1 + \Theta_t(x)) \text{ for all } (x_1, x') \in \tilde{B}_{2\delta} \setminus \{0\}, \tag{9.7}$$

where $\Theta_t(x_1, x') := e^{-\frac{1}{2}x_1 H_t(x')} - 1$ for all $x = (x_1, x') \in \tilde{B}_{2\delta}$ and all $t \in (-2,2)$. Here, $H_t(x')$ is the mean curvature of $\partial\tilde{\Omega}_t$ at the point $\varphi_t(0, x')$. Note that $\lim_{t\to 0} \Theta_t = \Theta_0$ in $C^k(U)$. Arguing as is Section 4, we get that

$$\left\{\begin{array}{lll} (-\Delta - V) u_{\alpha_+,t} & = O(d(x, \partial\tilde{\Omega}_t)|x|^{-\alpha_+(\gamma)-1}) & \text{in } \tilde{\Omega}_t \cap \tilde{B}_\delta \\ u_{\alpha_+,t} & > 0 & \text{in } \tilde{\Omega}_t \cap \tilde{B}_\delta \\ u_{\alpha_+,t} & = 0 & \text{on } \partial\tilde{\Omega}_t \setminus \{0\}, \end{array}\right.$$

and

$$u_{\alpha_+,t}(x) = \frac{d(x, \partial\tilde{\Omega}_t)}{|x|^{\alpha_+(\gamma)}}(1 + O(|x|) \text{ as } x \to 0.$$

The construction in Section 4 also yields

$$\lim_{t\to 0} u_{\alpha_+,t} \circ \Phi_t = u_{\alpha_+,0} \text{ in } C^2_{loc}(\overline{\tilde{\Omega}} \setminus \{0\}). \tag{9.8}$$

Note also that all these estimates are uniform in $t \in (-1,1)$. In particular, defining

$$f_t := -\Delta u_{\alpha_+,t} - V u_{\alpha_+,t}, \tag{9.9}$$

then there exists $C > 0$ such that

$$|f_t(x)| \leq C d(x, \partial\tilde{\Omega}_t)|x|^{-\alpha_+(\gamma)-1} \leq C|x|^{-\alpha_+(\gamma)} \tag{9.10}$$

for all $t \in (-1,1)$ and all $x \in \tilde{\Omega}_t \cap \tilde{B}_\delta$. Therefore, since $\gamma > \frac{n^2-1}{4}$, it follows from (9.8) and this pointwise control that $f_t \in L^{\frac{2n}{n+2}}(\tilde{\Omega}_t)$ for all $t \in (-1,1)$ and that

$$\lim_{t\to 0} \|f_t \circ \Phi_t - f_0\|_{L^{\frac{2n}{n+2}}(\tilde{\Omega})} = 0. \tag{9.11}$$

For any $t \in (-1,1)$, we let $v_t \in D^{1,2}(\tilde{\Omega}_t)$ be such that

$$-\Delta v_t - V v_t = f_t \text{ weakly in } D^{1,2}(\tilde{\Omega}_t).$$

The existence follows from the coercivity of $-\Delta - V$ on $\tilde{\Omega}_t$, which follows itself from the coercivity on $\tilde{\Omega} = \tilde{\Omega}_0$. We then get from (9.11) and the uniform coercivity on $\tilde{\Omega}_t$ that

$$\lim_{t\to 0} v_t \circ \Phi_t = v_0 \text{ in } D^{1,2}(\tilde{\Omega}) \text{ and } C^1_{loc}(\overline{\tilde{\Omega}} \setminus \{0, x_0\}).$$

It follows from the construction of the mass in Section 7 (see the proof of Theorem 7.1) that around 0, $|v_t(x)|$ is bounded by $|x|^{1-\alpha_-(\gamma)}$. Around x_0, $-\Delta v_t - V v_t = 0$ and the regularity Theorem 4.1 yields a control by $|x-x_0|^{1-\alpha_-(\gamma)}$. These controls are uniform with respect to $t \in (-1,1)$. Therefore, there exists $C > 0$ such that

$$|v_t(x)| \leq C d(x, \partial\tilde{\Omega}_t) \left(|x|^{-\alpha_-(\gamma)} + |x - x_0|^{-\alpha_-(\gamma)}\right)$$

for all $t \in (-1,1)$ and all $x \in \tilde{\Omega}_t$. Now define $u_t(x) := u_{\alpha_+,t}(x) - v_t(x)$ for all $t \in (-1,1)$ and $x \in \tilde{\Omega}_t$. This function satisfies all the requirements of Lemma 9.3. $\square$

Proof of Proposition 9.2: Let $\tilde{\Omega}_{t,r} = \tilde{\Omega}_t \setminus \tilde{D}_r$, and note that for $r \in (0, \delta/2)$, we have $\tilde{\Omega}_{t,r} \cap B_\delta(0) = \tilde{\Omega} \cap B_\delta(0)$. We shall define a mass associated to the potential V as in Proposition 7.4 and prove its continuity.

Step 1: The function $f_t : \tilde{\Omega}_t \to \mathbb{R}$ defined in (9.9) has compact support in $B_{2\delta}(0)$, therefore, it is well-defined also on $\tilde{\Omega}_{t,r}$. Let $v_{t,r} \in D^{1,2}(\tilde{\Omega}_{t,r})$ be such that

$$-\Delta v_{t,r} - V v_{t,r} = f_t \text{ weakly in } D^{1,2}(\tilde{\Omega}_{t,r}). \tag{9.12}$$

Since the operator $-\Delta - V$ is uniformly coercive on $\tilde{\Omega}_t$, it is also uniformly coercive on $\tilde{\Omega}_{t,r}$ with respect to (t,r), so that the definition of $v_{t,r}$ via (9.12) makes sense. The uniform coercivity and (9.9)-(9.10) yield the existence of $C > 0$ such that $\|v_{t,r}\|_{D^{1,2}(\tilde{\Omega}_{t,r})} \leq C$ for all t, r. Since $x_0 \notin \tilde{\Omega}_{t,r}$, (9.9)-(9.10) and regularity theory yield $v_{t,r} \in C^1(\overline{\tilde{\Omega}_{t,r}} \setminus \{0\})$ and for all $\rho > 0$, there exists $C(\rho) > 0$ independent of t and r such that

$$\|v_{t,r}\|_{C^1(\tilde{\Omega}_{t,r} \setminus (B_\rho(0) \cup B_\rho(x_0)))} \leq C(\rho). \tag{9.13}$$

Step 2: There exists $C > 0$ such that for all $t \in (-1,1)$ and all $x \in \tilde{\Omega}_{t,r}$,

$$|v_{t,r}(x)| \leq C d(x, \partial\Omega_t) \left(|x|^{-\alpha_-(\gamma)} + |x - x_0|^{-\alpha_-(\gamma)}\right). \tag{9.14}$$

Indeed, around 0, $\tilde{\Omega}_{t,r}$ coincides with $\tilde{\Omega}_t$, and the proof of the control goes as in the construction of the mass in Section 7 (see the proof of Proposition 7.1). The argument is different around x_0. We let $r_0 > 0$ be such that $\tilde{\Omega}_t \cap B_{2r_0}(x_0) = \tilde{\Omega} \cap B_{2r_0}(x_0)$. Therefore, for $r \in (0, r_0)$, we have that

$$\tilde{\Omega}_{t,r} \cap B_{2r_0}(x_0) = (\tilde{\Omega} \setminus \tilde{D}_r) \cap B_{2r_0}(x_0).$$

Arguing as in the proof of Proposition 4.3, there exists $\tilde{u}_{\alpha_-} \in C^\infty(\overline{\tilde{\Omega}} \setminus \{0\})$ and $\tau' > 0$ such that

$$\left\{\begin{array}{rll} \tilde{u}_{\alpha_-} & > 0 & \text{in } \tilde{\Omega} \cap B_{2r_0}(x_0) \\ \tilde{u}_{\alpha_-} & = 0 & \text{in } (\partial\tilde{\Omega}) \cap B_{2r_0}(x_0) \\ -\Delta \tilde{u}_{\alpha_-} - V \tilde{u}_{\alpha_-} & > 0 & \text{in } \tilde{\Omega} \cap B_{2r_0}(x_0). \end{array}\right.$$

Moreover, we have that

$$\tilde{u}_{\alpha_-}(x) = \frac{d(x, \partial\tilde{\Omega})}{|x - x_0|^{\alpha_-}} (1 + O(|x - x_0|)) \quad \text{as } x \to x_0,\ x \in \tilde{\Omega}. \tag{9.15}$$

Therefore, since $v_{t,r}$ vanishes on $B_{2r_0}(x_0)\cap\partial(\tilde{\Omega}\setminus\tilde{D}_r)$, it follows from (9.13) and the properties of $\tilde{u}_{\alpha_-}$ that there exists $C>0$ such that $v_{t,r}\leq C\tilde{u}_{\alpha_-}$ on the boundary of $(\tilde{\Omega}\cap\tilde{D}_r)\cap B_{2r_0}(x_0)$. Since in addition $(-\Delta-V)v_{t,r}=0<(-\Delta-V)(C\tilde{u}_{\alpha_-})$, it follows from the comparison principle that $v_{t,r}\leq C\tilde{u}_{\alpha_-}$ in $(\tilde{\Omega}\setminus\tilde{D}_r)\cap B_{2r_0}(x_0)$. Arguing similarly with $-v_{t,r}$ and using the asymptotic (9.15), we get (9.14).

Step 3: We have that

$$\lim_{t,r\to 0} v_{t,r}\circ\Phi_t=v_0 \text{ in } D^{1,2}(\tilde{\Omega})_{loc,\{x_0\}^c}\cap C^1_{loc}(\overline{\tilde{\Omega}}\setminus\{0,x_0\}), \tag{9.16}$$

where v_0 was defined in (7.20), and the convergence in $D^{1,2}(\tilde{\Omega})_{loc,\{x_0\}^c}$ means that $\lim_{t,r\to 0}\eta v_{t,r}\circ\Phi_t=\eta v_0$ in $D^{1,2}(\tilde{\Omega})$ for all $\eta\in C^\infty(\mathbb{R}^n)$ vanishing around x_0. Indeed, $v_{t,r}\circ\Phi_t\in D^{1,2}(\tilde{\Omega}\setminus\tilde{D}_r)\subset D^{1,2}(\tilde{\Omega})$. Uniform coercivity yields weak convergence in $D^{1,2}(\tilde{\Omega})$ to $\tilde{v}\in D^{1,2}(\tilde{\Omega})$. Passing to the limit, one gets $(-\Delta-V)\tilde{v}=f_0$, so that $\tilde{v}=v_0$. Uniqueness then yields convergence in $C^1_{loc}(\overline{\tilde{\Omega}}\setminus\{0,x_0\})$. With a change of variable, equation (9.12) yields an elliptic equation for $v_{t,r}\circ\Phi_t$. Multiplying this equation by $\eta^2\cdot(v_{t,r}\circ\Phi_t-v_0)$ for $\eta\in C^\infty(\mathbb{R}^n)$ vanishing around x_0, one gets convergence of $\eta v_{t,r}\circ\Phi_t$ to ηv_0 in $D^{1,2}(\tilde{\Omega})$. This proves the claim.

It follows from the construction of the mass (see Theorem 7.1) and the regularity Theorem 4.1, that there exists $K_0\in\mathbb{R}$ and for all (t,r) small, there exists $K_{t,r}\in\mathbb{R}$ such that

$$v_{t,r}(x)=K_{t,r}\frac{d(x,\partial\tilde{\Omega}_t)}{|x|^{\alpha_-(\gamma)}}+o\left(\frac{d(x,\partial\tilde{\Omega}_t)}{|x|^{\alpha_-(\gamma)}}\right) \text{ and } v_0(x)=K_0\frac{d(x,\partial\tilde{\Omega})}{|x|^{\alpha_-(\gamma)}}+o\left(\frac{d(x,\partial\tilde{\Omega})}{|x|^{\alpha_-(\gamma)}}\right) \tag{9.17}$$

as $x\in\tilde{\Omega}$ goes to 0. Note that around 0, $\tilde{\Omega}_{t,r}$ coincides with $\tilde{\Omega}_t$.

Step 4: We claim that

$$\lim_{t,r\to 0}K_{t,r}=K_0. \tag{9.18}$$

We only give a sketch. Noting $\tilde{v}_{t,r}:=v_{t,r}\circ\Phi_t$, the proof relies on (9.16) and the fact that

$$-\Delta_{\Phi_t^\star\mathrm{Eucl}}\tilde{v}_{t,r}-V\circ\Phi_t\tilde{v}_{t,r}=f_t\circ\Phi_t \text{ in } \tilde{\Omega}\cap B_\delta(0).$$

The comparison principle and the definitions (9.17) then yield (9.18).

Note that

$$m_\gamma(\Omega)=-K_0, \tag{9.19}$$

where the mass of a conformally bounded Ω is defined as in Proposition 7.4.

Step 5: convergence of the mass: We claim that

$$\lim_{t\to 0,R\to\infty}m_\gamma(\Omega_{t,R})=m_\gamma(\Omega). \tag{9.20}$$

We define $\tilde{H}_{t,r}:=u_{\alpha_+,t}-v_{t,r}$ so that

$$-\Delta\tilde{H}_{t,r}-V\tilde{H}_{t,r}=0 \text{ in } \tilde{\Omega}_{t,r}.$$

It follows from (9.6) and (9.17) that $\tilde{H}_{t,r}>0$ around 0. From the maximum principle, we deduce that $\tilde{H}_{t,r}>0$ on $\tilde{\Omega}_{t,r}$ and that it vanishes on $\partial\tilde{\Omega}_{t,r}\setminus\{0,x_0\}$.

It follows from (9.6) and (9.17) that

$$\tilde{H}_{t,r}(x)=\frac{d(x,\partial\tilde{\Omega}_{t,r})}{|x|^{\alpha_+}}-K_{t,r}\frac{d(x,\partial\tilde{\Omega}_{t,r})}{|x|^{\alpha_-}}+o\left(\frac{d(x,\partial\tilde{\Omega}_{t,r})}{|x|^{\alpha_-}}\right)$$

as $x \to 0$, $x \in \tilde{\Omega}_{t,r}$. Coming back to $\Omega_{t,R}$ with $R = r^{-1}$ via the inversion i with $H_{t,R}(x) := |x-x_0|^{2-n}\tilde{H}_{t,r}(i(x))$ for all $x \in \Omega_{t,R}$, we get that

$$\left\{\begin{array}{rll} -\Delta H_{t,R} - \frac{\gamma}{|x|^2}H_{t,R} & = 0 & \text{in } \Omega_{t,R}\\ H_{t,R} & > 0 & \text{in } \Omega_{t,R}\\ H_{t,R} & = 0 & \text{in } \partial\Omega_{t,R}\setminus\{0\}\end{array}\right.$$

and

$$H_{t,R}(x) = \frac{d(x,\partial\Omega_{t,R})}{|x|^{\alpha_+}} - K_{t,r}\frac{d(x,\partial\Omega_{t,R})}{|x|^{\alpha_-}} + o\left(\frac{d(x,\partial\Omega_{t,R})}{|x|^{\alpha_-}}\right)$$

as $x \to 0$, $x \in \Omega_{t,R}$. Therefore, it follows from the definition of the mass (see Theorem 7.1) that $m_\gamma(\Omega_{t,R}) = -K_{t,r}$ for all t, r, $R = r^{-1}$. Claim (9.20) then follows from (9.18) and (9.19). □

In order to prove Theorem 9.1, we need to exhibit prototypes of unbounded domains with either positive or negative mass.

Proposition 9.4. *Let Ω be a domain such that $0 \in \partial\Omega$ and Ω conformally bounded. Assume that $\gamma_H(\Omega) > \frac{n^2-1}{4}$ and fix $\gamma \in \left(\frac{n^2-1}{4}, \gamma_H(\Omega)\right)$. Then $m_\gamma(\Omega) > 0$ if $\mathbb{R}^n_+ \subsetneq \Omega$, and $m_\gamma(\Omega) < 0$ if $\Omega \subsetneq \mathbb{R}^n_+$.*

Proof of Proposition 9.4 : With H_0 defined as in (7.22), we set

$$\mathcal{U}(x) := H_0(x) - x_1|x|^{-\alpha_+} \text{ for all } x \in \Omega.$$

We first assume that $\mathbb{R}^n_+ \subsetneq \Omega$. We then have that

$$\left\{\begin{array}{ll} -\Delta\mathcal{U} - \frac{\gamma}{|x|^2}\mathcal{U} = 0 & \text{in } \mathbb{R}^n_+\\ \mathcal{U} \gneqq 0 & \text{in } \partial\mathbb{R}^n_+\setminus\{0\}.\end{array}\right. \tag{9.21}$$

We claim that

$$\int_{\mathbb{R}^n_+} |\nabla\mathcal{U}|^2\,dx < +\infty. \tag{9.22}$$

Indeed, at infinity, this is the consequence of the fact that $|\nabla\mathcal{U}|(x) \leq C|x|^{-\alpha_+}$ for all $x \in \mathbb{R}^n_+$ large, this latest bound being a consequence of (7.25) combined with elliptic regularity theory. At zero, the argument is different. Indeed, one first notes that $d(x,\partial\Omega') = x_1 + O(|x|^2)$ for $x \in \mathbb{R}^n_+$ close to 0, and therefore, $\mathcal{U}(x) = O(|x|^{1-\alpha_-})$ for $x \to 0$. The control on the gradient $|\nabla\mathcal{U}|(x) \leq C|x|^{-\alpha_-}$ at 0 follows from the construction of $\tilde{H}_0$. This yields integrability at 0 and proves (9.22).

We claim that $\mathcal{U} > 0$ in $\mathbb{R}^n_+$. Indeed, it follows from (9.21) and (9.22) that $\mathcal{U}_- \in D^{1,2}(\mathbb{R}^n_+)$. Multiplying equation (7.23) by $\mathcal{U}_-$, integrating by parts on $(B_R(0)\setminus B_\epsilon(0))\cap\mathbb{R}^n_+$, and letting $\epsilon \to 0$ and $R \to +\infty$ by using (9.22), one gets $\mathcal{U}_- \equiv 0$, and then $\mathcal{U} \geq 0$. The result follows from Hopf's maximum principle.

We now claim that

$$m_\gamma(\Omega) > 0. \tag{9.23}$$

Indeed, since $\mathcal{U} > 0$ in $\mathbb{R}^n_+$, there exists $c_0 > 0$ such that $\mathcal{U}(x) \geq c_0 x_1|x|^{-\alpha_-}$ for all $x \in \partial(B_1(0)_+)$. It then follows from (9.22), (9.21) and the comparison principle that $\mathcal{U}(x) \geq c_0 x_1|x|^{-\alpha_-}$ for all $x \in B_1(0)_+$. The expansion (7.24) then yields $-K_0 \geq c_0 > 0$. This combined with (9.19) proves the claim.

When $\Omega \subset \mathbb{R}^n_+$, the argument is similar except that one works on Ω (and not $\mathbb{R}^n_+$) and that $\mathcal{U} \lneqq 0$ in $\partial\Omega\setminus\{0\}$. This ends the proof of Proposition 9.4.

Proof of Theorem 9.1: Let ω be a smooth domain of $\mathbb{R}^n$ such that $0\in\partial\Omega$. Up to a rotation, there exists $\varphi\in C^\infty(\mathbb{R}^{n-1})$ such that $\varphi(0)=0$, $\nabla\varphi(0)=0$ and there exists $\delta_0>0$ such that

$$\omega\cap B_{\delta_0}(0)=\{x_1>\varphi(x')/\,(x_1,x')\in B_{\delta_0}(0)\}.$$

Let $\eta\in C_c^\infty(B_{\delta_0}(0))$ be such that $\eta(x)=1$ for all $x\in B_{\delta_0/2}(0)$, and define

$$\Phi_t(x):=\left(x_1+\eta(x)\frac{\varphi(tx')}{t},x'\right)\text{ for all }t>0\text{ and }x\in\mathbb{R}^n,$$

and $\Phi_0:=Id_{\mathbb{R}^n}$. It is easy to see that Φ_t satisfies the hypotheses of Proposition 9.2. Moreover, for $0<t<1$, we have that

$$\frac{\omega}{t}\cap\Phi_t(B_{\delta_0/2}(0))=\Phi_t(\mathbb{R}^n_+\cap B_{\delta_0/2}(0)).$$

We let Ω be a smooth domain at infinity such that

$$\Omega\cap B_1(0)=\mathbb{R}^n_+\cap B_1(0)\text{ and }\gamma_H(\Omega)>\frac{n^2-1}{4},\tag{9.24}$$

(for example, $\mathbb{R}^n_+$), and let $\Omega_{t,R}$ be as in Proposition 9.2. It is easy to see that

$$\omega\cap t\Phi_t(B_{\delta_0/2}(0))=t\Omega_{t,R}\cap t\Phi_t(B_{\delta_0/2}(0)).$$

Therefore, for $t>0$ small enough, we have that

$$\omega\cap B_{t\delta_0/3}(0)=t\Omega_{t,R}\cap B_{t\delta_0/3}(0).$$

Moreover, $\gamma_H(t\Omega_{t,R})=\gamma_H(\Omega_{t,R})>(n^2-1)/4$ as $t\to0$ and $R\to+\infty$ (see (9.4)). Concerning the mass, we have that

$$t^{\alpha_+(\gamma)-\alpha_-(\gamma)}m_\gamma(t\Omega_{t,R})=m_\gamma(\Omega_{t,R})\to m_\gamma(\Omega)\text{ as }t\to0,R\to+\infty.$$

We now choose Ω appropriately.
To get a negative mass, we choose Ω smooth at infinity such that $\Omega\cap B_1(0)=\mathbb{R}^n_+\cap B_1(0)$ and $\Omega\subsetneq\mathbb{R}^n_+$. Then $\gamma_H(\Omega)=n^2/4$, (9.24) holds and Proposition 9.4 yields $m_\gamma(\Omega)<0$. With this choice of Ω, we take $\Omega_-:=\Omega_{t,R}$ for t small and R large.

To get a positive mass, we choose $\mathbb{R}^n_+\subsetneq\Omega$ such that (9.24) holds (this is possible for any value of $\gamma_H(\Omega)$ arbitrarily close to $\frac{n^2}{4}$, see point (6) of Proposition 3.1). Then Proposition 9.4 yields $m_\gamma(\Omega)>0$. With this choice of Ω, we take $\Omega_+:=\Omega_{t,R}$ for t small and R large. This proves Proposition 9.1.

10. The Hardy singular interior mass and the remaining cases

The remaining situation not covered by Proposition 8.1 and Theorem 8.2 is $s=0$, $n=3$ and $\gamma\in(0,\frac{n^2}{4})$. If $\gamma\geq\gamma_H(\Omega)$, then Proposition 3.3 and Theorem 3.6 yield $\mu_{\gamma,0}(\Omega)\leq0<\mu_{\gamma,0}(\mathbb{R}^n_+)$ and the existence of extremals is guaranteed. When $\mu_{\gamma,0}(\mathbb{R}^n_+)$ does have an extremal U, then Proposition 8.3 and Theorem 3.6 provide sufficient conditions for the existence of extremals. The rest of this section addresses the remaining case, that is when $\gamma\in(0,\gamma_H(\Omega))$ and when $\mu_{\gamma,0}(\mathbb{R}^n_+)$ has no extremal, and therefore $\mu_{\gamma,0}(\mathbb{R}^3_+)=K(3,2)^{-2}$ according to Theorem 1.3.

We first define the "interior" mass in the spirit of Schoen-Yau [27].

Proposition 10.1. *Let $\Omega\subset\mathbb{R}^3$ be an open smooth bounded domain such that $0\in\partial\Omega$. Fix $x_0\in\Omega$. If $\gamma\in(0,\gamma_H(\Omega))$, then the equation*

$$\left\{\begin{array}{rl}-\Delta G-\frac{\gamma}{|x|^2}G=0 & \text{in }\Omega\setminus\{x_0\}\\ G>0 & \text{in }\Omega\setminus\{x_0\}\\ G=0 & \text{on }\partial\Omega\setminus\{0\}\end{array}\right.$$

has a solution $G \in C^2(\overline{\Omega} \setminus \{0, x_0\}) \cap D_1^2(\Omega \setminus \{x_0\})_{loc,0}$, that is unique up to multiplication by a constant. Moreover, for any $x_0 \in \Omega$, there exists a unique $R_\gamma(x_0) \in \mathbb{R}$ independent of the choice of G and $c_G > 0$ such that

$$G(x) = c_G \left(\frac{1}{|x - x_0|} + R_\gamma(x_0) \right) + o(1) \text{ as } x \to x_0.$$

Proof of Proposition 10.1. Since $\gamma < \gamma_H(\Omega)$, the operator $-\Delta - \gamma|x|^{-2}$ is coercive and we can consider G to be its Green's function at x_0 on Ω with Dirichlet boundary condition. In particular, for any $\varphi \in C_c^\infty(\Omega)$, we have that

$$\varphi(x) = \int_\Omega G_x(y) \left(-\Delta\varphi(y) - \gamma \frac{\varphi(y)}{|y|^2} \right) dy \quad \text{for } x \in \Omega,$$

where $G_x := G(x, \cdot)$. Fix $x_0 \in \Omega$ and let $\eta \in C_c^\infty(\Omega)$ be such that $\eta(x) = 1$ around x_0. Define the distribution $\beta_{x_0} : \Omega \to \mathbb{R}$ as

$$G_{x_0}(x) = \frac{1}{\omega_2} \left(\frac{\eta(x)}{|x-x_0|} + \beta_{x_0}(x) \right) \quad \text{for all } x \in \Omega,$$

where $\omega_2 := 4\pi$ is the volume of the canonical $2-$sphere. As one checks,

$$\left(-\Delta - \frac{\gamma}{|x|^2} \right) \beta_{x_0} = -\left(-\Delta - \frac{\gamma}{|x|^2} \right) \left(\frac{\eta(x)}{|x - x_0|} \right) := f = O(|x - x_0|^{-1})$$

in the distributional sense. Since $f \in L^2(\Omega)$ and, by uniqueness of the Green's function (since the operator is coercive), we have that $\beta_{x_0} \in D^{1,2}(\Omega)$. It follows from standard elliptic theory that $\beta_{x_0} \in C^\infty(\overline{\Omega} \setminus \{0, x_0\}) \cap C^{0,\theta}(\overline{\Omega} \setminus B_\delta(0))$ for all $\theta \in (0, 1)$ and $\delta > 0$. Since f vanishes around 0, it follows from Theorem 4.1 and Lemma 5.2 that

$$\beta_{x_0}(x) = O(|x|^{1-\alpha_-(\gamma)}) \quad \text{and} \quad |\nabla\beta_{x_0}(x)| = O(|x|^{-\alpha_-(\gamma)}) \quad \text{when } x \to 0. \tag{10.1}$$

We can therefore define the *mass of* Ω at x_0 associated to the operator L_γ by $R_\gamma(\Omega, x_0) := \beta_{x_0}(x_0)$. As one checks, $\beta_{x_0}(x_0)$ is independent of the choice of η.

The uniqueness is proved as in Theorem 7.1. The behavior on the boundary is given by Theorem 4.1 and the interior behavior around x_0 is classical. □

Lemma 10.2. *Let $\Omega \subset \mathbb{R}^3$ be an open smooth bounded domain such that $0 \in \partial\Omega$ and $x_0 \in \Omega$. Assume that $\gamma \in (0, \gamma_H(\Omega))$ and that $\mu_{\gamma,0}(\mathbb{R}^3_+) = K(3, 2)^{-2}$. Then, there exists a family $(u_\epsilon)_\epsilon$ in $D^{1,2}(\Omega)$ such that*

$$J^\Omega_{\gamma,0}(u_\varepsilon) = \frac{1}{K(n,2)^2} \left(1 - \frac{\omega_2 R_\gamma(x_0)}{3 \int_{\mathbb{R}^3} U^{2^\star} dx} \varepsilon + o(\varepsilon) \right) \text{ as } \varepsilon \to 0, \tag{10.2}$$

where $U(x) := (1 + |x|^2)^{-1/2}$ for all $x \in \mathbb{R}^3$ and $2^\star = 2^\star(0) = \frac{2n}{n-2}$.

Proof of Lemma 10.2: The proof is very similar to what was performed by Schoen [26] (see Druet [9, 10] and Jaber [21]). For $\varepsilon > 0$, define the functions

$$u_\varepsilon(x) := \eta(x) \left(\frac{\varepsilon}{\varepsilon^2+|x-x_0|^2} \right)^{\frac{1}{2}} + \varepsilon^{\frac{1}{2}} \beta_{x_0}(x) \quad \text{for all } x \in \Omega.$$

As one checks, $u_\varepsilon \in D^{1,2}(\Omega)$. Proceeding as in the case $\gamma > \frac{n^2-1}{4}$ of Section 8, we get (10.2). We omit the details that are standard. This proves Lemma 10.2. □

We finally get the following.

Theorem 10.3. *Let Ω be a bounded smooth domain of $\mathbb{R}^3$ such that $0 \in \partial\Omega$.*

(1) *If $\gamma \geq \gamma_H(\Omega)$, then there are extremals for $\mu_{\gamma,0}(\Omega)$.*
(2) *If $\gamma \leq 0$, then there are no extremals for $\mu_{\gamma,0}(\Omega)$.*

(3) *If* $0 < \gamma < \gamma_H(\Omega)$ *and there are extremals for* $\mu_{\gamma,0}(\mathbb{R}^n_+)$*, then there are extremals for* $\mu_{\gamma,0}(\Omega)$ *under either one of the following conditions:*
- $\gamma \leq \frac{n^2-1}{4}$ *and the mean curvature of* $\partial\Omega$ *at* 0 *is negative.*
- $\gamma > \frac{n^2-1}{4}$ *and the mass* $m_\gamma(\Omega)$ *is positive.*

(4) *If* $0 < \gamma < \gamma_H(\Omega)$ *and there are no extremals for* $\mu_{\gamma,0}(\mathbb{R}^n_+)$*, then there are extremals for* $\mu_{\gamma,0}(\Omega)$ *if there exists* $x_0 \in \Omega$ *such that* $R_\gamma(\Omega, x_0) > 0$.

Proof of Theorem 10.3: The two first points of the theorem follow from Proposition 8.1 and Theorem 3.6. The third point follows from Proposition 8.3. For the fourth point, in this situation, it follows from Theorem 1.3 that $\mu_{\gamma,0}(\mathbb{R}^n_+) = \frac{1}{K(n,2)^2}$, and then Lemma 10.2 yields $\mu_{\gamma,0}(\Omega) < \mu_{\gamma,0}(\mathbb{R}^n_+)$, which yields the existence of extremals by Theorem 3.6. This proves Theorem 10.3. □

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Nassif Ghoussoub, Department of Mathematics, 1984 Mathematics Road, The University of British Columbia, BC, Canada V6T 1Z2
E-mail address: nassif@math.ubc.ca

Frédéric Robert, Institut Élie Cartan, Université de Lorraine, BP 70239, F-54506 Vandœuvre-lès-Nancy, France
E-mail address: frederic.robert@univ-lorraine.fr